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NON COMMUTATIVE INTEGRATION FOR MONOTONE SEQUENTIALLY CLOSED C^* ALGEBRAS

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Introduction.

Recall that a monotone sequentially closed C^* algebra (MSC for short) is a C^* algebra in which every norm-bounded increasing sequence has a least upper bound, and that a Baire* algebra is an MSC which has a separating family of σ-normal states.

Previously R. V. Kadison [6], [7], E. Kehlet [8] and G. K. Pedersen [7], [11] have investigated these algebras, and the general results on MSC, which are used in this paper, can all be found there. The use of [7] for our purpose needs the comment that the statements from p. 3 to p. 9, Definition C, are all true for an MSC. In particular [7] then tells us that the polar decomposition is possible in an MSC, that we can form range projections, and that the set of projections in an MSC is a lattice under $\wedge$ and $\vee$.

Unfortunately it is unknown whether the comparison lemma holds in general for an MSC. It is therefore impossible to construct the algebra of mesurable operators as S. K. Berberian and K. Saitô do it in [1] and [12]. However, assume that A is an MSC with unit, and that F is a non empty subset of the set A_p of projections in A satisfying the following four conditions:

$$e,f \in F \;\Rightarrow\; e \vee f \in F .$$

$$e \in A_p, f \in F, e \leqq f \;\Rightarrow\; e \in F .$$

$$e \in A_p, f \in F, e \sim f \;\Rightarrow\; e \in F .$$

For e in A_p and (f_n) a sequence of elements in F decreasing to 0, $e \prec f_n$ for all n implies $e=0$.

It is then possible to construct an algebra of measurable operators with almost the same properties as the algebras considered in [1], [2] and [12]. Since the algebra so constructed is abstract, we have to give another definition of integrability than Segal gives in [13], but a theorem,

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which in essence is also proved by G. K. Pedersen [10, Prop. 4.3], makes it possible to prove the same integration theorems as Segal does.

Before we state the Fubini theorem, we prove a theorem on the tensor product of Baire* algebras.

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1. A center-valued trace for a certain MSC.

In this section A will denote an MSC with unit.

1.1. Definition. For x,y in A^+ we write $x \approx y$ if there is a sequence of elements $(z_n)_{n\in \mathbf{N}}$ in A such that

$$x = \sum z_n{}^* z_n \quad \text{and} \quad y = \sum z_n z_n{}^* .$$

We write $x \lessapprox y$ if there is a z in A^+ such that $x \approx z \leqq y$.

1.2. Definition. For x in A^+ we say that x is countably finite if $y \leqq x$ and $y \approx x$ imply $y = x$. When 1 is finite, we say that A is countably finite.

From [7] we obtain that $\approx$ is a countably additive equivalence relation and that $x \lessapprox y$ and $y \lessapprox x$ imply $x \approx y$. Note that A is said to be countably generated if A has a countable subset B such that A is the smallest sub-MSC containing B.

1.3. Theorem. *Let A be a countably generated, countably finite* MSC *with unit, then there exists a unique finite normalized, σ-normal faithfull center-valued trace on A.*

Proof. The proof is divided into some lemmas, which follow below.

1.4. Notation. For x in A, $L[x]$, $R[x]$ and $N[x]$ denote left, right and null projection of x, respectively. Further let $\prec$ and $\sim$ denote the usual preorder and equivalence in A_p induced by partial isometries. For x in A^+, $C[x]$ denotes the central support of $L[x]$. (Cf. [3], [7].)

1.5. Lemma. *For x,y in A^+ with $x \leqq 1$ and $y \leqq 1$ there exist x_1, x_2, y_1, y_2 in A^+ such that*

$$x = x_1 + x_2, \quad y = y_1 + y_2 ,$$
$$x_1 \approx y_1, \quad C[x_2]C[y_2] = 0 .$$

PROOF. (Cf. [3, Theorem 2.6].) We can suppose that the generating set U consists of a sequence $(u_n)_{n\in\mathsf{N}}$ of unitaries, which is closed under multiplication and *. Define

$$v_{11} = x^{\frac12}u_1yu_1^*x^{\frac12}, \quad z_{11} = y^{\frac12}u_1^*xu_1y^{\frac12},$$

then

$$v_{11} \approx z_{11}, \quad v_{11} \leqq x, \quad z_{11} \leqq y;$$

define

$$v_{12} = (x-v_{11})^{\frac12}u_1(y-z_{11})u_1^*(x-v_{11})^{\frac12},$$
$$z_{12} = (y-z_{11})^{\frac12}u_1^*(x-v_{11})u_1(y-z_{11})^{\frac12},$$

then

$$v_{11}+v_{12} \approx z_{11}+z_{12}, \quad v_{11}+v_{12} \leqq x, \quad z_{11}+z_{12} \leqq y,$$

etc. For

$$v_1 = \sum v_{1n}, \quad z_1 = \sum z_{1n},$$

we then have

$$v_1 \approx z_1, \quad v_1 \leqq x, \quad z_1 \leqq y.$$

Define now a by $a=(x-v_1)^{\frac12}u_1(y-z_1)u_1^*(x-v_1)^{\frac12}$. Then for each $m \in \mathsf{N}$,

$$a \leqq (x-v_1)^{\frac12}u_1\left(y-\textstyle\sum_{i=1}^{m-1} z_{1i}\right)u_1^*(x-v_1)^{\frac12} \lesssim z_{1m}.$$

Choose a_m such that $a \approx a_m \leqq z_{1m}$. Then $\sum_{m\in\mathsf{N}} a_m \leqq \sum_{m\in\mathsf{N}} z_{1m} = z_1$, so the countable finiteness of A implies that $a=0$.

Continuing in the same way with $u_2, x-v_1, y-z_1$ instead of u_1, x, y we find v_2 and z_2 with

$$v_2 \approx z_2, \quad v_2 \leqq x-v_1, \quad z_2 \leqq y-z_1,$$

etc. Define $x_1=\sum v_n$, $y_1=\sum z_n$. Then $x_1 \approx y_1$, $x_1 \leqq x$, $y_1 \leqq y$, and, for $u_n \in U$,

$$(x-x_1)^{\frac12}u_n(y-y_1)u_n^*(x-x_1)^{\frac12} = 0,$$

hence $(x-x_1)u_n(y-y_1)=0$ and, as U is closed under multiplication and *,

$$(x-x_1)A(y-y_1) = 0.$$

Thus, by [8, lemma 4.3]

$$C[x-x_1]C[y-y_1] = 0,$$

and the lemma is proved.

1.6. LEMMA. *The equivalence classes under $\approx$ are uniformly closed.*

PROOF. Let $(x_n)_{n\in\mathsf{N}}$ be a sequence in A^+ of pairwise equivalent elements which converges uniformly to y in A^+. Lemma 1.5 gives the existence of a central projection p in A such that

$$(1-p)y \lessapprox (1-p)x_n \quad \text{and} \quad px_n \lessapprox py\,.$$

Assume further that $\|y-x_n\| \leqq 2^{-n}$ for all $n \in \mathsf{N}$. Then there exists an r in A^+ with

$$px_n \approx r \leqq py \leqq p(x_n+2^{-n}1)$$

and thus

$$0 \leqq py-r \leqq px_n+2^{-n}p-r \approx r+2^{-n}p-r = 2^{-n}p$$

because A is assumed to be countably finite. Proposition 2.6 in [7] gives then that there exist $q_n, s_n \in A^+$ such that

$$q_n+s_n = 2^{-n}p, \quad q_n \approx py-r, \quad s_n \approx (px_n+2^{-n}p-r)-(py-r)\,.$$

Since $\sum q_n \leqq \sum 2^{-n}p = p$ and $q_{n+1} \approx q_n$, we conclude $q_n = 0$ and $py = r \approx px_n$, so $y \lessapprox x_n$.

Let $t = \sup_n \|x_n\|$. Then $t1-x_n \approx t1-x_{n+1}$ and $t1-x_n$ converges uniformly to $t1-y$, so $t1-y \lessapprox t1-x_n$ and $x_n \lessapprox y$ because 1 is countably finite.

1.7. Lemma. *For positive central elements x,y in A, $x \approx y$ implies $x = y$.*

Proof. Cf. [7, Theorem 3.5].

1.8. We want now to use the approximation theorem in [4, III, § 5]. We then ought to state and prove the Lemmas 1 to 5 for our algebra. However, as we have a unit and "range projections" in our algebra, it is easily seen that the proofs work as well in our case.

Combining this with Lemmas 1.6 and 1.7 we obtain a faithful map Φ from A^+ into the positive part of the center of A.

From the definition of $\approx$ it follows that Φ is a σ-normal, finite, normalized ($\Phi(1)=1$), center-valued trace. Every center-valued trace Ψ with these properties will be constant on the equivalence classes of $\approx$. Thus, for x in A^+

$$\Psi(x) = \Psi(\Phi(x)) = \Phi(x)\Psi(1) = \Phi(x)\,,$$

and we have finished the proof of Theorem 1.3.

2. The algebra of measurable operators.

In this section A will denote an MSC with unit, and F a subset of the set of projections A_p of A. The idea is then to let F act as the set of finite projections does in Berberian's and Saitô's construction of the algebra of measurable operators for an AW^* algebra. Since any set of orthogonal projections in an AW^* algebra has a least upper bound in the algebra, there is no reason, in the AW^* case, to consider the situation where we have "null-sets".

In the MSC case we cannot form uncountable sums of projections inside the algebra and, roughly speaking, in this way divide the "null-operators" out, so we will assume that we have a subset N of F representing the "null-sets".

2.1. Definition. A pair (F,N) of non empty subsets of A_p is called an $(\mathscr{F},\mathscr{N})$ system if and only if

1) $N \subseteq F$,
2) $e,f \in F \;\Rightarrow\; e \vee f \in F$,
3) $e \in A_p,\, f \in F,\, e \leqq f \;\Rightarrow\; e \in F$,
4) $e \in A_p,\, f \in F,\, e \sim f \;\Rightarrow\; e \in F$,
5) $e,f \in N \;\Rightarrow\; e \vee f \in N$,
6) $e \in A_p,\, f \in N,\, e \leqq f \;\Rightarrow\; e \in N$,
7) $e \in A_p,\, f \in N,\, e \sim f \;\Rightarrow\; e \in N$,
8) $e \in A_p,\, e \prec f_n,\, f_n \in F;\, f_n \downarrow f \in N \;\Rightarrow\; e \in N$,
9) $e_n \in N,\, e_n \uparrow e \;\Rightarrow\; e \in N$.

If $(F,0)$ is an $(\mathscr{F},\mathscr{N})$ system, F is said to be an $\mathscr{F}$ system.

We will now assume that A has an $(\mathscr{F},\mathscr{N})$ system (F,N).

2.2. Definition. A sequence $(e_n)_{n\in\mathsf{N}}$ of projections from A is called an SDD (strongly dense domain) if and only if e_n is increasing, $(1-e_n) \in F$ and $\inf(1-e_n) \in N$.

2.3. Definition. A pair of sequences (a_n, e_n) is called an EMO (essential measurable operator) if and only if (e_n) is an SDD, $a_n \in A$ and $a_n e_m = a_m e_m$, $a_n{}^* e_m = a_m{}^* e_m$ for $n \geqq m$.

2.4. Definition. Let x be in A and e a projection of A. Then

$$x^{-1}[e] = N[(1-e)x]\,.$$

2.5. Lemma. *Let* (x_n, e_n) *be an* EMO *and* (f_n) *an* SDD. *Then*

$$g_n = e_n \wedge x_n{}^{-1}[f_n]$$

is an SDD.

PROOF. Put $h_n = e_n \wedge (x_n e_n)^{-1}[f_n]$. Then $h_n \leqq e_n$ and

$$(1-f_n)x_n h_n = (1-f_n)x_n e_n h_n = 0$$

so $h_n \leqq g_n$, but

$$0 = (1-f_n)x_n g_n = (1-f_n)x_n e_n g_n$$

so $g_n \leqq h_n$ and $g_n = h_n$.

Since

$$\begin{aligned} g_n &= e_n \wedge N[(1-f_n)x_n e_n] = e_n \wedge N[(1-f_n)x_{n+1} e_n] \\ &= e_n \wedge N[(1-f_n)x_{n+1}] \leqq e_{n+1} \wedge N[(1-f_{n+1})x_{n+1}] = g_{n+1}, \end{aligned}$$

g_n is increasing. We also get $1-g_n \in F$ because

$$(1-g_n) = (1-e_n) \vee R[(1-f_n)x_n e_n] = (1-e_n) + R[(1-f_n)x_n e_n].$$

Put $g = \sup g_n$. Then for a natural number k

$$e_k(1-g)e_k = \inf_{n \geqq k} e_k(R[(1-f_n)x_n e_n])e_k$$

and for $n \geqq k$,

$$\begin{aligned} L[e_k(1-g)e_k] &\leqq L[e_k(R[(1-f_n)x_n e_n])e_k] \\ &\prec R[(1-f_n)x_n e_n] \prec 1-f_n \end{aligned}$$

so 8) in 2.1 gives $L[e_k(1-g)e_k] \in N$. Now let $e = \sup e_n$. Then

$$L[(1-g)e] = L[(1-g)e(1-g)] = \sup_k L[(1-g)e_k(1-g)],$$

but as

$$L[(1-g)e_k(1-g)] \sim L[e_k(1-g)e_k] \in N,$$

9) in 2.1 gives that $L[(1-g)e] \in N$. Finally

$$L[(1-g)(1-e)] \prec (1-e) \in N$$

and

$$(1-g) \leqq L[(1-g)e] \vee L[(1-g)(1-e)] \in N,$$

so the lemma is proved.

2.6. COROLLARY. *If* (e_n), (f_n) *are* SDD, *then* $(e_n \wedge f_n)$ *is* SDD.

PROOF. Put $x_n = 1$ for all n.

2.7. COROLLARY. *If* (f_n) *is* SDD *and* x *is in* A, *then* $(x^{-1}[f_n])$ *is* SDD.

PROOF. Put $x_n = x$; $e_n = 1$ for all n.

2.8. DEFINITION. EMO's (x_n, e_n), (y_n, z_n) are said to be equivalent, and we write $(x_n,e_n) \equiv (y_n,z_n)$ if and only if there exists an SDD (g_n) such that

$$x_n g_n = y_n g_n, \quad x_n{}^* g_n = y_n{}^* g_n .$$

2.9. Lemma. $\equiv$ *is an equivalence relation.*

Proof. Follows from Corollary 2.6.

2.10. Definition. The equivalence classes for $\equiv$ are called measurable operators MO, and the equivalence class of an EMO (x_n,e_n) is denoted $[x_n,e_n]$. The set of all MO's is denoted $M(A,F,N)$.

2.11. Theorem. $M(A,F,N)$ *is an associative algebra with involution* $*$ *over the complex numbers with respect to the operations*

$$[x_n,e_n]+[y_n,f_n] = [x_n+y_n, e_n \wedge f_n] ,$$
$$\lambda[x_n,e_n] = [\lambda x_n, e_n] ,$$
$$[x_n,e_n]^* = [x_n{}^*, e_n] ,$$
$$[y_n,f_n][x_n,e_n] = [y_n x_n, g_n] ,$$

where

$$g_n = (e_n \wedge x_n{}^{-1}[f_n]) \wedge (f_n \wedge y_n{}^{*-1}[e_n]) .$$

Proof. Use Lemma 2.5 and Corollary 2.6.

2.12. Lemma. *Define* $I = \{x \in A \mid L[x] \in N\}$. *Then* I *is a monotone sequentially, and hence uniformly, closed twosided ideal in* A.

Proof. For x in A we have $L[x] \sim R[x]$ so the properties of N imply that I is monotone sequentially closed and a twosided ideal in A.

From [6, p. 317] we then obtain that the selfadjoint part $I_{\text{s.a.}}$ of I is uniformly closed, so I is uniformly closed.

2.13. Lemma. A/I *is an* MSC *with unit and the canonical homomorphism* k *is* σ*-normal.*

Proof. Let (x_n) be a bounded increasing sequence of positive operators in A/I, and let (u_n) be selfadjoint operators in A such that $k(u_n) = x_n$.

Let $\alpha = \sup \|x_n\|$, $u_0 = 0$,

$$p_n = L[(u_n - u_{n-1})^-] \vee L[(\alpha 1 - u_n)^-] ,$$
$$p = V_n\, p_n ; \; y_n = (1-p) u_n (1-p) .$$

Then

$$0 \leqq y_1 \leqq \ldots \leqq y_n \leqq y_{n+1} \leqq \ldots \leqq \alpha 1$$

and

$$k(y_n) = x_n .$$

By continuing with this technique one obtains the lemma.

2.14. LEMMA. *For x in A and e, f in A_p,*

$$k(L[x]) = Lk([x]) \quad \text{and} \quad k(e \vee f) = k(e) \vee k(f) .$$

PROOF. k is σ-normal and $L[x] = \sup(1-(1-xx^*/\|x\|^2)^n)$. Cf. [7, pp. 8–9].

2.15. LEMMA. *$k(F)$ is an $\mathcal{F}$ system.*

PROOF. The two first conditions follow from 2.14 and 2.13.

Suppose $p,q \in (A/I)_p$, $p \sim q$, and $q = k(e)$ for some e in F. Choose u in A such that $k(u)^*k(u) = q$ and $k(u)k(u)^* = p$, write $e = u^*u + i$. Then $(1-L[i])e \in F$,

$$(1-L[i]) \wedge e = ((1-L[i]) \wedge e)u^*u((1-L[i]) \wedge e) \sim u((1-L[i]) \wedge e)u^* = f ,$$

$f \in F$ and $k(f) = p$.

Let (q_n) be a decreasing sequence from $k(F)$ with $\inf q_n = 0$, and let $p \in (A/I)_p$ with $p \prec q_n$ for all n. Using the technique from 2.13 we can find a decreasing sequence (f_n) from F with $k(f_n) = q_n$. The proof just before gives the existence of a sequence (e_n) from F such that $e_n \prec f_n$ and $k(e_n) = p$. The σ-normality of k implies that $\inf f_n \in N$ and $k(\bigwedge_n e_n) = p$. Hence 8) in 2.1 implies that $\bigwedge_n e_n \in N$, so $p = 0$ and $k(F)$ is an $\mathcal{F}$ system.

2.16. THEOREM. *The map K: $M(A,F,N) \to M(A/I,k(F),0)$ defined by*

$$K([x_n, e_n]) = [k(x_n), k(e_n)]$$

is an isomorphism.

PROOF. It is easy to verify that K is well defined and a homomorphism. Suppose $X = [x_n, e_n]$ is an MO and $K(X) = 0$. Then

$$k(x_n e_n) = k(x_n)k(e_n) = K(X)k(e_n) = 0 ,$$

so $R[x_n e_n] \in N$ and $f = \bigvee_n R[x_n e_n] \in N$. Hence $(e_n \wedge (1-f))$ is an SDD and $X = [x_n, e_n \wedge (1-f)] = 0$. Thus it is proved that K is faithful.

For $Y = [y_n, p_n] \in M(A/I, k(F), 0)$ we choose an SDD (e_n) in A such that $k(e_n) = p_n$, (2.13; 2.14; 2.15), and a sequence (x_n) in A with $k(x_n) = y_n$.

It is now clear that we can find $f \in N$ such that $(x_n, e_n \wedge (1-f))$ is an EMO and $K([x_n, e_n \wedge (1-f)]) = Y$.

3. The algebra $M(A, F, O)$.

In this section A will denote an MSC with unit and F an $\mathscr{F}$ system in A.

The theory for $M(A,F,O)$ is analogous to the theory developed in the papers [1], [2] and [12] by S. K. Berberian and K. Saitô, except for the cases mentioned below.

3.1. Theorem 3.2 in [12] must be divided into the following two theorems.

3.1.1. Let φ be a σ-normal isomorphism of A onto an MSC B, then there exists a unique extension Φ of φ to a σ-normal isomorphism of $M(A,F,O)$ onto $M(B,\varphi(F),O)$.

3.1.2. Let B be as in 3.1.1 and let G be an $\mathscr{F}$ system in B. If Φ is a σ-normal isomorphism of $M(A,F,O)$ onto $M(B,G,O)$, then $\Phi|A$ is a σ-normal isomorphism of A onto B.

3.2. Let u be a unitary in A and let $A(u)$ denote the C^* subalgebra spanned by u.

By $\left((A(u))_{\text{s.a.}}\right)_s{}^m$ we mean the monotone sequential closure of the real part of $A(u)$ in A. By [11, p. 222] it follows that $A(u)_s{}^m$ defined by

$$A(u)_s{}^m = \left(A(u)_{\text{s.a.}}\right)_s{}^m + i\left(A(u)_{\text{s.a.}}\right)_s{}^m$$

is the smallest MSC subalgebra of A containing u. That $A(u)_s{}^m$ is commutative follows from [7, Lemma 2.1].

3.2.1. Lemma. *Let Ω denote the spectrum of $A(u)_s{}^m$ and let H and G be closed disjoint subsets of Ω. Then there exists an open and closed subset E of Ω such that $H \subseteqq E$ and $E \cap G = \emptyset$.*

Proof. Since Ω is normal we can find $h, g \in C(\Omega)$ such that $h(H), g(G) \subseteqq \{1\}$ and $hg = 0$. Taking E as the support of $R[h]$ yields the lemma.

Now by writing $A(u)_s{}^m$ instead of $\{u\}''$ in the theorems of [2] and [12], we only have to make the following few remarks on the validity of these theorems in our case.

3.3. Theorem 6.4 of [12] is false, that is, M is not a Baer *ring in general, but in Berberian's formulation [2, Theorem 2.3], namely that if A is a Baer *ring, then M is too, the theorem is true here.

3.4. Theorem 6.3 of [12] is true, but we want to give a shorter proof.

3.4.1. THEOREM. *Let $x \in M$. Then there exists exactly one partial isometry $w \in M$ such that*

$$x = w|x|, \quad ww^* = L[x], \quad w^*w = R[x] .$$

PROOF. Write $|x| = [y_n, e_n]$ with $y_n \geqq 0$, $y_n e_n = y_n$. Then

$$y_n = y_n e_n = |x| e_n ,$$

so $y_n = |xe_n|$. Thus xe_n is bounded, and has a polar decomposition $xe_n = w_n y_n$ such that $w_n w_n^* = L[xe_n]$ and $w_n^* w_n = L[y_n]$. For $n \geqq m$ we have

$$w_n y_m = w_n y_m e_m = w_n y_n e_m = x e_n e_m = x e_m = w_m y_m$$

and $(w_n - w_m) y_m = 0$, so $(w_n - w_m) L[y_m] = 0$. Now [8, Corollary 5.2] gives the existence of a partial isometry w such that

$$ww^* = \sup w_n w_n^*, \quad w^*w = \sup w_n^* w_n$$

and $wL[y_m] = w_m$. For each n we then obtain

$$xe_n = w_n y_n e_n = w y_n e_n = w|x| e_n ,$$

so $x = w|x|$. It is obvious that $ww^* \geqq L[x]$ and $w^*w \geqq R[x]$, but

$$w_n w_n^* = L[xe_n] = L[xe_n x^*] \leqq L[xx^*] = L[x] ,$$

and

$$w_n^* w_n = L[y_n] = L[|x| e_n |x|] \leqq R[x] ,$$

so $ww^* = L[x]$ and $w^*w = R[x]$.

Suppose u is another partial isometry with the same properties as w, and let v be the Cayley transform of $|x|$. Then $u|x| = w|x|$ so $u(1+v) = w(1+v)$ hence

$$uL(1+v) = wL(1+v) \quad \text{and} \quad uR[x] = wR[x] ,$$

which implies $u = w$.

3.5. Theorem 6.2 of [12] has to be divided into the following two.

3.5.1. THEOREM. *If 1 is in F, then M is regular.*

PROOF. [1, Corollary 7.1].

3.5.2. THEOREM. *If M is regular, then 1 is finite.*

PROOF. Suppose that there exists a partial isometry w such that $w^*w=1$ and $ww^* \neq 1$.

Define p_n by $p_n=(w)^n(w^*)^n$ for $n \in \{0,1,2,\ldots\}$ and q_n by $q_n=p_n-p_{n-1}$ for $n \in \{1,2,\ldots\}$. Then (p_n) is a decreasing sequence of equivalent projections, so it has an infimum q. The projections (q_n) are pairwise orthogonal and equivalent and $\sum q_n=1-q$.

Let u be the unitary in A defined by

$$u = \exp(i(2-\pi))q+\textstyle\sum_{n=1}^{\infty} \exp(i(n^{-1}-\pi))q_n \ .$$

Then $(1-u)$ is invertible in A and $x=i(1+u)(1-u)^{-1}$ is positive. Since M is regular and

$$R[x] = L[x] = L[1+u] = 1 \ ,$$

it follows that x is invertible in M.

It is now easy to see that Theorem 6.1 of [12] and the construction of u yield a contradiction.

3.6. In [2] it is proved, that if A is an MSC and x_n is an increasing sequence of MO's which is majorized by an MO y, then $x=\sup x_n$ exists in the algebra of measurable operators.

3.6.1. LEMMA. *Let (x_n) be an increasing sequence in $M(A,F,O)$ for which $x=\sup x_n$ exists in $M(A,F,O)$, and let $y \in A$. Then $\sup y^*x_ny$ exists and $\sup y^*x_ny=y^*\sup x_ny$.*

PROOF. (Cf. [7, Lemma 2.1].) The sequence (y^*x_ny) is majorized by y^*xy, so the existence of $\sup y^*x_ny$ follows. Write $y=\sum_{k=1}^{4}\lambda_k u_k$ where u_k is unitary. Then

$$0 \leqq y^*(x-x_n)y \leqq 4 \textstyle\sum_{k=1}^{4} |\lambda_k|^2 u_k^*(x-x_n)u_k$$

and the lemma follows.

3.6.2. THEOREM. *Let (x_n) be an increasing sequence in $M(A,F,O)$ for which $x=\sup x_n$ exists in $M(A,F,O)$, and let $y \in M(A,F,O)$. Then*

$$\sup y^*x_ny = y^*\sup x_ny \ .$$

PROOF. Obviously $y^*xy \geqq \sup y^*x_ny=z$. Since $y \in M$, there exists an SDD (e_k) such that $ye_k \in A$ for all k. The lemma then gives that

$$e_k(y^*xy-z)e_k = 0 \ ,$$

so $(y^*xy-z)^{\frac{1}{2}}e_k=0$.

Lemma 4.5 in [12] then implies that $z=y^*xy$, and the theorem is proved.

3.7. In continuation of Section 9 of [1] we want to consider the monotone sequential closure in M of a sub-MSC B of A.

3.7.1. THEOREM. *If B is a sub-*MSC *of A with $1 \in B$, then $M(B, B \cap F, O)$ is isomorfic to a subalgebra M_B of M such that $M_B \cap A = B$.*

PROOF. [1, Theorem 9.1].

3.7.2. DEFINITION. Let $B_{\mathrm{s.a.}}$ be the real part of a sub-MSC B of A. Then $(B_{\mathrm{s.a.}})_s{}^M$ is defined as the smallest real subspace of $M_{\mathrm{s.a.}}$ which contains the limit of every increasing sequence of elements in $B_{\mathrm{s.a.}}$ majorized by an element of $M_{\mathrm{s.a.}}$.

The monotone sequential closure of B in M is denoted by $B_s{}^M$ and defined by $B_s{}^M = (B_{\mathrm{s.a.}})_s{}^M + i(B_{\mathrm{s.a.}})_s{}^M$.

3.7.3. THEOREM. *Under the assumptions of* 3.7.1 *we have $B_s{}^M = M_B$, where M_B is constructed as in* [1, Theorem 9.1].

PROOF. Let b_n be an increasing sequence of positive elements from B majorized by an element from M. Then $Y = \sup b_n$ exists and

$$(1+Y)^{-1} = \inf_A (1+b_n)^{-1} \in B .$$

If u is the Cayley transform of $1+Y$, then $-u^*$ is the Cayley transform of $(1+Y)^{-1}$, so $u \in B$, hence $1+Y \in M_B$ and $Y \in M_B$.

Thus we have proved that $B_s{}^M \subseteq M_B$ and will finish the proof by showing that the positive part $M_B{}^+$ of M_B is contained in $B_s{}^M$.

Let $x \in M_B{}^+$. Then x can be written $[x_n, e_n]$ with $x_n \geqq 0$, $x_n e_n = x_n$, where $x_n, e_n \in B$. Since $x e_n = x_n e_n = e_n x$,

$$x_n = x^{\frac{1}{2}} e_n x^{\frac{1}{2}} \leqq x ,$$

Theorem 3.6.2 then gives the result.

3.7.4. THEOREM. *Let B be a* sub-MSC *of A. Then $B_s{}^M$ is a selfadjoint subalgebra of M with $B_s{}^M \cap A = B$.*

PROOF. For any sequence (x_n) of elements of $B_s{}^M$ there exists a projection $e \in B$ such that $ex_n = x_n e = x_n$ for all n. Verifying first that if (z_n) is a monotone absolutely majorized sequence, then

$$L[\lim z_m] \leqq \mathsf{V}_m L[z_m] \in B ,$$

one obviously obtains the existence of such an e. By standard technique it follows that $eB_s{}^M e = (eBe)_s{}^M$, but taking eBe as a subalgebra of eAe, and using that $M(eAe, eFe, O)$ can be identified with eMe [1, Theorem 9.4], we infer from Theorem 3.7.3 that $eB_s{}^M e$ is a self-adjoint algebra. This completes the proof.

3.8. THEOREM. *There exists a one-to-one correspondence between the* MSC *ideals I of A and the monotone sequentially closed ideals J of M. The correspondence is obtained by*

$$J = (I)_s{}^M, \quad I = J \cap A .$$

PROOF. Simple but not very short.

4. Integration.

In accordance with Section 2 of this paper we shall only consider the case where A is an MSC with unit, φ a σ-normal faithful semifinite trace on A, and F an $\mathscr{F}$ system containing the set $\{p \in A_p \mid \varphi(p) < \infty\}$. We let m denote the defining ideal of φ and suppose that φ is extended to m. As usual,

$$m^+ = A^+ \cap m; \quad m^{\frac{1}{2}} = \{x \in A \mid x^* x \in m\} .$$

For $x \in m$, $\varphi(\cdot x)$ is a functional on A with norm $\varphi(|x|)$, and if $x \in m^+$ then $\varphi(\cdot x)$ is positive, σ-normal, and the family of all such functionals is separating, so A is a Baire *algebra.

4.1. LEMMA. *Let $x \in M, x \geqq 0$, and let $(e_n), (g_n)$ be* SDD *with $g_n \leqq e_n$, $e_n x e_n \in A$ and $g_n x g_n \in m^+$. If $\sup_n \varphi(g_n x g_n) < \infty$, then*

$$e_n x e_n \in m^+ \quad \textit{and} \quad \sup_n \varphi(e_n x e_n) = \sup_n \varphi(g_n x g_n) .$$

PROOF. Put $\alpha = \sup \varphi(g_n x g_n)$. Then

$$\alpha \geqq \varphi(g_n x g_n) = \varphi(x^{\frac{1}{2}} g_n x^{\frac{1}{2}}) \geqq \varphi(x^{\frac{1}{2}}(g_n \wedge e_k) x^{\frac{1}{2}}) ,$$

so

$$\alpha \geqq \sup_n \varphi(x^{\frac{1}{2}}(g_n \wedge e_k) x^{\frac{1}{2}}) = \varphi(x^{\frac{1}{2}} e_k x^{\frac{1}{2}}) = \varphi(e_k x e_k) ,$$

but $\varphi(e_n x e_n) = \varphi(x^{\frac{1}{2}} e_n x^{\frac{1}{2}}) \geqq \varphi(x^{\frac{1}{2}} g_n x^{\frac{1}{2}}) = \varphi(g_n x g_n)$, and the lemma follows.

4.2. DEFINITION. Let $x \in M$ and $x \geqq 0$. If there exists an SDD (e_n)

with $e_n x e_n \in m^+$ and $\sup \varphi(e_n x e_n) = \alpha < \infty$, then x is said to be integrable with integral α.

4.3. Proposition. *Let $x \in M$, $x \geqq 0$, be integrable with integral α. Then for each* SDD (e_n) *with $e_n x e_n \in A$, we have $\alpha = \sup \varphi(e_n x e_n)$ and $\alpha \geqq 0$.*

Proof. Use 4.1.

4.4. Definition. L^{1+} is defined as the set of all positive integrable operators and L^p for $1 \leqq p < \infty$ by

$$L^p = \{x \in M \mid |x|^p \in L^{1+}\}.$$

It is easy to verify that $L^{p+} = L^p \cap M^+$ for $1 \leqq p < \infty$ is an invariant linear system (Definition 1.3 in [9]), so L^p becomes a subspace of M, and for $x \in A$, $y \in L^p$ we have $xy \in L^p$. For L^1 we get that there exists a unique extension of φ to L^1. In the following we therefore suppose that φ is defined on L^1.

We will now give the theorem on which we base the proofs of the integration theory.

4.5. Lemma. *Let $x \in m^{\frac{1}{2}}$ and $e = R[x]$. Then there exists an increasing sequence (e_n) of projections in m such that $e = \sup e_n$.*

Proof. Put $e_n = R[(x^* x - e/n)^+]$, and the lemma follows.

4.6. Theorem. *Equipped with the inner product $(x|y) = \varphi(y^* x)$, $m^{\frac{1}{2}}$ is an achieved Hilbert algebra. Let H be the completion of $m^{\frac{1}{2}}$ as Hilbert space, and let $\Phi(x)$ for $x \in A$ be the closure of the operator $y \to xy$ defined on $m^{\frac{1}{2}}$. Then Φ is a σ-normal isomorphism of A onto a concrete Baire* algebra $\mathfrak{A}$ on H.*

Proof. Let $x \in m^{\frac{1}{2}}$, and choose a sequence (e_n) having the properties of Lemma 4.5. Then $xe_n \in m$ and

$$\begin{aligned}(x - xe_n | x - xe_n) &= \varphi(x^* x) - \varphi(x^* x e_n) - \varphi(e_n x^* x) + \varphi(e_n x^* x e_n) \\ &= \varphi(|x|^2) - \varphi(|x| e_n |x|) = \varphi(|x|(e - e_n)|x|),\end{aligned}$$

so by the σ-normality of φ it follows that $(m^{\frac{1}{2}})^2 = m$ is dense in $m^{\frac{1}{2}}$. By [4, I. 6.2 Theorem 2] it then follows that $m^{\frac{1}{2}}$ is a Hilbert algebra, and that Φ is a σ-normal homomorphism. It has already been mentioned that the functionals $\varphi(\cdot y)$, $y \in m^+$, are separating. If x is in A and $\Phi(x) = 0$, we get for each $y \in m^+$

$$0 = (\Phi(x)\, y^{\frac{1}{2}}|y^{\frac{1}{2}}) = \varphi(xy)\,,$$

and we can conclude that $x=0$, so Φ is a σ-normal isomorphism of A onto $\Phi(A)=\mathfrak{A}$. Theorem 1 of [4, I. 6.2] implies, that $\overline{\Phi(m^{\frac{1}{2}})}$ (the weak closure of $\Phi(m^{\frac{1}{2}})$) is a semifinite von Neumann algebra on H. Since $\Phi(m^{\frac{1}{2}})$ is a twosided ideal in $\mathfrak{A}$, $\overline{\Phi(m^{\frac{1}{2}})}$ is a twosided ideal in the von Neumann algebra $\overline{\mathfrak{A}}$, but $1 \in \overline{\Phi(m^{\frac{1}{2}})}$, so $\overline{\Phi(m^{\frac{1}{2}})} = \overline{\mathfrak{A}}$. Let w be the canonical trace on $\overline{\mathfrak{A}}$. Then if $x,y \in m^{\frac{1}{2}}$, we have

$$w(\Phi(y^*)\Phi(x)) = (x|y) = \varphi(y^*x)\,,$$

and if e is a projection in m, then $w(\Phi(e))=\varphi(e)<\infty$, so $\Phi(e)$ is σ-finite with respect to $\overline{\mathfrak{A}}$.

Suppose x is a bounded element of H [4, I. 5. Definition 2.3] and $x=x^*$ and let U_x be defined as in [4, I. 5]. Then Proposition 4 of [4, I. 5] gives the existence of a sequence (x_n), $x_n \in m^{\frac{1}{2}}$, such that $\|x-x_n\| \to 0$, $\sup\|\Phi(x_n)\|<\infty$, and $\Phi(x_n)$ converges strongly to U_x.

Lemma 4.5 applied to $\mathfrak{A}$, and the remark just before show that $R[x_n]=e_n$ is the countable supremum of σ-finite projections. Hence e_n is σ-finite, $e=\bigvee_{n=1}^{\infty} e_n$ is σ-finite and $e \in \overline{\mathfrak{A}}$. It is then obvious that $U_x e = U_x$, so a theorem due to Kadison [6, p. 322–23] gives the existence of a selvadjoint z in $\overline{\mathfrak{A}}$ such that

$$U_x = U_x e = ze \in \overline{\mathfrak{A}}\,.$$

Now take $y=\Phi^{-1}(U_x) \in A$ and suppose $y \notin m^{\frac{1}{2}}$. Then, since φ is semifinite, for each $n \in \mathsf{N}$ we can choose $z_n \in m$ such that $z_n \leqq y^*y$ and

$$n < \varphi(z_n) = w(\Phi(z_n)) \leqq w(\Phi(y^*y)) = w(U^*U) < \infty\,,$$

so $y \in m^{\frac{1}{2}}$, and $m^{\frac{1}{2}}$ is achieved.

4.7. Corollary. *The trace induced by Φ on $\mathfrak{A}$ can be extended to a normal semifinite faithful trace w on $\overline{\mathfrak{A}}$ and*

$$\{x \in \overline{\mathfrak{A}}^+ \mid w(x)<\infty\} = \{x \in \mathfrak{A}^+ \mid \varphi\circ\Phi^{-1}(x)<\infty\}\,.$$

4.8. Let E be the set of metrically finite projections in A, $E=\{p \in A_p \mid \varphi(p)<\infty\}$. Then E is an $\mathscr{F}$ system and $M(A,E,O) \subseteqq M(A,F,O)$. The definition of integrability implies that $L^p \subseteqq M(A,E,O)$ for $1 \leqq p < \infty$. If $x=[x_n,e_n]$ is in $M(A,E,O)$, then $1-\Phi(e_n)$ is metrically finite with respect to the faithful trace w, so $1-\Phi(e_n)$ is finite with respect to $\overline{\mathfrak{A}}$. According to Lemma 1.4 of [9], $\Phi(x_ne_n)$ converges n.e. to a measurable operator X affiliated with $\overline{\mathfrak{A}}$. Suppose $x=[z_n,f_n]$ with $1-f_n \in E$. Then $\Phi(z_nf_n)$ converges n.e. to a measurable operator Z affiliated with $\overline{\mathfrak{A}}$, but obviously $Z\Phi(e_n \wedge f_n) = X\Phi(e_n \wedge f_n)$, so Z and X agree on a strongly dense domain, hence $Z = X$ [13, Corollary 5.3]. It is then possible to extend Φ to a faithful homomorphism of $M(A,E,O)$ into the algebra of measurable operators affiliated with $\overline{\mathfrak{A}}$. (Cf. [9, Theorem 1].)

4.9. Lemma. *For* $x \in M(A,E,O)$ *we have* $\Phi(|x|^\alpha) = |\Phi(x)|^\alpha$.

Proof. Corollary 1.1 of [9].

4.10. Lemma. *The image of* $L^1(A,\varphi)$ *by* Φ *is* $L^1(\overline{\mathfrak{A}},H,w)$ (Segal's notation), *and* $w\Phi(x) = \varphi(x)$ *for* $x \in L^1(A,\varphi)$.

Proof. It is enough to consider the case where $X \in L^1(\overline{\mathfrak{A}},H,w)^+$. Then $x = \int_0^\infty \lambda d(E_\lambda)$ and $w(1-E_\lambda) < \infty$ for $\lambda > 0$ [9, 2.2], so $1-E_0 = \sup(1-E_{1/n})$ is σ-finite and an element of $\mathfrak{A}$.

Using the theorem in [6, p. 322–23] again and defining $X_n = XE_n = X(E_n - E_0)$ we get $X_n, E_n \in \mathfrak{A}$. Now

$$y = [\Phi^{-1}(X_n), \Phi^{-1}(E_n)] \in M(A,E,O) \quad \text{and} \quad \Phi(y) = X .$$

On the other hand,

$$\varphi(y) = \sup \varphi(ye_n) = \sup w(\Phi(ye_n)) = \sup w(x_n) = w(X) .$$

For $1 \leqq p < \infty$ and $x \in L^p(A,\varphi)$ we define as usual $\|x\|_p = \varphi(|x|^p)$.

4.11. Theorem. *For* $1 \leqq p < \infty$ *the restriction of* Φ *to* $L^p(A)$ *is an isometry of* $L^p(A)$ *onto* $L^p(\overline{\mathfrak{A}},H,w)$.

Proof. For $x \in L^p(A)$ Lemmas 4.9 and 4.10 give that

$$w(|\Phi(x)|^p) = w(\Phi(|x|^p)) = \varphi(|x|^p) ,$$

so Φ is an isometry, and an argument similar to that used in 4.10 shows that Φ is onto.

4.12. Theorem. *If* (x_n) *is an increasing sequence of positive operators in* L^p, $1 \leqq p < \infty$, *such that* $\{\|x_n\|_p\}$ *is bounded, then there exists an* $x \in L^p(A)^+$ *such that* $x = \sup_M x_n$ *and* $\|x - x_n\|_p \to 0$ *as* $n \to \infty$.

Proof. Since Φ is a * homomorphism, it preserves order, so by using 4.11 and Theorem 9 of [9], we get the existence of $x \in L^p(A)^+$ such that $\Phi(x) = \sup \Phi(x_n)$ and $\|\Phi(x) - \Phi(x_n)\|_p \to 0$. Hence $y = \sup_M x_n$ exists and $\Phi(x_n) \leqq \Phi(y) \leqq \Phi(x)$, which gives the theorem.

We will now proceed to the study of the Radon–Nikodym theorem. Let π denote the universal σ-normal representation of A (cf. [8]), E the corresponding Hilbert space, and put $B=\pi(A)$. Then there exists exactly one normal representation Ω of $\bar{B}$ such that $\Phi=\Omega\circ\Pi$ [8, Theorem 2.7]. Corollary 2 of [4, I. 4, Théorème 2] implies that $\Omega(\bar{B})=\bar{\mathfrak{A}}$. Finally let μ be the normal trace on $\bar{B}$ defined by $\mu=w\circ\Omega$.

4.13. Lemma. *Let G be the support of Ω and let $y\in\Pi(m^+)$. Then $Gy=y$.*

Proof. Let μ_y be the positive normal functional on $\bar{B}$ defined by $\mu_y(b)=\mu(yb)$. Then the support of μ_y is $GL[y]$. Suppose now that $L[y](1-G)\neq 0$ and choose $\xi\neq 0$, $\xi\in L[y](1-G)E$; then for the support g of ω_ξ we have $g\leqq L[y](1-G)$. Since g and $GL[y]$ are σ-finite with respect to $\bar{B}$, [6] implies

$$(\bar{B}_{\mathrm{s.a.}})(g+GL[y]) = (B_{\mathrm{s.a.}})(g+GL[y]) .$$

So there exists $p\in B_p$ such that $pg=g$ and $pGL[y]=0$. Now

$$0 = \mu_y(p) = w(\bar{\Omega}(py)) = w(\Omega(py)) ,$$

so $py=0$ since $\Omega|B$ is faithful, and we have got a contradiction.

4.14. Theorem. *Let f be a positive σ-normal functional on A. Then f can be written as a sum $g+h$, where g has the form $\varphi(\cdot x)$ for an x in $L^1(A,\varphi)^+$ and h is a positive σ-normal functional on A with the property $h(y)=0$ for all $y\in m^+$.*

Proof. Lifting f to $\bar{B}$, f becomes a normal positive functional $\sum\omega_{\xi_n}$, with $\xi_n\in E$, $\sum\|\xi_n\|^2<\infty$. Define

$$g' = \sum \omega_{G\xi_n} \quad \text{and} \quad h' = \sum \omega_{(1-G)\xi_n} .$$

Then Lemma 4.13 gives that $h'(y)=0$ for $y\in\Pi(m^+)$. Since G is the support of Ω, $g''=g'\circ\Omega^{-1}$ is a well-defined normal functional on $\bar{\mathfrak{A}}$. Therefore Corollary 3.1 in [9] gives the existence of X in $L^1(\bar{\mathfrak{A}},H,w)^+$ such that $g''=w(\cdot X)$. Let h, g on A be defined by

$$h = h'\circ\Pi, \quad g = g''\circ\Phi .$$

Then it is easy to verify that $f=g+h$ and that h is positive σ-normal with the desired property. If $z=\Phi^{-1}(X)$, then $z\in L^1(A,\varphi)^+$, and for $y\in A$ we have $yz\in L^1(A,\varphi)$ and $\varphi(yz)=w(\Phi(y)\,\bar{\cdot}\,X)=g(y)$ so the theorem is proved. (Here $\bar{\cdot}$ denotes strong product cf. [13].)

5. Tensorproduct of Baire* algebras.

5.1. DEFINITION. Let A and B be two concrete Baire* algebras acting on the Hilbert spaces H and K. We define the tensor product $A \otimes_s^m B$ as the monotone sequential closure of the algebraic tensor product $A \otimes B$ on the Hilbert space tensor product $H \otimes^h K$.

REMARK. From Theorem 1 in [11] it follows that $A \otimes_s^m B$ is a C^* and hence a Baire* algebra.

5.2. THEOREM. *Let A, B be concrete Baire* algebras and $\mathfrak{A}, \mathfrak{B}$ the universal σ-normal representations. Then there exists a σ-normal isomorphism of $A \otimes_s^m B$ onto $\mathfrak{A} \otimes_s^m \mathfrak{B}$.*

PROOF. As usual let $\bar{A}$, $\bar{B}$ be the weak closures of A and B and $\bar{A} \otimes^c \bar{B}$ the weak closure of the algebraic tensor product $\bar{A} \otimes \bar{B}$. Theorem 2.7 of [8] gives the existence of normal homomorphisms Φ of $\bar{\mathfrak{A}}$ onto $\bar{A}$ and Ψ of $\bar{\mathfrak{B}}$ onto $\bar{B}$ such that the restrictions $\Phi|\mathfrak{A}$ and $\Psi|\mathfrak{B}$ are σ-normal isomorphisms of $\mathfrak{A}$ onto A and $\mathfrak{B}$ onto B. If $\Phi \otimes^c \Psi$ is the canonical normal homomorphism of $\bar{\mathfrak{A}} \otimes^c \bar{\mathfrak{B}}$ onto $\bar{A} \otimes^c \bar{B}$, the restriction $\Phi \otimes^c \Psi | \mathfrak{A} \otimes_s^m \mathfrak{B}$ is a σ-normal homomorphism of $\mathfrak{A} \otimes_s^m \mathfrak{B}$ onto $A \otimes_s^m B$, and the theorem will be proved when it is shown that $\Phi \otimes^c \Psi | \mathfrak{A} \otimes_s^m \mathfrak{B}$ is faithful. Suppose that $x \in \mathfrak{A} \otimes_s^m \mathfrak{B}$ and $\Phi \otimes^c \Psi(x) = 0$. Then by using Tomiyama's Fubini maps [14, Theorem 1] the theorem follows as sketched below. If $\varphi \in \bar{\mathfrak{A}}_*$ and $\varphi \geqq 0$, then R_φ is positive and normal. Hence R_φ maps $\mathfrak{A} \otimes_s^m \mathfrak{B}$ into $\mathfrak{B}$. For $f \in \bar{A}_*$ we have

$$\Psi R_{f \circ \Phi}(x) = R_f(\Phi \otimes \Psi(x)) = 0 .$$

Since $\Psi|\mathfrak{B}$ is faithful and $R_{f \circ \Phi}(x) \in \mathfrak{B}$, we have $R_{f \circ \Phi}(x) = 0$ for $f \in \bar{A}_*$; but $f \circ \Phi(L_\psi(x)) = \psi(R_{f \circ \Phi}(x)) = 0$ for $\psi \in \bar{\mathfrak{B}}_*$ so $L_\psi(x) = 0$ for all $\psi \in \bar{\mathfrak{B}}_*$, and we can conclude that $x = 0$.

5.3. In the following A (resp. B) is a Baire* algebra with unit and a semifinite σ-normal faithful trace φ (resp. ω). Let m (resp. n) denote the definition ideal of φ (resp. ω), and let Φ (resp. Ω) be the induced representations. Then setting $\mathfrak{A} = \Phi(A)$, $\mathfrak{B} = \Omega(B)$ and using Proposition 9 of [4, I. § 5] and Theorem 4.6 we obtain

$$\mathfrak{A} \otimes_s^m \mathfrak{B} \subseteqq \bar{\mathfrak{A}} \otimes^c \bar{\mathfrak{B}} = \mathscr{U}(m^{\frac{1}{2}}) \otimes^c \mathscr{U}(n^{\frac{1}{2}}) = \mathscr{U}(m^{\frac{1}{2}} \otimes n^{\frac{1}{2}}) \subseteqq \overline{\mathfrak{A} \otimes_s^m \mathfrak{B}} .$$

5.4. LEMMA. *Let α be the canonical trace on $\mathscr{U}(m^{\frac{1}{2}} \otimes n^{\frac{1}{2}})$. Then*

$$h^+ = \{x \in \mathscr{U}(m^{\frac{1}{2}} \otimes n^{\frac{1}{2}})^+ \mid \alpha(x) < \infty\}$$

is contained in $\mathfrak{A} \otimes_s^m \mathfrak{B}$.

PROOF. If $x \in h^+$, then x has the form U_y for some bounded y in the completion of $m^{\frac{1}{2}} \otimes n^{\frac{1}{2}}$ as a Hilbert space. The arguments given in the proof of Theorem 4.6 together with the remarks of 5.3 show that $x \in \mathfrak{A} \otimes_s^m \mathfrak{B}$.

5.5. COROLLARY. *$A \otimes_s^m B$ has a canonical semifinite σ-normal faithful trace π such that $\pi(a \otimes b) = \varphi(a) w(b)$ for $a \in m$, $b \in n$. Let Π be the representation induced by π. Then $\Pi(A \otimes_s^m B)$ is spatial isomorphic to $\mathfrak{A} \otimes_s^m \mathfrak{B}$.*

PROOF. Theorem 5.2 gives that there exists a faithful σ-normal representation Ψ of $A \otimes_s^m B$ onto $\mathfrak{A} \otimes_s^m \mathfrak{B}$ such that

$$\Psi(\textstyle\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n \Phi(a_i) \otimes \Omega(b_i) .$$

Define then $\pi = \alpha \circ \Psi^{-1}$. If h is the definition ideal of α and k the definition ideal of π, it is obvious that Ψ induces an isometric isomorphism of the completion of $h^{\frac{1}{2}}$ onto the completion of $k^{\frac{1}{2}}$ as Hilbert spaces, so the corollary follows.

5.6. THEOREM. *There exists a continuous linear map*

$$\xi_1 \colon L^1(A \otimes_s^m B, \pi) \to L^1(B, w)$$

such that $w(\xi_1(x)) = \pi(x)$ for x in $L^1(A \otimes_s^m B, \pi)$.

PROOF. Consider the diagram

$$L^1(A \otimes_s^m B, \pi) \xrightarrow{(1)} L^1(\overline{\Pi(A \otimes_s^m B)}, \pi \circ \Pi^{-1}) \xrightarrow{(2)} L^1(\overline{\mathfrak{A} \otimes_s^m \mathfrak{B}}, \alpha)$$

$$\xrightarrow{(3)} L^1(\mathscr{U}(m^{\frac{1}{2}} \otimes n^{\frac{1}{2}}), \alpha) \xrightarrow{\tilde{\xi}_1} L^1(\mathscr{U}(n^{\frac{1}{2}}), w \circ \Omega^{-1}) \xrightarrow{(4)} L^1(B, w) .$$

Theorem 4.11 tells us that (1) and (4) are isometries. By Corollary 5.5, (2) is an isometry. By the Remark 5.3, (3) is an isometry. $\tilde{\xi}_1$ is a continuous linear map such that for $x \in L^1(\mathscr{U}(m^{\frac{1}{2}} \otimes n^{\frac{1}{2}}), \alpha)$ we have $\alpha(x) = w \circ \Omega^{-1}(\tilde{\xi}_1(x))$. The existence of $\tilde{\xi}_1$ is shown in [13, Corollary 22.1].

Define $\xi_1 = (4) \circ \tilde{\xi}_1 \circ (3) \circ (2) \circ (1)$. Then it is clear that ξ_1 has the desired properties.

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