

Non-Compact Symplectic Toric Manifolds[★]

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Abstract. A key result in equivariant symplectic geometry is Delzant’s classification of compact connected symplectic toric manifolds. The moment map induces an embedding of the quotient of the manifold by the torus action into the dual of the Lie algebra of the torus; its image is a unimodular (“Delzant”) polytope; this gives a bijection between unimodular polytopes and isomorphism classes of compact connected symplectic toric manifolds. In this paper we extend Delzant’s classification to non-compact symplectic toric manifolds. For a non-compact symplectic toric manifold the image of the moment map need not be convex and the induced map on the quotient need not be an embedding. Moreover, even when the map on the quotient is an embedding, its image no longer determines the symplectic toric manifold; a degree two characteristic class on the quotient makes an appearance. Nevertheless, the quotient is a manifold with corners, and the induced map from the quotient to the dual of the Lie algebra is what we call a unimodular local embedding. We classify non-compact symplectic toric manifolds in terms of manifolds with corners equipped with degree two cohomology classes and unimodular local embeddings into the dual of the Lie algebra of the corresponding torus. The main new ingredient is the construction of a symplectic toric manifold from such data. The proof passes through an equivalence of categories between symplectic toric manifolds and symplectic toric bundles over a fixed unimodular local embedding. This equivalence also gives a geometric interpretation of the degree two cohomology class.

Key words: Delzant theorem; symplectic toric manifold; Hamiltonian torus action; completely integrable systems

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1 Introduction

In the late 1980s, Delzant classified compact connected symplectic toric manifolds [7] by showing that the map

$$\text{symplectic toric manifold} \mapsto \text{its moment map image}$$

is a bijection onto the set of unimodular (also referred to as “smooth” or “Delzant”) polytopes. This beautiful work has been widely influential. The goal of this paper is to extend Delzant’s classification theorem to non-compact manifolds.

Delzant’s classification is built upon convexity and connectedness theorems of Atiyah and Guillemin–Sternberg [1, 10]. Compactness plays a crucial role in the proof of these theorems. Indeed, for a non-compact symplectic toric manifold the moment map image need not be convex

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and the fibers of the moment map need not be connected. And even when the fibers of the moment map are connected the moment map image need not uniquely determine the corresponding symplectic toric manifold. Thus, the passage to noncompact symplectic toric manifolds requires a different approach. As a first step we make the following observation (the proof is given in Appendix B):

Proposition 1.1. *Let (M, ω, μ) be a symplectic toric G -manifold. Then the quotient M/G is naturally a manifold with corners and the induced map*

$$\bar{\mu}: M/G \rightarrow \mathfrak{g}^*, \quad \bar{\mu}(G \cdot x) := \mu(x)$$

is a unimodular local embedding. (See Definitions A.16 and 2.5.)

Definition 1.2. Given a symplectic toric G -manifold (M, ω, μ) and a G -quotient map $\pi: M \rightarrow W$, we refer to the map $\psi: W \rightarrow \mathfrak{g}^*$ that is defined by $\mu = \psi \circ \pi$ as the *orbital moment map*.

See Remarks 1.4 and 1.5 for the origin of the notion of orbital moment map and its relation to developing map in affine geometry. The fact that the quotient M/G is a manifold with corners is closely related to the fact that for a completely integrable system with elliptic singularities the space of tori is a manifold with corners [3, 31].

Our classification result can be stated as follows.

Theorem 1.3. *Let \mathfrak{g}^* be the dual of the Lie algebra of a torus G and $\psi: W \rightarrow \mathfrak{g}^*$ a unimodular local embedding of a manifold with corners. Then*

1. *There exists a symplectic toric G -manifold (M, ω, μ) with G -quotient map $\pi: M \rightarrow W$ and orbital moment map ψ .*
2. *The set of isomorphism classes of symplectic toric G -manifolds M with G -quotient map $\pi: M \rightarrow W$ and orbital moment map ψ is in bijective correspondence with the set of cohomology classes*

$$H^2(W, \mathbb{Z}_G \times \mathbb{R}) \simeq H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R}),$$

where $\mathbb{Z}_G := \ker\{\exp: \mathfrak{g} \rightarrow G\}$ denotes the integral lattice of the torus G .

The main difficulty in proving Theorem 1.3 lies in establishing part (1). Results similar to part (2) hold in a somewhat greater generality for completely integrable systems with elliptic singularities (under a mild properness assumption) [3, 31, 32]: once one knows that there is *one* completely integrable system with the space of tori W , the space of isomorphism classes of all such systems is classified by the second cohomology of W with coefficients in an appropriate sheaf (q.v. op. cit.). The existence part for completely integrable systems, called the realization problem by Zung [32], is much more difficult. For instance in [32] the realization problem is only addressed for 2-dimensional spaces of tori. The solution to the realization problem announced in [3] and a similar solution in [31] is difficult to apply in practice. The solution, is, roughly, as follows. Given an integral affine manifold with corners W one shows first that there is an open cover $\{U_\alpha\}$ of W such that over each U_α the realization problem has a solution M_α . Then, *if* there exists a collection of isomorphisms $\varphi_{\alpha,\beta}: M_\beta|_{U_\alpha \cap U_\beta} \rightarrow M_\alpha|_{U_\alpha \cap U_\beta}$ satisfying the appropriate cocycle condition, the realization problem has a solution for W . Compare this with Theorem 1.3(1) which asserts that the realization problem for completely integrable torus actions *always* has a solution. We believe that the realization problem for completely integrable systems with elliptic singularities also always has a solution. We will address this elsewhere.

Our proof of part (1) of Theorem 1.3 proceeds as follows. We define symplectic toric G -bundles over ψ : these are symplectic principal G -bundles over manifolds with corners with

orbital moment map ψ . They form a category, which we denote by $\text{STB}_\psi(W)$. This category is always non-empty: it contains the pullback $\psi^*(T^*G \rightarrow \mathfrak{g}^*)$. We then construct a functor

$$\mathfrak{c}: \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W) \quad (1.1)$$

from the category of symplectic toric G -bundles over W to the category $\text{STM}_\psi(W)$ of symplectic toric G -manifolds over W . The functor \mathfrak{c} trades corners for fixed points; it is a version of a symplectic cut [16]. It follows, since $\text{STB}_\psi(W)$ is non-empty, that there always exist symplectic toric G -manifolds over a given unimodular local embedding $\psi: W \rightarrow \mathfrak{g}^*$ of a manifold with corners W .

More is true. We show that the functor \mathfrak{c} is an equivalence of categories. Hence, it induces a bijection, $\pi_0(\mathfrak{c})$, between the isomorphism classes of objects of our categories:

$$\pi_0(\mathfrak{c}): \pi_0(\text{STB}_\psi(W)) \rightarrow \pi_0(\text{STM}_\psi(W)).$$

The geometric meaning of the cohomology classes in $H^2(W; \mathbb{Z}_G \times \mathbb{R})$ that classify symplectic toric G -manifolds over W now becomes clear: *the elements of $H^2(W; \mathbb{Z}_G)$ classify principal G -bundles, and the elements of $H^2(W; \mathbb{R})$ keep track of the “horizontal part” of the symplectic forms on these bundles.*

We note that, for compact symplectic toric G -manifolds, the idea to obtain their classification by expressing these manifolds as the symplectic cuts of symplectic toric G -manifolds with free G actions is due to Eckhard Meinrenken (see [23, Chapter 7, Section 5]).

The paper is organized as follows. In Section 2, after introducing our notation and conventions, we construct the functor (1.1). In Section 3 we show that any two symplectic toric G bundles over the same unimodular local embedding are locally isomorphic (Lemma 3.1). In Section 4 we prove that the functor \mathfrak{c} in (1.1) is an equivalence of categories. In Section 5, we give the classification of symplectic toric G -bundles over a fixed unimodular local embedding $\psi: W \rightarrow \mathfrak{g}^*$ in terms of two characteristic classes, the Chern class c_1 , which is in $H^2(W, \mathbb{Z}_G)$ and encodes the “twistedness” of the G bundle, and the horizontal class c_{hor} , which is in $H^2(W, \mathbb{R})$ and encodes the “horizontal part” of the symplectic form on the bundle. We show that the map

$$(c_1, c_{\text{hor}}): \pi_0(\text{STB}_\psi(W)) \rightarrow H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R})$$

is a bijection. Since the map $\pi_0(\mathfrak{c}): \pi_0(\text{STB}_\psi(W)) \rightarrow \pi_0(\text{STM}_\psi(W))$ is a bijection, the composite

$$\pi_0(\text{STM}_\psi(W)) \xrightarrow{(c_1, c_{\text{hor}}) \circ \pi_0(\mathfrak{c})^{-1}} H^2(W, \mathbb{Z}_G \times \mathbb{R})$$

is a bijection as well. This classifies (isomorphism classes of) symplectic toric G -manifolds over $\psi: W \rightarrow \mathfrak{g}^*$.

Finally, in Section 6 we discuss those symplectic toric manifolds that are determined by their moment map images. In Proposition 6.5 we use Theorem 1.3 to derive Delzant’s classification theorem and its generalization in the case of symplectic toric G -manifolds that are not necessarily compact but whose moment maps are proper as maps to convex subsets of \mathfrak{g}^* . (In fact, already in [15] it was noted that, with the techniques of Condevaux–Dazord–Molino [4], Delzant’s proof should generalize to non-compact manifolds if the moment map is proper as a map to a convex open subset of the dual of the Lie algebra.) In Theorem 6.7, which was obtained in collaboration with Chris Woodward, we characterize those symplectic toric manifolds that are symplectic quotients of the standard \mathbb{C}^N by a subtorus of the standard torus \mathbb{T}^N . In Example 6.9 we construct a symplectic toric manifold that cannot be obtained by such a reduction.

The paper has two appendices. Appendix A contains background on manifolds with corners. In Appendix B, we recall the local normal form for neighborhoods of torus orbits in symplectic toric manifolds, and we use it to prove the following facts, which are known but maybe hard to find in the literature:

- 1) orbit spaces of symplectic toric manifolds are manifolds with corners;
- 2) orbital moment maps of symplectic toric manifolds are unimodular local embeddings; and
- 3) any two symplectic toric manifolds over the same unimodular local embedding are locally isomorphic.

In the remainder of this section, following referees' suggestions, we describe some relations of our work to existing literature on integral affine structures and Lagrangian fibrations.

Remark 1.4 (orbital moment maps). An *equivariant* moment map $\nu: N \rightarrow \mathfrak{h}^*$ for an action of a Lie group H on a symplectic manifold N descends to a continuous map $\bar{\nu}: N/H \rightarrow \mathfrak{h}^*/H$ between orbit spaces. This map was introduced by Montaldi [26] under the name of *orbit momentum map* and was used to study stability and persistence of relative equilibria in Hamiltonian systems. An analogue of this map in contact geometry was used by Lerman to classify contact toric manifolds [17].

The content of Proposition 1.1 is that for a symplectic toric manifold (M, ω, μ) the orbit space M/G is not just a topological space. It has a natural structure of a C^∞ manifold with corners and that the induced orbital moment map is C^∞ .

Symplectic toric manifolds, in addition to being examples of symplectic manifolds with Hamiltonian torus actions, are also a particularly nice class of completely integrable systems with elliptic singularities. Viewed this way $\bar{\mu}: M/G \rightarrow \mathfrak{g}^*$ is a developing map for an integral affine structure on the manifold with corners M/G (see also Remark 1.5 below).

Remark 1.5 (integral affine structures). An integral affine structure on a manifold with corners is usually defined in terms of an atlas of coordinate charts with integral affine transition maps; see, for example, [31]. It is not hard to see that such an atlas on a manifold W defines a Lagrangian subbundle \mathcal{L} of the cotangent bundle $T^*W \rightarrow W$ with two properties:

- 1) the fiber $\mathcal{L}_w \subset T_w^*W$ is a lattice;
- 2) if $w \in W$ lies in a stratum of W of codimension k then there is a local frame $\{\alpha_1, \dots, \alpha_n\}$ of T^*W defined near w ($n = \dim W$) so that the first k 1-forms $\alpha_1, \dots, \alpha_k$ annihilate the vectors tangent to the stratum.

Conversely, any such Lagrangian subbundle defines on W an atlas of coordinate charts with integral affine transition maps.

In general the bundle $\mathcal{L} \rightarrow W$ may have no global frame. And even if it does have a global frame $\{\alpha_1, \dots, \alpha_n\}$ the one forms α_j (which are necessarily closed) need not be exact. But if there is a global exact frame $\{df_1, \dots, df_n\}$ of $\mathcal{L} \rightarrow W$, then we have a smooth map $f = (f_1, \dots, f_n): W \rightarrow \mathbb{R}^n$. Such a map f is a developing map for the integral affine structure on W .

Observe that a unimodular local embedding $\psi: W \rightarrow \mathfrak{g}^*$ defines an integral affine structure on W as follows. Since ψ is a local embedding, the cotangent bundle T^*W is the pullback by ψ of the cotangent bundle $T^*\mathfrak{g}^*$. Consequently the standard Lagrangian lattice $\mathcal{L}_{\text{can}} = \mathfrak{g}^* \times \mathbb{Z}_G \subset \mathfrak{g}^* \times \mathfrak{g} \simeq T^*\mathfrak{g}^*$ pulls back to a Lagrangian subbundle of T^*W . A choice of a basis of $\{e_1, \dots, e_n\}$ of the integral lattice \mathbb{Z}_G defines a map $f: W \rightarrow \mathbb{R}^n$. It is given by

$$f(w) = (\langle \psi, e_1 \rangle, \dots, \langle \psi, e_n \rangle).$$

The map f is a developing map for $\psi^*\mathcal{L}_{\text{can}} \rightarrow W$.

Symplectic toric manifolds and proper Lagrangian fibrations

Let (M, ω, μ) be a symplectic toric G -manifold with a G quotient map $\pi: M \rightarrow W$. Restricting to the interior \mathring{W} of W (as a manifold with corners), we get a completely integrable system in the sense that was studied by Duistermaat [8], namely, a proper Lagrangian fibration with connected fibers. These were revisited and generalized by Dazord and Delzant [6]. For a detailed exposition see [20].

Remark 1.6 (the integral affine structure and the monodromy). As Duistermaat explains, a proper Lagrangian fibration with connected fibers $\pi: M \rightarrow B$ defines an integral affine structure on the base B . Each covector $\beta \in T_b^*B$ determines a vector field ξ_β along $\pi^{-1}(b)$ by the equation $\iota(\xi_\beta)\omega = \pi^*\beta$, and the Lagrangian lattice sub-bundle is

$$\mathcal{L} = \{\beta \mid \text{the flow of } \xi_\beta \text{ is } 2\pi \text{ periodic}\}.$$

Duistermaat's monodromy measures the non-triviality of the Lagrangian lattice sub-bundle $\mathcal{L} \rightarrow B$. When it is trivial, the bundle of tori $T^*B/\mathcal{L} \rightarrow B$ becomes a trivial bundle with fiber, say, G , T^*B and \mathcal{L} become trivial bundles with fibers \mathfrak{g}^* and \mathbb{Z}_G^* , and $\pi: M \rightarrow B$ becomes a G principal bundle. In this case, an orbital moment map is also a developing map for the integral affine structure. Having a moment map in this context exactly means that the integral affine structure on B is globally developable.

Remark 1.7 (the characteristic classes). Let $\pi: M \rightarrow B$ be a proper Lagrangian fibration with connected fibers. The fibers of the bundle of tori T^*B/\mathcal{L} act freely and transitively on the fibers of $\pi: M \rightarrow B$. Moreover, every point in B has a neighborhood over which $\pi: M \rightarrow B$ and $T^*B/\mathcal{L} \rightarrow B$ are isomorphic; this is Duistermaat's formulation of the Arnold–Liouville theorem on the local existence of action angle variables. Globally, such fibrations $\pi: M \rightarrow B$ are classified by the first cohomology group

$$H^1(C_{\text{Lagr}}^\infty(\cdot, T^*B/\mathcal{L}))$$

of the sheaf of Lagrangian sections of T^*B/\mathcal{L} .

The short exact sequence of sheaves

$$0 \rightarrow C^\infty(\cdot, \mathcal{L}) \rightarrow C_{\text{Lagr}}^\infty(\cdot, T^*B) \rightarrow C_{\text{Lagr}}^\infty(\cdot, T^*B/\mathcal{L}) \rightarrow 0$$

gives an exact sequence

$$\cdots \rightarrow H^1(C_{\text{Lagr}}^\infty(\cdot, T^*B)) \rightarrow H^1(C_{\text{Lagr}}^\infty(\cdot, T^*B/\mathcal{L})) \rightarrow H^2(B, \mathcal{L}) \rightarrow \cdots.$$

Noting that Lagrangian sections of T^*B are the same as closed one-forms, and identifying the H^1 of their sheaf with $H^2(B, \mathbb{R})$, we get an exact sequence

$$\cdots \rightarrow H^2(B, \mathbb{R}) \rightarrow H^1(C_{\text{Lagr}}^\infty(\cdot, T^*B/\mathcal{L})) \xrightarrow{c_1} H^2(B, \mathcal{L}) \rightarrow \cdots. \quad (1.2)$$

The second of these maps is Duistermaat's Chern class. When the monodromy is trivial, Duistermaat's Chern class is the Chern class of $\pi: M \rightarrow B$ as a principle G bundle. If $\pi: M \rightarrow W$ is the G -quotient map of a symplectic toric G -manifold, then Duistermaat's Chern class for $M|_{\mathring{W}}$ coincides with ours under the identification $H^2(W; \mathbb{Z}_G) \xrightarrow{\cong} H^2(\mathring{W}; \mathbb{Z}_G)$.

If the monodromy and Chern class both vanish, Duistermaat defines a class in $H^2(B, \mathbb{R})$, which is often called the *Lagrangian class*; it is the cohomology class of the pullback of ω by a global smooth section. If $\pi: M \rightarrow B$ is the G quotient map of a symplectic toric G -manifold, and if additionally the Chern class vanishes, then Duistermaat's Lagrangian class for $M|_{\mathring{W}}$ coincides with our horizontal class under the identification $H^2(W; \mathbb{R}) \xrightarrow{\cong} H^2(\mathring{W}; \mathbb{R})$.

Remark 1.8. If $\pi: M \rightarrow W$ is the G -quotient map for a symplectic toric G -manifold and $B = \overset{\circ}{W}$, our characteristic class gives a splitting

$$H^1(C_{\text{Lagr}}^\infty(B, T^*B/\mathcal{L})) \cong H^2(B; \mathbb{Z}_G^*) \oplus H^2(B; \mathbb{R})$$

that is consistent with (1.2). Moreover, our construction provides a geometric meaning to the Lagrangian class in $H^2(B, \mathbb{R})$.

In this case

- every element of $H^2(W; \mathbb{Z}_G)$ gives rise to a symplectic toric G -manifold, and
- distinct elements of $H^2(W; \mathbb{R})$ represent non-isomorphic symplectic toric G -manifolds.

Both of these facts are not necessarily true in the more general situation that is addressed by Duistermaat and Dazord–Delzant.

2 A functor from symplectic toric bundles to symplectic toric manifolds

The purpose of this section is to construct a functor

$$c: \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W)$$

from the category of symplectic toric G -bundles to the category of symplectic toric G -manifolds over a given unimodular local embedding $\psi: W \rightarrow \mathfrak{g}^*$ of a manifold with corners W . Once this functor is constructed, we deduce Theorem 1.3(1) almost immediately. In Section 4 we prove that c is an equivalence of categories. We start by establishing our notation and recording a few necessary definitions.

Notation and conventions. A *torus* is a compact connected abelian Lie group. A torus of dimension n is isomorphic, as a Lie group, to $(S^1)^n$ and to $\mathbb{R}^n/\mathbb{Z}^n$. We denote the Lie algebra of a torus G by \mathfrak{g} , the dual of the Lie algebra, $\text{Hom}(\mathfrak{g}, \mathbb{R})$, by \mathfrak{g}^* , and the integral lattice, $\ker(\exp: \mathfrak{g} \rightarrow G)$, by \mathbb{Z}_G . The *weight lattice* of G is the lattice dual to \mathbb{Z}_G ; we denote it by \mathbb{Z}_G^* . When a torus G acts on a manifold M , we denote the action of an element $g \in G$ by $m \mapsto g \cdot m$ and the vector field induced by a Lie algebra element $\xi \in \mathfrak{g}$ by ξ_M ; by definition

$$\xi_M(m) = \left. \frac{d}{dt} \right|_{t=0} (\exp(t\xi) \cdot m).$$

We write the canonical pairing between \mathfrak{g}^* and \mathfrak{g} as $\langle \cdot, \cdot \rangle$. Our sign convention for a moment map $\mu: M \rightarrow \mathfrak{g}^*$ for a Hamiltonian action of a torus G on a symplectic manifold (M, ω) is that it satisfies

$$d\langle \mu, \xi \rangle = -\omega(\xi_M, \cdot) \quad \text{for all } \xi \in \mathfrak{g}. \quad (2.1)$$

For us a *symplectic toric G -manifold* is a triple (M, ω, μ) where M is a manifold, ω is a symplectic form and $\mu: M \rightarrow \mathfrak{g}^*$ is a moment map for an effective Hamiltonian action of a torus G with $\dim M = 2 \dim G$.

Definition 2.1. A *unimodular cone* in the dual \mathfrak{g}^* of the Lie algebra of a torus G is a subset C of \mathfrak{g}^* of the form

$$C = \{ \eta \in \mathfrak{g}^* \mid \langle \eta - \epsilon, v_i \rangle \geq 0 \text{ for all } 1 \leq i \leq k \},$$

where ϵ is a point in \mathfrak{g}^* and $\{v_1, \dots, v_k\}$ is a basis of the integral lattice of a subtorus of G . We record the dependence of the cone C on the data $\{v_1, \dots, v_k\}$ and ϵ by writing

$$C = C_{\{v_1, \dots, v_k\}, \epsilon}.$$

Remark 2.2. The set $C = \mathfrak{g}^*$ is a unimodular cone defined by the empty basis of the integral lattice $\{0\}$ of the trivial subtorus $\{1\}$ of G .

Remark 2.3. A unimodular cone is a manifold with corners. Moreover, it is a manifold with faces (q.v. Definition A.10).

For a unimodular cone $C = C_{\{v_1, \dots, v_k\}, \epsilon}$ the facets are the sets

$$F_i = \{\eta \in C \mid \langle \eta - \epsilon, v_i \rangle = 0\}, \quad 1 \leq i \leq k.$$

The vector v_i in the formula above is the inward pointing primitive normal to the facet F_i . (Recall that a vector v in the lattice \mathbb{Z}_G is *primitive* if for any $u \in \mathbb{Z}_G$ the equation $v = nu$ for $n \in \mathbb{Z}$ implies that $n = \pm 1$.)

Lemma 2.4. *The primitive inward pointing normal v_i to a facet F_i of a unimodular cone $C_{\{v_1, \dots, v_k\}, \epsilon}$ is uniquely determined by any open neighborhood \mathcal{O} of a point x of F_i in C .*

Proof. The affine hyperplane spanned by F_i is uniquely determined by the intersection $\mathcal{O} \cap F_i$. Up to sign, such a hyperplane has a unique primitive normal. The sign of the normal is determined by requiring that at the point x the normal points into \mathcal{O} . ■

Definition 2.5 (unimodular local embedding (u.l.e.)). Let W be a manifold with corners and \mathfrak{g}^* the dual of the Lie algebra of a torus. A smooth map $\psi: W \rightarrow \mathfrak{g}^*$ is a *unimodular local embedding* (a *u.l.e.*) if for each point w in W there exists an open neighborhood $\mathcal{T} \subset W$ of the point and a unimodular cone $C \subset \mathfrak{g}^*$ such that $\psi(\mathcal{T})$ is contained in C and $\psi|_{\mathcal{T}}: \mathcal{T} \rightarrow C$ is an *open embedding*. That is, $\psi(\mathcal{T})$ is open in C and $\psi|_{\mathcal{T}}: \mathcal{T} \rightarrow \psi(\mathcal{T})$ is a diffeomorphism.

Remark 2.6. In Definition 2.5, the cone C is not uniquely determined by the point w ; for instance it can have facets that do not pass through $\psi(w)$. For example, let $G = (S^1)^2$, let $\psi: W \rightarrow \mathfrak{g}^* = \mathbb{R}^2$ be the inclusion map of the positive quadrant, and let $w = (1, 0)$. If the neighborhood \mathcal{T} of w meets the non-negative y axis, then the cone C must be the positive quadrant too. Otherwise, the natural choice for C is the closed upper half plane, but for suitable choices of \mathcal{T} the cone C can also be the intersection of the closed upper half plane with a half plane of the form $\{x + ny \geq c\}$ for $n \in \mathbb{Z}$ and $c < 1$ or of the form $\{x + ny \leq c\}$ for $n \in \mathbb{Z}$ and $c > 1$.

Remark 2.7. Proposition 1.1 shows that the orbital moment map of a symplectic toric manifold is a unimodular local embedding.

Example 2.8. It is easy to construct examples where the orbital moment map is not an embedding. Consider, for instance, a 2-dimensional torus G . Removing the origin from the dual of its Lie algebra \mathfrak{g}^* gives us a space that is homotopy equivalent to a circle. Thus the fibers of the universal covering map $p: W \rightarrow \mathfrak{g}^* \setminus \{0\}$ have countably many points. The pullback $p^*(T^*G)$ along p of the principal G -bundle $\mu: T^*G \rightarrow \mathfrak{g}^*$ is a symplectic toric G -manifold with orbit space W and orbital moment map p , which is certainly not an embedding.

Similarly, let S^2 be the unit sphere in \mathbb{R}^3 with the standard area form, and equip $S^2 \times S^2$ with the standard toric action of $(S^1)^2$ with moment map $\mu((x_1, x_2, x_3), (y_1, y_2, y_3)) = (x_3, y_3)$. Its image is the square $I^2 = [-1, 1] \times [-1, 1]$. Remove the origin, and let $p: W \rightarrow I^2 \setminus \{0\}$ be the universal covering. Then the fiber product $W \times_{I^2 \setminus \{0\}} (S^2 \times S^2)$ is a symplectic toric manifold; it is a \mathbb{Z} -fold covering of $(S^2 \times S^2) \setminus (\text{the equator} \times \text{the equator})$. As in the previous example, the orbital moment map is not an embedding. Unlike the previous example, the torus action here is not free.

Definition 2.9. Let W be a manifold with corners and $\psi: W \rightarrow \mathfrak{g}^*$ a unimodular local embedding. A *symplectic toric manifold over $\psi: W \rightarrow \mathfrak{g}^*$* is a symplectic toric G -manifold (M, ω, μ) , equipped with a quotient map $\pi: M \rightarrow W$ for the action of G on M (q.v. Definition A.16), such that

$$\mu = \psi \circ \pi.$$

Remark 2.10. Since the moment map $\mu: M \rightarrow \mathfrak{g}^*$ together with the symplectic form ω encodes the action of the group G on M and since the quotient map $\pi: M \rightarrow W$ together with $\psi: W \rightarrow \mathfrak{g}^*$ encode μ , we may regard a symplectic toric G -manifold over $\psi: W \rightarrow \mathfrak{g}^*$ as a triple $(M, \omega, \pi: M \rightarrow W)$.

We now fix a u.l.e.

$$\psi: W \rightarrow \mathfrak{g}^*$$

of a manifold with corners W into the dual of the Lie algebra of a torus G , and proceed to define the category $\text{STM}_\psi(W)$ of *symplectic toric G -manifolds over $W \xrightarrow{\psi} \mathfrak{g}^*$* and the category $\text{STB}_\psi(W)$ of *symplectic toric G -bundles over $W \xrightarrow{\psi} \mathfrak{g}^*$* .

Definition 2.11 (the category $\text{STM}_\psi(\mathbf{W})$ of symplectic toric G -manifolds over $\psi: \mathbf{W} \rightarrow \mathfrak{g}^*$). We define an *object* of the category $\text{STM}_\psi(W)$ to be a symplectic toric G -manifold $(M, \omega, \pi: M \rightarrow W)$ over W . A *morphism* φ from (M, ω, π) to (M', ω', π') is a G -equivariant symplectomorphism $\varphi: M \rightarrow M'$ such that $\pi' \circ \varphi = \pi$.

Notation 2.12. Informally, we may sometimes write M for an object of $\text{STM}_\psi(W)$ and $\varphi: M \rightarrow M'$ for a morphism between two such objects. Also, we may write W as shorthand for $\psi: W \rightarrow \mathfrak{g}^*$.

Definition 2.13 (the category $\text{STB}_\psi(\mathbf{W})$ of symplectic toric G -bundles over $\psi: \mathbf{W} \rightarrow \mathfrak{g}^*$). An *object* of the category $\text{STB}_\psi(W)$ is a principal G -bundle $\pi: P \rightarrow W$ over a manifold with corners (cf. Definition A.17) together with a G -invariant symplectic form ω so that $\mu := \psi \circ \pi$ is a moment map for the action of G on (P, ω) . We call the triple $(P, \omega, \pi: P \rightarrow W)$ a *symplectic toric G -bundle over the u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$* , or a symplectic toric G -bundle over W for short. A *morphism* φ from (P, ω, π) to (P', ω', π') is a G -equivariant symplectomorphism $\varphi: P \rightarrow P'$ with $\pi' \circ \varphi = \pi$.

Remark 2.14. The categories $\text{STM}_\psi(W)$ and $\text{STB}_\psi(W)$ are groupoids, that is, all of their morphisms are invertible.

If $\psi: W \rightarrow \mathfrak{g}^*$ is a u.l.e., (M, ω, π) is a symplectic toric G manifold over W and $U \subset W$ is open, then the restriction $\psi|_U: U \rightarrow \mathfrak{g}^*$ is also a u.l.e., and

$$(M|_U := \pi^{-1}(U), \omega|_{M|_U}, \pi|_{M|_U})$$

is a symplectic toric G -manifold over U . The restriction map extends to a functor

$$|_U^W: \text{STM}_\psi(W) \rightarrow \text{STM}_\psi(U).$$

Given an open subset V of U we get the restriction $|_V^U: \text{STM}_\psi(U) \rightarrow \text{STM}_\psi(V)$. The three restriction functors are compatible:

$$|_V^W = |_V^U \circ |_U^W.$$

In other words, the assignment

$$U \mapsto \mathbf{STM}_\psi(U)$$

is a (strict) presheaf of groupoids.

A reader familiar with stacks will have little trouble checking that the presheaf $\mathbf{STM}_\psi(\cdot)$ satisfies the descent condition with respect to any open cover of W and that thus $\mathbf{STM}_\psi(\cdot)$ is a stack on the site $\text{Open}(W)$ of open subsets of W with the cover topology. The stack \mathbf{STM}_ψ is not a geometric stack.

Similarly, a symplectic toric bundle over a manifold with corners W restricts to a symplectic toric bundle over an open subset of W . These restrictions define a presheaf of groupoids $\mathbf{STB}_\psi(\cdot)$. A reader familiar with stacks can check that $\mathbf{STB}_\psi(\cdot)$ is also a stack; see also Lemma 4.7 below.

Remark 2.15. If $\psi: W \rightarrow \mathfrak{g}^*$ is a u.l.e. and W is a manifold without corners (i.e., a manifold) then

$$\mathbf{STM}_\psi(W) = \mathbf{STB}_\psi(W).$$

If W is an arbitrary manifold with corners, then its interior $\overset{\circ}{W}$ (q.v. Definition A.3) is a manifold, and so

$$\mathbf{STM}_\psi(\overset{\circ}{W}) = \mathbf{STB}_\psi(\overset{\circ}{W}).$$

The functor $\mathbf{c}: \mathbf{STB}_\psi(W) \rightarrow \mathbf{STM}_\psi(W)$

Next we outline the construction of the functor $\mathbf{c}: \mathbf{STB}_\psi(W) \rightarrow \mathbf{STM}_\psi(W)$ from the category of symplectic toric G -bundles to the category of symplectic toric G -manifolds over a u.l.e. ψ .

Step 1: characteristic subtori. We show that ψ attaches to each point $w \in W$ a subtorus K_w of G together with a choice of a basis $\{v_1^{(w)}, \dots, v_k^{(w)}\}$ of its integral lattice \mathbb{Z}_{K_w} .

A basis of the integral lattice \mathbb{Z}_K of a torus K defines a linear symplectic representation $\rho: K \rightarrow \text{Sp}(V, \omega_V)$, which we may regard as a symplectic toric K -manifold (V, ω_V, μ_V) (here $\mu_V: V \rightarrow \mathfrak{k}^*$ is the associated moment map with $\mu_V(0) = 0$). Thus for each point $w \in W$ we also have a symplectic toric K_w -manifold (V_w, ω_w, μ_w) .

Step 2: a topological version \mathbf{c}_{top} of the functor \mathbf{c} . The collection of the subtori $\{K_w\}_{w \in W}$ defines for each principal G -bundle $\pi: P \rightarrow W$ an equivalence relation \sim in a functorial manner. We show that

1. Each quotient $\mathbf{c}_{\text{top}}(P) := P/\sim$ is a topological G -space with orbit space W and the action of G on $\mathbf{c}_{\text{top}}(P)$ is free over the interior $\overset{\circ}{W}$. (Here, \mathbf{c}_{top} stands for “topological cut”.)
2. For every map $\varphi: P \rightarrow P'$ of principal G -bundles over W we naturally get a G -equivariant homeomorphism $\mathbf{c}_{\text{top}}(\varphi): \mathbf{c}_{\text{top}}(P) \rightarrow \mathbf{c}_{\text{top}}(P')$.
3. These data define a functor

$$\mathbf{c}_{\text{top}}: \mathbf{STB}_\psi(W) \rightarrow \text{topological } G\text{-spaces over } W.$$

4. Moreover, \mathbf{c}_{top} is a map of presheaves of groupoids. In particular, for every open subset U of W ,

$$\mathbf{c}_{\text{top}}(P|_U) = \mathbf{c}_{\text{top}}(P)|_U.$$

Step 3: the actual construction of \mathbf{c} . We show that for every point $w \in W$ there is an open neighborhood U_w so that for every symplectic toric G -bundle $(P, \omega, \pi: P \rightarrow W)$ the symplectic quotient

$$\text{cut}(P|_{U_w}) := (P|_{U_w} \times V_w) //_0 K_w$$

is a symplectic toric G -manifold over U_w .

As in Step 2 the mapping $\text{cut}(\cdot|_{U_w})$ (i.e., the restriction to U_w followed by cut) from symplectic toric G -bundles over W to symplectic toric manifolds over U_w extends to a functor. In particular for every map $\varphi: P \rightarrow P'$ of symplectic toric G -bundles over W we have a map $\text{cut}(\varphi|_{U_w}): \text{cut}(P|_{U_w}) \rightarrow \text{cut}(P'|_{U_w})$ of symplectic toric G -manifolds over U_w .

At the same time, for each symplectic toric G -bundle $P \rightarrow W$ we construct a collection

$$\{\alpha_w^P: \mathbf{c}_{\text{top}}(P|_{U_w}) \rightarrow \text{cut}(P|_{U_w})\}_{w \in W}$$

of equivariant homeomorphisms that have the following two compatibility properties:

1. For a fixed bundle $P \in \text{STB}_\psi(W)$ and any two points w_1, w_2 the map

$$(\alpha_{w_2}^P) \circ (\alpha_{w_1}^P)^{-1}: \text{cut}(P|_{U_{w_1}})|_{U_{w_1} \cap U_{w_2}} \rightarrow \text{cut}(P|_{U_{w_2}})|_{U_{w_1} \cap U_{w_2}}$$

is a map of symplectic toric G -manifolds over $U_{w_1} \cap U_{w_2}$.

2. For a point $w \in W$ and a map $\varphi: P_1 \rightarrow P_2$ of symplectic toric bundles over W the diagram

$$\begin{array}{ccc} \mathbf{c}_{\text{top}}(P_1)|_{U_w} & \xrightarrow{\alpha_w^{P_1}} & \text{cut}(P_1|_{U_w}) \\ \downarrow \mathbf{c}_{\text{top}}(\varphi)|_{U_w} & & \downarrow \text{cut}(\varphi|_{U_w}) \\ \mathbf{c}_{\text{top}}(P_2)|_{U_w} & \xrightarrow{\alpha_w^{P_2}} & \text{cut}(P_2|_{U_w}) \end{array} \quad (2.2)$$

commutes.

The first property tells us that the family $\{\alpha_w^P\}_{w \in W}$ of homeomorphisms defines on $\mathbf{c}_{\text{top}}(P)$ the structure of a symplectic toric G -manifold over $\psi: W \rightarrow \mathfrak{g}^*$. We denote this manifold, which is an object of $\text{STM}_\psi(W)$, by $\mathbf{c}(P)$. The second property tells us that $\mathbf{c}_{\text{top}}(\varphi)$ defines a map $\mathbf{c}(\varphi): \mathbf{c}(P_1) \rightarrow \mathbf{c}(P_2)$ of symplectic toric G -manifolds over ψ . This gives rise to the desired functor \mathbf{c} .

We now proceed to fill in the details of the construction.

Details of Step 1. We start by proving

Lemma 2.16. *Given a unimodular local embedding $\psi: W \rightarrow \mathfrak{g}^*$ and a point $w \in W$ there exists a unique subtorus K_w of G and a unique basis $\{v_1^{(w)}, \dots, v_k^{(w)}\}$ of its integral lattice \mathbb{Z}_{K_w} such that the following holds. There exists an open neighborhood U_w of w in W so that*

$$\psi|_{U_w}: U_w \rightarrow C_w := \{\eta \in \mathfrak{g}^* \mid \langle \eta - \psi(w), v_j^{(w)} \rangle \geq 0 \text{ for } 1 \leq j \leq k\}$$

is an open embedding of manifolds with corners.

Proof. By definition of a u.l.e., there exists an open neighborhood $\mathcal{T} \subset W$ of w and a unimodular cone $C = C_{\{u_1, \dots, u_n\}, \epsilon} \subset \mathfrak{g}^*$ such that $\psi(\mathcal{T})$ is contained in C and $\psi|_{\mathcal{T}}: \mathcal{T} \rightarrow C$ is an open embedding of manifolds with corners. Since $\psi|_{\mathcal{T}}$ is an open embedding it maps the interior of \mathcal{T} to an open subset of the interior of C . We may assume that \mathcal{T} is a neighborhood with faces. Then the stratum S of \mathcal{T} containing w lies in exactly k facets $\mathcal{F}_1, \dots, \mathcal{F}_k$ of \mathcal{T} , where k

is the codimension of S . For each j the image $\psi(\mathcal{F}_j)$ is an open subset of a unique facet $F_{i(j)}$ of C and $\psi(\mathcal{T})$ is an open neighborhood of $\psi(\mathcal{F}_j)$ in C . By Lemma 2.4 the pair $(\psi(\mathcal{T}), \psi(\mathcal{F}_j))$ uniquely determines the primitive inward pointing normal $u_{i(j)}$ of the facet $F_{i(j)}$ of C . Since $\{u_1, \dots, u_n\}$ is a basis of an integral lattice of a subtorus of G , its subset $\{u_{i(j)}\}_{j=1}^k$ is also a basis of an integral lattice of a possibly smaller subtorus K_w of G . We set $v_j^{(w)} := u_{i(j)}$, $1 \leq j \leq k$. We note that

$$K_w = \exp(\text{span}_{\mathbb{R}} \{v_1^{(w)}, \dots, v_k^{(w)}\}).$$

To obtain the neighborhood \mathcal{U}_w we delete from the manifold with faces \mathcal{T} all the faces that *do not* contain w . ■

Remark 2.17. The basis $\{v_1^{(w)}, \dots, v_k^{(w)}\}$ and the corresponding torus K_w do not depend on our choice of the cone C : by construction $v_j^{(w)}$ is the primitive normal to the affine hyperplane spanned by $\psi(\mathcal{F}_j)$ that points into $\psi(\mathcal{T})$. In fact the only way we use the existence of the unimodular cone C is to insure that the set $\{v_1^{(w)}, \dots, v_k^{(w)}\}$ of normals to the facets of $\psi(\mathcal{T})$ forms a basis of an integral lattice of a subtorus of the torus G .

Similarly, the basis $\{v_i^{(w)}\}_{i=1}^k$ does not depend on the choice of \mathcal{T} either.

Remark 2.18. For each stratum of W the function $w \mapsto K_w$ is locally constant, hence constant. Consequently the subtorus K_w depends only on the stratum of W containing the point w and not on the point w itself. Similarly the basis $\{v_1^{(w)}, \dots, v_k^{(w)}\}$ depends only on the stratum of W containing w .

Remark 2.19. For $w' \in \mathcal{U}_w$ we can read off the group $K_{w'}$ from the face structure of \mathcal{U}_w and the set $\{v_1^{(w)}, \dots, v_k^{(w)}\}$. Namely

$$K_{w'} = \exp(\text{span}_{\mathbb{R}} \{v_i^{(w)} \mid \langle \psi(w') - \psi(w), v_i^{(w)} \rangle = 0\}).$$

We also note that the subset

$$\{v_i^{(w)} \mid \langle \psi(w') - \psi(w), v_i^{(w)} \rangle = 0\}$$

of $\{v_1^{(w)}, \dots, v_k^{(w)}\}$ forms a basis of the integral lattice of $K_{w'}$.

Lemma 2.20. *A manifold with corners W that admits a u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$ is a manifold with faces (q.v. [13] and Definition A.10 below). In particular, for any symplectic toric G -manifold (M, ω, μ) , the quotient M/G is a manifold with faces.*

Proof. The map $\psi: W \rightarrow \mathfrak{g}^*$ sends a neighborhood of a point in a codimension 1 stratum S of W to a relatively open subset of an affine hyperplane $H \subset \mathfrak{g}^*$ whose normal lies in the integral lattice \mathbb{Z}_G of G . Consequently ψ sends all of S to H and $\psi|_S: S \rightarrow H$ is a local diffeomorphism. The lemma follows from this observation. ■

Remark 2.21. It follows from the proof of Lemma 2.20 that the map $\psi: W \rightarrow \mathfrak{g}^*$ attaches to every (connected) codimension 1 stratum S of W a primitive vector $\lambda(S) \in \mathbb{Z}_G$ (namely, the corresponding primitive inward normal). The function $S \mapsto \lambda(S)$ is the analogue of the characteristic function of Davis and Januszkiewicz [5] and of the characteristic bundle of Yoshida [31].

Recall that any symplectic representation of a torus is complex hence has well-defined weights. These weights do not depend on a choice of an invariant complex structure compatible with the symplectic form since the space of such structures is path connected.

Lemma 2.22. *Let $\rho_i: K \rightarrow \mathrm{Sp}(V_i, \omega_i)$, $i = 1, 2$, be two symplectic representations of a torus K with the same set of weights. Then there exists a symplectic linear isomorphism of representations $\varphi: (V_1, \omega_1) \rightarrow (V_2, \omega_2)$.*

Proof. Choose K -invariant compatible complex structures on V_1 and V_2 . As complex K representations, each of V_1 and V_2 decomposes into one-dimensional complex representations. Because the weights are the same, it is enough to consider the case that V_1 and V_2 are the same complex vector space and its complex dimension is one. In this case, because ω_1 and ω_2 are both compatible with the complex structure, one must be a positive multiple of the other: $\omega_2 = \lambda^2 \omega_1$ for some scalar $\lambda > 0$. We may then take $\varphi(v) := \lambda v$. ■

Lemmas 2.16 and 2.22 imply that to any point w of a manifold with corners W a u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$ unambiguously attaches a symplectic toric K_w -manifold (V_w, ω_w, μ_w) : the weights of the representation V_w is the basis $\{v_j^*\}$ of the weight lattice $\mathbb{Z}_{K_w}^*$ dual to the basis $\{v_j^{(w)}\}$. If V'_w is another symplectic representation of K_w with the same set of weights as V_w then the symplectic toric K_w -manifolds (V_w, ω_w, μ_w) and $(V'_w, \omega'_w, \mu'_w)$ are linearly isomorphic as symplectic toric manifolds. ■

Details of Step 2. Given a principal G -bundle $\pi: P \rightarrow W$ we define \sim to be the smallest equivalence relation on P such that $p \sim p'$ whenever $\pi(p) = \pi(p')$ and p, p' lie on the same $K_{\pi(p)}$ orbit. We give the set P/\sim the quotient topology. Since the action of K_w on the fiber of P above w commutes with the action of G , the topological space

$$\mathfrak{c}_{\mathrm{top}}(P) := P/\sim$$

is naturally a G -space. For the same reason $\pi: P \rightarrow W$ descends to a quotient map $\bar{\pi}: \mathfrak{c}_{\mathrm{top}}(P) \rightarrow W$. Since for the points w in the interior of W the groups K_w are trivial, the action of G on $\mathfrak{c}_{\mathrm{top}}(P)|_{\bar{W}}$ is free.

If $\varphi: P \rightarrow P'$ is a map of principal G -bundles over W , then it maps fibers to fibers and K_w -orbits to K_w orbits thereby inducing $\mathfrak{c}_{\mathrm{top}}(\varphi): \mathfrak{c}_{\mathrm{top}}(P) \rightarrow \mathfrak{c}_{\mathrm{top}}(P')$. Explicitly $\mathfrak{c}_{\mathrm{top}}(\varphi)$ is given by

$$\mathfrak{c}_{\mathrm{top}}(\varphi)([p]) = [\varphi(p)].$$

Here, as before $[p] \in P/\sim = \mathfrak{c}_{\mathrm{top}}(P)$ denotes the equivalence class of $p \in P$ and $[\varphi(p)]$ denotes the corresponding class in $\mathfrak{c}_{\mathrm{top}}(P')$.

It is easy to check that the map

$$\begin{aligned} \mathfrak{c}_{\mathrm{top}}: \mathrm{STB}_{\psi}(W) &\rightarrow \text{topological } G\text{-spaces over } W, \\ (P \xrightarrow{\varphi} P') &\mapsto (\mathfrak{c}_{\mathrm{top}}(P) \xrightarrow{\mathfrak{c}_{\mathrm{top}}(\varphi)} \mathfrak{c}_{\mathrm{top}}(P')) \end{aligned}$$

is a functor that commutes with restrictions to open subsets of W . ■

Details of Step 3. We start by extending the symplectic reduction theorem of Marsden–Weinstein and Meyer [22, 24] to manifolds with corners.

Theorem 2.23. *Suppose (M, σ) is a symplectic manifold with corners with a proper Hamiltonian action of a Lie group K and an associated equivariant moment map $\Phi: M \rightarrow \mathfrak{k}^*$. Suppose further:*

1. *For any point $x \in \Phi^{-1}(0)$ the stabilizer K_x of x is trivial;*
2. *there is an extension $\tilde{\Phi}$ of Φ to a manifold \tilde{M} containing M as a domain (q.v. Definition A.8) with $\Phi^{-1}(0) = \tilde{\Phi}^{-1}(0)$.*

Then $\Phi^{-1}(0)$ is a manifold (without corners) and the quotient

$$M//_0K := \Phi^{-1}(0)/K$$

is naturally a symplectic manifold.

Remark 2.24. The main issue in proving the theorem is in showing that $\Phi^{-1}(0)$ is actually a manifold and that it has the right dimension. In other words the issue is transversality for manifolds with corners. To be more specific if Q is a manifold with corners, $f: Q \rightarrow \mathbb{R}^k$ is a smooth function and 0 is a regular value of f , then it is not true in general that $f^{-1}(0)$ is a manifold, with or without corners. Take, for example,

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$$

and $f(x, y, z) = z - x^2 + y^2$. Then 0 is a regular value of f but

$$f^{-1}(0) = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2, z \geq 0\},$$

which is clearly not a manifold, with or without boundary.

The standard approach to transversality for manifolds with corners [28] is to impose an additional requirement that the kernel of the differential of f is transverse to the strata of Q . However, in the situation we care about we have tangency instead. Moreover, it is easy to write down an example of a smooth function $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ so that the graph of h is tangent to the x - y plane but the set

$$\{(x, y, z) \mid z - h(x, y) = 0, z \geq 0\}$$

is not a manifold. This is why we make an awkward assumption on the level set $\Phi^{-1}(0)$ in Theorem 2.23. On the other hand, this assumption is easy to check in practice.

Before proving the theorem we first prove

Lemma 2.25. *Let $f: Q \rightarrow \mathbb{R}^n$ be a smooth function on a manifold with corners Q . Suppose \tilde{Q} is a manifold (without corners) containing Q as a domain, and $\tilde{f}: \tilde{Q} \rightarrow \mathbb{R}^n$ is an extension of f with*

$$f^{-1}(0) = \tilde{f}^{-1}(0).$$

If 0 is a regular value of f (that is, if for all $x \in f^{-1}(0)$ the map $d_x f: T_x Q \rightarrow \mathbb{R}^n$ is onto), then $f^{-1}(0)$ is naturally a smooth manifold of dimension $\dim Q - n$ in the sense of Definition A.14.

Proof. Since 0 is a regular value of f , and since $f^{-1}(0) = \tilde{f}^{-1}(0)$, the value 0 is also regular for \tilde{f} . Consequently, $\tilde{f}^{-1}(0)$ is naturally a manifold of dimension $\dim Q - n$. Since $\tilde{f}^{-1}(0) = f^{-1}(0) \subset Q$, we conclude that $f^{-1}(0)$ is naturally a manifold. ■

Remark 2.26. Note that the assumptions of the lemma force $\ker d_x f$ to be tangent to the strata of Q : otherwise $f^{-1}(0) = \tilde{f}^{-1}(0)$ cannot hold.

Proof of Theorem 2.23. Once we know that $\Phi^{-1}(0)$ is actually a manifold, the classical arguments of Marsden–Weinstein [22] and of Meyer [24] apply to show that $\sigma|_{\Phi^{-1}(0)}$ is basic and that its kernel is precisely the directions of the G orbits. Consequently the restriction $\sigma|_{\Phi^{-1}(0)}$ descends to a closed nondegenerate 2-form σ_0 on the manifold $\Phi^{-1}(0)/K$.

By Lemma 2.25 it is enough to show that 0 is a regular value of Φ . This will follow from our assumption that the K action on $\Phi^{-1}(0)$ is free. Again, the argument is standard. Indeed, let

$x \in \Phi^{-1}(0)$. To show that the differential $d_x\Phi: T_xM \rightarrow \mathfrak{k}^*$ is surjective, we need to show that the annihilator of its image is zero. Let $X \in \mathfrak{k}$ be in the annihilator of this image:

$$\langle d_x\Phi(v), X \rangle = 0 \quad \text{for all } v \in T_xM.$$

By the definition of the moment map, we may rewrite this as

$$\sigma_x(v, X_M(x)) = 0 \quad \text{for all } v \in T_xM.$$

Since σ_x is a nondegenerate form on T_xM , we conclude that $X_M(x) = 0$. Because the stabilizer of x is trivial, this implies that X is the zero vector in \mathfrak{k} . \blacksquare

Remark 2.27. If additionally there is a Hamiltonian action of a Lie group G on (M, σ) with a moment map $\mu: M \rightarrow \mathfrak{g}^*$ so that the actions of G and K commute and the moment map μ is K invariant and Φ is G invariant then

- the induced action of G on $(M//_0K, \sigma_0)$ is Hamiltonian and
- $\mu|_{\Phi^{-1}(0)}$ descends to a moment map $\tilde{\mu}: M//_0K \rightarrow \mathfrak{g}^*$ for the induced action of G .

Lemma 2.28. *Let $\psi: W \rightarrow \mathfrak{g}^*$ be a u.l.e. and $(\pi: P \rightarrow W, \omega)$ a symplectic toric G -bundle. Then for every point $w \in W$ there is a neighborhood U_w so that the symplectic quotient $(P|_{U_w} \times V_w)//_0K_w$ is a symplectic toric G -manifold.*

Proof. By Step 1 we have a neighborhood U of $w \in W$ with faces, a subtorus $K = K_w$ of G , a basis $\{v_1, \dots, v_k\}$ of the integral lattice of \mathbb{Z}_K , the dual basis $\{v_1^*, \dots, v_k^*\}$ of the weight lattice and a symplectic representation $K \rightarrow \text{Sp}(V_w, \omega_w)$ with the weights $\{v_1^*, \dots, v_k^*\}$ so that the map

$$\psi|_U: U \rightarrow C_w := \{\eta \in \mathfrak{g}^* \mid \langle \eta - \psi(w), v_i \rangle \geq 0, 1 \leq i \leq k\}$$

is an open embedding of manifolds with corners. It will be convenient to take $V_w = \mathbb{C}^k$, $\omega_w = \frac{\sqrt{-1}}{2\pi} \sum dz_j \wedge d\bar{z}_j$ with the action of K on \mathbb{C}^k given by

$$\exp(X) \cdot z := (e^{2\pi\sqrt{-1}\langle v_1^*, X \rangle} z_1, \dots, e^{2\pi\sqrt{-1}\langle v_k^*, X \rangle} z_k).$$

We may do so by Lemma 2.22. Then the moment map $\mu_w: \mathbb{C}^k \rightarrow \mathfrak{k}^*$ is given by the formula

$$\mu_w(z) = - \sum |z_j|^2 v_j^*.$$

Let $\iota: \mathfrak{k} \hookrightarrow \mathfrak{g}$ denote the canonical inclusion and $\iota^*: \mathfrak{g}^* \rightarrow \mathfrak{k}^*$ the dual map. Note that the kernel of ι^* is the annihilator \mathfrak{k}° of \mathfrak{k} in \mathfrak{g}^* . Set

$$\xi_0 := \iota^*(\psi(w)).$$

The cone

$$C'_w := \{\xi \in \mathfrak{k}^* \mid \langle \xi - \xi_0, v_i \rangle \geq 0, 1 \leq i \leq k\}$$

contains no nontrivial affine subspaces. If we identify \mathfrak{k}^* with a subspace of \mathfrak{g}^* , the cone C_w becomes the product $\mathfrak{k}^\circ \times C'_w$. Since $\psi|_U$ is an open embedding, we may assume (by shrinking U further if necessary) that U is a product:

$$U = \mathcal{O} \times U',$$

where \mathcal{O} is a neighborhood of w in the stratum containing it, $U' = \mathcal{V} \cap C'_w$ with \mathcal{V} a neighborhood in \mathfrak{k}^* of the apex of the cone C'_w , and such that \mathcal{O} and U' are contractible. We take U_w to be

this U . Then, since U is contractible the restriction of any principal bundle $P \rightarrow W$ to U is trivial.

Let $\nu: P|_U \rightarrow \mathcal{V} \cap C'_w \subset \mathfrak{k}^*$ be the composite

$$\nu := \iota^* \circ \psi \circ \pi.$$

We observe that (1) ν is a moment map for the action of K on $(P|_U, \omega)$ and (2) $\nu: P|_U \rightarrow \mathcal{V} \cap C'_w$ is a trivial fiber bundle (the typical fiber is $\mathcal{O} \times G$). Observation (2) implies that ν can be extended to a trivial $\mathcal{O} \times G$ fiber bundle $\tilde{\nu}: \tilde{P} \rightarrow \mathcal{V}$ so that $P|_U$ embeds into \tilde{P} as a domain. Observation (1) tells us that the diagonal action of K on $(P|_U \times \mathbb{C}^k, \omega \oplus \omega_w)$ is Hamiltonian with a corresponding moment map $\Phi: P|_U \times \mathbb{C}^k \rightarrow \mathfrak{k}^*$ given by

$$\Phi(p, z) = \nu(p) - \xi_0 + \mu_w(z) = \nu(p) - \xi_0 - \sum |z_j|^2 v_j^*.$$

Clearly

$$\tilde{\Phi}(p, z) := \tilde{\nu}(p) - \xi_0 + \mu_w(z)$$

is an extension of Φ . Since

$$\tilde{\Phi}^{-1}(0) = \{(p, z) \mid \tilde{\nu}(p) = \xi_0 - \mu_w(z)\}$$

and since

$$\xi_0 - \mu_w(z) \in C'_w \quad \text{for all } z \in \mathbb{C}^k,$$

we have

$$\Phi^{-1}(0) = \tilde{\Phi}^{-1}(0).$$

Since the action of G on P is free, so is the action of K on $P|_U \times \mathbb{C}^k$. Therefore we can apply Theorem 2.23 and conclude that

$$\text{cut}(P|_U) := (P|_U \times \mathbb{C}^k) //_0 K$$

is a symplectic manifold (without corners).

The action of G on P extends trivially to a Hamiltonian action of G on $P|_U \times \mathbb{C}^k$. This action of G is Hamiltonian, commutes with the action of K and satisfies the rest of the conditions of Remark 2.27. Consequently $\text{cut}(P|_U)$ is a Hamiltonian G -space. Note that

$$\dim \text{cut}(P|_U) = \dim (P|_U \times \mathbb{C}^k) - 2k = \dim P = 2 \dim G.$$

Thus to show that $\text{cut}(P|_U)$ is toric, it is enough to show that the action of G is free at some point of $\text{cut}(P|_U)$. Now take any point $\xi \in \mathcal{V}$ that also lies in the interior of the cone C'_w . Pick any point $p \in P|_U$ with $\nu(p) = \xi$ and $z \in \mathbb{C}^k$ with $\mu_w(z) = -\xi + \xi_0$. Then

$$\Phi(p, z) = \nu(p) + \mu_w(z) = \xi - \xi_0 + (-\xi + \xi_0) = 0.$$

On the other hand, since ξ is in the interior of the cone, the stabilizer of z is trivial. Hence the stabilizer of $(p, z) \in P|_U \times \mathbb{C}^k$ for the action of $G \times K$ is trivial as well. Consequently the stabilizer of the image of (p, z) in $\text{cut}(P|_U)$ for the action of G is trivial.

We leave it to the reader to check that $\text{cut}(P|_U)$ is a toric manifold over $\psi|_U: U \rightarrow \mathfrak{g}^*$. \blacksquare

Remark 2.29. If $(\pi_i: P_i \rightarrow W, \omega_i)$, for $i = 1, 2$, are two symplectic toric G -bundles over a u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$ and $\varphi: P_1 \rightarrow P_2$ is a morphism in $\text{STB}_\psi(W)$, i.e., a G -equivariant symplectomorphism with $\pi_2 \circ \varphi = \pi_1$, then for any $w \in W$

$$\varphi \times \text{id}: P_1|_{U_w} \times \mathbb{C}^k \rightarrow P_2|_{U_w} \times \mathbb{C}^k$$

is a $G \times K$ -equivariant symplectomorphism with $\Phi_2 \circ (\varphi \times \text{id}) = \Phi_1$. Hence $\varphi \times \text{id}$ maps $\Phi_1^{-1}(0)$ onto $\Phi_2^{-1}(0)$ and descends to an isomorphism of toric manifolds

$$\text{cut}(\varphi): \text{cut}(P_1|_{U_w}) \rightarrow \text{cut}(P_2|_{U_w}).$$

It is not hard to check that

$$\text{cut}: \text{STB}_\psi(U_w) \rightarrow \text{STM}_\psi(U_w)$$

is a functor for every $w \in W$. (Strictly speaking we have a family of functors parameterized by the points w of W ; we suppress this dependence in our notation.)

We now proceed to construct the natural G -equivariant homeomorphisms

$$\alpha_w^P: \mathfrak{c}_{\text{top}}(P|_{U_w}) \rightarrow \text{cut}(P|_{U_w}).$$

The construction depends on the fact that $(\mathbb{C}^k, \omega_w, \mu_w)$ is a symplectic toric K -manifold over the cone $\mu_w(\mathbb{C}^k) = \{\eta \in \mathfrak{k}^* \mid \langle \eta, v_i \rangle \leq 0 \text{ for } 1 \leq i \leq k\}$. Moreover,

1) the map $\mu_w: \mathbb{C}^k \rightarrow \mu_w(\mathbb{C}^k)$ has a continuous (Lagrangian) section

$$s: \mu_w(\mathbb{C}^k) \rightarrow \mathbb{C}^k, \quad s(\eta) = (\sqrt{\langle -\eta, v_1 \rangle}, \dots, \sqrt{\langle -\eta, v_k \rangle})$$

which is smooth over the interior of the cone $\mu_w(\mathbb{C}^k)$;

2) the stabilizer K_z of $z \in \mathbb{C}^k$ depends only on the face of the cone $\mu_w(\mathbb{C}^k)$ containing $\mu_w(z)$ in its interior:

$$K_z = \exp(\text{span}_{\mathbb{R}}\{v_i \in \{v_1, \dots, v_k\} \mid \langle \mu_w(z), v_i \rangle = 0\});$$

cf. Remark 2.19.

We continue with the notation above: $\xi_0 = \iota^*(\psi(w)) \in \mathfrak{k}^*$ is a point and $\nu = \iota^* \circ \mu: P|_U \rightarrow \mathfrak{k}^*$ the K -moment map. Then for any point $p \in P|_U$

$$\xi_0 - \nu(p) \in \mu_w(\mathbb{C}^k)$$

and

$$\begin{aligned} s(\xi_0 - \nu(p)) &= (\sqrt{\langle \nu(p) - \xi_0, v_1 \rangle}, \dots, \sqrt{\langle \nu(p) - \xi_0, v_k \rangle}) \\ &= (\sqrt{\langle \mu(p) - \psi(w), v_1 \rangle}, \dots, \sqrt{\langle \mu(p) - \psi(w), v_k \rangle}), \end{aligned}$$

where $\mu = \psi \circ \pi: P \rightarrow \mathfrak{g}^*$ is the moment map for the action of G on P . This gives us a continuous proper map

$$\phi: P|_U \rightarrow \Phi^{-1}(0) \subset P|_U \times \mathbb{C}^k, \quad \phi(p) = (p, s(\xi_0 - \nu(p))).$$

The image of ϕ intersects every K orbit in $\Phi^{-1}(0)$. Hence the composite

$$f = \tau \circ \phi: P|_U \rightarrow \Phi^{-1}(0)/K,$$

where $\tau: \Phi^{-1}(0) \rightarrow \Phi^{-1}(0)/K$ is the orbit map, is surjective. Next we argue that the fibers of f are precisely the equivalence classes of the relation \sim defined in Step 2. Two points $p_1, p_2 \in P|_U$ are equivalent with respect to \sim if and only if $\pi(p_1) = \pi(p_2)$ and there is an $a \in K_{\pi(p_1)}$ with $a \cdot p_2 = p_1$. On the other hand $f(p_1) = f(p_2)$ if and only if there is an $a \in K$ with

$$(p_1, s(\xi_0 - \nu(p_1))) = (a \cdot p_2, a \cdot s(\xi_0 - \nu(p_2))).$$

For any point $\eta \in \mu_w(\mathbb{C}^k)$

$$a \cdot s(\eta) = s(\eta) \quad \Leftrightarrow \quad a \text{ lies in the stabilizer } K_{s(\eta)} \text{ of } s(\eta).$$

For $\eta = \xi_0 - \nu(p_2) = \xi_0 - \nu(p_1) = \iota^*(\psi(w) - \psi(\pi(p_1)))$,

$$\begin{aligned} K_{s(\eta)} &= \exp(\text{span}_{\mathbb{R}}\{v_i \in \{v_1, \dots, v_k\} \mid \langle \xi_0 - \nu(p_1), v_i \rangle = 0\}) \\ &= \exp(\text{span}_{\mathbb{R}}\{v_i \in \{v_1, \dots, v_k\} \mid \langle \iota^*(\psi(w) - \psi(\pi(p_1))), v_i \rangle = 0\}) \\ &= \exp(\text{span}_{\mathbb{R}}\{v_i \in \{v_1, \dots, v_k\} \mid \langle \psi(w) - \psi(\pi(p_1)), v_i \rangle = 0\}) \\ &= K_{\pi(p_1)}. \end{aligned}$$

We conclude that the fibers of f are precisely the equivalence classes of the relation \sim . Therefore f descends to a continuous bijection

$$\alpha_w^P: \mathfrak{c}_{\text{top}}(P|_U) = (P|_U)/\sim \rightarrow \Phi^{-1}(0)/K = \text{cut}(P|_U), \quad \alpha_w^P([p]) = [p, s(\xi_0 - \nu(p))].$$

The properness of f implies that α_w^P is a homeomorphism. This follows from Lemma 2.30 below.

Lemma 2.30. *Let $f: A \rightarrow B$ be a continuous and proper bijection between topological spaces. Suppose that B is Hausdorff and compactly generated. That is, B is Hausdorff and a subset E of B is closed if and only if for every compact K the intersection $E \cap K$ is compact. Then f is a homeomorphism.*

Proof. Omitted. ■

The commutativity of (2.2) is easy: Since $\nu_1(p) = \nu_2(\varphi(p))$,

$$\begin{aligned} \text{cut}(\varphi)(\alpha_w^{P_1}[p]) &= \text{cut}(\varphi)([p, s(\xi_0 - \nu_1(p))]) = [\varphi(p), s(\xi_0 - \nu_2(\varphi(p)))] \\ &= \alpha_w^{P_2}([\varphi(p)]) = \alpha_w^{P_2}(\mathfrak{c}_{\text{top}}(\varphi)([p])). \end{aligned}$$

To finish the construction of the functor \mathfrak{c} it remains to show that $v := (\alpha_{w_2}^P) \circ (\alpha_{w_1}^P)^{-1}$ is a map of symplectic toric G -manifolds. Since $\alpha_{w_1}^P$ and $\alpha_{w_2}^P$ are G -equivariant homeomorphisms, so is v . It is enough to produce a smooth symplectic map ϑ satisfying

$$\vartheta \circ \alpha_{w_1}^P = \alpha_{w_2}^P.$$

Indeed, this implies that $\vartheta = v$, and hence that v is a smooth symplectic map; reversing the roles of w_1 and w_2 , we conclude that the inverse of v is also a smooth symplectic map, and so v is a G -equivariant diffeomorphism. Since the intersection $U_{w_1} \cap U_{w_2}$ can be covered by sets of the form U_{w_3} , it suffices to consider the case when U_{w_1} is contained in U_{w_2} .

Consider first the special case when $K_{w_1} = K_{w_2} = K$. Then the collections of the corresponding weights $\{v_j^{(w_1)}\}_{j=1}^k, \{v_j^{(w_2)}\}_{j=1}^k$ are the same set. Hence by Lemma 2.22 there exists a symplectic linear isomorphism $\vartheta: \mathbb{C}^k \rightarrow \mathbb{C}^k$ which permutes coordinates and intertwines the two representations and the corresponding moment maps. Consequently $\text{id} \times \vartheta: P|_{U_{w_1}} \times \mathbb{C}^k \rightarrow P|_{U_{w_1}} \times \mathbb{C}^k$ induces a symplectic isomorphism of symplectic quotients

$$\bar{\vartheta}: (P|_{U_{w_1}} \times \mathbb{C}^k) //_0 K \rightarrow (P|_{U_{w_1}} \times \mathbb{C}^k) //_0 K, \quad [p, z] \mapsto [p, \vartheta(z)].$$

It is easy to check that $\bar{\vartheta} \circ \alpha_{w_1}^P = \alpha_{w_2}^P$, hence $(\alpha_{w_2}^P) \circ (\alpha_{w_1}^P)^{-1}$ is a symplectomorphism in *this* case.

More generally we have a strict inclusion $\{v_j^{(w_1)}\}_{j=1}^{k_1} \subset \{v_j^{(w_2)}\}_{j=1}^{k_2}$. By the discussion of the special case above, it is not a loss of generality to assume that $v_j^{(w_1)} = v_j^{(w_2)}$ for all $1 \leq j \leq k_1$. We may then reduce the clutter in the notation by dropping the superscripts (w_1) and (w_2) and setting $K_i := K_{w_i}$, $i = 1, 2$.

By construction of the neighborhoods U_{w_i} (q.v. Lemma 2.16 and subsequent remarks) we have

- $\langle \psi(w_1) - \psi(w_2), v_i \rangle = 0$ for $i = 1, \dots, k_1$, and,
- for all $w \in U_{w_1}$,

$$\langle \psi(w) - \psi(w_2), v_i \rangle > 0 \quad \text{for } i = k_1 + 1, \dots, k_2.$$

Consequently for any point $p \in P|_{U_{w_1}}$ the functions

$$p \mapsto \sqrt{\langle \mu(p) - \psi(w_2), v_i \rangle}$$

are smooth for $i = k_1 + 1, \dots, k_2$. Also, for $p \in P|_{U_{w_1}}$

$$\langle \mu(p) - \psi(w_2), v_i \rangle = \langle \mu(p) - \psi(w_1), v_i \rangle$$

for $i = 1, \dots, k_1$. Now consider the map

$$\vartheta: P|_{U_{w_1}} \times \mathbb{C}^{k_1} \rightarrow P|_{U_{w_2}} \times \mathbb{C}^{k_2}$$

given by

$$\vartheta(p, z_1, \dots, z_{k_1}) = \left(p, z_1, \dots, z_{k_1}, \sqrt{\langle \mu(p) - \psi(w_2), v_{k_1+1} \rangle}, \dots, \sqrt{\langle \mu(p) - \psi(w_2), v_{k_2} \rangle} \right).$$

The map ϑ is smooth and K_1 -equivariant. Since $\vartheta^*(dz_j \wedge d\bar{z}_j) = 0$ for $j > k_1$, it is symplectic.

Next observe that

$$\vartheta^{-1}(\Phi_2^{-1}(0)) = \Phi_1^{-1}(0),$$

where $\Phi_j: P|_{U_{w_1}} \times \mathbb{C}^{k_j} \rightarrow \mathfrak{k}_j^*$, $j = 1, 2$ are the corresponding moment maps. This is because

$$(p, z) \in \Phi_j^{-1}(0) \Leftrightarrow \langle \psi(\pi(p)) - \psi(w_j), v_i \rangle = |z_i|^2 \quad \text{for all } i = 1, \dots, k_j.$$

Consequently ϑ descends to a well-defined smooth symplectic map

$$\bar{\vartheta}: \Phi_1^{-1}(0)/K_1 \rightarrow \Phi_2^{-1}(0)/K_2$$

given by

$$\bar{\vartheta}([p, z_1, \dots, z_{k_1}]) = \left[p, z_1, \dots, z_{k_1}, \sqrt{\langle \mu(p) - \psi(w_2), v_{k_1+1} \rangle}, \dots, \sqrt{\langle \mu(p) - \psi(w_2), v_{k_2} \rangle} \right].$$

Evidently,

$$\bar{\vartheta}(\alpha_{w_1}^P([p])) = \alpha_{w_2}^P([p]).$$

This finishes Step 3 of the construction. We have thus constructed the desired functor \mathbf{c} . ■

We are now in position to prove part (1) of Theorem 1.3.

Proof of Theorem 1.3(1). It is enough to show that the category $\text{STB}_\psi(W)$ is nonempty. For then for any object P of $\text{STB}_\psi(W)$, the object $\mathfrak{c}(P)$ is the desired symplectic toric manifold.

Consider the cotangent bundle T^*G with the action of G given by the lift of multiplication on the left. This action is Hamiltonian and the moment map $\mu: T^*G \rightarrow \mathfrak{g}^*$ makes T^*G into a symplectic principal G -bundle over \mathfrak{g}^* . The pullback of this bundle by $\psi: W \rightarrow \mathfrak{g}^*$ is an object of $\text{STB}_\psi(W)$.

Alternatively, for any principal G -bundle $\pi: P \rightarrow W$ and any choice of a connection 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ the closed 2-form $\sigma := d\langle \psi \circ \pi, A \rangle$ is symplectic and $\mu := \psi \circ \pi$ is a corresponding moment map (see Lemma 3.2 below). Then $(P \xrightarrow{\pi} W, \sigma)$ is an object of $\text{STB}_\psi(W)$. ■

We end the section with a lemma that will be used to prove that $\mathfrak{c}: \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W)$ is an equivalence of categories.

Lemma 2.31. *The functor $\mathfrak{c}: \text{STB}_\psi \rightarrow \text{STM}_\psi$ is a map of presheaves of groupoids. Moreover over the interior \mathring{W} the functor $\mathfrak{c}_{\mathring{W}}: \text{STB}_\psi(\mathring{W}) \rightarrow \text{STM}_\psi(\mathring{W})$ is isomorphic to the identity functor.*

Proof. Since $\mathfrak{c}_{\text{top}}$ is a map of presheaves then so is \mathfrak{c} . Over the interior \mathring{W} the functor $\mathfrak{c}_{\text{top}}$ is isomorphic to the identity functor, since we divide out by the relation whose equivalence classes are singletons.

Moreover, for any point $w \in \mathring{W}$ the corresponding group K_w is trivial. Hence

$$\text{cut}(P|_{U_w}) = (P|_{U_w} \times \{0\}) //_0 \{1\} \simeq P|_{U_w}$$

as symplectic toric manifolds. ■

3 Local trivializations of symplectic toric G -bundles

The purpose of this section is to prove the “local uniqueness” for symplectic toric G bundles. Here is the statement:

Lemma 3.1. *Let $P_0 = (\pi_0: P_0 \rightarrow W, \omega_0)$ and $P_1 = (\pi_1: P_1 \rightarrow W, \omega_1)$ be two symplectic toric G -bundles over a unimodular local embedding (u.l.e.) $\psi: W \rightarrow \mathfrak{g}^*$. Then for any open subset U of W with $H^2(U, \mathbb{Z}) = 0$ the restrictions $P_0|_U$ and $P_1|_U$ are isomorphic in $\text{STB}_\psi(U)$. Consequently, any two symplectic principal G -bundles over the same u.l.e. are locally isomorphic.*

Before proving Lemma 3.1, we need to establish two facts about symplectic forms on principal G -bundles over our u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$. Recall that the notion of a moment map does not *a priori* require a 2-form to be nondegenerate, we only need (2.1) to hold.

Note that it makes sense to pair a connection 1-form A on P with the moment map μ . This results in a real-valued G -invariant 1-form $\langle \mu, A \rangle \in \Omega^1(P)^G$.

Lemma 3.2. *Let $\psi: W \rightarrow \mathfrak{g}^*$ be a u.l.e. and $\pi: P \xrightarrow{\pi} W$ a principal G -bundle.*

- Any closed G -invariant 2-form on P with moment map $\mu := \psi \circ \pi$ is automatically symplectic.
- A connection 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ defines a bijection from the space of closed forms on W and the space of invariant symplectic forms on P with moment map μ , by

$$\beta \mapsto d\langle \mu, A \rangle + \pi^* \beta.$$

Consequently, any two closed G -invariant 2-forms on P with moment map μ differ by a basic closed 2-form.

Proof. We argue first that $d\langle\mu, A\rangle$ is nondegenerate.

For any vector $X \in \mathfrak{g}$ the Lie derivative $L_{X_P}\langle\mu, A\rangle$ with respect to the induced vector field X_P is zero. By Cartan's formula we then have

$$0 = d\iota(X_P)\langle\mu, A\rangle + \iota(X_P)d\langle\mu, A\rangle = d\langle\mu, X\rangle + \iota(X_P)d\langle\mu, A\rangle.$$

Therefore μ is a moment map for the action of G on $(P, d\langle\mu, A\rangle)$. Also,

$$d\langle\mu, A\rangle = \langle d\mu \wedge A \rangle + \langle \mu, dA \rangle.$$

Moreover, for any point $p \in P$ we have an isomorphism

$$T_p P = \mathcal{H}_p \oplus \mathcal{V}_p \xrightarrow{(d\mu_p \oplus A_p)} \mathfrak{g}^* \oplus \mathfrak{g},$$

where \mathcal{H}_p and \mathcal{V}_p are the horizontal and vertical subspace of $T_p P$ respectively. The map $d\mu_p|_{\mathcal{H}_p} \oplus A_p|_{\mathcal{V}_p}$ is an isomorphism because $d\pi_P: \mathcal{H}_p \rightarrow T_{\pi(p)}W$ is an isomorphism, $d\psi_{\pi(p)}: T_{\pi(p)}W \rightarrow \mathfrak{g}^*$ is an isomorphism since ψ_p is an embedding, and $A_p: \mathcal{V}_p \rightarrow \mathfrak{g}$ is an isomorphism too. The isomorphism $d\mu_p|_{\mathcal{H}_p} \oplus A_p|_{\mathcal{V}_p}$ identifies $\langle d\mu \wedge A \rangle$ with the canonical pairing $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$. On the other hand $\langle \mu, dA \rangle$ is basic. It follows that $d\langle\mu, A\rangle$ is nondegenerate.

Similarly, for any closed form β on W the form

$$\omega = \omega_{A,\beta} := d\langle\mu, A\rangle + \pi^*\beta$$

is a closed, nondegenerate G -invariant form on P and μ is a moment map for the action of G on (P, ω) .

Finally, if $\sigma \in \Omega^2(P)^G$ is a closed G -invariant 2-form with moment map μ and $A \in \Omega^1(P, \mathfrak{g})^G$ is a connection 1-form, then for any $X \in \mathfrak{g}$

$$\iota(X_P)(\sigma - d\langle\mu, A\rangle) = -d\langle\mu, X\rangle + d\langle\mu, X\rangle = 0.$$

Hence $\sigma - d\langle\mu, A\rangle$ is basic and there is a 2-form β on W with

$$\sigma = d\langle\mu, A\rangle + \pi^*\beta;$$

β is necessarily closed. Note that this also proves that σ is nondegenerate. ■

Lemma 3.3. *Let $\psi: W \rightarrow \mathfrak{g}^*$ be a u.l.e. and $\pi: P \rightarrow W$ a principal G -bundle as above. Suppose that $\omega \in \Omega^2(P)^G$ is a closed G -invariant form with moment map μ and $\gamma \in \Omega^1(W)$ a 1-form. Then (P, ω) and $(P, \omega + \pi^*d\gamma)$ are isomorphic in $STB_\psi(W)$. That is, there exists a gauge transformation $f: P \rightarrow P$ with $f^*(\omega + \pi^*d\gamma) = \omega$.*

Proof. We apply Moser's deformation method [27]. By Lemma 3.2 the forms

$$\omega_t = \omega + t\pi^*d\gamma, \quad t \in [0, 1],$$

are symplectic. They have the same moment map μ . Let X_t be the time-dependent vector field on P that satisfies

$$\iota(X_t)\omega_t = -\pi^*\gamma.$$

Note that X_t is G -invariant. For every $\xi \in \mathfrak{g}$, we have

$$\iota(X_t)d\langle\mu, \xi\rangle = -\omega_t(\xi_P, X_t) = -\iota(\xi_P)\pi^*\gamma = 0.$$

Hence $d\mu(X_t) = 0$. Since $\mu = \psi \circ \pi$ and ψ is a local embedding we have

$$d\pi(X_t) = 0.$$

That is, X_t is tangent to the fibers of $P \rightarrow W$, which are tori. Consequently we can integrate the vector field X_t to obtain a G -equivariant isotopy $\phi_t: P \rightarrow P$ which exists for all $t \in [0, 1]$ and projects to the identity map on the base W . Then

$$\frac{d}{dt}(\phi_t^* \omega_t) = \phi_t^* \left(L_{X_t} \omega_t + \frac{d}{dt} \omega_t \right) = d\phi_t^*(\iota(X_t)\omega_t + \pi^* \gamma) = 0.$$

Consequently $f := \phi_1: (P, \omega) \rightarrow (P, \omega + \pi^* d\gamma)$ is an isomorphism of symplectic toric G -bundles over $\psi: W \rightarrow \mathfrak{g}^*$, as desired. \blacksquare

Proof of Lemma 3.1. Recall that for a torus $G = \mathfrak{g}/\mathbb{Z}_G$ the principal G -bundles over a manifold with corners N are classified by $H^2(N, \mathbb{Z}_G)$. Since $H^2(U, \mathbb{Z}) = 0$ by assumption, $H^2(U, \mathbb{Z}_G) = 0$ as well. Consequently there exists a G -equivariant diffeomorphism

$$h: P_0|_U \rightarrow P_1|_U$$

inducing the identity map on U . By Lemma 3.2

$$h^* \omega_1 = \omega_0 + \pi^* \beta$$

for some closed 2-form β on U . Since $H^2(U, \mathbb{R}) = 0$, there is a 1-form γ on U with $\beta = d\gamma$. By Lemma 3.3 there is a gauge transformation $f: P_0|_U \rightarrow P_0|_U$ with

$$f^*(\omega_0 + \pi^* \beta) = \omega_0.$$

Therefore

$$(h \circ f)^* \omega_1 = f^*(h^* \omega_1) = f^*(\omega_0 + \pi^* \beta) = \omega_0. \quad \blacksquare$$

Remark 3.4. In the language of stacks Lemma 3.1 asserts that the stack STB_ψ is a gerbe: any two objects are locally isomorphic.

4 Equivalence of categories of symplectic toric bundles and symplectic toric manifolds

In this section we show that the functor c is an equivalence of categories. This reduces the classification of symplectic toric G -manifolds to that of symplectic toric G -bundles. More specifically we prove

Theorem 4.1. *Let $\psi: W \rightarrow \mathfrak{g}^*$ be a unimodular local embedding (u.l.e). The functor*

$$c: \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W)$$

constructed in Section 2 is an equivalence of categories.

Remark 4.2. We actually show that for any open set $U \subset W$ the functor

$$c_U: \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$$

is an equivalence of categories. In other words c is an isomorphism of presheaves of groupoids.

The proof of Theorem 4.1 proceeds in a series of lemmas. Our first goal is to prove that the functor c is full and faithful.

Lemma 4.3 (c is faithful). *For any open subset U of W and for any two objects $P_1, P_2 \in \text{STB}_\psi(U)$ the map*

$$c = c_U: \text{Hom}(P_1, P_2) \rightarrow \text{Hom}(c(P_1)c(P_2)), \quad \phi \mapsto c(\phi)$$

is injective.

Proof. The idea is easy: if two isomorphisms of symplectic toric G bundles over W map to the same isomorphism of symplectic toric G manifolds over W , then they must coincide over the interior of W . By continuity, they must coincide over all of W .

(a) Recall that the functor

$$c_{U \cap \overset{\circ}{W}}: \text{STB}_\psi(U \cap \overset{\circ}{W}) \rightarrow \text{STM}_\psi(U \cap \overset{\circ}{W})$$

is isomorphic to the identity functor (q.v. Lemma 2.31): we have isomorphisms $\{\delta_Q: Q \rightarrow c(Q)\}_{Q \in \text{STB}_\psi(U \cap \overset{\circ}{W})}$ such that for any morphism $\phi: Q_1 \rightarrow Q_2$ in $\text{STB}_\psi(U \cap \overset{\circ}{W})$ the diagram

$$\begin{array}{ccc} Q_1 & \xrightarrow{\delta_{Q_1}} & c(Q_1) \\ \phi \downarrow & & \downarrow c(\phi) \\ Q_2 & \xrightarrow{\delta_{Q_2}} & c(Q_2) \end{array}$$

commutes. Hence

$$c: \text{Hom}(Q_1, Q_2) \rightarrow \text{Hom}(c(Q_1), c(Q_2))$$

is invertible with the inverse c^{-1} given by

$$c^{-1}(\varphi) = \delta_{Q_2}^{-1} \circ \varphi \circ \delta_{Q_1}. \quad (4.1)$$

(b) If $\phi_1, \phi_2 \in \text{Hom}(P_1, P_2)$ are two morphisms with $c(\phi_1) = c(\phi_2)$ then

$$c(\phi_1|_{P_1|_{U \cap \overset{\circ}{W}}}) = c(\phi_1)|_{P_1|_{U \cap \overset{\circ}{W}}} = c(\phi_2)|_{P_1|_{U \cap \overset{\circ}{W}}} = c(\phi_2|_{P_1|_{U \cap \overset{\circ}{W}}}).$$

By (b) above

$$\phi_1|_{P_1|_{U \cap \overset{\circ}{W}}} = \phi_2|_{P_1|_{U \cap \overset{\circ}{W}}}.$$

Since the restriction $P_1|_{U \cap \overset{\circ}{W}}$ is dense in $P_1|_U$,

$$\phi_1 = \phi_2. \quad \blacksquare$$

Next, we need to check that, for every two objects P_1 and P_2 , every morphism $\varphi: c(P_1) \rightarrow c(P_2)$ comes from a morphism $P_1 \rightarrow P_2$. Again, the idea is easy: φ gives a morphism $\overset{\circ}{\phi}$ between open dense subsets of P_1 and P_2 , (namely, the preimages of the interior of W), and we need to check that $\overset{\circ}{\phi}$ extends smoothly to the boundary. It is enough to check that $\overset{\circ}{\phi}$ extends locally; local extensions will coincide on the overlaps of their domains. Locally, P_1 and P_2 are isomorphic, so it remains to consider the case that $P_1 = P_2$. For this case we will use the following theorem of Haefliger, Salem and Schwartz:

Theorem 4.4 ([12, Theorem 3.1]). *Let M be a manifold with an action of a torus G and $h: M \rightarrow M$ a G -equivariant diffeomorphism with $h(x) \in G \cdot x$ for all points $x \in M$. Let $\pi: M \rightarrow M/G$ be the orbit map. Then there exists a map $f: M/G \rightarrow G$ such that*

$$h(x) = f(\pi(x)) \cdot x$$

for all $x \in M$ and such that $f \circ \pi$ is smooth.

We continue with the proof that the functor \mathbf{c} is full.

Lemma 4.5. *For any open subset U of W and for any $P \in STB_\psi(U)$ the map*

$$\mathbf{c}: \text{Hom}(P, P) \rightarrow \text{Hom}(\mathbf{c}(P), \mathbf{c}(P))$$

is onto.

Proof. By Theorem 4.4, given $\varphi \in \text{Hom}(\mathbf{c}(P), \mathbf{c}(P))$ there is a smooth function $f: U \rightarrow G$ so that

$$\varphi(x) = f(\pi(x)) \cdot x,$$

where $\pi: \mathbf{c}(P) \rightarrow U$ is the quotient map. By Step (a) of the proof of Lemma 4.3 and (4.1),

$$\varphi|_{P|_{U \cap \hat{W}}} = \mathbf{c}(\mathring{\phi}),$$

where $\mathring{\phi}$ is given by

$$\mathring{\phi} = (\delta_P)^{-1} \circ \varphi|_{P|_{U \cap \hat{W}}} \circ \delta_P.$$

Hence for $p \in P|_{U \cap \hat{W}}$,

$$\begin{aligned} \mathring{\phi}(p) &= (\delta_P)^{-1} (f(\pi(\delta_P(p))) \cdot \delta_P(p)) = (\delta_P)^{-1} (f(\pi(p)) \cdot \delta_P(p)) \\ &= f(\pi(p)) \cdot \delta_P^{-1}(\delta_P(p)) = f(\pi(p)) \cdot p. \end{aligned}$$

Define the map $\phi: P \rightarrow P$ by

$$\phi(p) = f(\pi(p)) \cdot p \quad \text{for all } p \in P.$$

This map is G -equivariant and commutes with the orbit map $\pi: P \rightarrow U$. Since $f \circ \pi$ is smooth, the map ϕ is a diffeomorphism. Moreover since the restriction of ϕ to $P|_{U \cap \hat{W}}$ is $\mathring{\phi}$, the map ϕ is symplectic on $P|_{U \cap \hat{W}}$. Since $P|_{U \cap \hat{W}}$ is dense in P , we conclude that ϕ is symplectic on all of P , i.e., $\phi \in \text{Hom}(P, P)$. It remains to check that $\mathbf{c}(\phi) = \varphi$. But the functor \mathbf{c} commutes with restrictions to $P|_{U \cap \hat{W}}$ and

$$\mathbf{c}(\phi)|_{P|_{U \cap \hat{W}}} = \mathbf{c}(\mathring{\phi}) = \varphi|_{P|_{U \cap \hat{W}}}$$

by construction. Hence, by Lemma 4.3, $\mathbf{c}(\phi) = \varphi$. ■

Lemma 4.6. *Suppose $U \subset W$ is an open subset with $H^2(U, \mathbb{Z}) = 0$. Then for any $P_1, P_2 \in STB_\psi(U)$ the map*

$$\mathbf{c} = \mathbf{c}_U: \text{Hom}(P_1, P_2) \rightarrow \text{Hom}(\mathbf{c}(P_1), \mathbf{c}(P_2)),$$

is a bijection.

Proof. By Lemma 4.3, the map c is an injection. So we only need to check that c is onto. Let $\varphi \in \text{Hom}(c(P_1), c(P_2))$. By Lemma 3.1 there exists an isomorphism $\phi: P_1 \rightarrow P_2$. Then $c(\phi)^{-1} \circ \varphi \in \text{Hom}(c(P_1), c(P_1))$. By Lemma 4.5

$$c(\phi)^{-1} \circ \varphi = c(\nu)$$

for some $\nu \in \text{Hom}(P_1, P_1)$. Hence

$$\varphi = c(\phi) \circ c(\nu) = c(\phi \circ \nu). \quad \blacksquare$$

We are now in position to finish the proof that c is fully faithful by observing that for any two objects $P_1, P_2 \in \text{STB}_\psi(W)$ the functions $\underline{\text{Hom}}(P_1, P_2)$ and $\underline{\text{Hom}}(c(P_1), c(P_2))$ from the collection of open subset of W to sets given respectively by

$$\underline{\text{Hom}}(P_1, P_2)(U) := \text{Hom}(P_1|_U, P_2|_U).$$

and

$$\underline{\text{Hom}}(c(P_1), c(P_2))(U) := \text{Hom}(c(P_1)|_U, c(P_2)|_U)$$

are sheaves. Moreover

$$c = c_U: \text{Hom}(P_1|_U, P_2|_U) \rightarrow \text{Hom}(c(P_1)|_U, c(P_2)|_U)$$

is a map of sheaves. By Lemma 4.6 the map c_U is a bijection for any contractible open set U . Hence $c: \underline{\text{Hom}}(P_1, P_2) \rightarrow \underline{\text{Hom}}(c(P_1), c(P_2))$ is an isomorphism of sheaves.

This proves that for *any* open subset $U \subset W$ the functor

$$c_U: \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$$

is fully faithful. It remains to prove that c is essentially surjective. As a first step in the proof of essential surjectivity we observe that the objects on $\text{STB}_\psi(W)$ satisfy descent in the sense of Grothendieck:

Lemma 4.7. *Let $\{U_i\}_{i \in I}$ be an open cover of the manifold with corners W , $U_{ij} := U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ for all $i, j, k \in I$. Suppose we have a collection of objects $P_i \in \text{STB}_\psi(U_i)$ and isomorphisms $\Phi_{ij}: P_j|_{U_{ij}} \rightarrow P_i|_{U_{ij}}$ defining a (normalized) cocycle: $\Phi_{ii} = \text{id}$, $\Phi_{ji} = \Phi_{ij}^{-1}$ and*

$$\Phi_{ij}|_{U_{ijk}} \circ \Phi_{jk}|_{U_{ijk}} \circ \Phi_{ki}|_{U_{ijk}} = \text{id}$$

for all triples $i, j, k \in I$. Then there exists an object $P \in \text{STB}_\psi(W)$ and isomorphisms $\gamma_i: P|_{U_i} \rightarrow P_i$ so that

$$\begin{array}{ccc} P_j|_{U_{ij}} & \xleftarrow{\gamma_j} & P|_{U_{ij}} \\ \Phi_{ij} \downarrow & & \parallel \\ P_i|_{U_{ij}} & \xleftarrow{\gamma_i} & P|_{U_{ij}} \end{array} \quad (4.2)$$

commutes.

Proof. We may take $P = (\bigsqcup_{i \in I} P_i) / \sim$ where \sim is the equivalence relation defined by the Φ_{ij} s. Then P is a principal G -bundle over W and the symplectic G -invariant forms on the P_i s define a G -invariant symplectic form on P . The maps $\gamma_i^{-1}: P_i \rightarrow P|_{U_i}$ are induced by the inclusions $P_i \hookrightarrow \bigsqcup_{j \in I} P_j$. \blacksquare

Lemma 4.8. *For any open subset U of W the functor $c: \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$ is essentially surjective.*

Proof. Given $M \in \text{STM}_\psi(U)$, we want to show that it is isomorphic to $c(P)$ for some $P \in \text{STB}_\psi(U)$.

Since $\text{STB}_\psi(U)$ is nonempty, we may choose an object $P' \in \text{STB}_\psi(U)$. By Lemma B.4 $c(P')$ and M are locally isomorphic. Therefore there is a cover $\{U_i\}_{i \in I}$ of U and a family of isomorphisms $\{\varphi_i: c(P')|_{U_i} \rightarrow M|_{U_i}\}$. Set

$$P_i := P'|_{U_i}.$$

Consider the collection of isomorphisms

$$\varphi_{ij} := (\varphi_i|_{U_{ij}})^{-1} \circ \varphi_j|_{U_{ij}}: c(P_j)|_{U_{ij}} \rightarrow c(P_i)|_{U_{ij}}, \quad i, j \in I.$$

Since c is fully faithful, there are unique isomorphisms

$$\Phi_{ij}: P_j|_{U_{ij}} \rightarrow P_i|_{U_{ij}}$$

with $c(\Phi_{ij}) = \varphi_{ij}$. Since c commutes with restrictions to open subsets and since $\{\varphi_{ij}\}_{i,j \in I}$ form a cocycle and the Φ_{ij} are unique, $\{\Phi_{ij}\}_{i,j \in I}$ form a cocycle as well. By Lemma 4.7 there is $P \in \text{STB}_\psi(W)$ and a family $\{\gamma_i: P|_{U_i} \rightarrow P_i\}$ of isomorphisms so that (4.2) commutes. Then

$$\begin{array}{ccccc} M|_{U_{ij}} & \xleftarrow{\varphi_j} & c(P_j)|_{U_{ij}} & \xleftarrow{c(\gamma_j)} & c(P)|_{U_{ij}} \\ \parallel & & \downarrow \varphi_{ij} & & \parallel \\ M|_{U_{ij}} & \xleftarrow{\varphi_i} & c(P_i)|_{U_{ij}} & \xleftarrow{c(\gamma_i)} & c(P)|_{U_{ij}} \end{array}$$

commutes as well. Consequently

$$\varphi_i \circ c(\gamma_i)|_{c(P)|_{U_{ij}}} = \varphi_j \circ c(\gamma_j)|_{c(P)|_{U_{ij}}}.$$

Therefore the family $\{\varphi_i \circ c(\gamma_i): c(P)|_{U_i} \rightarrow M|_{U_i}\}$ gives rise to a well defined isomorphism $c(P) \rightarrow M$. \blacksquare

This completes our proof of Theorem 4.1. In fact, we have proved more:

Theorem 4.9. *Let $\psi: W \rightarrow \mathfrak{g}^*$ be a u.l.e. Then the functor*

$$c: \text{STB}_\psi \rightarrow \text{STM}_\psi$$

is an isomorphism of stacks over the site of open subsets of the manifold with corners W .

Proof. Recall that $c: \text{STB}_\psi \rightarrow \text{STM}_\psi$ commutes with restrictions, hence, it is a map of stacks. By Theorem 4.1, for every open subset $U \subset W$, the functor $c_U: \text{STB}_\psi(U) \rightarrow \text{STM}_\psi(U)$ is an equivalence of categories. Hence $c: \text{STB}_\psi \rightarrow \text{STM}_\psi$ is an isomorphism of stacks. \blacksquare

5 Characteristic classes and classification of symplectic toric G -manifolds

As we have seen in the previous section the functor $c: \text{STB}_\psi(W) \rightarrow \text{STM}_\psi(W)$ is an equivalence of categories. Hence it defines a bijection $\pi_0(c): \pi_0(\text{STB}_\psi(W)) \rightarrow \pi_0(\text{STM}_\psi(W))$ between the

sets of equivalence classes. Thus to finish the proof of Theorem 1.3(2) it is enough to construct a bijection

$$\pi_0(\text{STB}_\psi(W)) \leftrightarrow H^2(W, \mathbb{R}) \times H^2(W, \mathbb{Z}_G).$$

Recall that for a torus G with integral lattice \mathbb{Z}_G and a manifold with corners N there is a bijection

$$c_1: \pi_0(BG(N)) \rightarrow H^2(N, \mathbb{Z}_G),$$

where $BG(N)$ denotes the category of principal G -bundles over N and c_1 assigns to each isomorphism class $[P] \in \pi_0(BG(N))$ of a bundle P its first Chern class $c_1(P)$. Recall also that the map c_1 is an isomorphism of presheaves. Namely if $V \xrightarrow{i} U \hookrightarrow N$ are two open subsets of N then the diagram

$$\begin{array}{ccc} \pi_0(BG(U)) & \xrightarrow{c_1} & H^2(U, \mathbb{Z}_G) \\ i^* \downarrow & & \downarrow i^* \\ \pi_0(BG(V)) & \xrightarrow{c_1} & H^2(V, \mathbb{Z}_G) \end{array}$$

commutes. Pre-composing with the map $\pi_0(\text{STB}_\psi(\cdot)) \rightarrow \pi_0(BG(\cdot))$ that is induced by the forgetful functor, we get a homomorphism

$$c_1: \pi_0(\text{STB}_\psi(\cdot)) \rightarrow H^2(\cdot, \mathbb{Z}_G) \tag{5.1}$$

of presheaves on W .

Proposition 5.1. *Fix a u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$. The homomorphism (5.1) extends to an isomorphism of presheaves*

$$(c_1, c_{\text{hor}}): \pi_0(\text{STB}_\psi(\cdot)) \rightarrow H^2(\cdot, \mathbb{Z}_G) \times H^2(\cdot, \mathbb{R}). \tag{5.2}$$

Definition 5.2. We call the second component of the isomorphism (5.2) the *horizontal class*. We say informally that $c_{\text{hor}}([(P, \omega)])$ is the horizontal class of the symplectic toric bundle $(P \rightarrow W, \omega)$.

Proof of Proposition 5.1. Fix an open set $U \subset W$. We construct a bijection

$$F = F_U: H^2(U, \mathbb{Z}_G) \times H^2(U, \mathbb{R}) \rightarrow \pi_0(\text{STB}_\psi(U))$$

that commutes with pullbacks by open inclusions $i: V \hookrightarrow U$ and which is the inverse of the map (5.2) on U . Here, we take $H^2(\cdot, \mathbb{R})$ to be the second de Rham cohomology.

Given $(c, [\beta]) \in H^2(U, \mathbb{Z}_G) \times H^2(U, \mathbb{R})$ choose a principal G -bundle P with $c_1(P) = c$. By Lemma 3.2 a choice of a connection 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ defines a symplectic form

$$\omega_{A, \beta} = d\langle \mu, A \rangle + \pi^* \beta.$$

If $\beta' \in [\beta]$ is another closed 2-form representing the class $[\beta] \in H^2(U, \mathbb{R})$ then $\beta' = \beta + d\gamma$ for some $\gamma \in \Omega^1(U)$. By Lemma 3.3 the objects $(P, \omega_{A, \beta})$ and $(P, \omega_{A, \beta'})$ are isomorphic in $\text{STB}_\psi(U)$. If A' is a different choice of a connection on P then $A - A' = \pi^* a$ for some $a \in \Omega^1(U, \mathfrak{g})$. Consequently by Lemma 3.3 the symplectic bundles $(P, \omega_{A, \beta})$ and $(P, \omega_{A', \beta})$ are also isomorphic. We conclude that the map $F: H^2(U, \mathbb{Z}_G) \times H^2(U, \mathbb{R}) \rightarrow \pi_0(\text{STB}_\psi(U))$ that assigns to a pair $(c, [\beta])$ the isomorphism class of $(P, \omega_{A, \beta})$ with $c_1(P) = c$ is well-defined.

Given any $(P, \omega) \in \text{STB}_\psi(U)$, Lemma 3.3 implies that $\omega = \omega_{A, \beta}$ for some closed 2-form β on U . Hence F is onto.

Suppose $F(c, [\beta]) = [(P, \omega)] = F(c', [\beta'])$ for some $(c, [\beta]), (c', [\beta']) \in H^2(U, \mathbb{Z}_G) \times H^2(U, \mathbb{R})$. Then

$$c = c_1(P) = c'.$$

Now $F(c, [\beta]) = [(P, \omega_{A, \beta})]$ and $F(c', [\beta']) = [(P, \omega_{A', \beta'})]$ for some connections $A, A' \in \Omega^1(P, \mathfrak{g})$. Since $[(P, \omega_{A, \beta})] = [(P, \omega_{A', \beta'})]$ there is a gauge transformation $f: P \rightarrow P$ with $f^*\omega_{A', \beta'} = \omega_{A, \beta}$. Since A and f^*A' are both connections on P ,

$$f^*A' - A = \pi^*a$$

for some $a \in \Omega^1(U, \mathfrak{g})$. Since $\mu \circ f = \mu$ and $\pi \circ f = \pi$,

$$f^*(\omega_{A', \beta'}) = f^*(d\langle \mu, A' \rangle + \pi^*\beta') = d\langle \mu \circ f, f^*A' \rangle + f^*\pi^*\beta' = d\langle \mu, f^*A' \rangle + \pi^*\beta'.$$

Consequently,

$$0 = f^*(\omega_{A', \beta'}) - \omega_{A, \beta} = d\langle \mu, f^*A' \rangle + \pi^*\beta' - d\langle \mu, A \rangle - \pi^*\beta.$$

Therefore

$$\pi^*(\beta - \beta') = d\langle \mu, f^*A' - A \rangle = d\langle \mu, \pi^*a \rangle = \pi^*(d\langle \psi, a \rangle).$$

Hence

$$\beta - \beta' = d\langle \psi, a \rangle.$$

Therefore $[\beta] = [\beta']$, and F is one-to-one. ■

As an immediate consequence we have

Proof of Theorem 1.3(2). It follows from Proposition 5.1 that the composite

$$\pi_0(\text{STM}_\psi(W)) \xrightarrow{\pi_0(\mathfrak{c})^{-1}} \pi_0(\text{STB}_\psi(W)) \xrightarrow{(c_1, c_{\text{hor}})} H^2(W, \mathbb{Z}_G) \times H^2(W, \mathbb{R})$$

is a bijection. ■

Definition 5.3 (Chern and horizontal classes of a symplectic toric manifold). Let $(\pi: M \rightarrow W, \omega) \in \text{STB}_\psi(W)$ be a symplectic toric manifold over a u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$. We define its *Chern class* to be the first Chern class of the corresponding principal torus bundle:

$$c_1(M, \omega, \pi) := c_1 \circ \pi_0(\mathfrak{c})^{-1}([M, \omega, \pi]).$$

Similarly its *horizontal class* is the horizontal class of the corresponding bundle:

$$c_{\text{hor}}(M, \omega, \pi) := c_{\text{hor}} \circ \pi_0(\mathfrak{c})^{-1}([M, \omega, \pi]).$$

Another consequence of Proposition 5.1 is

Corollary 5.4. *Fix a u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$. If $H^2(W, \mathbb{Z}) = 0$ then any two objects of $\text{STM}_\psi(W)$ are isomorphic.*

This corollary significantly strengthens Lemma B.4.

Remark 5.5. The Chern and horizontal classes for a symplectic toric manifold over W have a nice geometric interpretation in terms of the restriction to the interior $\overset{\circ}{W}$.

If W is a manifold with corners and $\overset{\circ}{W}$ is its interior then the inclusion map $\iota: \overset{\circ}{W} \hookrightarrow W$ is a smooth homotopy equivalence. (A homotopy inverse is obtained from the flow of a vector field on W that is supported in a small neighborhood of the topological boundary ∂W , is transverse to all the codimension 1 strata of W , and points inward along the boundary ∂W .) So the restriction maps $\iota^*: H^2(W; \mathbb{Z}_G) \rightarrow H^2(\overset{\circ}{W}; \mathbb{Z}_G)$ and $\iota^*: H^2(W; \mathbb{R}) \rightarrow H^2(\overset{\circ}{W}; \mathbb{R})$ are isomorphisms. By Proposition 5.1 the diagram

$$\begin{array}{ccc} \pi_0(\text{STM}_\psi(W)) & \xrightarrow{(c_1, c_{\text{hor}})} & H^2(W, Z_G) \times H^2(W, \mathbb{R}) \\ \iota^* \downarrow & & \downarrow \iota^* \\ \pi_0(\text{STM}_\psi(\overset{\circ}{W})) & \xrightarrow{(c_1, c_{\text{hor}})} & H^2(\overset{\circ}{W}, Z_G) \times H^2(\overset{\circ}{W}, \mathbb{R}) \end{array}$$

commutes. Since $c: \text{STB}_\psi(\overset{\circ}{W}) \rightarrow \text{STM}_\psi(\overset{\circ}{W})$ is isomorphic to the identity, the induced map $\pi_0(c)$ is the identity. It follows that for a symplectic toric manifold $(M, \omega, \pi) \in \text{STM}_\psi(W)$

$$\iota^* c_1(M, \omega, \pi) = c_1(M|_{\overset{\circ}{W}}, \omega, \pi) = c_1(M|_{\overset{\circ}{W}}),$$

where $c_1(M|_{\overset{\circ}{W}})$ is the Chern class of the principal G -bundle $M|_{\overset{\circ}{W}} \rightarrow \overset{\circ}{W}$. Note that this in particular relates our definition of the Chern class of a symplectic toric manifold to Duistermaat's definition of the Chern class of a completely integrable system [8] (cf. Remarks 1.7 and 1.8).

Similarly

$$\iota^* c_{\text{hor}}(M, \omega, \pi) = c_{\text{hor}}(M|_{\overset{\circ}{W}}, \omega, \pi),$$

where the class on the right is the horizontal class of the symplectic principal G -bundle $M|_{\overset{\circ}{W}} \rightarrow \overset{\circ}{W}$.

6 Toric manifolds determined by their moment map images

As we mentioned in the introduction, in general the image of the moment map doesn't tell us much about the symplectic toric manifold. There are two reasons for this. First of all, the orbital moment map may not be an embedding – see Example 2.8. Secondly, even when the orbital moment map is an embedding the second integral cohomology of the orbit space, which then has to be (isomorphic to) the image of the moment map, may not be trivial.

Example 6.1. Consider a three-dimensional torus G . Then $H^2(\mathfrak{g}^* \setminus \{0\}, \mathbb{Z}) = \mathbb{Z}$. By Theorem 1.3 there are $\mathbb{Z}_G \times \mathbb{R}$ isomorphism classes of symplectic toric manifolds over $\mathfrak{g}^* \setminus \{0\}$ (they are all principal G -bundles over $\mathfrak{g}^* \setminus \{0\}$). For all of these manifolds the orbital moment map is an embedding and the moment map image is $\mathfrak{g}^* \setminus \{0\}$. This family of manifolds has been studied by Bates [2]. He proves that any principal G -bundle $P \rightarrow \mathfrak{g}^* \setminus \{0\}$ admits a symplectic form making the fibers of $P \rightarrow \mathfrak{g}^* \setminus \{0\}$ Lagrangian and the action of G on P Hamiltonian.

Example 6.2. It is also easy to construct examples of symplectic toric manifolds where the orbital moment map is an embedding, the torus action is not free and the second integral cohomology of the orbit space is nontrivial. For instance let G again be a three-dimensional torus, let $\Delta \subset \mathfrak{g}^*$ be a unimodular simplex (or any other unimodular polytope) and let $W = \Delta \setminus \{w_0\}$, where w_0 is a point in the interior of Δ . Then again $H^2(W \setminus \{w_0\}, \mathbb{Z}) = \mathbb{Z}$ and consequently there are $\mathbb{Z}_G \times \mathbb{R}$ isomorphism classes of symplectic toric manifolds over W . For every symplectic toric manifold M over $W \hookrightarrow \mathfrak{g}^*$ the points over the vertices of W are the fixed points of the G action.

Theorem 1.3 implies that (the isomorphism class of) a symplectic toric manifold (M, ω, μ) is uniquely determined by $\mu(M)$ if (1) the orbital moment map is an embedding and (2) $H^2(\mu(M), \mathbb{Z}) = 0$. The connectedness and convexity theorems of Atiyah, Guillemin and Sternberg imply that any compact connected symplectic toric manifold falls into this class of toric manifolds. But there is more. Recall that the assumption of compactness in connectedness and convexity theorems can be weakened. Namely,

Theorem 6.3 (cf. [18, Theorem 4.3]). *Let $\mu: M \rightarrow \mathfrak{g}^*$ be a moment map for an action of a torus G on a symplectic manifold (M, ω) . Suppose there exists a convex open subset $U \subset \mathfrak{g}^*$ such that $\mu(M) \subset U$ and $\mu: M \rightarrow U$ is proper. Then the image $\mu(M)$ is convex and each fiber of μ is connected.*

It will be convenient to have the following definition:

Definition 6.4. A map $F: M \rightarrow V$ from a space M to a finite-dimensional vector space V is proper as a map into a convex open set if there is a convex open set $U \subset V$ so that $F(M) \subset U$ and $F: M \rightarrow U$ is proper.

Proposition 6.5. *Let (M, ω, μ) be a connected symplectic toric G -manifold whose moment map μ is proper as a map into a convex open set. Then the image $\mu(M)$ determines (M, ω, μ) up to isomorphism.*

Proof. Theorem 6.3 and Lemma 2.30 imply that the orbital moment map $\bar{\mu}$ is an embedding. Consequently, $\bar{\mu}: M/G \rightarrow \mu(M)$ is a diffeomorphism of manifolds with corners and $\mu: M \rightarrow \mu(M)$ is a quotient map. That is (M, ω, μ) is a symplectic toric manifold over $\mu(M)$. By Theorem 6.3, $\mu(M) \simeq M/G$ is convex, hence $H^2(M/G, \mathbb{Z}) = 0$. It now follows from Theorem 1.3 that any two symplectic toric manifolds over $\mu(M)$ are isomorphic. ■

Remark 6.6. Proposition 6.5 can be used for “coordinatization” of compact symplectic toric manifolds in the sense of Duistermaat and Pelayo [9]. Namely, let (M, ω, μ) be a compact connected symplectic toric G -manifold with momentum polytope $\Delta = \mu(M)$. Express Δ as the intersection of half-spaces H_1, \dots, H_N whose boundaries are the affine spans of the facets of Δ . For each vertex ϵ of Δ , let C_ϵ be the intersection of those H_j whose boundary contains ϵ (this is the tangent cone to Δ at ϵ), and let \mathcal{T}_ϵ be the intersection of $\text{interior}(H_j)$ over those j such that $\epsilon \in \text{interior}(H_j)$.

Let U_ϵ denote the preimage in M of the convex open subset \mathcal{T}_ϵ of \mathfrak{g}^* . The sets U_ϵ form a covering of M by G -invariant open dense subsets. By Proposition 6.5, each of these subsets U_ϵ is equivariantly symplectomorphic to a G -invariant open subset of \mathbb{C}^n , where G acts on \mathbb{C}^n through the isomorphism $G \rightarrow (S^1)^n$ for which the momentum map image is C_ϵ .

The reader familiar with Delzant’s paper may wonder which of the symplectic toric manifolds that we classify can be obtained as symplectic quotients of some standard \mathbb{C}^N by an action of a subtorus of the standard torus $\mathbb{T}^N = (S^1)^N$. The following theorem and its proof are the result of our discussion with Chris Woodward. We thank Chris for bringing up the question and helping us prove the answer.

Theorem 6.7. *A symplectic toric G -manifold $(M, \omega, \mu: M \rightarrow \mathfrak{g}^*)$ is isomorphic to a regular symplectic quotient of \mathbb{C}^N by a subtorus of the standard torus \mathbb{T}^N if and only if its orbital moment map $\bar{\mu}: M/G \rightarrow \mathfrak{g}^*$ is an embedding and its image is a closed convex polyhedral subset of \mathfrak{g}^* with at least one vertex and at most N facets.*

Remark 6.8. We already know that the orbital momentum map is locally an embedding as a manifold with corners. So it is a global embedding as a manifold with corners if and only if it is a global embedding topologically.

Proof of Theorem 6.7. We first argue that the conditions of the theorem are necessary: if a symplectic toric manifold (M, ω, μ) is a non-singular symplectic quotient of \mathbb{C}^N by a subtorus $K \hookrightarrow \mathbb{T}^N$, then the orbital moment map $\bar{\mu}$ is an embedding and the moment map image $\mu(M)$ is a unimodular polyhedral subset of \mathfrak{g}^* with at least one vertex.

Recall that for the standard action of \mathbb{T}^N on \mathbb{C}^N the map $\psi: \mathbb{C}^N \rightarrow \mathbb{R}^N$, $\psi(z_1, \dots, z_N) = (|z_1|^2, \dots, |z_N|^2)$, is a moment map (with an appropriate identification of the dual of the Lie algebra of \mathbb{T}^N with \mathbb{R}^N). The image of ψ is the positive orthant $\mathbb{R}_+^N := \{t \in \mathbb{R}^N \mid t_i \geq 0 \forall i\}$, and the orbital moment map $\bar{\psi}: \mathbb{C}^N/\mathbb{T}^N = \mathbb{R}_+^N \rightarrow \mathbb{R}^N$ is an embedding. The inclusion $K \hookrightarrow \mathbb{T}^N$ induces an inclusion $i^T: \mathfrak{k} \hookrightarrow \text{Lie}(\mathbb{T}^N)$ of Lie algebras and, dually, the projection $i^T: \text{Lie}(\mathbb{R}^N)^* = \mathbb{R}^N \rightarrow \mathfrak{k}^*$. Then $\varphi := i^T \circ \psi: \mathbb{C}^N \rightarrow \mathfrak{k}^*$ is a moment map for the action of K on \mathbb{C}^N . Suppose $\nu \in \mathfrak{k}^*$ is a point such that the action of K on its preimage $\varphi^{-1}(\nu)$ is free. Then ν is necessarily a regular value of φ . Moreover, the symplectic quotient $M := \varphi^{-1}(\nu)/K$ is a symplectic toric G -manifold for $G = \mathbb{T}^N/K$ (cf. [7]). The moment map $\mu: M \rightarrow \mathfrak{g}^*$ has the following description. Since \mathfrak{g}^* is canonically isomorphic to the annihilator \mathfrak{k}° of \mathfrak{k} in $\text{Lie}(\mathbb{T}^N)^* = \mathbb{R}^N$ and since $(i^T)^{-1}(\nu) = \lambda + \mathfrak{k}^\circ$ for some $\lambda \in (i^T)^{-1}(\nu)$, we can identify \mathfrak{g}^* with the affine plane $(i^T)^{-1}(\nu)$. With this identification, the restriction

$$\psi|_{\varphi^{-1}(\nu)} = \psi|_{\psi^{-1}((i^T)^{-1}(\nu))}: \psi^{-1}((i^T)^{-1}(\nu)) \rightarrow (i^T)^{-1}(\nu)$$

descends to $\mu: M = \varphi^{-1}(\nu)/K \rightarrow (i^T)^{-1}(\nu) \simeq \mathfrak{g}^*$. Since all the fibers of ψ are \mathbb{T}^N -orbits, the fibers of μ are $\mathbb{T}^N/K = G$ -orbits. Since $\bar{\psi}$ is an open map to its image, so is $\bar{\mu}$. Hence the orbital moment map $\bar{\mu}: M/G \rightarrow (i^T)^{-1}(\nu) \simeq \mathfrak{g}^*$ is an embedding. Since the image of M under μ is $\psi(\mathbb{C}^N) \cap (i^T)^{-1}(\nu) = \mathbb{R}_+^N \cap (i^T)^{-1}(\nu)$, it has at most N facets. Now we argue that the image has at least one vertex. We give a symplectic (as opposed to a convex geometry) argument.

The function $f(z) = \sum |z_j|^2: \mathbb{C}^N \rightarrow \mathbb{R}$ is proper, non-negative, \mathbb{T}^N -invariant, and its Hamiltonian vector field generates a circle action. Hence $f|_{\varphi^{-1}(\nu)}$ descends to a proper non-negative G -invariant periodic Hamiltonian \bar{f} on M . Since \bar{f} is non-negative and proper, it achieves a minimum somewhere on M . Since \bar{f} is periodic and G -invariant, the set of points where \bar{f} achieves its minimum is a G -invariant symplectic submanifold M' of M . Since \bar{f} is proper, M' is compact. Since the action of G on M' is Hamiltonian, it has to have a fixed point. The image of this fixed point under the moment map μ is a vertex of the polyhedral set $\mu(M)$.

Now suppose that the orbital moment map $\bar{\mu}: M/G \rightarrow \mathfrak{g}^*$ of a symplectic toric G -manifold $(M, \omega, \mu: M \rightarrow \mathfrak{g}^*)$ is an embedding, that its image $A = \bar{\mu}(M/G)$ is an intersection of N closed half-spaces, and that A has a vertex $*$. Let H_1, \dots, H_n denote the half-spaces whose supporting hyperplanes form the facets meeting at $*$. Since A is unimodular, $n = \dim G$ and the intersection $H_1 \cap \dots \cap H_n$ is isomorphic (as a unimodular cone) to \mathbb{R}_+^n . (In particular, $N \geq \dim G$.) Consequently there is an isomorphism $G \rightarrow \mathbb{T}^n$ so that the image of a corresponding moment map $\psi: \mathbb{C}^n \rightarrow \mathfrak{g}^*$ is $H_1 \cap \dots \cap H_n$. Let H_{n+1}, \dots, H_N denote the remaining half-spaces, so that $A = H_1 \cap \dots \cap H_N$. We now successively apply the symplectic cut construction to \mathbb{C}^n using the half-spaces H_{n+1}, \dots, H_N [16]. This amounts to taking a symplectic quotient of $\mathbb{C}^n \times \mathbb{C}^{N-n}$ by an action of $(S^1)^{N-n}$. Since A is unimodular, the quotient is smooth [18]. This symplectic quotient of \mathbb{C}^N is a symplectic toric G -manifold with moment map image A and whose orbital moment map is an embedding. Because A is convex, its second cohomology is trivial; by Theorem 1.3, this symplectic quotient of \mathbb{C}^N is isomorphic to $(M, \omega, \mu: M \rightarrow \mathfrak{g}^*)$. ■

We end this section with an example of a noncompact symplectic toric G -manifold that is *not* isomorphic to a symplectic reduction of any \mathbb{C}^N :

Example 6.9. Let $G = (S^1)^2$. Let W be the closed region in $\mathfrak{g}^* = \mathbb{R}^2$ that is bounded on the bottom by the positive x axis and on the top by the polygonal path that is obtained by connecting, in this order, the points $(k(k-1)/2, k)$ for $k \in \{0, 1, 2, \dots\}$. See Fig. 1. The set W

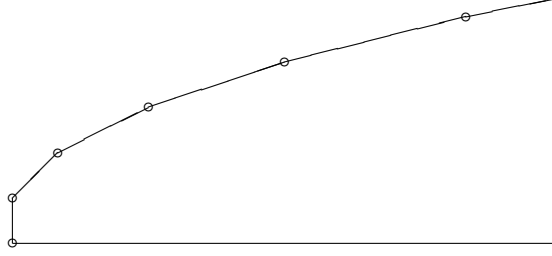


Figure 1. A noncompact symplectic toric manifold that is not a reduction of \mathbb{C}^N .

is locally unimodular: near the vertex $(0, 0)$ it coincides with the positive orthant, and near the vertex $v = (k(k-1)/2, k)$ for $k \geq 1$ it coincides with the cone $v + \mathbb{R}_+(-k-1, -1) + \mathbb{R}_+(k, 1)$, which is unimodular because $\det \begin{bmatrix} -(k-1) & -1 \\ k & 1 \end{bmatrix} = 1$. By Theorem 1.3, there exists a symplectic toric G -manifold M with moment image W . Because W has infinitely many facets (edges) and by Theorem 6.7, M is not isomorphic to a symplectic quotient of any \mathbb{C}^N .

A Manifolds with corners

We quote Joyce [14]:

“[manifolds with corners] were first developed by Cerf [1] and Douady [2] in 1961, who were primarily interested in their Differential Geometry. Jänich [5] used manifolds with corners to classify actions of transformation groups on smooth manifolds. Melrose [12, 13] and others study analysis of elliptic operators on manifolds with corners. . . . How one sets up the theory of manifolds with corners is not universally agreed, but depends on the applications one has in mind. . . . there are at least four inequivalent definitions of manifolds with corners, two inequivalent definitions of boundary, and (including ours) four inequivalent definitions of smooth map in use in the literature.”

The purpose of the appendix is to spell out our approach to manifolds with corners and their maps. In particular we spell out what we mean for a subset Y of a manifold with corners X to be naturally a smooth manifold (Definition A.14) and what we mean by an embedding of a manifold with corners into a manifold (Definition A.5).

Definition A.1 (manifold with corners). Let V be an (arbitrary) subset of \mathbb{R}^n . A map $\varphi: V \rightarrow \mathbb{R}^m$ is *smooth* if for every point p of V there exist an open subset Ω in \mathbb{R}^n containing p and a smooth map from Ω to \mathbb{R}^m whose restriction to $\Omega \cap V$ coincides with $\varphi|_{\Omega \cap V}$. A map φ from V to a subset of \mathbb{R}^m is *smooth* if it is smooth as a map to \mathbb{R}^m . A map φ from a subset of \mathbb{R}^n to a subset of \mathbb{R}^m is a *diffeomorphism* if it is a bijection and both it and its inverse are smooth.

A *sector* is the set $[0, \infty)^k \times \mathbb{R}^{n-k}$ where n is a non-negative integer and k is an integer between 0 and n . Let X be a Hausdorff second countable topological space. A *chart* on an open subset U of X is a homeomorphism φ from U to an open subset V of a sector. Charts $\varphi: U \rightarrow V$ and $\varphi': U' \rightarrow V'$ are *compatible* if $\varphi' \circ \varphi^{-1}$ is a diffeomorphism from $\varphi(U \cap U')$ to $\varphi'(U \cap U')$. An *atlas* on X is a set of pairwise compatible charts whose domains cover X . Two atlases are *equivalent* if their union is an atlas. A *manifold with corners* is a Hausdorff second countable topological space equipped with an equivalence class of atlases.

We sometimes refer to an ordinary manifold as a “manifold without boundary or corners”.

Definition A.2 (smooth map). Let X and Y be manifolds with corners. A map $h: X \rightarrow Y$ is *smooth* if for every point in X there exists an open neighbourhood U in X and an open subset U' of Y and charts $\varphi: U \rightarrow V$ and $\varphi': U' \rightarrow V'$ of X and Y such that $h(U) \subset U'$ and such that $\varphi' \circ h \circ \varphi^{-1}: V \rightarrow V'$ is smooth.

A map f from an (arbitrary) subset $A \subset X$ to Y is smooth if for every point in A there exists a neighbourhood \mathcal{O} in X and a smooth map to Y whose restriction to $A \cap \mathcal{O}$ coincides with f ; a map from A to a subset B of Y is smooth if it is smooth as a map to Y , and it is a *diffeomorphism* if it is smooth and has a smooth inverse.

It is easy to check that the composition of two smooth maps is again smooth. Hence manifolds with corners form a category. The isomorphisms in this category are *diffeomorphisms*. We may refer to a smooth map between two manifolds with corners as a *map of manifolds with corners*.

Definition A.3. The *dimension* of a manifold with corners X is n if the charts take values in sectors in \mathbb{R}^n . A point x of X has *index* k if there exists a chart φ from a neighbourhood of x to $[0, \infty)^k \times \mathbb{R}^{n-k}$ such that $\varphi(x) = 0$; the index of a point is well defined. The k -*boundary*, $X^{(k)}$, of X is the set of points of index $\geq k$. The (topological) *boundary* of X is the 1-boundary, $\partial X := X^{(1)}$. The *interior* of X is the complement of the boundary: $\overset{\circ}{X} := X \setminus \partial X$; it is the set of points of index 0. We refer to the connected components of the sets

$$\partial^{(k)} X := X^{(k)} \setminus X^{(k+1)}$$

as the *strata* of X .

Definition A.4. The *tangent space* $T_x X$ of a manifold with corners X at a point $x \in X$ is the space of derivations at x of germs at x of smooth functions defined near x . Thus, the tangent space is a vector space even if the point x is in the boundary of X .

Similarly the tangent bundle TX of a manifold with corners X is a vector bundle over X as is the cotangent bundle T^*X and its exterior powers. The total spaces of TX and T^*X are manifolds with corners (cf. [25, p. 19]).

A *differential k -form* on a manifold with corners X is a smooth section of the k th exterior power of its cotangent bundle.

Exterior derivative d makes sense on manifolds with corners. So do closed forms and symplectic forms. Thus, the usual notions of a Hamiltonian action and a moment map extend to manifolds with corners without change.

We note that, on an open subset of a sector, a closed form locally extends as a *closed* form, because it locally has a primitive and the primitive has a local smooth extension.

Definition A.5. A smooth map $f: N \rightarrow M$ of manifolds with corners is an *embedding* if it is a topological embedding and the differential $df_x: T_x M \rightarrow T_{f(x)} N$ of f is injective at every point $x \in M$. Equivalently, f is an embedding if $f: N \rightarrow f(N)$ is a diffeomorphism.

Example A.6. The inclusion $[0, \infty)^k \hookrightarrow \mathbb{R}^k$ is an embedding.

One can prove (q.v. [25, p. 21]):

Lemma A.7. *A manifold with corners M can be embedded in a manifold \tilde{M} (without corners) of the same dimension.*

Definition A.8 (domain). If a manifold with corners M is embedded in a manifold \tilde{M} (without corners) and if $\dim M = \dim \tilde{M}$, we say that M is a *domain* in \tilde{M} .

Remark A.9. If $M \subset \tilde{M}$ is a domain and $f: M \rightarrow V$ is a smooth map to some finite-dimensional vector space V , then f extends to a smooth map \tilde{f} from some open neighbourhood of M in \tilde{M} . We refer to \tilde{f} as an *extension* of f to \tilde{M} .

It is not true that the closure of a stratum of a manifold with corners is a manifold with corners. See for example Fig. 2.1 in [14]. Nor is it true a stratum of codimension k lies in the closure of exactly k codimension 1 strata.

Definition A.10. A manifold with corners X is a *manifold with faces* (q.v. [13]) if every point of X of index k lies in the closure of exactly k codimension 1 strata.

We refer to the closures of the strata of a manifold with faces X as *faces* and the codimension 1 faces as *facets*.

One can show that for a manifold with faces the closure of a stratum is a manifold with faces (op. cit.).

Example A.11. A unimodular cone is a manifold with faces.

Example A.12. A sector $[0, \infty)^k \times \mathbb{R}^{n-k}$ is a manifold with faces.

Remark A.13. An open subspace of a manifold with faces is again a manifold with faces. Consequently any manifold with corners is a manifold with faces locally: for any manifold with corners N and any point $x \in N$ there is an open neighbourhood U of x in N such that U is a manifold with faces. We will refer to such neighbourhood U as a *neighbourhood with faces*.

The following definition is nonstandard but is essential for the purposes of this paper.

Definition A.14. We say that a subset Y of a manifold with corners X is *naturally a smooth manifold* if it has a manifold structure such that the inclusion map $Y \hookrightarrow X$ is an embedding in the sense of Definition A.5. If such a manifold structure on Y exists, then it is unique.

Example A.15. With Definition A.14, the parabola $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$, as a subset of the upper half plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, is naturally a smooth manifold.

Definition A.16. An *action* of a Lie group G on a manifold with corners X is a homomorphism ρ from G to the group of diffeomorphisms of X such that the map $G \times X \rightarrow X$ given by $(a, x) \mapsto \rho(a)(x)$ is smooth.

Given an action of a compact Lie group G on a manifold with corners X , we say that a smooth map $\pi: X \rightarrow W$ from X to another manifold with corners W is a *quotient map* if for every G -invariant smooth map $f: X \rightarrow Y$ there exists a unique smooth map $\bar{f}: W \rightarrow Y$ such that $f = \bar{f} \circ \pi$. Such a map π identifies W with the set X/G of G orbits.

We say that X/G is *naturally a manifold with corners* if it has a manifold with corners structure such that the map $X \rightarrow X/G$ that takes each point to its orbit is a quotient map in the above sense. If such a structure exists then it is unique. We say that X/G is *naturally a smooth manifold*, or, simply, that it is a manifold, if it has such a structure without boundary or corners.

Definition A.17. Let W be a manifold with corners and G a Lie group. A *principal G -bundle* over W is a manifold with corners P equipped with a right action of G and with a map $\pi: P \rightarrow W$ making it into a topological principal G -bundle in which local trivializations can be chosen to be equivariant diffeomorphisms of manifolds with corners.

On a manifold with corners M , the de Rham cohomology is well defined and invariant under homotopy. The Poincaré lemma holds: every closed form is locally exact. The proofs are exactly as for ordinary manifolds. A key step is that if β is a k -form on $[0, 1] \times M$ then $i_1^* \beta - i_0^* \beta = \pi_* d\beta + d\pi_* \beta$ where $i_t: M \rightarrow M \times [0, 1]$ is $i_t(m) = (m, t)$ and where π_* is fiber integration.

B Local structure of symplectic toric manifolds

The purpose of this section is to set our notation and to recall the “local uniqueness” result, that symplectic toric manifolds over the same unimodular local embedding (u.l.e.) are locally isomorphic. The results of the appendix are adapted from [7]; see also [19]. We begin by recalling the symplectic slice representation.

Definition B.1. Let a compact Lie group G act on a symplectic manifold (M, ω) with a moment map $\mu: M \rightarrow \mathfrak{g}^*$. The *symplectic slice representation* at a point x of M is the linear symplectic action of the stabilizer G_x on the symplectic vector space $V_x := T_x(G \cdot x)^\omega / (T_x(G \cdot x) \cap (T_x(G \cdot x))^\omega)$ that is induced from the linearization at x of the G_x action on M . Here, $T_x(G \cdot x)^\omega$ denotes the symplectic perpendicular to $T_x(G \cdot x)$ in $T_x M$.

We recall that the orbits of a Hamiltonian torus action are isotropic. In Definition B.1, if the orbit $G \cdot x$ is isotropic, then the symplectic slice is simply $V_x = T_x(G \cdot x)^\omega / T_x(G \cdot x)$.

The following theorem is a consequence of the equivariant constant rank embedding theorem of Marle [21]; also see [30, Section 2]. The proof in the case where the group is a torus was given earlier by Guillemin and Sternberg [11].

Theorem B.2. *Let a compact Lie group G act on symplectic manifolds (M, ω) and (M', ω') with moment maps $\mu: M \rightarrow \mathfrak{g}^*$ and $\mu': M' \rightarrow \mathfrak{g}^*$. Fix a point x of M and a point x' of M' . Suppose that x and x' have the same stabilizer, their symplectic slice representations are linearly symplectically isomorphic, and they have the same moment map value. Then there exists an equivariant symplectomorphism from an invariant neighbourhood of x in M to an invariant neighbourhood of x' in M' that respects the moment maps and that sends x to x' .*

We write points in the standard torus $\mathbb{T}^\ell = \mathbb{R}^\ell / \mathbb{Z}^\ell$ as ℓ -tuples $[t_1, \dots, t_\ell]$ with $(t_1, \dots, t_\ell) \in \mathbb{R}^\ell$. Alternatively, since $\mathbb{R}^\ell / \mathbb{Z}^\ell \simeq (\mathbb{R}/\mathbb{Z})^\ell$, we may think of a point on the standard ℓ -torus \mathbb{T}^ℓ as an ℓ -tuple (q_1, \dots, q_ℓ) with $q_i \in \mathbb{R}/\mathbb{Z}$. We think of the q_i s as coordinates. Then the cotangent bundle $T^*\mathbb{T}^\ell$ has canonical coordinates $(q_1, \dots, q_\ell, p_1, \dots, p_\ell)$ with $p_i \in \mathbb{R}^*$. The symplectic form on the cotangent bundle is given by $\omega = \sum dp_i \wedge dq_i$ in these coordinates. The lift of the action of \mathbb{T}^ℓ on itself by left multiplication to the action on $T^*\mathbb{T}^\ell$ is Hamiltonian with an associated moment map

$$T^*\mathbb{T}^\ell \rightarrow (\mathbb{R}^\ell)^* \simeq (\mathbb{R}^*)^\ell, \quad (q_1, \dots, q_\ell, p_1, \dots, p_\ell) \mapsto (p_1, \dots, p_\ell).$$

Recall the local normal form for neighbourhoods of orbits in a symplectic toric G -manifold:

Lemma B.3. *Let $(M, \omega, \mu: M \rightarrow \mathfrak{g}^*)$ be a symplectic toric G -manifold. Consider a point x in M ; denote its stabilizer by K .*

1. *There exists an isomorphism $\tau_K: K \rightarrow \mathbb{T}^k$ such that the symplectic slice representation at x is isomorphic to the action of K on \mathbb{C}^k obtained from the composition of τ_K with the standard action of \mathbb{T}^k on \mathbb{C}^k , which is*

$$[t_1, \dots, t_k] \cdot (z_1, \dots, z_k) = (e^{2\pi\sqrt{-1}t_1} z_1, \dots, e^{2\pi\sqrt{-1}t_k} z_k). \quad (\text{B.1})$$

2. *Let $\tau: G \rightarrow \mathbb{T}^\ell \times \mathbb{T}^k$ be an isomorphism of Lie groups such that $\tau(a) = (1, \tau_K(a))$ for all $a \in K$. Then there exists a G -invariant open neighbourhood U of x in M and a τ -equivariant open symplectic embedding*

$$j: U \hookrightarrow T^*\mathbb{T}^\ell \times \mathbb{C}^k$$

with $j(G \cdot x) = \mathbb{T}^\ell \times \{0\}$. Here \mathbb{T}^ℓ acts on $T^*\mathbb{T}^\ell$ by the lift of the left multiplication and \mathbb{T}^k acts on \mathbb{C}^k by (B.1). Our normalization for the symplectic form on \mathbb{C}^k is $\omega_{\mathbb{C}^k} = \frac{\sqrt{-1}}{2\pi} \sum dz_j \wedge d\bar{z}_j$, so that

$$\mu|_U = \mu(x) + \tau^* \circ \phi \circ j, \quad (\text{B.2})$$

where

$$\phi((q_1, \dots, q_\ell, p_1, \dots, p_\ell), (z_1, \dots, z_k)) = \left((p_1, \dots, p_\ell), \sum |z_j|^2 e_j^* \right),$$

e_1^*, \dots, e_k^* is the canonical basis of the weight lattice $(\mathbb{Z}^k)^*$, and $\tau^*: (\mathbb{R}^*)^\ell \times (\mathbb{R}^*)^k \rightarrow \mathfrak{g}^*$ is the isomorphism on duals of Lie algebras that is induced by τ .

Part (1) of Lemma B.3 follows from the facts that every linear symplectic action of a compact group preserves some compatible Hermitian structure and that every k -dimensional abelian subgroup of $U(k)$ is conjugate to the subgroup of diagonal matrices. Part (2) follows from Theorem B.2.

We are now ready to prove Proposition 1.1. For the reader's convenience we recall its statement:

Proposition 1.1. *Let (M, ω, μ) be a symplectic toric G -manifold. Then the quotient M/G is naturally a manifold with corners, and the orbital moment map $\bar{\mu}: M/G \rightarrow \mathfrak{g}^*$ is a u.l.e. (q.v. Definitions A.16 and 2.5).*

Proof. By Lemma B.3 we may assume that $M = T^*\mathbb{T}^\ell \times \mathbb{C}^k$ with the action of $G = \mathbb{T}^\ell \times \mathbb{T}^k$ as in the lemma and that the moment map is the map $\mu: M \rightarrow (\mathbb{R}^*)^\ell \times (\mathbb{R}^*)^k$ given by

$$\mu((q_1, \dots, q_\ell, p_1, \dots, p_\ell), (z_1, \dots, z_k)) = \left((p_1, \dots, p_\ell), \sum |z_j|^2 e_j^* \right),$$

where, as before, e_1^*, \dots, e_k^* is the canonical basis of the weight lattice $(\mathbb{Z}^k)^*$. Then

$$\mu(M) = (\mathbb{R}^\ell)^* \times \left\{ \sum \eta_j e_j^* \mid \eta_j \geq 0 \text{ for all } j \right\}.$$

Hence, $\mu(M)$ is a unimodular cone; in particular, it is a manifold with corners. We now argue that $\mu: M \rightarrow \mu(M)$ is a quotient map in the category of manifolds with corners; see Definition A.16. The fibers of μ are precisely the G -orbits. So it remains to show that for any manifold with corners N and any G -invariant smooth map $f: M \rightarrow N$ there exists a unique smooth map $\tilde{f}: \mu(M) \rightarrow N$ such that

$$f = \tilde{f} \circ \mu.$$

Clearly, there exists a unique map \tilde{f} with the above property, and our task is to show that \tilde{f} is actually smooth. Without loss of generality we assume that $N = \mathbb{R}$. The smoothness of \tilde{f} then follows from a special case of a theorem of Schwarz [29]. The key point is that since the functions $|z_1|^2, \dots, |z_k|^2$ generate the ring of \mathbb{T}^k invariant polynomials on \mathbb{C}^k , for any smooth \mathbb{T}^k -invariant function h on \mathbb{C}^k there is a smooth function \tilde{h} on \mathbb{R}^k with $h(z_1, \dots, z_k) = \tilde{h}(|z_1|^2, \dots, |z_k|^2)$. ■

To finish the section, we prove that symplectic toric G -manifolds over the same u.l.e. are locally isomorphic.

Lemma B.4. *Let (M, ω, π) and (M', ω', π') be two symplectic toric G -manifolds over the same u.l.e. $\psi: W \rightarrow \mathfrak{g}^*$. Then for any point $w \in W$ there is a neighbourhood U_w of w in W and an isomorphism $\varphi: \pi^{-1}(U_w) \rightarrow (\pi')^{-1}(U_w)$ of symplectic toric G -manifolds over $U_w \xrightarrow{\psi} \mathfrak{g}^*$.*

Proof. Fix $w \in W$. Let x be a point in $\pi^{-1}(w)$. Every invariant neighbourhood of $G \cdot x$ in M is a subset of M of the form $\pi^{-1}(U_w)$ where U_w is a neighbourhood of w in W .

Let K be the stabilizer of x in G , and let k be the dimension of K . Lemma B.3 implies that the symplectic slice representation at x is linearly symplectically isomorphic to the action of K on \mathbb{C}^k through an isomorphism $\tau|_K: K \xrightarrow{\cong} (S^1)^k$. Let v_1^*, \dots, v_k^* denote the basis of the weight lattice \mathbb{Z}_K^* that corresponds under $\tau|_K$ to the standard basis of the weight lattice $(\mathbb{Z}^k)^*$. Thus, v_1^*, \dots, v_k^* represent the weights of the K action on \mathbb{C}^k . Let v_1, \dots, v_k be the dual basis, in \mathbb{Z}_K , and let $\epsilon = \mu(x)$. The equation (B.2) for the moment map implies that every neighbourhood of x contains a smaller invariant neighbourhood of x whose moment map image is a neighbourhood of ϵ in the cone $C_{\{v_1, \dots, v_k\}, \epsilon}$ (cf. Definition 2.1).

By combining the above discussion with Lemma 2.4 we see that the symplectic slice representation is determined up to linear symplectic isomorphism by the image of an arbitrary sufficiently small invariant neighbourhood of $G \cdot x$. This image is exactly $\psi(U_w)$, where U_w is an arbitrary sufficiently small neighbourhood of w in W . So the germ of ψ at w determines the symplectic slice representation up to linear symplectic isomorphism. Clearly, this germ also determines the moment map value, $\mu(x)$, which is equal to $\psi(w)$. The result then follows from Theorem B.2. ■

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