NON-COHERENT SCATTERING

I. The Redistribution Functions with Doppler Broadening

David G. Hummer

(Received 1962 July 12)

Summary

The redistribution in frequency of radiation scattered from moving atoms is examined in some generality, allowing for the different types of scattering which occur in the atom's rest frame under different circumstances. Some general formulae are obtained and a number of explicit results are given. Finally some attention is devoted to the properties of the redistribution functions and to the methods which may be used for computing them.

1.0. *Introduction.*—Spitzer (**1**) gives a good resumé of the various physical causes for noncoherence in frequency in the scattering of light by atoms. The effects of noncoherence arising from radiation damping have been studied some time ago by Spitzer (**2**) and Woolley (**3**) on the basis of the Wigner-Weisskopf theory. The nature of the redistribution of frequencies for impact broadening was discussed by Zanstra (**4**) from a classical point of view. The situation for statistical broadening is considered by Edmonds (**5**) and more satisfactorily by Holstein (**6**). In all of the above, Doppler broadening was ignored. The redistribution function for Doppler broadening of a line with zero natural width has been obtained by Unno (**7**), Thomas (**8**) and Field (**9**), although Zanstra (**10**) first discussed the problem using an approximate form. Field generalized the results to include the effects of atom recoil and dipole scattering. The redistribution function for combined Doppler and natural broadening was first obtained by Henyey (**11**) and subsequently by Unno (**12**) and Sobolev (**30**).

In this paper we obtain a very general redistribution function for the physically realistic situations in which scattering, according to an arbitrary redistribution function and an arbitrary phase function in the atom's rest frame, is further modified by the Doppler effect. We obtain explicit formulae for the redistribution functions in four cases. They are, with the Roman numeral which will subsequently identify them:

- Zero natural line width (I).
- Radiation damping with coherence in the atom's rest frame (II).
- Radiation and collision damping with complete redistribution in the atom's frame (III).
- Resonance scattering (IV).

When it becomes necessary to distinguish between the different phase functions, we will append A and B to indicate isotropic and dipole scattering respectively. Finally some attention is given to the properties of the redistribution functions and to methods which may be used for computing them.

2.0. *Formulation.*—Let the observer's frame of reference be the frame in which the gas of a homogeneous atmosphere has random thermal motion but no mass motion. In the frame let a photon have frequency in the range ν to $\nu+d\nu$ and direction $\mathbf{n}$ within an element of solid angle $d\Omega$. Consider its absorption and re-emission in the same spectrum line. Then the probability that this photon is absorbed *and* that the re-emitted photon has frequency ν' to $\nu'+d\nu'$ and direction $\mathbf{n}'$ in $d\Omega'$ is

$$R(\nu, \mathbf{n};\ \nu', \mathbf{n}')\, d\nu\, d\Omega\, d\nu' d\Omega'.$$

This is normalized to unity:

$$\iiiint R(\nu, \mathbf{n};\ \nu', \mathbf{n}')\, d\nu\, d\Omega\, d\nu' d\Omega' = 1. \qquad (2.0.1)$$

Integration over ν' and Ω' gives the absorption probability $\phi(\nu)$:

$$\phi(\nu)\, d\nu\, d\Omega = 4\pi\, d\nu\, d\Omega \int\int_{-\infty}^{\infty} R(\nu, \mathbf{n};\ \nu', \mathbf{n}')\, d\nu' d\Omega', \qquad (2.0.2)$$

with normalization

$$\int\int_{-\infty}^{\infty} \phi(\nu)\, d\nu\, d\Omega = 4\pi. \qquad (2.0.3)$$

For convenience we consider all integrals over frequency to run from $-\infty$ to ∞.

Let ν_0 be the central frequency of the line and let $\mathbf{v}$ be the velocity of the absorbing atom, in the observer's frame. In the rest frame of the atom the photons $(\nu, \mathbf{n})$ and $(\nu', \mathbf{n}')$ have frequencies ξ and ξ', where to first order in ν/c,

$$\left.\begin{aligned} \nu &= \xi + \frac{1}{c}\nu_0 \mathbf{n}\cdot\mathbf{v}, \\ \nu' &= \xi' + \frac{1}{c}\nu_0 \mathbf{n}'\cdot\mathbf{v}. \end{aligned}\right\} \qquad (2.0.4)$$

To the same order, $\mathbf{n}$ and $\mathbf{n}'$ are unchanged in going from one frame to the other. We take the absorption probability in the atom's frame, $f(\xi)$, to be independent of direction and to have the normalization

$$\int_{-\infty}^{\infty} f(\xi)\, d\xi = 1.$$

The redistribution function $p(\xi, \xi')$ in the atom's frame is defined somewhat differently from R. Let $g(\mathbf{n}, \mathbf{n}')$ be the phase function, then

$$p(\xi, \xi') g(\mathbf{n}, \mathbf{n}')\, d\xi' d\Omega'$$

gives the probability that *if* a photon $(\xi, \mathbf{n})$ has been absorbed, a photon $(\xi', \mathbf{n}')$ is emitted. The normalization conditions for these functions are

$$\int_{-\infty}^{\infty} p(\xi, \xi')\, d\xi' = \int g(\mathbf{n}, \mathbf{n}')\, d\Omega' = 1.$$

The probability that a photon $(\xi, \mathbf{n})$ will be absorbed is

$$f(\xi)\, d\xi \frac{d\Omega}{4\pi}$$

and the probability that the event will be followed by emission of a photon $(\xi', \mathbf{n}')$ is

$$f(\xi) p(\xi, \xi') g(\mathbf{n}, \mathbf{n}') \frac{d\Omega}{4\pi} d\Omega' d\xi\, d\xi'.$$

But this compound event corresponds to the scattering of $(\nu, \mathbf{n})$ to $(\nu', \mathbf{n}')$ by an atom of velocity $\mathbf{v}$. Thus we have

$$R_{\mathbf{v}}(\nu, \mathbf{n};\, \nu'\, \mathbf{n}') = f\left[\nu - \frac{\nu_0}{c}\mathbf{n}\,.\,\mathbf{v}\right] P\left[\nu - \frac{\nu_0}{c}\mathbf{n}\,.\,\mathbf{v}, \nu' - \frac{\nu_0}{c}\mathbf{n}'\,.\,\mathbf{v}\right] \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi}. \tag{2.0.5}$$

The ease with which this is averaged over the Maxwell velocity distribution depends on selecting a convenient coordinate system.

2.1. *General result.*—We introduce a set of mutually orthogonal unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ such that $\mathbf{n}_1$ and $\mathbf{n}_2$ lie in the plane defined by $\mathbf{n}$ and $\mathbf{n}'$. It will simplify later work if $\mathbf{n}_1$ is chosen to bisect the acute angle formed by $\mathbf{n}$ and $\mathbf{n}'$; this angle we call γ. Then

$$\mathbf{n} = a\mathbf{n}_1 + b\mathbf{n}_2 \tag{2.1.1}$$

and

$$\mathbf{n}' = a\mathbf{n}_1 - b\mathbf{n}_2, \tag{2.1.2}$$

where

$$a = \cos\frac{\gamma}{2}, \quad b = \sin\frac{\gamma}{2}. \tag{2.1.3}$$

Let

$$v_i = \mathbf{v}\,.\,\mathbf{n}_i, \quad i = 1, 2, 3;$$

then the probability that an atom of mass M has velocity components v_1 to $v_1 + dv_1$, etc. is

$$dP(v_1, v_2, v_3) = \pi^{-3/2}\alpha^{-3}\, e^{-\alpha^{-2}[v_1^2 + v_2^2 + v_3^2]}\, dv_1\, dv_2\, dv_3, \tag{2.1.4}$$

where

$$\alpha = (2kT/M)^{1/2}. \tag{2.1.5}$$

Introduce the reduced velocity

$$\mathbf{u} = \mathbf{v}/\alpha \tag{2.1.6}$$

and the Doppler width

$$\Delta = \nu_0\alpha/c. \tag{2.1.7}$$

With these results we sum (2.0.5) to obtain our general result,

$$R(\nu, \mathbf{n};\, \nu', \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u_1^2 - u_2^2} f[\nu - \Delta(au_1 + bu_2)] \times p[\nu - \Delta(au_1 + bu_2),\, \nu' - \Delta(au_1 - bu_2)]\, du_1\, du_2. \tag{2.1.8}$$

For certain cases, however, it will be convenient to choose $\mathbf{n}_1$ to coincide with $\mathbf{n}$.

Then

$$\mathbf{v}\,.\,\mathbf{n}' = a'v_1 + b'v_2, \tag{2.1.9}$$

where

$$a' = \cos\gamma, \quad b' = \sin\gamma. \tag{2.1.10}$$

The alternative general form is then

$$R(\nu, \mathbf{n};\, \nu'\, \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi^2} \int_{-\infty}^{\infty} e^{-u_1^2} f(\nu - \Delta u_1) \int_{-\infty}^{\infty} e^{-u_2^2} \times p[\nu - \Delta u_1,\, \nu' - \Delta(a'u_1 + b'u_2)]\, du_1\, du_2. \tag{2.1.11}$$

This form is useful when $p(\xi, \xi')$ is independent of ξ, as happens, for instance, in cases of complete redistribution.

Since the atmosphere is homogeneous, these results depend only on the angle between $\mathbf{n}$ and $\mathbf{n}'$, as expected.

2.2. *Coherence in atom's frame.*—If the scattering can be assumed to be coherent in the rest frame of the atom, we can write

$$p(\xi, \xi') = \delta(\xi - \xi'), \tag{2.2.1}$$

where $\delta(x)$ is the singular function introduced by Dirac. Writing out the argument of the delta function, we obtain

$$p(\xi, \xi') = \delta(\nu - \nu' - 2b\Delta u_2). \tag{2.2.2}$$

Now u_1 no longer appears in the function p, which greatly simplifies the subsequent calculations. This is a consequence of choosing $\mathbf{n}_1$ to bisect γ; this choice is due to Henyey (11).

In carrying out the integration over u_2, we must use $2b\Delta u_2$ as a variable of integration to maintain the normalization of the delta function.

For $b \neq 0$, we obtain

$$R(\nu, \mathbf{n}; \nu', \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{8\pi^2 \Delta b} e^{-\frac{(\nu-\nu')^2}{4b^2\Delta^2}} \int_{-\infty}^{\infty} e^{-u^2} f\left(\frac{\nu+\nu'}{2} - \Delta a u\right) du. \tag{2.2.3}$$

When $b = 0$, that is $\mathbf{n} = \mathbf{n}'$, we have instead

$$R(\nu, \mathbf{n}; \nu' \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi^{3/2}} \delta(\nu - \nu') \int_{-\infty}^{\infty} e^{-u^2} f(\nu - \Delta u)\, du. \tag{2.2.4}$$

2.21. *Zero natural line width.*—The first case of interest is that of zero natural line width,

$$f(\xi)\, d\xi = \delta(\xi - \nu_0)\, d\xi. \tag{2.21.1}$$

Integrating over Δu, we obtain, after some algebra

$$R_{\mathrm{I}}(x, \mathbf{n}; x', \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi^2 \sin\gamma} \exp\left[-x'^2 - (x - x'\cos\gamma)^2 \csc^2\gamma\right], \tag{2.21.2}$$

where

$$x = \frac{\nu - \nu_0}{\Delta}.$$

For isotropic scattering the phase function is

$$g_{\mathrm{A}}(\mathbf{n}, \mathbf{n}') = \frac{1}{4\pi}, \tag{2.21.3}$$

while for dipole scattering it is

$$g_{\mathrm{B}}(\mathbf{n}, \mathbf{n}') = \frac{3}{16\pi}(1 + \cos^2\gamma). \tag{2.21.4}$$

The result for isotropic scattering was first obtained by Thomas (8). For R_{I}, the normalization conditions (2.0.1) and (2.0.3) can be verified directly.

2.22. *Radiation damping.*—We turn now to consideration of radiation damping, where we have

$$f(\xi)\, d\xi = \frac{\delta}{\pi} \frac{1}{(\xi - \nu_0)^2 + \delta^2} d\xi, \tag{2.22.1}$$

where $4\pi\delta$ is the sum of the transition probabilities out of the states concerned. From (2.23) we obtain

$$R_{\mathrm{II}}(x, \mathbf{n}; x', \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi^2 \sin\gamma} \exp\left[-\left(\frac{x - x'}{2}\right)^2 \csc^2\left(\frac{\gamma}{2}\right)\right] H\left(\sigma \sec\frac{\gamma}{2}, \frac{x + x'}{2} \sec\frac{\gamma}{2}\right), \tag{2.22.2}$$

where

$$\sigma = \delta/\Delta. \tag{2.22.3}$$

Here we have introduced the well-known Voigt function

$$H(a,u) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}\,dy}{(u-y)^2 + a^2}. \tag{2.22.4}$$

R_{II} was first obtained by Henyey (**11**) who gives graphs of

$$R_{\text{II}}(x, \mathbf{n};\ x', \mathbf{n}')/\phi(x) \quad \text{for} \quad \sigma = \cdot 01.$$

2.3. *Noncoherence in rest frame.*—We consider first the case in which the emission is distributed according to the frequency dependence of the absorption coefficient in the atom's rest frame, here taken to be given by the dispersion formula. This situation corresponds, at least approximately, to impact broadening. We have

$$p(\xi, \xi')\,d\xi' = f(\xi')\,d\xi = \frac{\delta}{\pi} \frac{1}{(\xi' - \nu_0)^2 + \delta^2}\,d\xi', \tag{2.3.1}$$

where δ now includes the effects of both natural and impact broadening. Using the alternative general form, this gives

$$R_{\text{III}}(x, \mathbf{n};\ x', \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi^2 \sin\gamma} \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(x-y)^2 + \sigma^2} H(\sigma \csc\gamma, x' \csc\gamma - y \operatorname{ctn}\gamma)\,dy. \tag{2.3.2}$$

2.31. *Resonance scattering.*—By resonance scattering, we mean an absorption-emission process in which the atomic transitions are $i \to j \to i$ where, generally, the levels i and j are both broadened by either spontaneous or induced transitions from them. For these transitions in which the final and initial states are the same, one cannot calculate the rest frame redistribution function by assuming the Weisskopf-Wigner (**13**) energy distribution, as do Woolley (**3**), Woolley and Stibbs (**14**), and Henyey (**15**). This point is discussed by Rosseland (**16**). Instead, the later results of Weisskopf (**17**), (**18**) must be used; the problem is treated as well by Heitler (**19**).

This distinction is to some extent academic, since Weisskopf's and Heitler's treatments are strictly applicable to scattering from the ground state, and then, unless the radiation field is very strong, the level width is for all practical purposes zero and the usual result (2.22.1) is obtained. None the less, we include this work as the best available result, even if its general validity is uncertain. Weisskopf's formula is also much simpler than that of Woolley and Stibbs.

In our notation Weisskopf's result is

$$f(\xi)p(\xi, \xi') = \frac{\delta_i \delta_j}{\pi^2} \frac{1}{[(\xi - \xi')^2 + \delta_i^2][(\xi' - \nu_0)^2 + \delta_j^2]}, \tag{2.31.1}$$

where $4\pi\delta_i$ and $4\pi\delta_j$ are the total transition probabilities from the lower and upper levels respectively. The fact that resonance scattering is a single quantum process makes it impossible to separate $f(\xi)$ and $p(\xi, \xi')$.

Using the first general form, we obtain

$$R_{\text{IV}}(x, \mathbf{n};\ x', \mathbf{n}') = \frac{g(\mathbf{n}, \mathbf{n}')}{2\pi^2 \sin\gamma} \frac{\sigma_i \sec\gamma/2}{\pi} \times \int_{-\infty}^{\infty} \frac{e^{-y^2} H(\sigma_j \csc\gamma/2, y \operatorname{ctn}\gamma/2 - x \csc\gamma/2)\,dy}{[(x-x')\sec\gamma/2 - 2y]^2 + (\sigma_i \sec\gamma/2)^2}. \tag{2.31.2}$$

3.0. *Integrated redistribution functions.*—In view of the great difficulty in solving the transfer equation for noncoherent scattering, the scattering is assumed to be isotropic and one needs $R(x, \mathbf{n}; x', \mathbf{n}')$ integrated over $\mathbf{n}$ and $\mathbf{n}'$, or

$$R(x, x') = 8\pi^2 \int_0^\pi R(x, \mathbf{n}; x', \mathbf{n}') \sin\gamma \, d\gamma. \tag{3.0.1}$$

The definition (2.0.2) gives

$$\phi(x) = \int_{-\infty}^{\infty} R(x, x') \, dx'. \tag{3.0.2}$$

The integrals in (3.0.1) are difficult to evaluate, and it is easiest to derive $R(x, x')$ afresh. We begin with expression (2.0.5) using (2.1.6) and (2.1.7),

$$R_{\mathbf{u}}(\nu, \mathbf{n}; \nu', \mathbf{n}') = f[\nu - \Delta\mathbf{n}.\mathbf{u}] p[\nu - \Delta\mathbf{n}.\mathbf{u}, \nu' - \Delta\mathbf{n}'.\mathbf{u}] \frac{g(\mathbf{n}, \mathbf{n}')}{4\pi}. \tag{3.0.3}$$

Holding ν, ν' and $\mathbf{u}$ fixed, integrate over all incoming and outgoing directions. It is convenient to use a coordinate system $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ in which

$$\mathbf{u} = u\mathbf{n}_3. \tag{3.0.4}$$

In these co-ordinates we have

$$\mathbf{n}.\mathbf{u} = \mu u$$

and

$$\mathbf{n}'.\mathbf{u} = \mu' u, \tag{3.0.5}$$

where μ is the cosine of the angle between $\mathbf{n}$ and $\mathbf{n}_3$; μ' is similarly defined. In this coordinate system, we have

$$d\Omega = d\mu \, d\phi \tag{3.0.6}$$

with a similar relation for primed quantities, so that carrying out the ϕ and ϕ' integrations we have

$$R_u(\nu, \nu') = \tfrac{1}{2} \int_{-1}^{1} \int_{-1}^{1} g(\mu, \mu') f[\nu - \Delta\mu u] p[\nu - \Delta\mu u, \nu' - \Delta\mu' u] \, d\mu \, d\mu', \tag{3.0.7}$$

where

$$g(\mu, \mu') = \int_0^{2\pi} g(\mu, \mu', \phi) \, d\phi. \tag{3.0.8}$$

Writing (2.21.4) in terms of μ, μ' and ϕ and integrating over ϕ, we have for dipole scattering,

$$g_{\mathrm{B}}(\mu, \mu') = \tfrac{3}{16} [3 - \mu^2 - \mu'^2 + 3\mu^2\mu'^2]. \tag{3.0.9}$$

The corresponding result for isotropic scattering is

$$g_{\mathrm{A}}(\mu, \mu') = \tfrac{1}{2}. \tag{3.0.10}$$

3.1. *Coherence in rest frame.*—When the scattering is coherent in the atom's frame, we have

$$p[\nu - \Delta\mu u, \nu' - \Delta\mu' u] = \delta[\nu - \nu' - \Delta u(\mu - \mu')]. \tag{3.1.1}$$

Since the range of integration for μ, μ' is finite, for certain values of the quantity $(\nu - \nu')/\Delta u$, the singularity of the δ-function will lie outside the range of integration and $R_{\mathbf{u}}$ will vanish. First consider the μ' integral which we call I. Let

$$y = \Delta u \mu', \tag{3.1.2}$$

then we have

$$I=\frac{1}{\Delta u}\int_{-\Delta u}^{\Delta u} g\left(\mu,\frac{y}{\Delta u}\right)\delta[y-(\nu'-\nu+\Delta u\mu)]\,dy$$

$$=\begin{cases}\dfrac{1}{\Delta u}g\left[\mu,\mu+\dfrac{\nu'-\nu}{\Delta u}\right], & \Delta u\geqslant\nu'-\nu+\Delta u\mu\geqslant-\Delta u\\ 0 & , \text{ otherwise.}\end{cases} \quad (3.1.3)$$

Let

$$\Lambda(x)=\begin{cases}1, & 1\geqslant x\geqslant-1\\ 0, & \text{otherwise,}\end{cases} \quad (3.1.4)$$

then

$$R_u(\nu,\nu')=\frac{1}{2\Delta u}\int_{-1}^{1} f(\nu-\Delta u\mu)g\left(\mu,\mu+\frac{\nu'-\nu}{\Delta u}\right)\Lambda\left[\mu+\frac{c}{\Delta u}(\nu'-\nu)\right]d\mu. \quad (3.1.5)$$

When u is sufficiently small, Λ will vanish for all values of μ and we obtain the minimum value of u for which scattering from ν to ν' can occur when the scattering is coherent in the frame of the atom. Detailed consideration of the range of μ for which $\Lambda=1$ yields the inequalities

$$\underline{\nu}+\Delta u\geqslant\nu-\Delta u\mu\geqslant\bar{\nu}-\Delta u, \quad (3.1.6)$$

where $\bar{\nu}, \underline{\nu}$ are the larger and smaller of ν and ν' respectively. The minimum value of u is obtained by equating the bounds of (3.1.6),

$$u_{\min}=\tfrac{1}{2}\frac{\bar{\nu}-\underline{\nu}}{\Delta}. \quad (3.1.7)$$

Let

$$\Theta(x,x_0)=\begin{cases}1, & x>x_0,\\ 0, & x<x_0,\end{cases} \quad (3.1.8)$$

then we can write

$$R_u(\nu,\nu')=\tfrac{1}{2}\left(\frac{1}{\Delta u}\right)^2\Theta\left(u-\tfrac{1}{2}\frac{|\nu-\nu'|}{\Delta},0\right)\int_{\bar{\nu}-\Delta u}^{\underline{\nu}+\Delta u} f(y)g\left(\frac{\nu-y}{\Delta u},\frac{\nu'-y}{\Delta u}\right)du\,dy. \quad (3.1.9)$$

Summing over the Maxwellian distribution of radial velocities,

$$dP(u)=\frac{4}{\sqrt{\pi}}e^{-u^2}u^2du, \quad (3.1.10)$$

we obtain a general formula for $R(\nu,\nu')$ when the scattering process is coherent in the atom's frame,

$$R(\nu,\nu')=\frac{2}{\Delta^2\sqrt{\pi}}\int_{u_{\min}}^{\infty} e^{-u^2}\int_{\bar{\nu}-\Delta u}^{\underline{\nu}+\Delta u} f(y)g\left(\frac{\nu-y}{\Delta u},\frac{\nu'-y}{\Delta u}\right)dy\,du. \quad (3.1.11)$$

3.11. *Zero natural line width.*—For vanishing line width, $f(y)$ is given by (2.2.1). We find that the integral over y in (3.1.11) will vanish unless the condition

$$\underline{\nu}+\Delta u\geqslant\nu_0\geqslant\bar{\nu}-\Delta u \quad (3.11.1)$$

is satisfied. Thus for zero line width, the smallest value of u for which scattering from ν to ν' can occur is

$$u'_{\min} = \frac{\overline{|\nu-\nu_0|}}{\Delta}, \tag{3.11.2}$$

where $\overline{|\nu-\nu_0|}$ is the larger of $|\nu-\nu_0|$, $|\nu'-\nu_0|$. In general

$$u'_{\min} \geqslant u_{\min},$$

so that no contradiction arises between (3.11.2) and (3.1.7). We obtain from (3.1.11), using (3.0.10), the result for isotropic scattering

$$R_{\text{I-A}}(x, x') = \tfrac{1}{2}\operatorname{erfc}(\overline{|x|}), \tag{3.11.3}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{3.11.4}$$

The corresponding result for dipole scattering is

$$R_{\text{I-B}}(x, x') = \frac{3}{8\sqrt{\pi}}\int_{\overline{|x|}}^\infty e^{-u^2}\left[3-\left(\frac{x}{u}\right)^2-\left(\frac{x'}{u}\right)^2+3\left(\frac{x}{u}\right)^2\left(\frac{x'}{u}\right)^2\right]du$$

$$= \frac{3}{8}\left\{\tfrac{1}{2}\operatorname{erfc}(\overline{|x|})[3+2(x^2+x'^2)+4x^2x'^2] - \frac{e^{-\overline{|x|}^2}}{\sqrt{\pi}}\overline{|x|}(2\underline{|x|}^2+1)\right\}. \tag{3.11.5}$$

In (3.11.3) and (3.11.5), $\overline{|x|}$ and $\underline{|x|}$ are the larger and smaller respectively of $|x|$ and $|x'|$. The result (3.11.3) was first obtained by Unno (7). Field (9) first obtained the result for dipole scattering. $R_{\text{I-A}}$ and $R_{\text{I-B}}$ are shown in Fig. 1. The effect of the dipole scattering is to increase the coherence of the scattering process, since it weights more heavily those scatterings in which the Doppler shift is small.

3.12. *Radiation damping.*—In the case of radiation damping, substitution of (2.22.1) into (3.1.11) gives, for isotropic scattering,

$$R_{\text{II-A}}(x, x') = \pi^{-3/2}\int_{\frac{1}{2}|\bar{x}-\underline{x}|}^\infty e^{-u^2}\left[\tan^{-1}\frac{\underline{x}+u}{\sigma} - \tan^{-1}\frac{\bar{x}-u}{\sigma}\right]du. \tag{3.12.1}$$

The corresponding result for the dipole phase function is

$$R_{\text{II-B}}(x, x') = \frac{3\pi^{-3/2}}{8}\sigma\int_{\frac{1}{2}|\bar{x}-\underline{x}|}^\infty e^{-u^2}\int_{\bar{x}-u}^{\underline{x}+u}\left[3-\left(\frac{x-t}{u}\right)^2-\left(\frac{x'-t}{u}\right)^2 +3\left(\frac{x-t}{u}\right)^2\left(\frac{x'-t}{u}\right)^2\right]\frac{dt\,du}{t^2+\sigma^2}. \tag{3.12.2}$$

The integrals over t are elementary but the resulting expression is so complicated that it will not be given here. The final integration over u in both (3.12.1) and (3.12.2) must be done numerically or by an expansion of some sort.

$R_{\text{II-A}}$ was first derived by Unno (12) and subsequently by Sobolev (30). Jefferies and White (20) plot $R_{\text{II-A}}(x, x')/\phi(x')$ for $\sigma = \cdot 001$, and Sobolev tabulates and plots $\sqrt{(\pi)}R_{\text{II-A}}(x, x')$ for $\sigma = \cdot 1$.

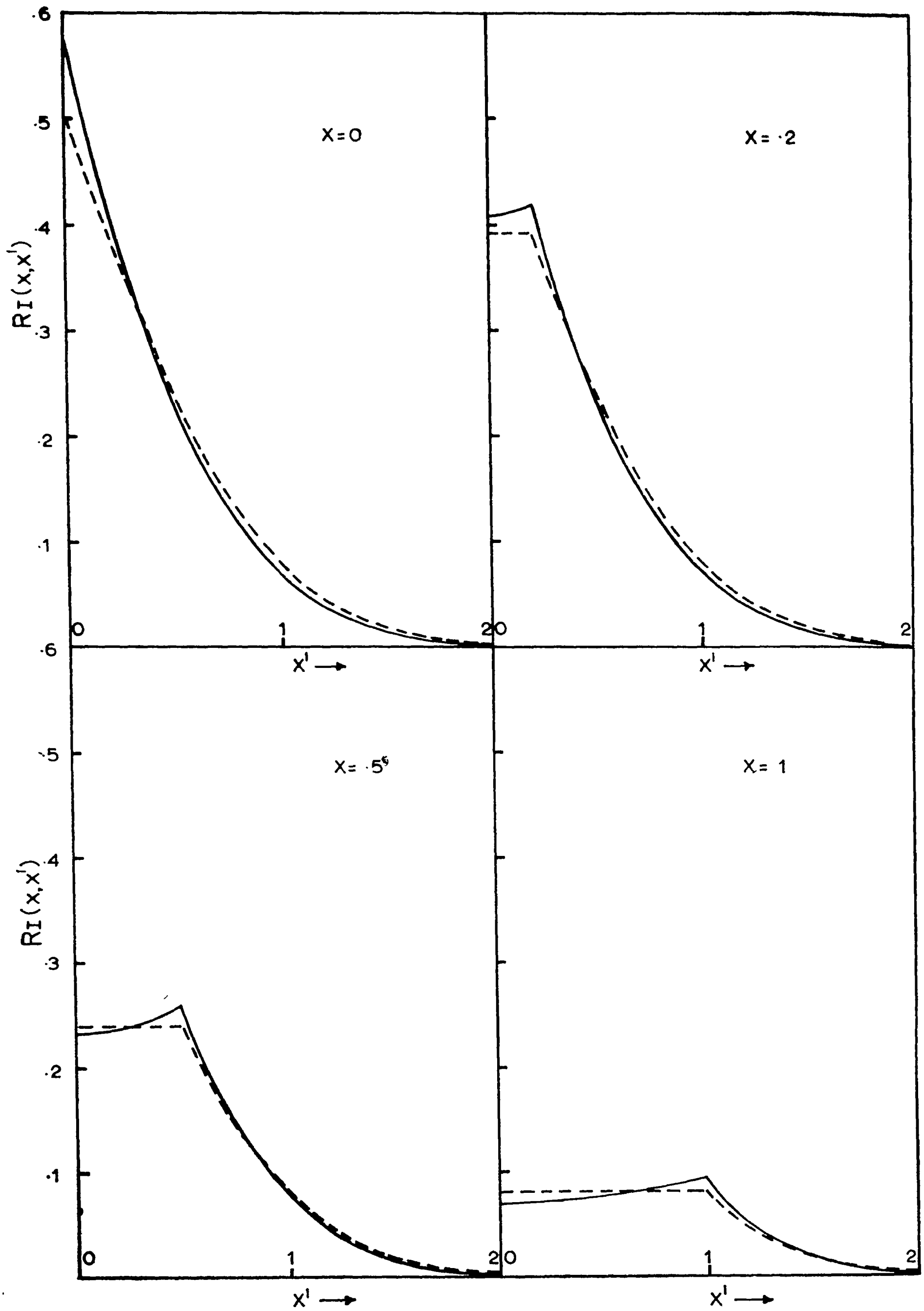

FIG. 1.—*Redistribution function for zero natural line width:* -------- *isotropic scattering;* ——— *dipole scattering. These functions are symmetric about* $x'=0$.

3.2. *Noncoherence in rest frame.*—Making the same assumptions as in (2.3), we get from (3.0.7)

$$R_u(\nu,\nu')=\tfrac{1}{2}\frac{\delta^2}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}g(\mu,\mu')\frac{1}{(\nu-\nu_0-\Delta\mu u)^2+\delta^2}\,\frac{1}{(\nu'-\nu_0-\Delta\mu' u)^2+\delta^2}\,d\mu\,d\mu'. \tag{3.2.1}$$

We no longer have any restriction on the range of μ, μ' or u. For $g=\frac{1}{2}$, the integrations are elementary; summing these results over the Maxwellian distribution we obtain

$$R_{\text{III-A}}(x, x') = \pi^{-5/2} \int_0^\infty e^{-u^2} \left[\tan^{-1}\frac{x+u}{\sigma} - \tan^{-1}\frac{x-u}{\sigma}\right] \times \left[\tan^{-1}\frac{x'+u}{\sigma} - \tan^{-1}\frac{x'-u}{\sigma}\right] du. \quad (3.2.2)$$

This result is new and is potentially useful when impact and Doppler broadening are both important, although it is valid only to the extent that the momentum of the scattering atom is unchanged by the collisions it experiences.

3.21. *Resonance scattering.*—Substituting (2.31.1) into (3.0.7) with $g=\frac{1}{2}$, and summing over the Maxwellian distribution, we obtain

$$R_{\text{IV-A}}(x, x') = \pi^{-5/2} \sigma_j \int_0^\infty e^{-u^2} u \int_{-1}^{1} \left[\tan^{-1}\frac{x'-x+u(1-\mu)}{\sigma_i} - \tan^{-1}\frac{x'-x-u(1-\mu)}{\sigma_i}\right] \times \frac{d\mu}{(x-\mu u)^2+\sigma_j^2} du. \quad (3.21.1)$$

This form should be used for all lines in which the width of the lower level is significantly different from zero.

3.3. *Direct integration of* $R_{\text{I-A}}$ $(x, \mathbf{n};\, x'\mathbf{n}')$.—As a check on the preceding, we will carry out the integration of $R(x, \mathbf{n};\, x', \mathbf{n}')$ for the case of zero line width and isotropic scattering. Substituting (2.21.2) into (3.0.1) and using some trigonometric identities yields

$$R_{\text{I-A}}(x, x') = \frac{1}{2\pi} \int_0^\pi \exp\left[-\frac{x'^2+x^2-2xx'\cos\gamma}{\sin^2\gamma}\right] d\gamma. \quad (3.3.1)$$

It is sufficient to take x and x' positive. Then

$$r = [x^2+x'^2-2xx'\cos\gamma]^{1/2}$$

is the third side of a triangle with sides x and x' enclosing angle γ. Let θ be the angle opposite $\bar{x}$; then by the law of sines we have

$$\frac{r}{\sin\gamma} = \frac{\bar{x}}{\sin\theta} = \frac{x}{\sin(\gamma+\theta)}, \quad (3.3.2)$$

which provides a simplification of (3.3.1). The angle θ as defined here is a monotonic function of γ, so that a straightforward calculation gives

$$R_{\text{I-A}}(x, x') = \frac{1}{2\pi} \int_0^\pi e^{-\left(\frac{\bar{x}}{\sin\theta}\right)^2} \left[1 + \frac{\underline{x}}{\bar{x}} \frac{\cos\theta}{\sqrt{1-\left(\frac{\underline{x}}{\bar{x}}\right)^2 \sin^2\theta}}\right] d\theta. \quad (3.3.3)$$

Substituting

$$y = \operatorname{ctn}\theta$$

into (3.3.3) gives

$$R_{\text{I-A}}(x, x') = \frac{e^{-\underline{x}^2}}{2\pi} \int_{-\infty}^{\infty} e^{-\bar{x}^2 y^2} \left[\frac{1}{1+y^2} + \text{odd function of } y\right] dy$$
$$= \tfrac{1}{2}\operatorname{erfc}(\underline{x}), \quad (3.3.4)$$

the final step utilizing a known result (**21**). Equation (3.3.4) agrees with (3.11.3). The equivalence of form in (3.11.3) to that in (3.3.1) was given without proof by Unno (7).

4.0. *Symmetry properties.*—Certain symmetries are apparent in the redistribution functions; these are examined here because of their use in simplification of the transfer equation and because of their value in computation of these functions. In order to discover the generality of these results, we begin by rewriting the general formula (2.1.8). Let

$$\tilde{f}(\nu-\nu_0)=f(\nu) \tag{4.0.1}$$

and

$$\tilde{p}(\nu-\nu_0, \nu'-\nu_0)=p(\nu,\nu'). \tag{4.0.2}$$

Then

$$R(x,\mathbf{n};\,x',\mathbf{n}')=\frac{g(\mathbf{n},\mathbf{n}')}{4\pi^2}\Delta^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-u_1^2-u_2^2}\tilde{f}[\Delta(x-au_1-bu_2)]$$
$$\times\tilde{p}[\Delta(x-au_1-bu_2),\Delta(x'-au_1+bu_2)]\,du_1\,du_2. \tag{4.0.3}$$

Changing the signs of x, x', u_1 and u_2 gives

$$R(-x,\mathbf{n};\,-x',\mathbf{n})=\frac{g(\mathbf{n},\mathbf{n}')\Delta^2}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-u_1^2-u_2^2}\tilde{f}[\Delta(-x+au_1+bu_2)]$$
$$\times\tilde{p}[\Delta(-x+au_1+bu_2),\Delta(-x'+au_1-bu_2)]\,du_1\,du_2. \tag{4.0.4}$$

If we assume the scattering process is symmetric about the line centre in the atom's frame, so that

$$\tilde{f}(\xi)\tilde{p}(\xi,\xi')=\tilde{f}(-\xi)\tilde{p}(-\xi,\,-\xi'), \tag{4.0.5}$$

we see immediately from (4.0.4) that

$$R(-x,\mathbf{n};\,-x',\mathbf{n}')=R(x,\mathbf{n};\,x',\mathbf{n}'), \tag{4.0.6}$$

and consequently

$$R(-x,\,-x')=R(x,x'). \tag{4.0.7}$$

These results are quite general, depending only on the assumption (4.0.5), and in particular, hold for all the cases considered in this paper. On the other hand, (4.0.5) is not true, for instance, in the asymmetrical results of statistical broadening.

The transformation $\mathbf{n}\rightarrow-\mathbf{n}$ changes the sign of a and b in the expressions for incoming frequencies. Keeping in mind that $g(\mathbf{n},\mathbf{n}')$ depends only on $\cos\gamma$, we have

$$R(-x,\,-\mathbf{n};\,x',\mathbf{n}')=\frac{g(\mathbf{n},\mathbf{n}')\Delta^2}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-u_1^2-u_2^2}\tilde{f}[\Delta(-x+au_1+bu_2)]$$
$$\times\tilde{p}[\Delta(-x+au_1+bu_2),\Delta(x'-au_1+bu_2)]du_1\,du_2. \tag{4.0.8}$$

In order to recover (4.0.3), it is sufficient that

$$\tilde{f}(\xi)\tilde{p}(\xi,\xi')=\tilde{f}(-\xi)\tilde{p}(-\xi,\,-\xi'). \tag{4.0.9}$$

This is obviously true when

$$\tilde{p}(\xi,\xi')=\tilde{p}(\xi')$$

and

$$\tilde{f}(\xi)=\tilde{f}(-\xi)$$

which hold for impact broadening. Another sufficient condition for the equivalence of (4.0.3) and (4.0.8) is

$$\tilde{f}(\xi)=\delta(\xi),$$

which, since it implies $p(\xi,\xi')=\delta(\xi-\xi')$, is applicable to R_{I} only. The same arguments hold for $R(-x,\mathbf{n}; x', -\mathbf{n}')$, so we have

$$R_i(-x, -\mathbf{n}; x',\mathbf{n}')=R_i(-x,\mathbf{n}; x', -\mathbf{n}')=R_i(x,\mathbf{n}; x',\mathbf{n}'), \; i=\mathrm{I, III}. \tag{4.0.10}$$

This may be verified from the explicit forms by making the substitution $\gamma\to\pi-\gamma$ and $x\to -x$. On the other hand, $p(\xi,\xi')=\delta(\xi-\xi')$ alone is not sufficient to insure that (4.0.10) will hold.

Since R depends only on the angle between $\mathbf{n}$ and $\mathbf{n}'$ we may obviously interchange them without changing R. The situation for x and x' is not so simple. It is easy to see from (4.0.3) that $p(\xi,\xi')=\delta(\xi-\xi')$ is a sufficient condition for

$$R_i(x,\mathbf{n}; x',\mathbf{n}')=R_i(x',\mathbf{n}; x,\mathbf{n}'), i=\mathrm{I, II}. \tag{4.0.11}$$

This result does not hold in the case of noncoherence in the atom's frame, but the integrated result is symmetric in x and x', hence

$$R_i(x, x')=R_i(x', x), \quad i=\mathrm{I, II, III}.$$

5.0. *The transfer equations.*—Since several different functions similar to R and p have been introduced into the literature, it appears desirable to write down the transfer equations containing R. The familiar Milne-Eddington equation becomes, in the usual notation

$$\frac{\mu}{k_0}\frac{dI_x(\mu,z)}{dz}=-\phi(x)I_x(\mu,z)+4\pi(1-\epsilon)\int_{4\pi}\int_{-\infty}^{\infty} I_{x'}(\mu',z)R(x',\mathbf{n}'; x,\mathbf{n})\,dx'd\Omega' + \epsilon\phi(x)B_x(T), \tag{5.0.1}$$

where k_0 is the absorption coefficient integrated over all frequencies.

If $I_{x'}(\mu',z)$ under the integral sign in (5.0.1) is assumed to be isotropic, then (5.0.1) reduces to the familiar form

$$\frac{\mu}{k_0}\frac{dI_x(\mu,z)}{dz}=-\phi(x)I_x(\mu,z)+(1-\epsilon)\int_{-\infty}^{\infty} J_{x'}(z)R(x', x)dx'+\epsilon\phi(x)B_x(T).$$

Unno (7) has shown that this substitution of J for I has the same accuracy as Eddington's approximation if $R(-x, -\mathbf{n}; x',\mathbf{n}')=R(x, \mathbf{n}; x',\mathbf{n}')$ and if $I(x)=I(-x)$. These conditions are obviously not very general.

6.0. *Computation of* $R(x, x')$.—The redistribution functions for non-zero line width are all expressed by definite integrals which apparently cannot be evaluated in terms of a finite number of elementary functions. The direct numerical integration is quite time-consuming because of both the step-function-like behaviour of the integrands and the large number of points at which $R(x, x')$ is required in the numerical solution of the transfer equation. In one case, III-A, we have been able to obtain an expansion, useful for at least some range of frequencies.

6.1. $R_{\mathrm{III\text{-}A}}(x, x')$.—We will work formally, as all of the inter-changes of limiting processes can be justified by appeal to standard theorems to be found, for instance, in chapters V and VI of Carslaw's book (22). Rewrite (3.2.2) as

$$R(x,x')=\pi^{-5/2}\int_0^{\infty} e^{-y^2}F_\sigma(x,y)F_\sigma(x',y)\,dy, \tag{6.1.1}$$

where

$$F_\sigma(x,y) \equiv \tan^{-1}\frac{x+y}{\sigma} - \tan^{-1}\frac{x-y}{\sigma}$$
$$= 2\int_0^\infty e^{-\sigma t}\cos xt \sin yt\, t^{-1}\, dt, \qquad (6.1.2)$$

the integral representation following from the well-known Laplace Transform of $t^{-1}\sin t$.

For $\sigma > 0$, we can substitute (6.1.2) into (6.1.1) twice and interchange the order of integration to obtain

$$R(x,x') = 4\pi^{-5/2}\int_0^\infty\int_0^\infty e^{-\sigma(s+t)}\cos xt \cos x's\, q(s,t) s^{-1}t^{-1}\, ds\, dt, \qquad (6.1.3)$$

where

$$q(s,t) = \int_0^\infty e^{-y^2}\sin yt \sin ys\, dy$$
$$= \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}(t^2+s^2)}\sinh(\tfrac{1}{2}st).$$

Replacing $\sinh(\frac{1}{2}st)$ by its series and integrating term by term in (6.1.3) we obtain finally

$$R_{\text{III-A}}(x,x') = \frac{2}{\pi^2}\sum_{n=0}^{\infty}[2^{2n+1}(2n+1)!]^{-1}K_{2n}(\sigma,x)K_{2n}(\sigma,x'), \qquad (6.1.4)$$

where

$$K_n(\sigma,x) \equiv \int_0^\infty e^{-\frac{1}{4}t^2-\sigma t}\cos xt\, t^n\, dt, \quad n = 0, 1, 2 \ldots .$$

The properties and methods of computation of these functions are discussed in the Appendix.

From the inequalities (A.2) we see that if we can prove the absolute convergence of the series (6.1.4) for $x = x' = \sigma = 0$, we can establish the absolute convergence and hence the convergence for all x and x', $\sigma \geqslant 0$. Using (A.2) for $K_{2n}(0, 0)$, we obtain the test series whose general term is

$$a_n = 2^{2n+1}\Gamma^2(n+\tfrac{1}{2})/\Gamma(2n+2). \qquad (6.1.5)$$

Using Sterling's approximation, we obtain

$$a_n \sim \frac{\sqrt{\pi}}{8}\frac{1}{(2n+1)n^{1/2}}(1+O(n^{-1})),$$

so that the series of terms (6.1.5) converges. Hence the series (6.1.4) converges but the rate of convergence is very slow, going as $n^{-3/2}$ for small x, x'. As x gets larger the functions $K_n(\sigma, x)$ for $\sigma > 0$ begin to decrease rapidly and the series is then useful.

In the limit $\sigma = 0$ the functions $K_{2n}(\sigma, x)$ become Hermite Polynomials and (6.1.4) reduces to the expansion of $R_{\text{I-A}}(x, x')$ obtained by Unno (**23**).

6.2. $R_{\text{II-A}}(x, x')$.—One can obtain a formal expansion of $R_{\text{II-A}}(x, x')$ by replacing the lower limit of integration in (3.12.1) by a step function, integrating from $-\infty$ to ∞, and going through the same operations as for $R_{\text{III-A}}$, but because of the non-uniform convergence of the representation of the step function, the procedure cannot be justified and the result will not be given here.

The best we can do in this case is to obtain a useful upper bound. We observe from (3.1.2.1) that

$$|R_{\text{II-A}}(x, x')| \leqslant \pi^{-3/2} \int_{\frac{1}{2}(\bar{x}-\underline{x})}^{\infty} e^{-u^2} \left| \tan^{-1}\frac{\underline{x}+u}{\sigma} - \tan^{-1}\frac{\bar{x}-u}{\sigma} \right| du \leqslant \pi^{-1/2} \int_{\frac{1}{2}(\bar{x}-\underline{x})}^{\infty} e^{-u^2}\, du$$

or

$$R_{\text{II-A}}(x, x') \leqslant \tfrac{1}{2}\,\text{erfc}\,(\tfrac{1}{2}(\bar{x}-\underline{x})). \tag{6.2.1}$$

7.0. *Asymptotic behaviour.*—Since the intensity of a line formed in a situation in which non-coherent scattering is important is determined primarily by the behaviour of the absorption coefficient and redistribution function in the wings of the line, it is useful to know the asymptotic behaviour of the redistribution functions. Using the well-known asymptotic expansion for erfc(x) we obtain, from (3.11.3) and (6.2.1) respectively, the asymptotic form for cases I and II:

$$R_{\text{I-A}}(x, x') \sim O\left[\frac{e^{-|\bar{x}|^2}}{|\bar{x}|}\right]$$

and

$$R_{\text{II-A}}(x, x') \sim O\left[\frac{e^{-\frac{1}{4}(\bar{x}-\underline{x})^2}}{(\bar{x}-\underline{x})}\right].$$

From the asymptotic formulae for $K_{2n}(\sigma, x)$ given in the Appendix, we obtain for $|x|$ and $|x'|$ large and $\sigma > 0$

$$R_{\text{III-A}}(x, x') \sim O\left(\frac{\sigma^2}{x^2 x'^2}\right).$$

The long range character of R_{III} should radically alter the solution of the transfer equation from those in the cases I and II, which are in turn differentiated by the short and long range behaviour of the absorption coefficients, respectively.

Acknowledgments.—I should like to express my gratitude to the Office of Naval Research of the United States Navy for support of this work under contract N 62558–3207. I also want to acknowledge my indebtedness to Dr M. Aub for some valuable suggestions on the work in Section 3.3. Finally, I wish to thank Dr M. J. Seaton for introducing me to the problems of non-coherent scattering and for critically reading the manuscript.

APPENDIX

$K_{2n}(\sigma, x)$ and related functions.

We have introduced the functions

$$K_n(\sigma, x) = \int_0^{\infty} e^{-\frac{1}{4}t^2 - \sigma t} \cos xt\, t^n\, dt, \quad n = 0, 1, 2, \ldots. \tag{A.1}$$

These functions can be expressed in terms of parabolic cylinder functions but since the appropriate functions have not been tabulated we give here some results to facilitate their computation. From (A.1) we see immediately that

$$|K_n(\sigma, x)| \leqslant |K_n(\sigma, 0)| \leqslant |K_n(0, 0)| = 2^n \Gamma\left(\frac{n+1}{2}\right). \tag{A.2}$$

which is used in the text to establish convergence of the expansions.

One can easily develop an extensive analytical apparatus for these functions, of which we give only two results, a recurrence relation and an expansion in powers of σ:

$$K_n(\sigma, x)+4\sigma K_{n-1}(\sigma, x)-2[2n-3-2\sigma^2-2x^2]K_{n-2}(\sigma, x)-8(n-2)\sigma K_{n-3}(\sigma, x)$$
$$+4(n-2)(n-3)K_{n-4}(\sigma, x)=0, \quad x\neq 0 \tag{A.3}$$

and

$$K_n(\sigma, x)=\sum_{m=0}^{\infty}\frac{(-\sigma)^m}{m!}I_{n+m}(x), \tag{A.4}$$

where

$$I_n(x)=K_n(0, x)=2^{n+1}\int_0^{\infty} e^{-t^2}\cos 2xt\, t^n\, dt. \tag{A.5}$$

The functions $I_n(x)$ have been introduced since they are functions of one variable and can thus be tabulated once for all. As well, we have explicit expressions for them. The expansion of $K_n(\sigma, x)$ in powers of σ is an extension of the familiar expansion of the Voigt function in the same way; in fact $K_0(\sigma, x)$ is, apart from a factor of $\sqrt{\pi}$, the Voigt function given in (2.22.4). The functions $I_n(x)$ are simply related to the $H_n(x)$ of Harris (24) and to the functions introduced by Plass and Fivel (25).

The functions $I_n(x)$ obey the recurrence relation (A.3) with $\sigma=0$; as well, they have the derivative formula

$$\frac{d^2}{dx^2}I_n(x)=-I_{n+2}(x). \tag{A.6}$$

Combining the two we obtain a differential equation

$$\frac{d^2}{dx^2}I_n(x)+2x\frac{d}{dx}I_n(x)+2(n+1)I_n(x)=0, \tag{A.7}$$

which can easily be transformed into Hermite's equation and Weber's equation. The series expansion for $I_n(x)$ valid for all x is

$$I_n(x)=2^n\sum_{m=0}^{\infty}\frac{(-1)^m(2x)^{2m}\Gamma\left(m+\frac{n+1}{2}\right)}{(2m)!}. \tag{A.8}$$

This series has limited computational use since the terms become very large before they begin to decrease, although then convergence is very rapid.

From the integral representation for Hermite Polynomials we obtain the explicit form

$$I_n(x)=(-1)^{n/2}\sqrt{\pi}\,e^{-x^2}H_n(x),\ n=0, 2, 4\ldots, \tag{A.9}$$

where $H_n(x)$ are the Hermite Polynomials defined by

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}).$$

The functions $I_n(x)$ for odd n have an entirely different behaviour from those of even order, and may be expressed in terms of Dawson's function

$$F(x)=\int_0^x e^{t^2-x^2}\,dt=\int_0^{\infty} e^{-t^2}\sin 2xt\,dt, \tag{A.10}$$

3*

which satisfies the differential equation

$$F'(x) = 1 - 2xF(x). \tag{A.11}$$

As shown by Plass and Fivel (**25**), the functions $I_n(x)$, $n = 1, 3, 5, \ldots$ are related to the nth derivative of Dawson's function. Using (A.6), (A.11) and the well-known properties of Hermite Polynomials, one can obtain explicit formulae for these functions, the first few of which are listed here:

$$\left.\begin{aligned}
I_1(x) &= 2[-F(x)H_1(x) + H_0(x)],\\
I_3(x) &= 2[F(x)H_3(x) - H_2(x) + 2H_0(x)],\\
I_5(x) &= 2[-F(x)H_5(x) + H_4(x) - 6H_2(x) + 8H_0(x)],\\
I_7(x) &= 2[F(x)H_7(x) - H_6(x) + 10H_4(x) - 48H_2(x) + 48H_0(x)],\\
I_9(x) &= 2[-F(x)H_9(x) + H_8(x) - 14H_6(x) + 120H_4(x)\\
&\qquad - 480H_2(x) + 384H_0(x)].
\end{aligned}\right\} \tag{A.12}$$

Tables of $F(x)$ are available: the two best are those of Lohmander and Rittsten (**31**), $x = 0(\cdot 01)3(\cdot 02)5$, $1/x = 0(\cdot 005)\cdot 2$, $10d$, and of Rosser (**26**), $x = 0(\cdot 05)4(\cdot 1)6\cdot 5(\cdot 5)12\cdot 5$, $10d$. Also useful is the well-known tabulation by Miller and Gordon (**27**), although it has some last figure errors. Good tables of $e^{x^2}F(x)$ are given by Terrill and Sweeny (**28**). For use in machine computing, a Chebyshev expansion of $F(x)$ capable of very high precision has been obtained (**29**). Unfortunately the formulae (A.12) suffer severely from cancellation and can only be used for small values of x and n, unless one computes $H_n(x)$ and $F(x)$ to a very large number of significant figures.

By a straightforward application of the method of Steepest Descents to the integral

$$\int_\Gamma e^{-t^2+2ixt} t^n \, dt, \quad n = 1, 3, 5, \ldots,$$

where Γ is the closed rectangular path in the complex t-plane with vertices (0, 0), (∞, 0), (∞, x), (0, x) we find that

$$I_n(x) \sim (-1)^{\frac{n+1}{2}} n!\, x^{-n-1} g_n\left(\frac{1}{x}\right), \tag{A.13}$$

where

$$g_n\left(\frac{1}{x}\right) = \sum_{i=0} A_i^n x^{-2i}, \quad A_0^n = 1. \tag{A.14}$$

Since it becomes increasingly tedious to obtain the higher coefficients in (A.14) by the method of Steepest Descents, we have used a series solution of the differential equation for $g_n(1/x)$ to obtain a recurrence relation for the higher coefficients

$$\frac{A_1^n}{A_0^n} = \frac{(n+1)(n+2)}{4}, \tag{A.15}$$

$$\frac{A_{i+1}^n}{A_i^n} = \frac{4i^2 + 2i(2n+3) + (n+1)(n+2)}{4i(+1)}. \tag{A.16}$$

Since the $I_n(x)$ for even n are exponentially decreasing, we need know no more about their asymptotic behaviour.

For all x, the even order $I_n(x)$ can be obtained using (A.9) for $n=0$, 2 and (A.3) for higher n. For x less than about 3, working to about eight significant figures, this process works for the odd order functions as well, starting from the first two formulae of (A.12). To bridge the gap between this region and the asymptotic region ($x \sim n$), the differential equation (A.7) must be used.

Tables of the functions

$$G_n(x) \equiv 2^{n+1} \int_0^\infty e^{-t^2+2ixt} t^n \, dt \equiv I_n(x) + iJ_n(x),$$

$n=0(1)12$, $x=0(\cdot 1)3$, $n=0(1)10$, $x=3(\cdot 1)12$, $6sf$, have been computed on the Mercury Computer at the University of London Computer Unit and will be deposited in the Library of the Royal Astronomical Society.

Physics Department,
University College,
London, W.C.1:
1962 *July*.

References

(1) L. Spitzer, Jr., *Ap. J.*, **99**, 1, 1944.
(2) L. Spitzer, Jr., *M.N.*, **96**, 794, 1936.
(3) R. v. d. R. Woolley, *M.N.*, **98**, 624, 1938.
(4) H. Zanstra, *M.N.*, **101**, 273, 1941; *M.N.*, **106**, 225, 1946.
(5) F. N. Edmonds, Jr., *Ap. J.*, **121**, 418, 1955.
(6) T. Holstein, *Westinghouse Research Laboratories Scientific Paper* No. 1501, Appendix A.
(7) W. Unno, *Pub. of the Astronomical Society of Japan*, **3**, 158, 1952.
(8) R. N. Thomas, *Ap. J.*, **125**, 260, 1947.
(9) G. B. Field, *Ap. J.*, **129**, 551, 1959.
(10) H. Zanstra, *B.A.N.*, **11**, 1, 1949.
(11) L. G. Henyey, *Proc. Nat. Acad. Sci.*, **26**, 50, 1941.
(12) W. Unno, *Pub. of the Astronomical Society of Japan*, **4**, 100, 1952.
(13) V. Weisskopf and E. Wigner, *Zs. f. Phys.*, **63**, 54, 1930.
(14) R. v. d. R. Woolley and D. W. N. Stibbs, *The Outer Layers of a Star*, Oxford, 1953.
(15) L. G. Henyey, *Ap. J.*, **103**, 332, 1946.
(16) S. Rosseland, *Theoretical Astrophysics*, Oxford, 1936, p. 90.
(17) V. Weisskopf, *Ann. d. Phys.*, **9**, 23, 1931.
(18) V. Weisskopf, *Observatory*, **56**, 291, 1933.
(19) W. Heitler, *Quantum Theory of Radiation*, 3rd Edition, Oxford 1954, p. 199.
(20) J. T. Jefferies and O. R. White, *Ap. J.*, **132**, 767, 1960.
(21) W. Grobner and N. Hofreiter, *Integraltafel* **2**, 2nd edition, Vienna, 1958.
(22) H. S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals*, 2nd edition, London, 1921.
(23) W. Unno, *Ap. J.*, **129**, 388, 1959.
(24) D. Harris III, *Ap. J.*, **109**, 53, 1949.
(25) G. N. Plass and D. I. Fivel, *Ap. J.*, **117**, 225, 1953.
(26) J. B. Rosser, *Theory and Application of* $\int_0^z e^{-x^2}\,dx$ *and* $\int_0^z e^{-p^2y^2}\,dy \int_0^y e^{-x^2}\,dx$, Mapleton House, Brooklyn, N.Y., 1948.
(27) W. Lash Miller and A. R. Gordon, *Journal of Phys. Chemistry*, **35**, 2878, 1931.
(28) H. M. Terrill and L. Sweeny, *Journal of the Franklin Institute*, **237**, 495, 1944; **238**, 220, 1944.
(29) D. G. Hummer, to be published.
(30) V. V. Sobolev, *Vestnik Leningradskogo Gosudarstvennogo Universiteta* No. 5, 85, 1955.
(31) B. Lohmander and S. Rittsten, *Kungl. Fysiografiska Sällskapets I Lund Förhandlingar*, Bd. 28, Nr. 6, 1958.