Granular Matter (2009) 11:129–137
DOI 10.1007/s10035-008-0124-0

Non-coaxial version of Rowe's stress-dilatancy relation

Marte Gutierrez · Jianfeng Wang

Received: 11 September 2007 / Published online: 23 January 2009

Abstract Non-coaxiality occurs when the directions of the principal plastic strain increments and the principal stresses deviate. Extensive experimental data have now conclusively shown that plastic flow in granular soils is non-coaxial particularly during loadings involving rotation of the principal stress directions. One way to integrate the effects of non-coaxiality is by modifying the expressions for energy dissipation and stress-dilatancy used in modeling plastic deformation of granular soils. In this regard, the paper's main objective is to derive a non-coaxial version of Rowe's stress-dilatancy relation, thereby making it more general and applicable to loadings involving principal stress rotation. The paper also applies Rowe's non-coaxial stress-dilatancy equation in the determination of the effects of principal stress rotation in granular soils during simple shear loading conditions. Previous experimental data from simple shear tests on sand are used to validate the proposed non-coaxial version of Rowe's stress-dilatancy relation.

Keywords Energy dissipation · Dilatancy · Flow rule · Non-coaxiality · Simple shear

M. Gutierrez (✉)
Division of Engineering, Colorado School of Mines,
1610 Illinois St., Golden, CO 80401, USA
e-mail: mgutierr@mines.edu

J. Wang
Department of Building and Construction,
City University of Hong Kong,
Kowloon, Hong Kong
e-mail: jefwang@cityu.edu.hk

1 Introduction

Coaxiality, which implies that the directions of the principal plastic strain increments and the principal stresses coincide, is a commonly used assumption in constitutive modeling of geomaterials. Gutierrez and Ishihara [1] showed that conventional plasticity models expressed in terms of the usual stress and plastic strain increment invariants implicitly assume coaxiality in the plastic flow rule. However, extensive experimental data have now conclusively shown that plastic flow in granular soils is non-coaxial particularly during loading involving principal rotation. Since loading conditions in geotechnical engineering invariably involve principal stress rotation, non-coaxiality is an important aspect of constitutive response of soils that needs to be considered. Non-coaxiality has also important implications on the post-localization response of granular soils as shown by Vardoulakis and Georgopolous [2], and Gutierrez and Vardoulakis [3].

The first objective of this paper is to further examine the effects of non-coaxiality on the stress-dilatancy response of granular soils. This is done by deriving a non-coaxial version of the stress-dilatancy relationship developed by Rowe [4]. In this manner, Rowe's stress-dilatancy relationship can be made more general and applicable to loadings involving principal stress rotation. Rowe's stress-dilatancy relation is one of the first rational attempts at characterizing the dilatancy of granular soils and has been used in the formulation of several constitutive models for granular soils [e.g. 5–8]. Another objective of the paper is to provide further experimental justifications to the validity of the approach by [1] in accounting for the effects of non-coaxiality in the stress-dilatancy and energy dissipation of granular soils. To this end, previous experimental data from simple shear tests on sand will be re-analyzed and compared to the non-coaxial version of Rowe's stress-dilatancy relation.

2 Non-coaxiality and energy dissipation in granular materials

Before deriving the non-coaxial version of Rowe's stress-dilatancy relationship, it is worthwhile to first briefly review the effects of non-coaxiality on the energy dissipation of granular soils. Gutierrez and Ishihara [1] have shown that one way to integrate the effects of non-coaxiality in the plastic deformation of granular soils is by modifying the expressions for energy dissipation and stress-dilatancy. Assuming that the elastic strains are negligible, and the total and plastic strain increments are the same (i.e., $\dot{\varepsilon}_{ij} = \dot{\varepsilon}^p_{ij}$), the rate of dissipated energy $\dot{W}$ by a material under applied stress σ_{ij} and subjected to a strain-rate $\dot{\varepsilon}_{ij}$ is calculated as $\dot{W} = \sigma_{ij}\dot{\varepsilon}_{ij}$. In two-dimensional plane-strain conditions ($\dot{\varepsilon}_{zz} = \dot{\varepsilon}_{zx} = \cdots = 0$), the dissipated energy can be written in terms of the components of the stress and strain-rate tensors as:

$$\dot{W} = \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{xx}\dot{\varepsilon}_{xx} + \sigma_{yy}\dot{\varepsilon}_{yy} + 2\sigma_{xy}\dot{\varepsilon}_{xy} \tag{1}$$

Equation (1) is always correct as it calculates the dissipated energy using stress and strain rate tensors that are referred to a common coordinate system. However, this equation is expressed in terms of the stress and strain increment tensors, instead of invariants, and is not easy to use in constitutive modeling.

If α is the orientation of the major principal stress σ_1, and β is the orientation of the major principal strain rate $\dot{\varepsilon}_1$, both referred to the same y-axis, then the following are obtained from the corresponding Mohr-circles for stress and strain increment:

$$\sigma_x = s - t\cos(2\alpha), \quad \sigma_y = s + t\cos(2\alpha), \quad \sigma_{xy} = t\sin(2\alpha) \tag{2}$$

$$\dot{\varepsilon}_x = \frac{1}{2}\dot{v} - \frac{1}{2}\dot{\gamma}\cos(2\beta), \quad \dot{\varepsilon}_y = \frac{1}{2}\dot{v} + \frac{1}{2}\dot{\gamma}\cos(2\beta), \quad \dot{\varepsilon}_{xy} = \frac{1}{2}\dot{\gamma}\sin(2\beta) \tag{3}$$

where

$$s = \frac{1}{2}(\sigma_1 + \sigma_3), \quad t = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{4}$$

$$\dot{v} = \dot{\varepsilon}_1 + \dot{\varepsilon}_3, \quad \dot{\gamma} = \dot{\varepsilon}_1 - \dot{\varepsilon}_3 \tag{5}$$

$$\tan 2\alpha = \frac{2\sigma_{xy}}{\sigma_y - \sigma_x}, \quad \tan 2\beta = \frac{2\dot{\varepsilon}_{xy}}{\dot{\varepsilon}_y - \dot{\varepsilon}_x} \tag{6}$$

Substituting Eqs. (2) and (3) in Eq. (1) yields the following correct expression for the dissipated energy in terms of invariants:

$$\dot{W} = s\dot{v} + ct\dot{\gamma}, \quad c = \cos 2\Delta \tag{7}$$

where Δ is the *non-coaxiality* angle [1] equal to the difference between the principal stress and the principal strain-rate directions:

$$\Delta = |\alpha - \beta| \tag{8}$$

and c is the corresponding Gutierrez–Ishihara *non-coaxiality parameter* [1]. In case of coaxial flow, $\Delta = 0$ and $c = 1.0$, and one obtains the coaxial expression for energy dissipation that has been commonly used in constitutive modeling.

Assuming, as in Critical State Soil Mechanics [9], that the energy dissipation in any state is the same as in critical state (i.e., $t = s \cdot \sin\phi_c$, where ϕ_c is the critical state friction angle, and $\dot{v} = 0$), then

$$\dot{W} = s\dot{v} + c \cdot t\dot{\gamma} = s \cdot \sin\phi_c\dot{\gamma} \tag{9}$$

Equation (7) yields the following stress-dilatancy relationship for two-dimensional loading:

$$\frac{\dot{v}}{\dot{\gamma}} = \sin\phi_c - c\frac{t}{s} \tag{10}$$

In terms of the mobilized dilation angle ψ and the mobilized friction angle ϕ, Eq. (10) can also be written as:

$$\sin\psi = \sin\phi_c - c\sin\phi \tag{11}$$

$$\sin\psi = \frac{\dot{v}}{\dot{\gamma}} = \frac{\dot{\varepsilon}_1 + \dot{\varepsilon}_3}{\dot{\varepsilon}_1 - \dot{\varepsilon}_3}, \quad \sin\phi = \frac{t}{s} = \frac{\sigma_1 - \sigma_3}{\sigma_1 + \sigma_3} \tag{12}$$

In Eq. (11), the volumetric strain increment is contractive when $c\sin\phi < \sin\phi_c$ and dilative when $c\sin\phi > \sin\phi_c$. It should be noted that recent publications on the behavior of sands and sand-structure interfaces [e.g., 10–12] have discussed the issue of the uniqueness of stress-dilatancy relations. The need to introduce a state parameter to account for the stress level and density dependency of the phase transformation and the ultimate state (both characterized by zero dilatancy condition) has been emphasized. For simplicity, this is not considered here but could be easily implemented in the proposed relations.

3 Non-coaxial version of Rowe's stress-dilatancy relation

Rowe [4,13] formulated a stress-dilatancy relationship for granular materials of the following form:

$$\dot{K} = \tan^2\left(\frac{\pi}{4} + \frac{\phi_\mu}{2}\right) \tag{13}$$

where ϕ_μ is the inter-particle friction angle and $\dot{K}$ is the ratio of the input work to output work, which in case of bi-axial loading condition is equal to:

$$\dot{K} = -\frac{\sigma_1\dot{\varepsilon}_1}{\sigma_3\dot{\varepsilon}_3} \tag{14}$$

The work ratio $\dot{K}$ can also be expressed in terms of the principal stress ratio $R = \sigma_1/\sigma_3$ and the strain rate ratio $D = -\dot{\varepsilon}_3/\dot{\varepsilon}_1$ as:

$$\dot{K} = \frac{R}{D} \tag{15}$$

It is noted from Eq. (14) that the direction of shearing is along the fixed direction of $(\pi/4 - \phi_\mu/2)$ from the major principal stress direction.

As pointed out by de Jong [14], Rowe's derivation of Eqs. (13)–(15) is based on a minimum energy principle although Rowe presented no proof as to the validity of this principle. Such a proof is essential because the energy function can be negative for frictional systems. Instead of using an energy principle, de Jong [14] showed the validity of the Rowe's stress-dilatancy relation based on friction law alone. Recently, Niiseki [15] re-derived Rowe's equation based on optimality theory. Niiseki showed that ϕ_μ on the plane of maximum strength mobilization depends not only on mineral surface of particles but also on deformation mechanisms. By relating the internal friction angle to the dilation angle, Niiseki also pointed out that particle movement directions, which are related to the strain rate directions, actually change during strain hardening deformation.

Rowe derived Eqs. (13)–(15) assuming coaxiality of the principal stress and principal strain increment directions. However, as noted by [14], this assumption cannot be validated, and it is not necessary to restrict Rowe's stress-dilatancy relation only to coaxial conditions. To account for non-coaxiality, De Jong [14] derived a version of Rowe's stress dilatancy relation in terms of the modified work ratio $\dot{K}^*$ defined as:

$$\dot{K}^* = \frac{R}{D^*} = -\frac{\sigma_1/\sigma_3}{\dot{\varepsilon}_x/\dot{\varepsilon}_y} = -\frac{\sigma_1\dot{\varepsilon}_y}{\sigma_3\dot{\varepsilon}_x} \tag{16}$$

where $D^* = -\dot{\varepsilon}_x/\dot{\varepsilon}_y$ is the modified strain rate ratio expressed in terms of the strain rates $\dot{\varepsilon}_x$ and $\dot{\varepsilon}_y$ along the x and y axes. The above equation is generally valid since $\dot{\varepsilon}_x$ and $\dot{\varepsilon}_y$ represent strain rates along the σ_1 and σ_3 directions which also coincide with the x and y axes in biaxial loading condition. Although the shear strain rate $\dot{\varepsilon}_{xy}$ on the $x - y$ axis is not zero, no work is done on $\dot{\varepsilon}_{xy}$ since $\sigma_{xy} = 0$ on the principal stress plane. The modified work ratio $\dot{K}^*$ represents the ratio of the work done by the major and minor principal stresses and is valid in case the principal stress and principal strain increment directions deviate. The definition of $\dot{K}^*$ is more general than Rowe's definition of the ratio of input work to output work $\dot{K}$, which assumes the planes of sample deformation do not deviate from the principal stress planes.

Although Eq. (16) is valid in case of non-coaxiality, it cannot be directly used to develop constitutive models since it is expressed in terms of $\dot{\varepsilon}_x$ and $\dot{\varepsilon}_y$, which are not invariant quantities. In the following, an invariant and non-coaxial version of Rowe's stress dilatancy equation will be derived. Using the non-coaxiality angle Δ (Eq. 8), the dissipated energy in 2D conditions (Eq. 1) can also be calculated as:

$$\dot{W} = \sigma_1\dot{\varepsilon}_1\cos^2\Delta + \sigma_1\dot{\varepsilon}_3\sin^2\Delta + \sigma_3\dot{\varepsilon}_3\cos^2\Delta + \sigma_3\dot{\varepsilon}_1\sin^2\Delta \tag{17}$$

The modified work ratio $\dot{K}^*$ can be calculated as:

$$\dot{K}^* = \frac{\sigma_1\dot{\varepsilon}_1\cos^2\Delta - \sigma_1\dot{\varepsilon}_3\sin^2\Delta}{-\sigma_3\dot{\varepsilon}_3\cos^2\Delta + \sigma_3\dot{\varepsilon}_1\sin^2\Delta} \tag{18}$$

Noting that $\dot{\varepsilon}_3$ is negative when $\dot{\varepsilon}_1$ is positive, both the numerator and denominator are positive. From the Mohr's circle of stress and strain increment, one obtains

$$\frac{\sigma_1}{\sigma_3} = \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{19}$$

$$-\frac{\dot{\varepsilon}_3}{\dot{\varepsilon}_1} = \tan^2\left(\frac{\pi}{4} + \frac{\psi}{2}\right) \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (18), yields:

$$K^* = \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\frac{1 + \tan^2\Delta\tan^2\left(\frac{\pi}{4} + \frac{\psi}{2}\right)}{\tan^2\Delta + \tan^2\left(\frac{\pi}{4} + \frac{\psi}{2}\right)} \tag{21}$$

Equation (21) can be transformed into:

$$\begin{aligned} K^* &= \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\frac{1 - \sin\psi^*}{1 + \sin\psi^*} \\ &= \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\Big/\tan^2\left(\frac{\pi}{4} + \frac{\psi^*}{2}\right) \end{aligned} \tag{22}$$

where ψ^* is the nominal dilation angle considering the non-coaxiality effect, and

$$\sin\psi^* = c\sin\psi \tag{23}$$

Using, Niiseki's [15] optimization technique, ϕ can be related to ψ^*, and using the condition that at the critical state, $\phi = \phi_c$ giving $\dot{v} = 0$ and $\psi^* = 0$, gives the following extended non-coaxial version of Rowe's stress-dilatancy:

$$\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \tan\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right)\tan\left(\frac{\pi}{4} + \frac{\psi^*}{2}\right) \tag{24}$$

Equation (24) can also be expressed in the following alternative forms:

$$\sin\phi = \frac{\sin\phi_c + \sin\psi^*}{1 + \sin\phi_c\sin\psi^*} \tag{25}$$

$$\frac{1 + \sin\phi}{1 - \sin\phi} = \left(\frac{1 + \sin\phi_c}{1 - \sin\phi_c}\right)\left(\frac{1 + \sin\psi^*}{1 - \sin\psi^*}\right) \tag{26}$$

$$\sin\psi^* = c\sin\psi = \frac{\sin\phi_c - \sin\phi}{(1 - \sin\phi\sin\phi_c)} \tag{27}$$

Different from the conventional Rowe's equation, the nominal dilation angle $\psi^* = \sin^{-1}(c \cdot \sin\psi)$ given in Eqs. (24)–(27) takes the place of the conventional dilation angle ψ, and is different from the dilation angle ψ due to the non-coaxiality effects since $c < 1.0$ for non-coaxial flow. Figure 1a, b illustrate the effects of non-coaxiality on the

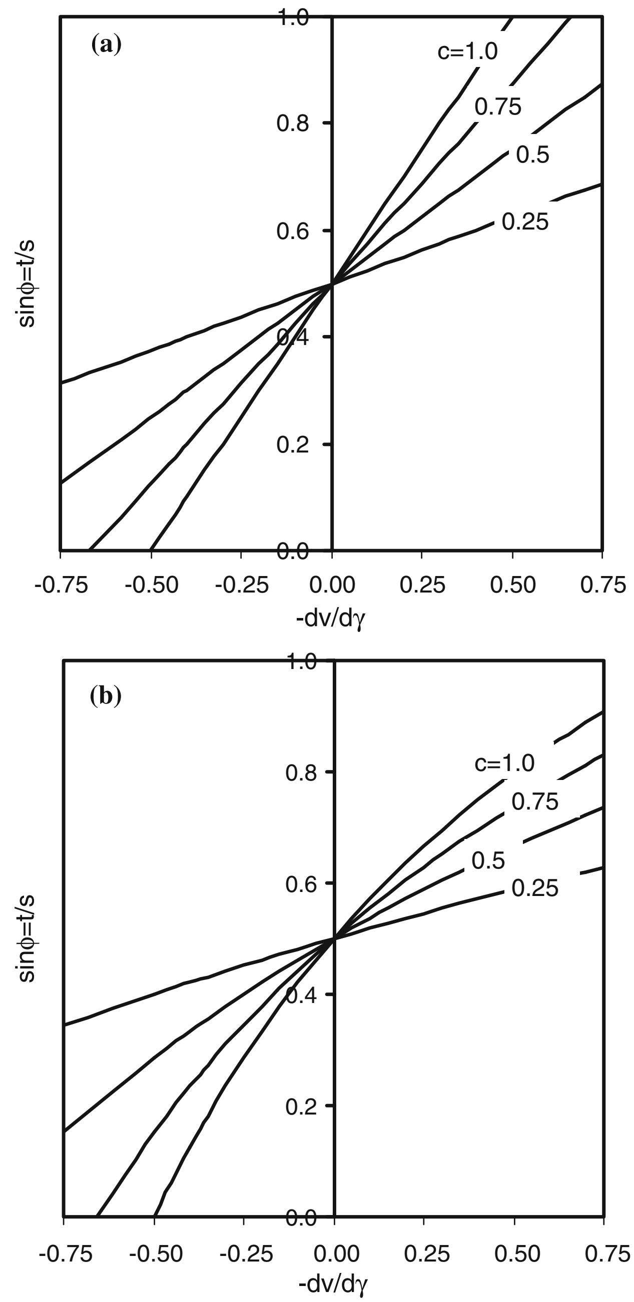

Fig. 1 Stress-dilatancy relationships from **a** Critical State Soil Mechanics, and **b** Rowe, both as function of the non-coaxiality parameter c

Critical State Soil Mechanics and Rowe's stress-dilatancy relationships as expressed in Eqs. (11) and (27), respectively. The stress-dilatancy relationships are plotted in the $-\dot{v}/\dot{\gamma}$ versus t/s axes for different constant values of the non-coaxiality parameter c. As indicated above, positive values of $\dot{v}/\dot{\gamma}$ correspond to contractive volumetric strain increment, and negative values correspond to dilative strain increment. As expected, the Critical State Soil Mechanics relationship (Eq. 11) is linear, while Rowe's relationship is curved in this plot. For the same value of c, the Critical State Soil Mechanics lines are steeper than the Rowe's curves. The effect of the parameter c is to lower the slope of the $\sin\psi$ versus $\sin\phi$ curves for both relationships.

4 Comparison with experimental simple shear data

To show its validity, the non-coaxial version of Rowe's stress dilatancy relationship (Eqs. 24–27) is compared with experimental data from direct simple shear (DSS) tests on sands. Simple shear is one of the most common modes of deformation of granular materials. For instance simple shear condition is predominant during shaking of level grounds during earthquakes when deformation is assumed to propagate vertically from the bedrock in the form of shear waves. Materials within localized failure zones also deform in simple shear. Simple shear deformation can be simulated in element tests using the direct simple shear (DSS) device, or the hollow cylindrical torsional simple shear device.

A major difficulty in interpreting direct simple shear test results is that the principal stress directions are not fixed but they rotate during simple shear loading. As a result, the orientations of the failure planes are not known and depend on the degree of principal stress rotation. In simple shear testing, principal stress rotation cannot be directly controlled and only limited rotation can be achieved. To fully use laboratory results from simple shear testing, it is therefore necessary to quantify the effects of principal stress rotation on the response of the material. In the following, the non-coaxial version of Rowe's stress-dilatancy relationship will be compared with experimental simple shear data on sand. The comparisons will be made: (1) to validate the non-coaxial version of Rowe's stress-dilatancy relationship, and (2) to analytically investigate the effects of principal stress rotation and non-coaxiality on the simple shear response of granular soils.

To use Rowe's stress-dilatancy equation, it is necessary to relate the parameters involved in the equation to those that are used in simple shear loading. Figure 2 shows the stress and strain conditions that are encountered in simple shear loading. The soil sample is restrained from deforming laterally (i.e., $\dot{\varepsilon}_x = 0$) and the sample is consolidated under vertical stress σ_y. After consolidation, the sample is sheared under a constant rate of shear displacement $\dot{\varepsilon}_{xy}$ until the peak or residual shear strength has been achieved. For drained test, the vertical stress is usually kept constant (i.e., $\dot{\sigma}_y = 0$). Undrained test is achieved by keeping the height of sample constant (i.e. $\dot{\varepsilon}_y = 0$) which, because of zero lateral strains, also keeps the sample volume constant.

The degree of shearing is measured by the shear stress ratio σ_{xy}/σ_y or the direct simple shear friction angle ϕ_{dss}:

$$\tan\phi_{dss} = \frac{\sigma_{xy}}{\sigma_y} \tag{28}$$

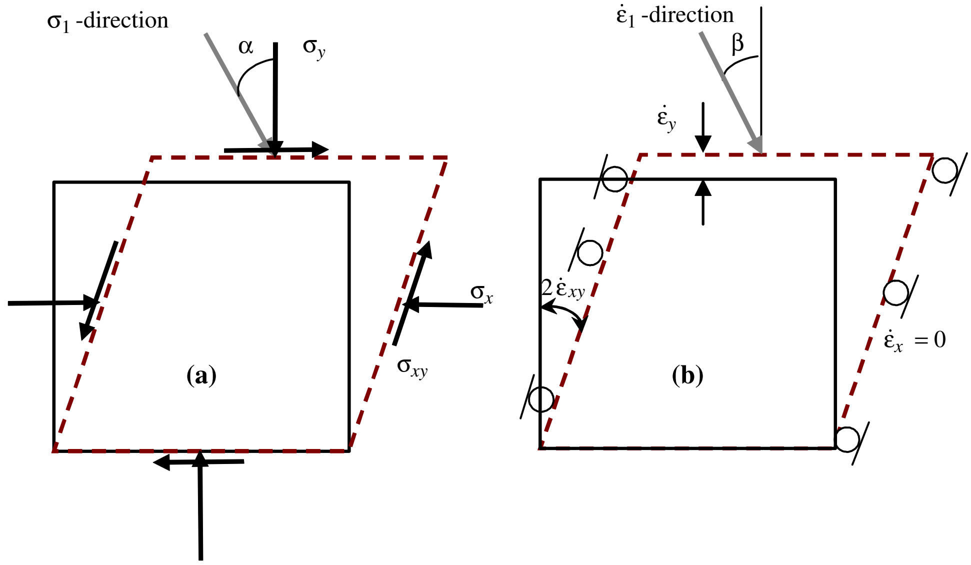

Fig. 2 **a** State of stress, and **b** state of strain under simple shear conditions

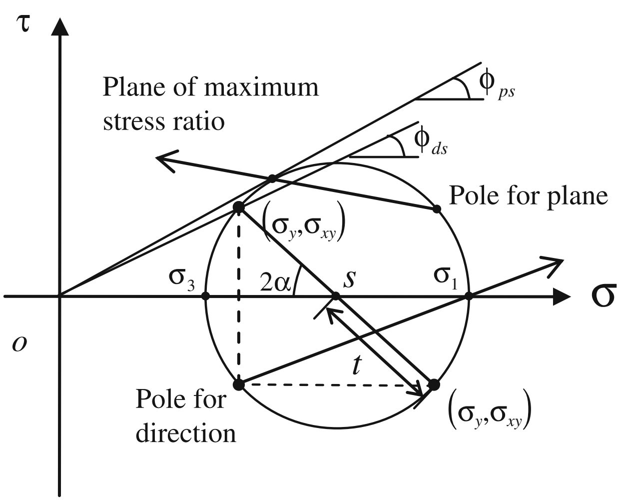

Fig. 3 Mohr's circle for state of stress in simple shear

From the Mohr-circle for stress (Fig. 3), the following relationship can be obtained between the plane strain friction angle ϕ (Eq. 12), the simple shear friction angle ϕ_{dss} and the principal stress direction α:

$$\sin\phi = \frac{\tan\phi_{dss}}{\sin 2\alpha - \cos 2\alpha \tan\phi_{dss}} \quad (29)$$

In addition to the evaluation of Eq. (28), the principal stress direction α is also needed in the evaluation of the non-coaxiality angle Δ (Eq. 8), which in turn is needed to determine the non-coaxiality parameter c in Eqs. (24)–(27). Oda and Konishi [16], Oda [17], and Ochiai [18] developed a simple expression for the degree of principal stress rotation that occurs during simple shear loading. They arrived at an expression which reads:

$$\tan(\phi_{dss}) = \frac{\sigma_{xy}}{\sigma_y} = \sin\phi_c \tan\alpha \quad (30)$$

This equation implies a straight-line relationship between the simple shear stress ratio σ_{xy}/σ_y on a horizontal plane and the tangent of the angle that σ_1 makes with the vertical axis. Equation (30) is compared with the simple shear experimental data of Cole [19] in Fig. 4. As can be seen, Eq. (30) provides a good approximation of the principal stress rotation during simple shear loading of sands. The experimental results show only slight differences in the degree of principal stress rotation between the loose, medium dense and dense samples.

The angle of non-coaxiality Δ is evaluated using the plastic flow rule developed by Gutierrez et al. [20] for plastic flow in the $(\sigma_y - \sigma_x)/2$ versus σ_{xy} stress rotation plane. This flow rule is described in the Appendix, and gives the following expression for the non-coaxiality angle:

$$\Delta = \alpha - \xi - \frac{1}{2}\sin^{-1}\left(\frac{\sin\phi}{\sin\phi_p}\sin(2\alpha - 2\xi)\right) \quad (31)$$

where ϕ_p is the peak friction angle, and ξ is the direction of the principal stress increment $\dot{\sigma}_1$ measured from the y-axis. This angle is defined as:

$$\tan 2\xi = \frac{2\dot{\sigma}_{xy}}{\dot{\sigma}_y - \dot{\sigma}_x} \quad (32)$$

To obtain an expression for the rotation of the principal stress increment direction ξ, it is only necessary to note that the stress increment vector should be tangential to stress path. The equation of the stress path given in Eq. (30) in terms of the stress ratio σ_{xy}/σ_y and the principal stress direction α can be differentiated to obtain this tangent. First, Eq. (30) is re-written in terms of 2α:

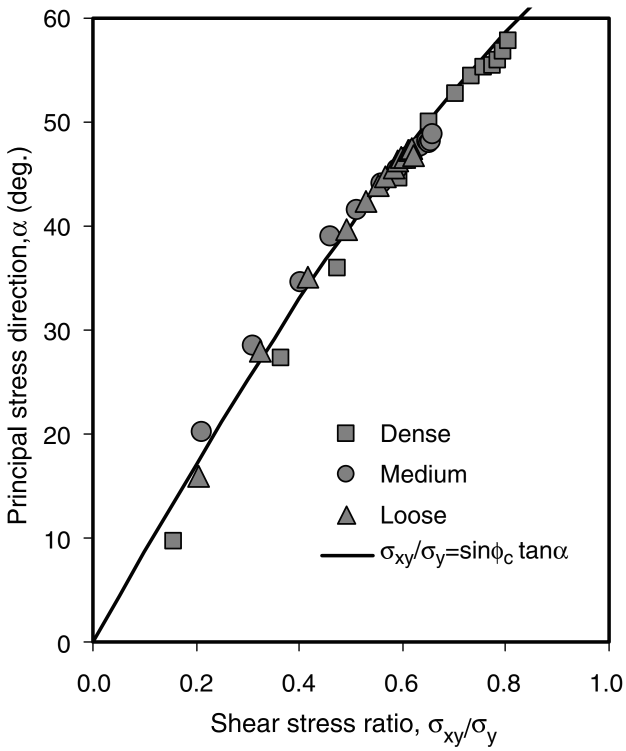

Fig. 4 Principal stress rotation α as function shear stress ratio σ_{xy}/σ_y from direct simple shear tests on Leighton–Buzzard sand

$$\frac{\sigma_{xy}}{\sigma_y} = \sin\phi_c \left(\frac{\sqrt{1+\tan^2 2\alpha} - 1}{\tan 2\alpha} \right) \tag{33}$$

Substituting Eq. (6), assuming a constant vertical stress (i.e., $\dot{\sigma}_y = 0$), and using Eq. (32), the principal stress direction ξ can be obtained as:

$$\tan 2\xi = -\frac{2\dot{\sigma}_{xy}}{\dot{\sigma}_x} \tag{34}$$

Combining Eqs. (33) and (34) gives the principal stress increment direction:

$$\tan 2\xi = -\frac{2\dot{\sigma}_{xy}}{\dot{\sigma}_x} = -\sin\phi_c \frac{\sigma_y}{\sigma_{xy}} = -\sin\phi_c \frac{1}{\tan\phi_{dss}} \tag{35}$$

As can be seen, similar to angle α, the direction ξ only depends on critical state friction angle ϕ_c, and the stress ratio σ_{xy}/σ_y (or the simple shear friction angle ϕ_{dss}).

The above equations fully describe the parameters required to derive the dilation angle ψ given in Eq. (27). Given the simple shear stress ratio σ_{xy}/σ_y (or ϕ_{dss}), the angles α and ξ are calculated from Eqs. (30) and (35) which are then used to calculate the non-coaxiality angle Δ (Eq. 31) and the non-coaxiality parameter c (Eq. 7). Also using ϕ_{dss}, the mobilized friction angle ϕ is calculated using Eq. (29). Given the values of c and ϕ, the dilation angle ψ can be calculated from Eq. (27).

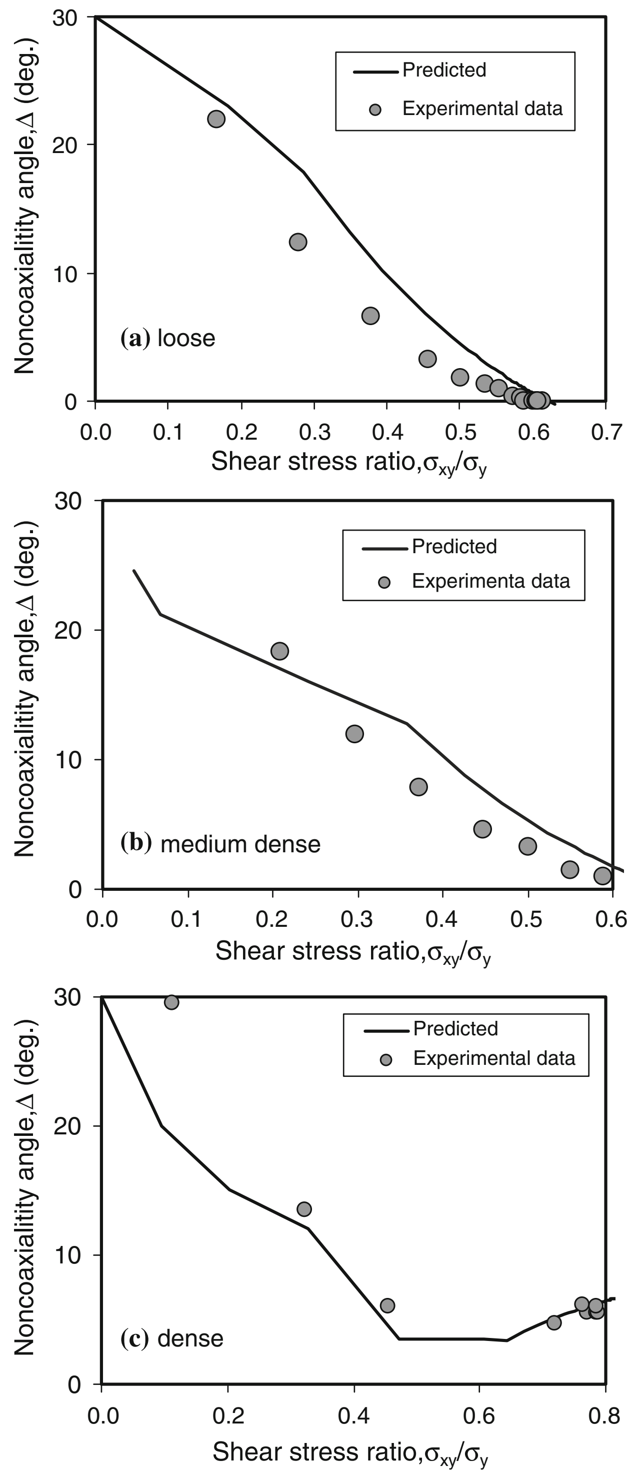

Fig. 5 Predicted and experimental data on non-coaxiality during simple shear deformation of Leighton–Buzzard sand. **a** dense, **b** medium dense and **c** dense samples

The above equations are compared to the experimental results obtained by Cole [19] from direct simple shear tests on Leighton–Buzzard sand using Cambridge University's Mark 5 DSS apparatus. Details of the apparatus and test

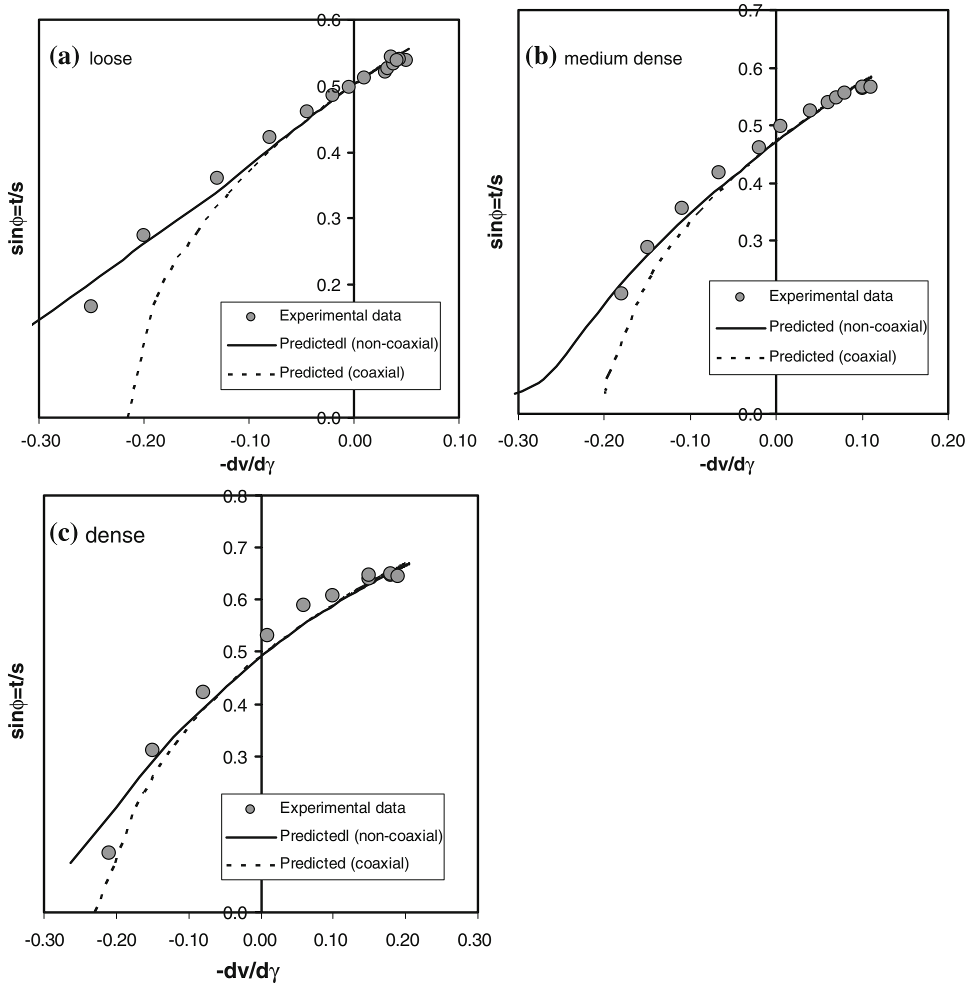

Fig. 6 Predicted and experimental data on stress-dilatancy during simple shear deformation of Leighton–Buzzard sand. **a** dense, **b** medium dense and **c** dense samples. Predictions from both coaxial and non-coaxial versions of Rowe's stress-dilatancy relation are shown

material are given in [19]. Comparisons of the non-coaxial version of Rowe's stress dilatancy relation are made with loose, medium dense and dense samples of Leighton–Buzzard sand, all consolidated and sheared at constant vertical stress of $\sigma_y = 400\,\text{kPa}$.

The comparisons are made first to verify the validity of the non-coaxiality angle Δ given in Eq. (31). Figure 5 shows predicted and measured values of Δ as function of σ_{xy}/σ_y for three densities of Leighton–Buzzard sand. As shown in Fig. 4, the major principal stress rotated by about 50–60° as the samples are sheared to failure and residual conditions. Due to the rotation, the major principal stress direction lagged behind the major principal strain increment direction resulting in non-coaxiality. The degree of non-coaxiality is about 30° at the start of shearing, and gradually reduces to almost zero as σ_{xy}/σ_y approaches failure condition.

As can be seen, Eq. (31) satisfactorily agrees with the measured non-coaxiality angles obtained from direct simple shear tests on Leigthon–Buzzard sand. Equation (31) also satisfactorily reflects the decrease in Δ as the shear stress ratio σ_{xy}/σ_y is increased. Note that the actual values of σ_{xy}/σ_y from the experiments were used in the calculation of Δ. For the test of the dense sample, significant strain softening was observed causing σ_{xy}/σ_y to decrease after the peak shear strength has been reached. As a result, the non-coaxiality increased slightly as σ_{xy}/σ_y exceeds a value of about 0.6.

Figure 6 shows the predicted and measured stress-dilatancy plots for the three densities of Leighton–Buzzard sand in the $-\dot{v}/\dot{\gamma}$ versus t/s axes. Predictions from both the coaxial and non-coaxial versions of Rowe stress-dilatancy relation are shown. For the non-coaxial version, the stress-dilatancy plots account for the variation in the non-coaxiality parameter c as the mobilized friction angle is increased, based on the predicted variations of the non-coaxiality angle Δ shown in Fig. 5. For the co-axial version, it is assumed that $c = 1.0$ for all values of ϕ.

As can be seen, both the coaxial and non-coaxial versions fit the experimental data at high values of mobilized friction angle ϕ. At low friction angles (or for contractive response), in general, the non-coaxial version fits the experimental data better, particularly for the loose and medium dense sands. For the loose and medium dense samples, the coaxial stress-dilatancy relation significantly deviates from both the experimental and non-coaxial version at low friction angle. For the dense samples, the difference between coaxial and non-coaxial versions is less significant. The deviation between the coaxial and non-coaxial versions is due to the fact that non-coaxiality induced a higher degree of contraction at low friction angles than the nominal dilation angle assuming coaxiality. As the mobilized friction angle is decreased, the non-coaxiality angle approaches zero, and the effects of non-coaxiality disappears.

5 Conclusions

The derivation of the non-coaxial version of Rowe's stress-dilatancy relationship was presented. The derivation followed the previous approach used by Gutierrez and Ishihara [1] in deriving a non-coaxial version of the Critical State Soil Mechanics stress-dilatancy relation by incorporating the Gutierrez–Ishihara non-coaxiality parameter c. Equations relating the parameters in Rowe's non-coaxial stress-dilatancy relationship to simple shear loading conditions were also derived. It was shown that the non-coaxial version agrees with the experimental data on the simple shear dilatancy of sands for contractive response at low values of mobilized friction angle ϕ for loose and medium dense. In contrast, the coaxial version significantly under predicts the experimental dilatancy angles at low mobilized friction angles for loose and medium dense sands. For the dense samples, the difference between coaxial and non-coaxial versions is less significant. The difference between the coaxial and non-coaxial versions is due to the fact that non-coaxiality caused by principal stress rotation in simple shear loading induces a higher degree of contraction at low friction angles than the nominal dilation angle assuming coaxiality. The non-coaxiality angle approaches zero as the mobilized friction angle is increased, and consequently, the effects of non-coaxiality on dilatancy becomes insignificant.

Appendix: Noncoaxiality angle due to principal stress rotation

The angle of non-coaxiality angle Δ required to determine the non-coaxiality parameter c in the stress-dilatancy relationships given in Eqs. (11) and (27) is evaluated using the flow rule developed by Gutierrez et al. [20] for plastic flow in the $(\sigma_y - \sigma_x)/2$ versus σ_{xy} stress rotation plane. This flow rule is shown in Fig. 7. In this figure, the strain increment components $(\dot{\varepsilon}_y - \dot{\varepsilon}_x)$ and $2\dot{\varepsilon}_{xy}$ have been superimposed on the stress plane. Point A is the current stress point. The failure surface is circular and centered at the origin of the $(\sigma_y - \sigma_x)/2$ versus σ_{xy} axes. The distances $\overline{OA}$ and $\overline{OB}$ are

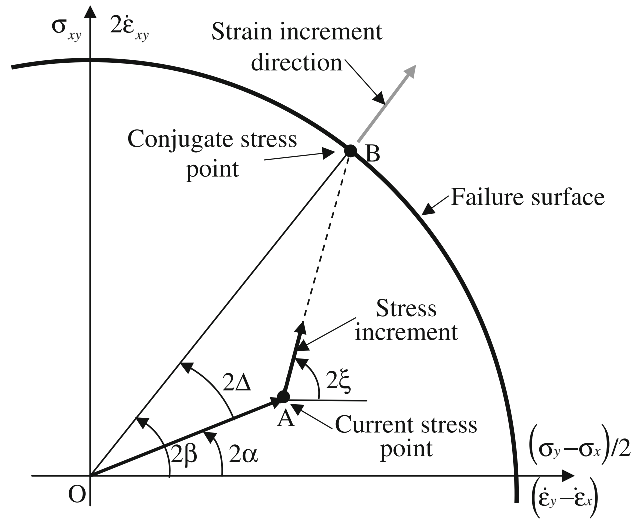

Fig. 7 Non-coaxial flow rule in the $(\sigma_y - \sigma_x)/2$ versus σ_{xy} stress rotation plane

equal to the current shear stress t and the radius of the circular failure surface, respectively. These distances can be related to the current mean stress s, the mobilized friction angle ϕ and the peak friction angle ϕ_p as follows:

$$\overline{OA} = s \sin\phi \quad \text{and} \quad \overline{OB} = s \sin\phi_p \tag{36}$$

On the stress plane, a stress vector makes an angle equal to 2α (Eq. 6) from the $(\sigma_y - \sigma_x)/2$ axis. Similarly, a stress increment vector makes an angle equal to 2ξ (Eq. 32) from the $(\sigma_y - \sigma_x)/2$ axis. On the strain increment plane, a strain increment vector has a length equal to the plastic shear strain increment $\dot{\gamma}$ and makes an angle equal to 2β (Eq. 6) from the $(\dot{\varepsilon}_y - \dot{\varepsilon}_x)$ axis. This plastic strain increment direction is evaluated as the normal to the failure surface at the conjugate point A which is the intersection of the failure surface and the stress increment vector extended from the current stress point. This flow rule is based on the experimental observations that plastic flow on the $(\sigma_y - \sigma_x)/2$ versus σ_{xy} stress plane is dependent on the stress increment direction [20]. This flow rule should be contrasted with conventional plasticity formulations where the plastic strain increment direction is evaluated at the current stress point independent of the stress increment direction. Details of the flow described in this appendix including the experimental verification are given in [20].

From triangle OAB in Fig. 7, the following angles can be obtained:

$$\angle OAB = \pi - (2\xi - 2\alpha) \tag{37}$$

$$\angle BOA = 2\Delta \tag{38}$$

$$2\Delta = \pi - \angle OAB - \angle ABO \tag{39}$$

Using the law of sines:

$$\frac{OA}{\sin \angle ABO} = \frac{OB}{\sin \angle OAB} \tag{40}$$

$$\angle ABO = \sin^{-1}\left(\frac{OA}{OB}\sin \angle OAB\right) \tag{41}$$

Substituting Eqs. (37) and (38) in Eq. (39):

$$\angle ABO = \sin^{-1}\left(\frac{\sin\phi}{\sin\phi_p}\sin(\pi + 2\alpha - 2\xi\right) \tag{42}$$

Substituting Eqs. (37) and (42) in Eq. (39) gives:

$$2\Delta = 2\alpha - 2\xi - \sin^{-1}\left(\frac{\sin\phi}{\sin\phi_p}\sin\left(\pi + 2\alpha - 2\xi\right)\right) \tag{43}$$

Simplifying the above equation results in the expression for Δ given in Eq. (31).

References

1. Gutierrez, M., Ishihara, K.: Non-coaxiality and energy dissipation in granular materials. Soils Found. **40**(2), 49–59 (2000)
2. Vardoulakis, I., Georgopoulos, I.G.: The "stress-dilatancy" hypothesis revisited: shearbanding related instabilities. Soils Found. **45**(2), 61–76 (2005)
3. Gutierrez, M., Vardoulakis, I.: Energy dissipation and post-bifurcation behavior of granular soils. Int. J. Numer. Anal. Meth. Geomech. **31**(3), 435–455 (2006)
4. Rowe, P.W.: The Stress dilatancy relation for static equilibrium of an assembly of particles in contact. Proc. R. Soc. Lond. Ser. A. **269**, 500–527 (1962)
5. Wan, R.G., Guo, P.J.: Stress dilatancy and fabric dependencies on sand behavior. J. Eng. Mech. ASCE. **13**(6), 635–645 (2004)
6. Nova, R., Wood, D.M.: A constitutive model for sand in triaxial compression. Int. J. Numer. Anal. Meth. Geomech. **3**(3), 255–278 (2005)
7. Ling, H.I., Liu, H.: Pressure-level dependency and densification behavior of sand through generalized plasticity model. Eng. Mech. ASCE. **8**, 851–860 (2003)
8. McDowell, G.R.: A simple non-associated flow model for sand. Granul. Matter **4**(2), 65–69 (2002)
9. Schofield, A., Wroth, C.P.: Critical State Soil Mechanics. McGraw-Hill, London (1967)
10. De Gennaro, V., Frank, R.: Elasto-plastic analysis of the interface behaviour between granular media and structure. Comp. Geotech. **29**(7), 547–572 (2002)
11. Ghionna, V.N., Mortara, G.: An elastoplastic model for sand-structure interface behaviour. Géotechnique **52**(1), 41–50 (2002)
12. Li, X.S., Dafalias, Y.F.: Dilatancy for cohesionless soils. Géotechnique **50**(4), 449–460 (2000)
13. Rowe, P.W.: The relation between the shear strength of sand in triaxial compression, plane strain and direct shear. Géotechnique **19**(1), 75–86 (1969)
14. de Jong, D.J.: Rowe's stress-dilatancy relation based on friction. Géotechnique **26**(3), 527–534 (1976)
15. Niiseki, S.: Formulation of Rowe's stress-dilatancy equation based on plane of maximum mobilization. Proceedings of Powders and Grains Conference, pp. 213–216 (2001)
16. Oda, M., Konishi, J.: Microscopic deformation of granular material in simple shear. Soils Found. **14**(4), 25–38 (1974)
17. Oda, M.: On the relation $\tau/\sigma_n = \kappa \tan\alpha$ in the simple shear test. Soils Found. **15**(4), 35–41 (1975)
18. Ochiai, H.: The behaviour of sand in the direct shear test (in Japanese). J. Jap. Soc. Soil Mech. Found. Eng. **15**(4), 93–100 (1975)
19. Cole, E.R.L.: The behavior of soils in the simple shear apparatus. Ph.D. Thesis, Cambridge University (1967)
20. Gutierrez, M., Ishihara, K., Towhata, I.: Flow theory for sand during rotation of principal stress direction. Soils Found. **31**(4), 121–132 (1993)