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NON-DEGENERATE MIXED FUNCTIONS

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Abstract

Mixed functions are analytic functions in variables $z_1, \dots, z_n$ and their conjugates $\bar{z}_1, \dots, \bar{z}_n$. We introduce the notion of Newton non-degeneracy for mixed functions and develop a basic tool for the study of mixed hypersurface singularities. We show the existence of a canonical resolution of the singularity, and the existence of the Milnor fibration under the strong non-degeneracy condition.

1. Introduction

Let $f(\mathbf{z})$ be a holomorphic function of n-variables $z_1, \dots, z_n$ such that $f(\mathbf{0}) = 0$. As is well-known, J. Milnor proved that there exists a positive number ε_0 such that the argument mapping $f/|f| : S_\varepsilon^{2n-1} \backslash K_\varepsilon \to S^1$ is a locally trivial fibration for any positive ε with $\varepsilon \le \varepsilon_0$ where $K_\varepsilon = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ ([12]). In the same book, he proposed to study the links coming from a pair of real-valued real analytic functions $g(\mathbf{x}, \mathbf{y})$, $h(\mathbf{x}, \mathbf{y})$ where $\mathbf{z} = \mathbf{x} + \mathbf{y}i$. Namely putting $f(\mathbf{x}, \mathbf{y}) := g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y}) : \mathbf{R}^{2n} \to \mathbf{C}$, he proposed to study the condition for $f/|f| : S_\varepsilon^{2n-1} \backslash K_\varepsilon \to S^1$ to be a fibration. This is an interesting problem. In fact, if one can find such a pair of analytic functions g, h, it may give an interesting link variety K_ε whose complement $S_\varepsilon^{2n-1} \backslash K_\varepsilon$ is fibered over S^1 where K_ε cannot come from any complex analytic links. The difficulty is that for an arbitrary choice of g, h, it is usually not a fibration. A breakthrough is given by the work of Ruas, Seade and Verjovsky [20]. After this work, many examples of pairs $\{g, h\}$ which give real Milnor fibrations have been investigated. However in most papers, certain restricted types of functions are mainly considered ([5, 6, 22, 19, 11, 18, 3]).

The purpose of this paper is to propose a wide class of pairs $\{g, h\}$ such that the corresponding mapping $f = g + ih$ defines a Milnor fibration. We consider a complex valued analytic function f expanded in a convergent power series of variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$$

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where $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$) as usual. Here $\bar{z}_j$ is the complex conjugate of z_j. We call $f(\mathbf{z}, \bar{\mathbf{z}})$ *a mixed analytic function* (or *a mixed polynomial, if* $f(\mathbf{z}, \bar{\mathbf{z}})$ *is a polynomial*) of $z_1, \dots, z_n$. We are interested in the topology of the hypersurface $V = \{\mathbf{z} \in \mathbf{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$, which we call *a mixed hypersurface*. Here we use the terminology *hypersurface* in order to point out the similarity with complex analytic hypersurfaces. We will see later that $\operatorname{codim}_{\mathbf{R}} V = 2$ if V is non-degenerate (Theorem 19). We denote the set of mixed functions of variables $\mathbf{z}$, $\bar{\mathbf{z}}$ by $\mathbf{C}\{\mathbf{z}, \bar{\mathbf{z}}\}$. This approach is equivalent to the original one. In fact, writing $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ with $z_j = x_j + iy_j$ $j = 1, \dots, n$, and using real variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, and dividing $f(\mathbf{z}, \bar{\mathbf{z}})$ in the real and the imaginary parts so that $f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$ where $g := \Re f$, $h := \Im f$, we can see that V is defined by two real-valued analytic functions $g(\mathbf{x}, \mathbf{y})$, $h(\mathbf{x}, \mathbf{y})$ of $2n$-variables $x_1, y_1, \dots, x_n, y_n$. Conversely, for a given real analytic variety $W = \{g(\mathbf{x}, \mathbf{y}) = h(\mathbf{x}, \mathbf{y}) = 0\}$ which is defined by two real-valued analytic functions g, h, we can consider W as a mixed hypersurface by introducing a mixed function $f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ where

$$f(\mathbf{z}, \bar{\mathbf{z}}) := g\left(\frac{\mathbf{z} + \bar{\mathbf{z}}}{2}, \frac{\mathbf{z} - \bar{\mathbf{z}}}{2i}\right) + ih\left(\frac{\mathbf{z} + \bar{\mathbf{z}}}{2}, \frac{\mathbf{z} - \bar{\mathbf{z}}}{2i}\right).$$

The advantage of our view point is that we can use rich techniques of complex hypersurface singularities. For complex hypersurfaces defined by holomorphic functions, the notion of the non-degeneracy in the sense of the Newton boundary plays an important role for the resolution of singularities and the determination of the Milnor fibration ([10, 23, 14, 15, 16]). We will introduce the notion of *non-degeneracy* for mixed functions or mixed polynomials and prove basic properties in §2 and §3.

In §4, we will give a canonical resolution of mixed hypersurface singularities. First we take an admissible toric modification $\hat{\pi} : X \to \mathbf{C}^n$. This does not resolve the singularities but it turns out that we only need a real modification or a polar modification after the toric modification to complete the resolution (Theorem 24).

In §5, we consider the Milnor fibration of a given mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$. It turns out that *the non-degeneracy* is not enough for the existence of the Milnor fibration of f. We need *the strong non-degeneracy* of $f(\mathbf{z}, \bar{\mathbf{z}})$ which guarantees the existence of the Milnor fibration (Theorem 33, Theorem 29). We show that the Milnor fibrations of the first type and of the second type,

$$f/|f| : S_\varepsilon \backslash K_\varepsilon \to S^1 \quad \text{and} \quad f : \partial E(r, \delta)^* \to S_\delta^1,$$

are equivalent (Theorem 36). We also show that for a polar weighted homogeneous polynomial, the global fibration is equivalent to the above two fibrations (Theorem 33).

In §6, we will see that the mixed singularities are much more complicated than the complex singularities and that the topological equivalence class is not a combinatorial invariant even in the easiest case of plane curves.

In §7, we discuss Milnor fibrations for non-isolated mixed singularities under the super strong non-degeneracy condition (Theorem 52).

In §8, we give an A'Campo type formula for the zeta function of the Milnor fibration in the case of mixed curves (Theorem 60).

This paper is a continuation of the previous one [17] and we use the same notations. This paper consists of the following sections. We hope this paper provides a systematical method to study mixed singularities.

Contents

Below are notations we use frequently in this paper:

$S_r^{2n-1},\ S_r = \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbf{C}^n \mid \|\mathbf{z}\| = r\}$, (sphere of the radius r)

$\|\mathbf{z}\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$

$B_r^{2n},\ B_r = \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbf{C}^n \mid \|\mathbf{z}\| \le r\}$ (ball of the radius r)

$\mathbf{C}^I = \{\mathbf{z} = (z_1, \dots, z_n) \mid z_j = 0, j \notin I\},\ B_r^I = \{\mathbf{z} \in \mathbf{C}^I \mid \|\mathbf{z}\| \le r\}$

$\mathbf{C}^{*I} = \{\mathbf{z} = (z_1, \dots, z_n) \mid z_j = 0 \Leftrightarrow j \notin I\}$

$\mathbf{C}^{*n} = \mathbf{C}^{*I},\ B^{*n} = B^{*I}$ with $I = \{1, \dots, n\}$

$\mathbf{R}^{+n} = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_j \ge 0, j = 1, \dots, n\}$

$(\mathbf{z}, \mathbf{w}) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$: hermitian inner product

$\Re(\mathbf{z}, \mathbf{w}) = \Re(z_1\bar{w}_1 + \cdots + z_n\bar{w}_n)$: real Euclidean inner product

$D(\delta) := \{\eta \in \mathbf{C} \mid |\eta| \le \delta\},\ D(\delta)^* := \{\eta \in \mathbf{C} \mid 0 < |\eta| \le \delta\}$

$S_\delta^1 := \{\eta \in \mathbf{C} \mid |\eta| = \delta\}$.

2. Newton boundary and non-degeneracy of mixed functions

2.1. Polar weighted homogeneous polynomials

2.1.1. *Radial degree and polar degree.* Let $M = \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ be a mixed monomial where $\nu = (\nu_1, \dots, \nu_n)$, $\mu = (\mu_1, \dots, \mu_n)$ and let $P = {}^t(p_1, \dots, p_n)$ be a weight vector. We define *the radial degree of M*, $\mathrm{rdeg}_P\, M$ and *the polar degree of M*, $\mathrm{pdeg}_P\, M$ with respect to P by

$$\mathrm{rdeg}_P\, M = \sum_{j=1}^n p_j(\nu_j + \mu_j), \quad \mathrm{pdeg}_P\, M = \sum_{j=1}^n p_j(\nu_j - \mu_j).$$

2.1.2. *Weighted homogeneous polynomials.* Recall that a polynomial $h(\mathbf{z})$ is called *a weighted homogeneous polynomial with weights* $P = {}^t(p_1, \dots, p_n)$ if $p_1, \dots, p_n$ are integers and there exists a positive integer d so that

$$f(t^{p_1} z_1, \dots, t^{p_n} z_n) = t^d f(\mathbf{z}), \quad t \in \mathbf{C}.$$

The integer d is called the degree of f with respect to the weight vector P.

A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^{\ell} c_i \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i}$ is called *a radially weighted homogeneous polynomial* if there exist integers $q_1, \dots, q_n \geq 0$ and $d_r > 0$ such that it satisfies the equality:

$$f(t^{q_1} z_1, \dots, t^{q_n} z_n, t^{q_1} \bar{z}_1, \dots, t^{q_n} \bar{z}_n) = t^{d_r} f(\mathbf{z}, \bar{\mathbf{z}}), \quad t \in \mathbf{R}^*.$$

Putting $Q = {}^t(q_1, \dots, q_n)$, this is equivalent to $\mathrm{rdeg}_Q\, \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i} = d_r$ for $i = 1, \dots, \ell$ with $c_i \neq 0$. Write $f = g + ih$ so that g, h are polynomials with real coefficients of $2n$-variables $(x_1, y_1, \dots, x_n, y_n)$. If f is a radially weighted homogeneous polynomial of type $(q_1, \dots, q_n; d_r)$, $g(\mathbf{x}, \mathbf{y})$ and $h(\mathbf{x}, \mathbf{y})$ are weighted homogeneous polynomials of type $(q_1, q_1, \dots, q_n, q_n; d_r)$ (i.e., $\deg x_j = \deg y_j = q_j$).

A polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *a polar weighted homogeneous polynomial* if there exists a weight vector $(p_1, \dots, p_n)$ and a non-zero integer d_p such that

$$f(\lambda^{p_1} z_1, \dots, \lambda^{p_n} z_n, \bar{\lambda}^{p_1} \bar{z}_1, \dots, \bar{\lambda}^{p_n} \bar{z}_n) = \lambda^{d_p} f(\mathbf{z}, \bar{\mathbf{z}}), \quad \lambda \in \mathbf{C}^*, \quad |\lambda| = 1$$

where $\gcd(p_1, \dots, p_n) = 1$. Usually we assume that $d_p > 0$. This is equivalent to

$$\mathrm{pdeg}_P\, \mathbf{z}^{\nu_i} \bar{\mathbf{z}}^{\mu_i} = d_p, \quad i = 1, \dots, \ell.$$

Here the weight p_i can be zero or a negative integer. The weight vector $(p_1, \dots, p_n)$ is called *the polar weights* and d_p is called the *polar degree* respectively. This notion was first introduced by Ruas-Seade-Verjovsky [20] and Cisneros-Molina [4]. In [17], we have assumed that a polar weighted homogeneous polynomial is also a radially weighted homogeneous polynomial. Although it is not necessary to be assumed, we will only consider such polynomials in this paper.

Recall that the radial weights and polar weights define $\mathbf{R}^*$-action and S^1-action on $\mathbf{C}^n$ respectively by

$$t \circ \mathbf{z} = (t^{q_1} z_1, \dots, t^{q_n} z_n), \quad t \circ \bar{\mathbf{z}} = (t^{q_1} \bar{z}_1, \dots, t^{q_n} \bar{z}_n), \quad t \in \mathbf{R}^*$$

$$\lambda \circ \mathbf{z} = (\lambda^{p_1} z_1, \dots, \lambda^{p_n} z_n), \quad \lambda \circ \bar{\mathbf{z}} = \overline{\lambda \circ \mathbf{z}}, \quad \lambda \in S^1 \subset \mathbf{C}$$

In other words, this is an $\mathbf{R}^* \times S^1$ action on $\mathbf{C}^n$.

Lemma 1. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a radially weighted homogeneous polynomial, $V = \{\mathbf{z} \in \mathbf{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ and $V^* = V \cap \mathbf{C}^{*n}$. Assume that $V \backslash \{O\}$ (respectively V^*) is smooth and* $\mathrm{codim}_{\mathbf{R}} V = 2$. *If the radial weight vector is strictly positive, namely $q_j > 0$ for any $j = 1, \dots, n$, the sphere S_r intersects transversely with $V \backslash \{O\}$ (resp. with V^*) for any $r > 0$.*

We are mainly considering the case that $V \setminus \{O\}$ has no mixed singularity in the sense of §3.1.

Proof. This is essentially the same with Proposition 4 in [17]. In Proposition 4, we have assumed that $f(\mathbf{z}, \bar{\mathbf{z}})$ is polar weighted homogeneous but we did not use this assumption in the proof. The radial action is enough as we will see below. Assume that three vectors dg, dh, $d\phi$ are linearly dependent at $\mathbf{z}_0 = (\mathbf{x}_0, \mathbf{y}_0) \in V^*$, where $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$ and $\phi(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n (x_j^2 + y_j^2)$. As $V \setminus \{O\}$ (resp. V^*) is non-singular, we can find real numbers α, β so that $d\phi(\mathbf{x}_0, \mathbf{y}_0) = \alpha\, dg(\mathbf{x}_0, \mathbf{y}_0) + \beta\, dh(\mathbf{x}_0, \mathbf{y}_0)$. Here $d\phi$, dg, dh are the respective gradient vectors of the functions ϕ, g, h. For example, $dg(\mathbf{x}, \mathbf{y}) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial y_1} \dots, \frac{\partial g}{\partial x_n}, \frac{\partial g}{\partial y_n}\right)$. Let $\ell(t) = (t \circ \mathbf{x}_0, t \circ \mathbf{y}_0)$, $t \in \mathbf{R}^+$ be the orbit of $\mathbf{z}_0$ by the radial action. Let $\mathbf{v}$ be the tangent vector of the orbit. Then we have:

$$\ell(t) = (t^{q_1} x_{01}, t^{q_1} y_{01}, \dots, t^{q_n} x_{0n}, t^{q_n} y_{0n})$$
$$\frac{d}{dt}\phi(\ell(t))|_{t=1} = \Re(d\phi(\mathbf{x}_0, \mathbf{y}_0), \mathbf{v}) = 2\sum_{i=1}^n q_i(x_{0i}^2 + y_{0i}^2) > 0.$$

On the other hand, we also have the equality:

$$\begin{aligned}\frac{d}{dt}\phi(\ell(t))|_{t=1} &= \alpha\Re(dg(\mathbf{x}_0, \mathbf{y}_0), \mathbf{v}) + \beta\Re(dh(\mathbf{x}_0, \mathbf{y}_0), \mathbf{v}) \\ &= \alpha \left.\frac{dg(\ell(t))}{dt}\right|_{t=1} + \beta \left.\frac{dh(\ell(t))}{dt}\right|_{t=1} = 0.\end{aligned}$$

This is an obvious contradiction to the above inequality. □

2.2. Newton boundary of a mixed function. Suppose that we are given a mixed analytic function $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu, \mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. We always assume that $c_{0,0} = 0$ so that $O \in f^{-1}(0)$. We call the variety $V = f^{-1}(0)$ *the mixed hypersurface*. The *radial Newton polygon* $\Gamma_+(f; \mathbf{z}, \bar{\mathbf{z}})$ (at the origin) of a mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ is defined by the convex hull of

$$\bigcup_{c_{\nu,\mu} \neq 0} (\nu + \mu) + \mathbf{R}^{+n}.$$

Hereafter we call $\Gamma_+(f; \mathbf{z}, \bar{\mathbf{z}})$ simply the Newton polygon of $f(\mathbf{z}, \bar{\mathbf{z}})$. *The Newton boundary* $\Gamma(f; \mathbf{z}, \bar{\mathbf{z}})$ is defined by the union of compact faces of $\Gamma_+(f)$. Observe that $\Gamma(f)$ is nothing but the ordinary Newton boundary if f is a complex analytic function. For a given positive integer vector $P = (p_1, \dots, p_n)$, we associate a linear function ℓ_P on $\Gamma(f)$ defined by $\ell_P(\nu) = \sum_{j=1}^n p_j \nu_j$ for $\nu \in \Gamma(f)$ and let $\Delta(P, f) = \Delta(P)$ be the face where ℓ_P takes its minimal value. In other words, P gives radial weights for variables $z_1, \dots, z_n$ by $\mathrm{rdeg}_P\, z_j = \mathrm{rdeg}_P\, \bar{z}_j = p_j$ and $\mathrm{rdeg}_P\, \mathbf{z}^\nu \bar{\mathbf{z}}^\mu = \sum_{j=1}^n p_j(\nu_j + \mu_j)$. To distinguish the points on the Newton boundary and weight vectors, we denote by N the set of integer weight vectors and denote a vector $P \in N$ by a column vectors. We denote by N^+, N^{++} the subset of positive or strictly positive weight vectors respectively. Thus $P = {}^t(p_1, \dots, p_n) \in N^{++}$ (respectively $P \in N^+$) if and only if $p_i > 0$ (resp. $p_i \geq 0$) for any $i = 1, \dots, n$. We denote the minimal value of ℓ_P by $d(P; f)$ or simply $d(P)$. Note that

$$d(P; f) = \min\{\mathrm{rdeg}_P\, \mathbf{z}^\nu \bar{\mathbf{z}}^\mu \mid c_{\nu,\mu} \neq 0\}.$$

For a positive weight P, we define *the face function* $f_P(\mathbf{z}, \bar{\mathbf{z}})$ by

$$f_P(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu+\mu \in \Delta(P)} c_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu.$$

Example 2. Consider a mixed function $f := z_1^3 \bar{z}_1^2 + z_1^2 z_2^2 + z_2^3 \bar{z}_2$. The Newton boundary $\Gamma(f; \mathbf{z}, \bar{\mathbf{z}})$ has two faces Δ_1, Δ_2 which have weight vectors $P := {}^t(2, 3)$ and $Q := {}^t(1, 1)$ respectively. The corresponding invariants are

$$f_P(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3 \bar{z}_1^2 + z_1^2 z_2^2, \quad d(P; f) = 10$$

$$f_Q(\mathbf{z}, \bar{\mathbf{z}}) = z_1^2 z_2^2 + z_2^3 \bar{z}_2, \quad d(Q; f) = 4.$$

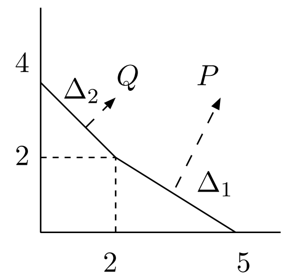

FIGURE 1. $\Gamma(f)$

It is sometimes important to consider the convex hull of vertices $\hat{\Delta}(P)$ in $\mathbf{R}^n \times \mathbf{R}^n$ which is defined by

$$\hat{\Delta}(P) = \text{convex hull of } \{(\nu, \mu) \in \mathbf{R}^n \times \mathbf{R}^n \mid c_{\nu,\mu} \neq 0, \nu + \mu \in \Delta(P)\}$$

Let $S : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be the map defined by $(\nu, \mu) \mapsto \nu + \mu$. Then $\Delta(P) = S(\hat{\Delta}(P))$ by the definition. We call $\hat{\Delta}(P)$ *the mixed face of* $\Gamma(f)$ and $\Delta(P)$ *the radial face of* $\Gamma(f)$ *with respect to* P respectively, when the distinction is necessary.

2.3. Non-degenerate functions. Suppose that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a given mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$. For $P \in N^{++}$, the face function $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a radially weighted homogeneous polynomial of type $(p_1, \dots, p_n; d)$ with $d = d(P; f)$.

DEFINITION 3. Let P be a strictly positive weight vector. We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *non-degenerate* for P, if the fiber $f_P^{-1}(0) \cap \mathbf{C}^{*n}$ contains no critical point of the mapping $f_P : \mathbf{C}^{*n} \to \mathbf{C}$. In particular, $f_P^{-1}(0) \cap \mathbf{C}^{*n}$ is a smooth real codimension 2 manifold or an empty set. We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *strongly non-degenerate* for P if the mapping $f_P : \mathbf{C}^{*n} \to \mathbf{C}$ has no critical points. If $\dim \Delta(P) \geq 1$, we further assume that $f_P : \mathbf{C}^{*n} \to \mathbf{C}$ is surjective onto $\mathbf{C}$.

A mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *non-degenerate* (respectively *strongly non-degenerate*) if f is non-degenerate (resp. *strongly non-degenerate*) for any strictly positive weight vector P.

Consider the function $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$. Then $V = f^{-1}(0)$ is a single point $\{O\}$. By the above definition, f is a non-degenerate mixed function. To avoid such an unpleasant situation, we say that a mixed function $g(\mathbf{z}, \bar{\mathbf{z}})$ is *a true non-degenerate function* if it satisfies further *the non-emptiness condition*:

(NE): For any $P \in N^{++}$ with $\dim \Delta(P, g) \geq 1$, the fiber $g_P^{-1}(0) \cap \mathbf{C}^{*n}$ is non-empty.

Remark 4. Assume that $f(\mathbf{z})$ is a holomorphic function. Then $f_P(\mathbf{z})$ is a weighted homogeneous polynomial and we have the Euler equality:

$$d(P; f) f_P(\mathbf{z}) = \sum_{i=1}^{n} p_i z_i \frac{\partial f_P}{\partial z_i}(\mathbf{z}).$$

Thus $f_P : \mathbf{C}^{*n} \to \mathbf{C}$ has no critical point over $\mathbf{C}^*$. Thus f is non-degenerate for P implies f is strongly non-degenerate for P. This is also the case if $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial.

Example 5. I. Consider the mixed function $f := z_1^3\bar{z}_1^2 + z_1^2 z_2^2 + z_2^3\bar{z}_2$ which we have considered in Example 2. Then f is strongly non-degenerate for each of the weight vectors $P = {}^t(2,3)$, $Q = {}^t(1,1)$.

II. Consider a mixed function

$$g(\mathbf{z}, \bar{\mathbf{z}}) = z_1\bar{z}_1 + \cdots + z_r\bar{z}_r - (z_{r+1}\bar{z}_{r+1} + \cdots + z_n\bar{z}_n), \quad 1 \leq r \leq n-1.$$

Then $V = g^{-1}(0)$ is a smooth real codimension one variety and thus it is degenerate for $P = {}^t(1, 1, \ldots, 1)$.

III. Consider a mixed function

$$f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^2 + a z_1\bar{z}_2 + \bar{z}_2^2, \quad a \in \mathbf{C}.$$

Then f is non-degenerate if and only if $a \neq \pm 2$.

IV. Finally we give an example of a mixed function which is non-degenerate but not strongly non-degenerate. Consider a mixed function

$$\begin{aligned} f(\mathbf{z}, \bar{\mathbf{z}}) &= 1/4 z_1^2 - 1/4 \bar{z}_1^2 + z_1\bar{z}_1 - (1+i)(z_1+z_2)(\bar{z}_1+\bar{z}_2) \\ &= g(x_1, x_2, y_1, y_2) + i h(x_1, x_2, y_1, y_2) \end{aligned}$$

$$\begin{aligned} \text{where } g(x_1, x_2, y_1, y_2) &= x_1^2 + y_1^2 - (x_1+x_2)^2 - (y_1+y_2)^2 \\ h(x_1, x_2, y_1, y_2) &= x_1 y_1 - (x_1+x_2)^2 - (y_1+y_2)^2 \end{aligned}$$

f is a radially homogeneous polynomial of degree 2 but it is not polar weighted homogeneous. One can check that f is non-degenerate for the weight vector $P = {}^t(1,1)$ but it is not strongly non-degenerate. In fact, it has two families of critical points

$$t \mapsto (x_1, x_2, y_1, y_2) = (t, -t, \pm t, \mp t), \quad 0 < t$$

with the critical values $(2 \pm i)t^2$.

Proposition 6. *Let $g(\mathbf{z}, \bar{\mathbf{z}})$ be a radially weighted homogeneous polynomial and let $M := \mathbf{z}^{\mathbf{a}}\bar{\mathbf{z}}^{\mathbf{b}}$ be a mixed monomial and put $h := Mg(\mathbf{z}, \bar{\mathbf{z}})$. Then 0 is a regular value of $g : \mathbf{C}^{*n} \to \mathbf{C}$ if and only if 0 is a regular value of $h : \mathbf{C}^{*n} \to \mathbf{C}$.*

The assertion is immediate from the definition because $g^{-1}(0) \cap \mathbf{C}^{*n} = h^{-1}(0) \cap \mathbf{C}^{*n}$ and the tangential map $dh_{\mathbf{w}} : T_{\mathbf{w}}\mathbf{C}^{*n} \to T_0\mathbf{C}$ is equal to $M\, dg_{\mathbf{w}}$ for any $\mathbf{w} \in g^{-1}(0) \cap \mathbf{C}^{*n}$.

Recall that for a subset $I \subset \{1, \dots, n\}$, we use the notations $\mathbf{C}^I = \{\mathbf{z} \in \mathbf{C}^n \mid z_j = 0, j \notin I\}$ and $f^I = f|_{\mathbf{C}^I}$.

Proposition 7. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a non-degenerate (respectively strongly non-degenerate) mixed function. Assume that f^I is not constantly zero for some $I \subset \{1, 2, \dots, n\}$. Then f^I is a non-degenerate (resp. strongly non-degenerate) function as a function of variables $\{z_i, \bar{z}_i, \mid i \in I\}$.*

Proof. The proof is exactly parallel to that of Proposition 1.5, [16]. Take a compact face Δ of $\Gamma(f^I)$. There is a strictly positive weight vector $P = {}^t(p_i)_{i \in I} \in N^I$ such that $\Delta = \Delta(P, f^I)$. We consider a strictly positive weight vector $Q = {}^t(q_1, \dots, q_n)$ such that $q_i = p_i$ for $i \in I$ and $q_i = \nu$ for $i \notin I$. It is easy to see that $f_Q(\mathbf{z}, \bar{\mathbf{z}}) = f_P^I(\mathbf{z}_I, \bar{\mathbf{z}}_I)$ if ν is sufficiently large. Here $f_P^I = (f^I)_P$. Now by the assumption, 0 is not a critical value of $f_Q : \mathbf{C}^{*n} \to \mathbf{C}$ (respectively $f_Q : \mathbf{C}^{*n} \to \mathbf{C}$ has no critical points). As f_Q contains only variables z_i, $i \in I$, 0 is not a critical value of $f_P^I : \mathbf{C}^{*I} \to \mathbf{C}$ (resp. $f_P^I : \mathbf{C}^{*I} \to \mathbf{C}$ has no critical points). □

For a complex valued mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$, we use the notation ([17]):

$$df(\mathbf{z}, \bar{\mathbf{z}}) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) \in \mathbf{C}^n, \quad \bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = \left(\frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n}\right) \in \mathbf{C}^n$$

We use freely the following convenient criterion for a given point to be a critical point as a function to $\mathbf{C}$ in this paper.

Proposition 8 (Proposition 1, [17]). *The following two conditions are equivalent. Let $\mathbf{w} \in \mathbf{C}^n$.*

(1) *$\mathbf{w}$ is a critical point of $f : \mathbf{C}^n \to \mathbf{C}$.*
(2) *There exists a complex number α with $|\alpha| = 1$ such that $\overline{df(\mathbf{w}, \bar{\mathbf{w}})} = \alpha\, \bar{d}f(\mathbf{w}, \bar{\mathbf{w}})$.*

Hereafter we use the simplified notation $\overline{df}(\mathbf{w}, \overline{\mathbf{w}})$ for $\overline{df(\mathbf{w}, \overline{\mathbf{w}})}$.

Example 9. Let us consider the following mixed polynomials

$$f_1 = z_1\bar{z}_1 - z_2^2, \quad f_2 = z_1\bar{z}_1 - z_2\bar{z}_2, \quad f_3 = z_1^2\bar{z}_1 - z_2^2\bar{z}_2$$

and the corresponding mixed varieties $V_i = f_i^{-1}(0)$, $i = 1, 2, 3$. Each of them has an isolated singularity at the origin. In fact, as real varieties, they are described as follows.

$$\begin{aligned} V_1 &= \{(\mathbf{x}, \mathbf{y}) \mid x_1^2 + y_2^2 = x_2^2 - y_2^2, x_2 y_2 = 0\} \\ &= \{(\mathbf{x}, \mathbf{y}) \mid x_1^2 + y_2^2 = x_2^2, y_2 = 0\} \\ V_2 &= \{(\mathbf{x}, \mathbf{y}) \mid x_1^2 + y_2^2 = x_2^2 + y_2^2\}, \quad \dim_{\mathbf{R}} V_2 = 3 \\ V_3 &= \{(z_1, z_2) \mid z_1 = r_1 \exp(i\theta_1), z_2 = r_2 \exp(i\theta_2), r_1 = r_2, \theta_1 = \theta_2\} \\ &= \{(z_1, z_2) \mid z_1 - z_2 = 0\}. \end{aligned}$$

V_3 is a special case of polynomials which has been considered in [20]. f_1, f_3 are non-degenerate but f_2 is a degenerate mixed function as it is not surjective (onto $\mathbf{C}$) and $\dim_{\mathbf{R}} V_2 = 3$. Note also that $\overline{df_2} = \bar{d}f_2 = (z_1, -z_2)$. f_1 is not a polar weighted homogeneous polynomial (as the monomial $z_1\bar{z}_1$ can not have a positive degree) while f_3 is a polar weighted polynomial of type $(1, 1; 1)$.

2.4. Some useful functions. Let J be a subset of $\{1, \dots, n\}$ and consider *the J-conjugation map* $\iota_J : \mathbf{C}^n \to \mathbf{C}^n$ defined by:

$$\iota_J : (z_1, \dots, z_n) \mapsto (w_1, \dots, w_n), \quad w_j = \begin{cases} z_j & j \notin J \\ \bar{z}_j & j \in J. \end{cases}$$

Of course, we define $\iota_J(\bar{z}_j) = \overline{\iota_J(z_j)}$.

Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function. We call that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *J-conjugate holomorphic* if f is an analytic function of the variables $\{z_j \mid j \notin J\}$ and $\{\bar{z}_k \mid k \in J\}$, or equivalently $f \circ \iota_J(\mathbf{z})$ is a holomorphic function.

A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called a *J-conjugate weighted homogeneous polynomial* if $f \circ \iota_J(\mathbf{z})$ is a weighted homogeneous polynomial. Let $P = (p_1, \dots, p_n)$ be the weight vector of $f \circ \iota_J(\mathbf{z})$ and let d be the degree. We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *a J-conjugate weighted homogeneous polynomial of the weight type* $(p_1, \dots, p_n; d)$. The following is obvious by the definition.

PROPOSITION 10. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a J-conjugate weighted homogeneous polynomial of the weight type $(p_1, \dots, p_n; d)$. Then $f(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted polynomial with the polar weight type $(\iota_J P; d)$ where*

$$\iota_J P = (p_1', \dots, p_n'), \quad p_j' = \begin{cases} -p_j & j \in J \\ p_j & j \notin J \end{cases}$$

Furthermore $f(\mathbf{z},\bar{\mathbf{z}})$ is also a radially weighted homogeneous polynomial of the radial weight type $(p_1,\dots,p_n;d)$.

Let $M=\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a mixed monomial and let $g(\mathbf{z},\bar{\mathbf{z}})=M\cdot f(\mathbf{z},\bar{\mathbf{z}})$ where $f(\mathbf{z},\bar{\mathbf{z}})$ is a J-conjugate weighted homogeneous polynomial. We say $g(\mathbf{z},\bar{\mathbf{z}})$ is *a pseudo J-conjugate weighted homogeneous polynomial* if $\mathrm{pdeg}_{P'}\, g\neq 0$ where $P'=\iota_J P$ is the polar weight vector of $f(\mathbf{z},\bar{\mathbf{z}})$. Note that $g\circ\iota_J(\mathbf{z})$ need not to be holomorphic. Further, if $J=\emptyset$, we say that g is *a pseudo weighted homogeneous polynomial*. Then g takes the form $Mf(\mathbf{z})$ where f a weighted homogeneous polynomial and M is a mixed monomial.

Proposition 11. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a J-conjugate weighted homogeneous polynomial of the weight type $(p_1,\dots,p_n;d)$. Let $M=\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a monomial and assume that $g(\mathbf{z},\bar{\mathbf{z}})=Mf(\mathbf{z},\bar{\mathbf{z}})$ is a pseudo J-conjugate weighted homogeneous polynomial, namely* $\mathrm{pdeg}_{P'}\, M+d\neq 0$. *Then $g:\mathbf{C}^{*n}\to\mathbf{C}$ has no critical points if and only if $f:\mathbf{C}^{*n}\to\mathbf{C}$ has no critical points.*

Proof. As $g(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted polynomial, the only possible singular fiber is $g^{-1}(0)$. Thus the assertion is immediate as $g^{-1}(0)=f^{-1}(0)$ in $\mathbf{C}^{*n}$. □

Example 12. Let $f(\mathbf{z},\bar{\mathbf{z}})=z_1^2+\cdots+z_{n-1}^2+\bar{z}_n^3$. Then f is a J-conjugate weighted homogeneous polynomial of the weight type $(3,\dots,3,2;6)$ with $J=\{n\}$. A mixed polynomial $g(\mathbf{z},\bar{\mathbf{z}})=\mathbf{z}^\nu\bar{\mathbf{z}}^\mu f(\mathbf{z},\bar{\mathbf{z}})$ is a pseudo J-conjugate weighted homogeneous polynomial if

$$3\sum_{j=1}^{n-1}(\nu_i-\mu_i)-2(\nu_n-\mu_n)+6\neq 0.$$

Definition 13. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed function. We say that f is a *Newton pseudo conjugate weighted homogeneous polynomial* if for any $P\in N^{++}$, there exists a subset $J(P)\subset\{1,\dots,n\}$ such that the face function $f_P(\mathbf{z},\bar{\mathbf{z}})$ is a $J(P)$-pseudo conjugate weighted homogeneous polynomial. Here $J(P)$ can differ for each P. For a Newton pseudo conjugate weighted homogeneous function, the non-degeneracy condition is easily checked by Proposition 11.

Example 14. I. Let $f(\mathbf{z},\bar{\mathbf{z}})=z_1^5+z_1^2\bar{z}_2^2+z_2^m\bar{z}_2^2$ with $m\geq 2$. Then the Newton boundary has two faces and the corresponding weights are $P=(2,3)$ and $Q=(m,2)$. The face functions are

$$f_P(\mathbf{z},\bar{\mathbf{z}})=z_1^2(z_1^3+\bar{z}_2^2),\quad f_Q(\mathbf{z},\bar{\mathbf{z}})=\bar{z}_2^2(z_1^2+z_2^m)$$

and f is a Newton pseudo conjugate weighted homogeneous polynomial if $m\neq 2$. Note that for $m=2$, the polar degree of $f_Q(\mathbf{z},\bar{\mathbf{z}})$ is 0. See also the next example. We give a class of functions which can not be non-degenerate.

II. Consider the radially weighted homogeneous polynomial

$$f(\mathbf{z},\bar{\mathbf{z}})=\sum_{i=1}^{n}c_jz_j^{a_j}\bar{z}_j^{a_j},\quad c_1,\dots,c_n\in\mathbf{C}^*.$$

where $a_1,\dots,a_n$ are positive integers. This is very special as $z_j^{a_j}\bar{z}_j^{a_j}=|z_j|^{2a_j}\geq 0$. Let $\Omega:=\{\sum_{j=1}^n\alpha_jc_j\,|\,\alpha_j>0\}$ be the open cone of the complex numbers $\mathbf{C}$ generated by $c_1,\dots,c_n$.

Proposition 15. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be as above. The image of $f:\mathbf{C}^{*n}\to\mathbf{C}$ is Ω and f is a submersion on Ω.*

Proof. As $z_j^{a_j}\bar{z}_j^{a_j}>0$ for $z_j\neq 0$, $f(\mathbf{C}^{*n})\subset\Omega$. For an $\eta\in\Omega$, write η as $\eta=\sum_{j=1}^n\alpha_jc_j$ with $\alpha_j>0$. Take w_j so that $|w_j|^{2a_j}=\alpha_j$. Then $\mathbf{w}=(w_1,\dots,w_n)\in f^{-1}(\eta)\cap\mathbf{C}^{*n}$. Thus the image of f is onto Ω. We identify $T_{f(\mathbf{w})}\mathbf{C}$ with $\mathbf{C}$ by $\alpha\dfrac{\partial}{\partial x}+\beta\dfrac{\partial}{\partial y}\leftrightarrow\alpha+i\beta$. Here the coordinates of $\mathbf{C}$ are $x+iy$. Then it is easy to see that the tangent vector of the j-th radial line $r_j(t,\mathbf{w})$ defined by $t\mapsto(w_1,\dots,tw_j,\dots,w_n)$ is mapped by $d_{\mathbf{w}}f$ to $2a_j|w_j|^{2a_j}c_j$. This implies that $f:\mathbf{C}^{*n}\to\mathbf{C}$ is a submersion onto Ω. $\square$

Corollary 16. *Let $f(\mathbf{z},\bar{\mathbf{z}})=\sum_{i=1}^n c_jz_j^{a_j}\bar{z}_j^{a_j}=\sum_{i=1}^n c_j|z_j|^{2a_j}$ as in Proposition* 15.

(1) *If $0\in\Omega$, $V=f^{-1}(0)\subset\mathbf{C}^{*n}$ is smooth and non-empty.*
(2) *$f(\mathbf{z},\bar{\mathbf{z}})$ is not a true non-degenerate mixed function.*

Proof. The first assertion is immediate from Proposition 15. For the second assertion, take any two dimensional subspace $\mathbf{C}^I$ of $\mathbf{C}^n$ with $I=\{i,j\}$, the open cone $\Omega(c_i,c_j)$ generated by c_i, c_j cannot be the whole $\mathbf{C}$. Considering the weight vector S so that $\deg_S z_k=N$, $k\neq i,j$ and $\deg_S z_k=1$ for $k=i,j$, we see that $f_S(\mathbf{z},\bar{\mathbf{z}})=c_i|z|^{2a_i}+c_j|z_j|^{2a_j}$, as long as N is sufficiently large. If $\dim_{\mathbf{R}}\Omega_{c_i,c_j}=2$, it is easy to see that $0\notin\Omega_{c_i,c_j}$. Thus $(f^I)^{-1}(0)\cap\mathbf{C}^{*I}=\emptyset$. If $\dim_{\mathbf{R}}\Omega_{c_i,c_j}=1$, either $0\notin\Omega_{c_i,c_j}$ or $0\in\Omega_{c_i,c_j}$. If $0\notin\Omega_{c_i,c_j}$, $(f^I)^{-1}(0)\cap\mathbf{C}^{*I}=\emptyset$ as above. If $0\in\Omega_{c_i,c_j}$, $\arg c_i+\arg c_j=0$ and the real dimension of $(f^I)^{-1}(0)\cap\mathbf{C}^{*2}$ is 3 and any point of $(f^I)^{-1}(0)$ is a critical point. Thus in any case f^I is not true non-degenerate. $\square$

Example 17. Consider

$$g(\mathbf{z},\bar{\mathbf{z}})=\sum_{j=1}^n|z_j|^{2a_j},\quad h(\mathbf{z},\bar{\mathbf{z}})=\sum_{j=1}^m|z_j|^{2a_j}-\sum_{j=m+1}^n|z_j|^{2a_j}$$

with $1<m<n$. Then the image of g and h are the strictly positive half real line $\{x\geq 0\}$ and the whole real line $\mathbf{R}$ respectively and $g^{-1}(0)\cap\mathbf{C}^{*n}=\emptyset$ and $\dim_{\mathbf{R}}h^{-1}(0)=2n-1$.

2.5. Pull-back of a polar weighted homogeneous polynomial. Let $\sigma = (p_{ij}) = (P_1, \dots, P_n)$ be a unimodular matrix where $P_j = {}^t(p_{1j}, \dots, p_{nj})$ is the j-th column vector. Consider the toric morphism

$$\psi_\sigma : \mathbf{C}^n \to \mathbf{C}^n, \quad \mathbf{w} \mapsto \mathbf{z} = (z_1, \dots, z_n)$$
$$z_j = w_1^{p_{j1}} \cdots w_n^{p_{jn}}, \quad j = 1, \dots, n.$$

See §4.1 for more details. Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^m c_{\mu,\nu} \mathbf{z}^{\mu_j} \bar{\mathbf{z}}^{\nu_j}$ be a polar weighted homogeneous polynomial of type $(p_1, \dots, p_n; d_p)$ and let $(q_1, \dots, q_n; d_r)$ be the radial weights. Then they satisfy the equality:

$$\sum_{j=1}^n (\mu_j - \nu_j) p_j = d_p, \quad \sum_{j=1}^n (\nu_j + \nu_j) q_j = d_r, \quad j = 1, \dots, m$$

where $P = {}^t(p_1, \dots, p_n)$ and $Q = {}^t(q_1, \dots, q_n)$. Consider the pull-back

$$\psi_\sigma^*(f)(\mathbf{w}, \overline{\mathbf{w}}) = \sum_{j=1}^m c_{\mu,\nu} \psi_\sigma^*(\mathbf{z}^{\mu_j} \bar{\mathbf{z}}^{\nu_j}) = \sum_{j=1}^m c_{\mu,\nu} \mathbf{w}^{\mu'_j} \overline{\mathbf{w}}^{\nu'_j}$$

where $\mu'_j = \mu_j \sigma$, $\nu'_j = \nu_j \sigma$ and μ_j, ν_j are considered raw vectors. We define $P' := \sigma^{-1} P$. Then we see that

$$(\mu'_j + \nu'_j) Q' = (\mu_j + \nu_j) \sigma \sigma^{-1} Q = d_r$$
$$(\mu'_j - \nu'_j) P' = (\mu_j - \nu_j) \sigma \sigma^{-1} P = d_p$$

for any $j = 1, \dots, m$. Thus

LEMMA 18. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted mixed polynomial of the radial weight type $(q_1, \dots, q_n; d_r)$ and of the polar weight type $(p_1, \dots, p_n; d_p)$. Then $g(\mathbf{w}, \overline{\mathbf{w}}) := \psi_\sigma^* f(\mathbf{w}, \overline{\mathbf{w}})$ is also a polar weighted homogeneous polynomial. The radial weight type and the polar weight type are $(q'_1, \dots, q'_n; d_r)$ and $(p'_1, \dots, p'_n; d_p)$ respectively where*

$$\begin{pmatrix} q'_1 \\ \vdots \\ q'_n \end{pmatrix} = \sigma^{-1} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad \begin{pmatrix} p'_1 \\ \vdots \\ p'_n \end{pmatrix} = \sigma^{-1} \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

Two fibrations are isomorphic by ψ_σ, using the following commutative diagram.

$$\begin{array}{ccc} \mathbf{C}^{*n} & \xrightarrow{f} & \mathbf{C}^* \\ \uparrow{\scriptstyle \psi_\sigma} & & \uparrow{\scriptstyle \mathrm{id}} \\ \mathbf{C}^{*n} & \xrightarrow{g} & \mathbf{C}^* \end{array}$$

(*The commutativity implies that ψ_σ is a fiber preserving diffeomorphism.*)

3. Isolatedness of the singularities

Let $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^{\nu}\bar{\mathbf{z}}^{\mu}$. As we are mainly interested in the topology of a germ of a mixed hypersurface at the origin, we always assume that f does not have the constant term so that $O \in f^{-1}(0)$. Put $V = f^{-1}(0) \subset \mathbf{C}^n$.

3.1. Mixed singular points. We say that $\mathbf{w} \in V$ is a *mixed singular point* if $\mathbf{w}$ is a critical point of the mapping $f : \mathbf{C}^n \to \mathbf{C}$. We say that V is *mixed non-singular* if it has no mixed singular points. If V is mixed non-singular, V is smooth variety of real codimension two. Note that a singular point of V (as a point of a real algebraic variety) is a mixed singular point of V but the converse is not necessarily true. For example, every point of the sphere $S = \{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n = 1\}$ is a mixed singular point.

3.2. Non-vanishing coordinate subspaces. For a subset $J \subset \{1,2,\dots,n\}$, we consider the subspace $\mathbf{C}^J$ and the restriction $f^J := f|_{\mathbf{C}^J}$. Consider the set

$$\mathcal{NV}(f) = \{I \subset \{1,\dots,n\} \mid f^I \not\equiv 0\}.$$

We call $\mathcal{NV}(f)$ *the set of non-vanishing coordinate subspaces for* f. Put

$$V^{\#} = \bigcup_{I \in \mathcal{NV}(f)} V \cap \mathbf{C}^{*I}.$$

THEOREM 19. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a true non-degenerate mixed function. Then there exists a positive number r_0 such that the following properties are satisfied.*

(1) (Isolatedness of the singularity) *The mixed hypersurface $V^{\#} \cap B_{r_0}$ is mixed non-singular. In particular*, $\mathrm{codim}_{\mathbf{R}}\, V^{\#} = 2$.

(2) (Transversality) *The sphere S_r with $0 < r \le r_0$ intersects $V^{\#}$ transversely.*

Proof. We prove that the origin is an isolated mixed singularity. Or $V^{\#} \cap B_{r_0}$ has no mixed singularity, if r is sufficiently small. Denote the mixed singular locus of V by $\Sigma_m(V)$. Assume the contrary. Using the Curve Selection Lemma ([12, 7]), we can find a real analytic curve $\mathbf{z}(t) \in \mathbf{C}^n$, $0 \le t \le 1$ so that $\mathbf{z}(t) \in \Sigma_m(V) \cap V^{\#}$ for $t \neq 0$ and $\mathbf{z}(0) = O$. Using Proposition 8 we can find a real analytic family $\lambda(t)$ in $S^1 \subset \mathbf{C}$ such that

$$\overline{df}(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = \lambda(t)\, \bar{d}f(\mathbf{z}(t),\bar{\mathbf{z}}(t)). \tag{1}$$

Put $I = \{j \mid z_j(t) \not\equiv 0\}$. As $\mathbf{z}(t) \in V^{\#}$, $I \in \mathcal{NV}(f)$, the restriction $f^I = f|_{\mathbf{C}^I}$ is not constantly zero. We may assume that $I = \{1,\dots,m\}$ and we consider f^I and the Taylor expansion of $\mathbf{z}(t)$:

$$\mathbf{z}_i(t) = b_i t^{a_i} + (\text{higher terms}), \quad b_i \neq 0 \quad i = 1,\dots,m$$
$$\lambda(t) = \lambda_0 + \lambda_1 t + (\text{higher terms}), \quad \lambda_0 \in S^1 \subset \mathbf{C}.$$

Put $A = (a_1, \dots, a_m)$ and we consider the face function f_A^I of $f^I(\mathbf{z}, \bar{\mathbf{z}})$. Let $d = d(A; f^I) > 0$ and $\mathbf{b} = (b_1, \dots, b_m) \in \mathbf{C}^{*m}$. Then we have

$$\frac{\partial f}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_A^I}{\partial z_j}(\mathbf{b}, \bar{\mathbf{b}}) t^{d-a_j} + (\text{higher terms}), \quad j = 1, \dots, m$$
$$\frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_A^I}{\partial \bar{z}_j}(\mathbf{b}, \bar{\mathbf{b}}) t^{d-a_j} + (\text{higher terms}) \quad j = 1, \dots, m.$$

Observe that by the equality (1), we have the following equality:

$$\mathrm{ord}_t \frac{\partial f^I}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \mathrm{ord}_t \frac{\partial f^I}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)), \quad j = 1, \dots, m.$$

Thus by (1), we get the equality:

$$\overline{df_A^I}(\mathbf{b}, \bar{\mathbf{b}}) = \lambda_0 \, \bar{d} f_A^I(\mathbf{b}, \bar{\mathbf{b}}).$$

On the other hand, the equality $f^I(\mathbf{z}(t)) \equiv 0$ implies that $f_A^I(\mathbf{b}, \bar{\mathbf{b}}) = 0$. This implies that $\mathbf{b} \in \mathbf{C}^{*I}$ is a critical point of $f_A^I : \mathbf{C}^{*I} \to \mathbf{C}$, which is a contradiction to the non-degeneracy of $f^I(\mathbf{z}, \bar{\mathbf{z}})$.

The second assertion is the result of a standard argument (Corollary 2.9, [12]). $\square$

We say that f is *k-convenient* if $J \in \mathcal{NV}(f)$ for any $J \subset \{1, \dots, n\}$ with $|J| = n - k$. We say that f is *convenient* if f is $(n-1)$-convenient. Note that $V^{\#} = V \backslash \{O\}$ if f is convenient. For a given ℓ with $0 < \ell \le n$, we put $W(\ell) = \{\mathbf{z} \in \mathbf{C}^n \mid |I(\mathbf{z})| \le \ell\}$ where $I(\mathbf{z}) = \{i \mid z_i = 0\}$. Thus $W(n-1) = \mathbf{C}^{*n}$. If f is ℓ-convenient, $V \cap W(\ell) \subset V^{\#}$.

COROLLARY 20. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient true non-degenerate mixed polynomial. Then $V = f^{-1}(0)$ has an isolated mixed singularity at the origin.*

Remark 21. The assumption "true" is to make sure that $V^* = f^{-1}(0) \cap \mathbf{C}^{*n}$ is non-empty.

4. Resolution of the singularities

We consider a mixed analytic function $f(\mathbf{z}, \bar{\mathbf{z}})$ and the corresponding mixed hypersurface $V = f^{-1}(0)$. We assume that $O \in V$ is an isolated mixed singularity, unless otherwise stated.

If f is complex analytic, a "resolution of f" is usually understood as a proper holomorphic mapping $\varphi : X \to \mathbf{C}^n$ so that

(i) $E := \varphi^{-1}(O)$ is a union of smooth (complex analytic) divisors which intersect transversely and $\varphi : X - E \to \mathbf{C}^n - \{O\}$ is biholomorphic,

(ii) the divisor $(\varphi^* f)$ is a union of smooth divisors intersecting transversely and we can write $(\varphi^* f) = \hat{V} \cup E$ where $\hat{V}$ is the strict transform of V (= the closure of $\varphi^{-1}(V - \{O\})$),

(iii) for any point $P \in E_I^* \cap \hat{V}$ with $I = \{i_1, \dots, i_s\}$, there exists an analytic coordinate chart $(u_1, \dots, u_n)$ so that the pull-back of f is written as $U \times u_1^{m_1} \cdots u_j^{m_j}$ where U is a unit in a neighborhood of P, $E_{i_k} = \{u_k = 0\}$ $(k = 1, \dots, s-1)$ and $\hat{V} = \{u_s = 0\}$. Here $E_I^* := \bigcap E_{i \in I} \setminus \bigcup_{j \notin I} E_j$.

For a mixed hypersurface, a resolution of this type does not exist in general. The main reason is that there is no complex structure in the tangent space of V. Nevertheless we will show that a suitable toric modification partially resolves such singularities.

4.1. Toric modification and resolution of complex analytic singularities. For the reader's convenience, we recall some basic facts about the toric modifications at the origin. We use the notations and the terminologies of [14, 15, 16] and §2.2.

4.1.1. *Toric modification.* Let $A = (a_{i,j}) \in GL(n, \mathbf{Z})$ with $\det A = \pm 1$. We call such a matrix *a unimodular matrix*. We associate to A a birational morphism

$$\psi_A : \mathbf{C}^{*n} \to \mathbf{C}^{*n}$$

which is defined by $\psi_A(\mathbf{z}) = (z_1^{a_{1,1}} \cdots z_n^{a_{1,n}}, \dots, z_1^{a_{n,1}} \cdots z_n^{a_{n,n}})$. If the coefficients of A are non-negative, ψ_A can be defined on $\mathbf{C}^n$. Note that ψ_A is a group homomorphism of the algebraic group $\mathbf{C}^{*n}$ and we have

$$\psi_A^{-1} = \psi_{A^{-1}}, \quad \psi_A \circ \psi_B = \psi_{AB}.$$

We consider the space of integer weight vectors N and we denote weight vectors by column vectors. Here the coordinates $\mathbf{z} = (z_1, \dots, z_n)$ is fixed. The space of the weight vectors with coefficients in $\mathbf{R}$ is denoted by $N_{\mathbf{R}}$.

Now we consider the subspace of positive weight vectors $N_{\mathbf{R}}^+$. Let $P_1, \dots, P_m$ be vectors in $N_{\mathbf{R}}^+$. The polyhedral cone generated by $P_1, \dots, P_m$ is defined by

$$\mathrm{Cone}(P_1, \dots, P_m) := \{t_1 P_1 + \cdots + t_m P_m \in N \mid t_i \in \mathbf{R}, t_i \geq 0, i = 1, \dots, m\}.$$

The interior of $\mathrm{Cone}(P_1, \dots, P_m)$ is called an open cone and it is defined as

$$\mathrm{IntCone}(P_1, \dots, P_m) := \{t_1 P_1 + \cdots + t_m P_m \in N \mid t_i \in \mathbf{R}, t_i > 0, i = 1, \dots, m\}.$$

The cone $\mathrm{Cone}(P_1, \dots, P_m)$ is called a *simplicial cone* if $\{P_1, \dots, P_m\}$ are linearly independent. We consider only the case where $P_1, \dots, P_m$ are integer vectors. We call $P_1, \dots, P_m$ *the vertices* of the cone, if $P_1, \dots, P_m$ are chosen to be primitive integer vectors, by multiplications of rational numbers if necessary. It is called a *regular simplicial cone* if $\{P_1, \dots, P_m\}$ can be a part of $\mathbf{Z}$-basis of N. For a regular simplicial cone $\sigma = \mathrm{Cone}(P_1, \dots, P_n)$ of dimension n with vertices $P_1, \dots, P_n$, we associate a unimodular matrix A whose j-th column is P_j. By an abuse of notation, we also denote A by σ. Let $E_1, \dots, E_n$ be the standard basis of N. ($E_j = {}^t(0,\dots,1,0,\dots,0)$ where 1 is at the j-coordinate.) Then $\mathrm{Cone}(E_1,\dots,E_n)$ is a regular simplicial cone and it is nothing but $N_{\mathbf{R}}^+$.

We consider a simplicial cone subdivision Σ^* of the cone $\mathrm{Cone}(E_1,\dots,E_n)$ for which every cone is regular. Such a subdivision is called a *regular fan*. Suppose that Σ^* is a regular fan. Let $\mathscr{S}$ be the set of n-dimensional cones and let $\mathscr{V}^+$ be the set of strictly positive vertices. For simplicity, we assume that the vertices of Σ^* are the union of $\{E_1,\dots,E_n\}$ and $\mathscr{V}^+$. For each $\sigma \in \mathscr{S}$, we consider a copy of a complex Euclidean space $\mathbf{C}_\sigma^n$ with coordinates $\mathbf{u}_\sigma = (u_{\sigma 1},\dots,u_{\sigma n})$ and the morphism $\pi_\sigma : \mathbf{C}_\sigma^n \to \mathbf{C}^n$ defined by $\pi_\sigma(\mathbf{u}_\sigma) = \psi_\sigma(\mathbf{u}_\sigma)$. Taking the disjoint sum $\coprod_{\sigma\in\mathscr{S}} \mathbf{C}_\sigma^n$, we glue together $\coprod_{\sigma\in\mathscr{S}} \mathbf{C}_\sigma^n$ under the following equivalence relation:

$$\mathbf{u}_\sigma \sim \mathbf{u}_\tau \quad \text{if } \psi_{\tau^{-1}\sigma} \text{ is well-defined at } \mathbf{u}_\sigma \text{ and } \psi_{\tau^{-1}\sigma}(\mathbf{u}_\sigma) = \mathbf{u}_\tau.$$

We denote the quotient space $\coprod_{\sigma\in\mathscr{S}} \mathbf{C}_\sigma^n/\!\sim$ by X_{Σ^*}. Then X_{Σ^*} is a complex manifold of dimension n and the morphisms $\pi_\sigma : \mathbf{C}_\sigma^n \to \mathbf{C}^n$, $\sigma \in \mathscr{S}$ are compatible with the identification and thus they define a birational proper holomorphic mapping

$$\hat{\pi} : X_{\Sigma^*} \to \mathbf{C}^n.$$

The restriction $\hat{\pi}$ to $X_{\Sigma^*}\backslash\hat{\pi}^{-1}(0)$ is a biholomorphic onto $\mathbf{C}^n\backslash\{O\}$. We call $\hat{\pi} : X_{\Sigma^*} \to \mathbf{C}^n$ *the toric modification* associated with the regular fan Σ^* [14, 16]. The irreducible exceptional divisors correspond bijectively to the vertices $P \in \mathscr{V}^+$ and we denote it by $\hat{E}(P)$. Then $\hat{\pi}^{-1}(O) = \bigcup_{P\in\mathscr{V}^+} \hat{E}(P)$.

The easiest non-trivial case is when $\mathscr{V}^+ = \{P = {}^t(1,\dots,1)\}$. In this case, X_{Σ^*} is nothing but the ordinary blowing-up at the origin of $\mathbf{C}^n$.

4.1.2. *Dual Newton diagram and admissible toric modifications*. Let $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ be a germ of mixed function in n variables $z_1,\dots,z_n$. We introduce an equivalence relation in $N_{\mathbf{R}}^+$ by

$$P \sim Q, \quad P, Q \in N_{\mathbf{R}}^+ \quad \Leftrightarrow \quad \Delta(P;f) = \Delta(Q;f).$$

The set of equivalence classes gives an open polyhedral cone subdivision of $N_{\mathbf{R}}^+$ and we denote it as $\Gamma^*(f;\mathbf{z})$ and we call it *the dual Newton diagram*. Let Σ^* be a regular fan which is a regular simplicial cone subdivision of $\Gamma^*(f)$. If Σ^* is a regular simplicial cone subdivision of $\Gamma^*(f)$, the toric modification $\hat{\pi} : X_{\Sigma^*} \to \mathbf{C}^n$ is called *admissible* for $f(\mathbf{z},\bar{\mathbf{z}})$. The basic fact for non-degenerate holomorphic functions is:

Theorem 22 ([14, 15, 16]). *Assume that $f(\mathbf{z})$ be a non-degenerate convenient analytic function with an isolated singularity at the origin. Let $\hat{\pi} : X_{\Sigma^*} \to \mathbf{C}^n$ be an admissible toric modification. Then it is a good resolution of the mapping $f : \mathbf{C}^n \to \mathbf{C}$ at the origin.*

This is a starting observation of the present paper.

4.2. Blowing up examples. We consider some examples.

Example 23. A. Let $C_1 = \{(z_1, z_2) \in \mathbf{C}^2 \mid z_1^2 - z_2^2 = 0\}\}$, $V_1 = \{(z_1, z_2) \in \mathbf{C}^2 \mid f_1(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ and $V_2 = \{(z_1, z_2) \in \mathbf{C}^2 \mid f_2(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ where $f_1(\mathbf{z}, \bar{\mathbf{z}}) = \bar{z}_1^2 - z_2^2$ and $f_2(\mathbf{z}, \bar{\mathbf{z}}) = z_1\bar{z}_1 - z_2^2$. C_1 is a union of two smooth complex line, V_1 is a union of two smooth real planes, $\bar{z}_1 \pm z_2 = 0$ and V_2 is an irreducible variety. Consider

$$\hat{\pi}_1 : X_1 \to \mathbf{C}^2$$

where $\hat{\pi}_1 : X_1 \to \mathbf{C}^2$ is the toric modification associated with the regular fan generated by vertices

$$\Sigma_1^* = \left\{ E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Geometrically, $\hat{\pi}_1$ is an ordinary blowing up. Note that for the complex curve C_1, the two components are separated by a single blowing up $\hat{\pi}_1$. We will see what happens to the two other mixed curves V_1, V_2. In the toric coordinate $\mathbf{C}_\sigma^2$ with $\sigma = \mathrm{Cone}(P, E_2)$ and the toric coordinates (u_1, u_2), the strict transform $\hat{V}_1$, $\hat{V}_2$ of V_1, V_2 are defined in the torus $\mathbf{C}_\sigma^{*2}$ as

$$\begin{aligned} \hat{C}_1 \cap \mathbf{C}_\sigma^{*2} &= \{(u_1, u_2) \in \mathbf{C}_\sigma^{*2} \mid u_1^2 - u_1^2 u_2^2 = u_1^2(1 - u_2^2) = 0\} \\ \hat{V}_1 \cap \mathbf{C}_\sigma^{*2} &= \{(u_1, u_2) \in \mathbf{C}_\sigma^{*2} \mid \bar{u}_1^2 - u_1^2 u_2^2 = 0\}, \\ \hat{V}_2 \cap \mathbf{C}_\sigma^{*2} &= \{(u_1, u_2) \in \mathbf{C}_\sigma^{*2} \mid u_1(\bar{u}_1 - u_1 u_2^2) = 0\}. \end{aligned}$$

The first expression shows that $\hat{C}_1$ is already smooth and separated into two peaces. Unlike the case of holomorphic functions, we observe that

$$\{(u_1, u_2) \in \mathbf{C}_\sigma^2 \mid \bar{u}_1^2 - u_1^2 u_2^2 = 0\} \supsetneq \hat{V}_1, \quad \{(u_1, u_2) \in \mathbf{C}_\sigma^2 \mid \bar{u}_1 - u_1 u_2^2 = 0\} \supsetneq \hat{V}_2$$

as $\hat{E}(P) = \{u_1 = 0\} \not\subset \hat{V}_i$, $i = 1, 2$. In both cases, we see that the 1-sphere $|u_2| = 1$ appears as their intersection with the exceptional divisor $\hat{E}(P)$. It is easy to see that for $\hat{V}_1$, both irreducible components $L_\pm^* = \{(u_1, u_2) \in \mathbf{C}_\sigma^{*2} \mid \bar{u}_1 \pm u_1 u_2 = 0\}$ satisfy the limit equality $\hat{L}_\varepsilon^* \cap \hat{E}(P) = \{(0, u_2) \mid |u_2| = 1\}$ with $\varepsilon = \pm$. Thus $\hat{L}_+ \cap \hat{L}_-$ is the 1-sphere $|u_2| = 1$ and the ordinary blowing up does not separate the two smooth components. For $\hat{V}_2$, we will see later that it has two link components. See §6 for the definition of the link components. This illustrates the complexity of the limit set of the tangent lines in the mixed varieties.

B. We consider an ordinary cusp (complex analytic) $C_2 = \{z_2^2 - z_1^3 = 0\}$ and a mixed curve $V_3 = \{z_2^2 - z_1^2\bar{z}_1 = 0\}$ with the same Newton boundary and an admissible toric blowing up $\hat{\pi} : X_2 \to \mathbf{C}^2$ which is associated with the regular simplicial fan:

$$\Sigma_2^* = \left\{ E_1, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, R = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, E_2 \right\}$$

Let (u_1, u_2) be the toric coordinate of $\mathbf{C}_\sigma^2$ with $\sigma = (Q, R) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Then the pull back of the defining polynomials are defined in this coordinate chart as

$$\hat{C}_2 \cap \mathbf{C}_\sigma^{*2} = \{(u_1, u_2) \in \mathbf{C}_\sigma^{*2} \mid u_1^6 u_2^3 (u_2 - 1) = 0\}$$
$$\hat{V}_3 \cap \mathbf{C}_\sigma^{*2} = \{(u_1, u_2) \in \mathbf{C}_\sigma^{*2} \mid u_1^4 u_2^2 (u_1^2 u_2^2 - \bar{u}_1^2 \bar{u}_2) = 0\}.$$

Observe that $\hat{C}_2$ is smooth and transverse to the exceptional divisor $\hat{E}(Q) = \{u_1 = 0\}$. The strict transform $\tilde{V}_3$ is defined by $u_1^2 u_2^2 - \bar{u}_1^2 \bar{u}_2 = 0$ in $\mathbf{C}_\sigma^{*2}$. We see again that for $\tilde{V}_3$, a sphere $|u_2| = 1$ appears as the intersection with the exceptional divisor. We observe that $\hat{V}_3 \cap \hat{E}(Q) = \{(0, u_2) \mid |u_2| = 1\}$.

The above examples show that the toric modification does not resolve the singularities of non-degenerate mixed hypersurfaces. To get a good resolution of a mixed hypersurface singularity, we need to compose a toric modification with a normal real blowing up or a normal polar modification which we introduce below.

4.3. Normal real blowing up and normal polar blowing up of C. Consider the complex plane with two coordinate systems $z = x + iy$ and $z = r\exp(i\theta)$. We can consider the following two modifications.

(I) Let $\imath_{\mathbf{R}} : \mathbf{C} \backslash \{O\} \to \mathbf{C} \times \mathbf{RP}^1$ defined by $z = x + iy \mapsto (z, [x : y])$ and let $\mathscr{R}\mathbf{C}$ be the closure of the image of $\imath_{\mathbf{R}}$. This is called the real blowing up. $\mathscr{R}\mathbf{C}$ is a real two dimensional manifold which has two coordinate charts $(U_0, (\tilde{x}, t))$ and $(U_1, (s, \tilde{y}))$. These coordinates are defined by $\tilde{x} = x$, $t = y/x$ and $\tilde{y} = y$, $s = x/y$. The canonical projection $\omega_{\mathbf{R}} : \mathscr{R}\mathbf{C} \to \mathbf{C}$ is given as $\omega_{\mathbf{R}}(\tilde{x}, t) = \tilde{x}(1 + it)$ and $\omega_{\mathbf{R}}(s, \tilde{y}) = \tilde{y}(s + i)$. Note that $\omega_{\mathbf{R}}^{-1}(O) = \mathbf{RP}^1$ and $\omega_{\mathbf{R}} : \mathscr{R}\mathbf{C} \backslash \{O\} \times \mathbf{RP}^1 \to \mathbf{C} \backslash \{O\}$ is diffeomorphism.

(II) Consider the polar embedding $\imath_p : \mathbf{C} \backslash \{O\} \to \mathbf{R}^+ \times S^1$ which is defined by $\imath_p(r\exp(\theta i)) = (r, \exp(\theta i))$. Here $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x \geq 0\}$. Let $\mathscr{P}\mathbf{C} = \mathbf{R}^+ \times S^1$ and $\omega_p : \mathscr{P}\mathbf{C} \to \mathbf{C}$ be the projection defined by $\omega_p(r, \exp(\theta i)) = r\exp(\theta i)$. We can see easily that $\omega_p^{-1}(O) = \{0\} \times S^1$ and $\omega_p : \mathscr{P}\mathbf{C} \backslash \{0\} \times S^1 \to \mathbf{C} \backslash \{O\}$ is a diffeomorphism. Note that $\mathscr{P}\mathbf{C}$ is a manifold with boundary.

4.3.1. *Canonical factorization.* There exists a canonical mapping $\psi : \mathscr{P}X \to \mathscr{R}\mathbf{C}$ which is defined by

$$\psi(r, \exp(\theta i)) = \begin{cases} (\tilde{x}, t) = (r\cos\theta, \tan\theta), & \theta \neq \pm\dfrac{\pi}{2} \\ (s, \tilde{y}) = (\cot\theta, r\sin\theta), & \theta \neq 0, \pi \end{cases}$$

It is obvious that ψ gives the commutative diagram

$$\begin{array}{ccc} \mathscr{P}\mathbf{C} & \xrightarrow{\psi} & \mathscr{R}\mathbf{C} \\ \Big\downarrow{\scriptstyle \omega_p} & & \Big\downarrow{\scriptstyle \omega_{\mathbf{R}}} \\ \mathbf{C} & = & \mathbf{C} \end{array}$$

Note that the restriction of ψ over the exceptional sets is a $2:1$ map:

$$\psi : \{O\} \times S^1 \to \{O\} \times \mathbf{RP}^1, \quad \exp(\theta i) \mapsto [\cos(\theta) : \sin(\theta)]$$

4.4. Resolution of a mixed function. Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function and let $V = f^{-1}(0)$ and we assume that V has an isolated mixed singularity at the origin and the real codimension of V is two. (Note that if V is non-degenerate, it has a real codimension two by the definition of non-degeneracy and Theorem 19.) Let Y be a real analytic manifold of dimension $2n$ and let $\Phi : Y \to \mathbf{C}^n$ be a proper real analytic mapping. We say that $\Phi : Y \to \mathbf{C}^n$ is *a resolution of a real type* (respectively *a resolution of a polar type*) of the mixed function f if

(1) Let $E = \Phi^{-1}(O)$ and let $E = E_1 \cup \cdots \cup E_r$ be the irreducible components. Each E_j is a real codimension one smooth subvariety.
(2) Y is a real analytic manifold of dimension $2n$. For a resolution of a real type, Y has no boundary while for a resolution of a polar type Y is a real analytic manifold with boundary and $\partial Y = E$.
(3) The restriction $\Phi : Y - E \to \mathbf{C}^n \backslash \{O\}$ is a real analytic diffeomorphism.
(4) Let $\tilde{V}$ be the strict transform of V ($=$ the closure of $\Phi^{-1}(V \backslash \{O\})$). Then $\tilde{V}$ is a smooth manifold of real codimension 2 in an open neighborhood of E.
(5) For $I = \{i_1, \dots, i_t\}$, put $E_I^* := \bigcap_{k=1}^t E_{i_k} \backslash \bigcup_{j \notin I} E_j$. For $P \in E_I^* \cap \tilde{V}$, there exists a local real analytic coordinate system $(U, (u_1, \dots, u_{2n}))$ centered at P such that

$$\Phi^* f(\mathbf{u}) = u_1^{m_1} \cdots u_t^{m_t} (u_{t+1} + i u_{t+2})$$

so that $U \cap E_{i_j} = \{u_j = 0\}$ for $j = 1, \dots, t$ and $U \cap \tilde{V} = \{u_{t+1} + i u_{t+2} = 0\}$. In the case of a resolution of a polar type, we assume also that $Y \cap U = \{u_1 \geq 0, \dots, u_t \geq 0\}$.

For example, assume that $t = 1$ for simplicity. Then the condition (5) says the following. If we are considering a resolution of a real type,

$$U \cong \mathbf{R}^{2n} \text{ or } B^{2n}, \quad E_{i_1} = \{u_1 = 0\}, \quad \Phi^* f(\mathbf{u}) = u_1^{m_1}(u_2 + i u_3),$$

if we are considering a resolution of polar type,

$$U \cong \mathbf{R}^{2n} \cap \{u_1 \geq 0\}, \quad E_{i_1} = \{u_1 = 0\}, \quad \Phi^* f(\mathbf{u}) = u_1^{m_1}(u_2 + i u_3).$$

See the next section for more details.

4.4.1. *Normal real blowing up.* Let X be a complex manifold of dimension n with a finite number of smooth complex divisors $E_1, \dots, E_\ell$ such that the union of divisors $E = \bigcup_{i=1}^\ell E_i$ has at most normal crossing singularities. Then we can consider the composite of real modifications for the normal complex 1-dimensional subspaces along the divisor $E_1, \dots, E_\ell$. Put it as $\omega_{\mathbf{R}} : \mathscr{R}X \to X$ and we call it *the normal real blowing up* along E. It is immediate from the definition that

(1) $\mathscr{R}X$ is a differentiable manifold and $\omega_{\mathbf{R}} : \mathscr{R}X \backslash \omega_{\mathbf{R}}^{-1}(E) \to Y \backslash E$ is a diffeomorphism.
(2) Inverse image $\tilde{E}_j := \omega_{\mathbf{R}}^{-1}(E_j)$ of E_j is a real codimension 1 variety which is fibered over E_j' with a fiber S^1. Here E_j' is the normal real blowing up of E_j along $\bigcup_{i \neq j} E_i \cap E_j$. Putting $E_I^* := \bigcap_{i \in I} E_i \backslash \bigcup_{j \notin I} E_j$, $\tilde{E}_I^* := \omega_{\mathbf{R}}^{-1}(E_I^*)$ is fibered over E_I^* with fiber $(S^1)^k$ where $k = |I|$.

Take a point $P \in E_I^*$ and choose a local coordinate $(W, (u_1, \dots, u_n))$ so that $I = \{1, \dots, m\}$ and $E_j = \{u_j = 0\}$, $j = 1, \dots, m$. Then $\omega_{\mathbf{R}}^{-1}(W)$ is isomorphic to $(\mathscr{R}\mathbf{C})^m \times \mathbf{C}^{n-m}$ covered by 2^m coordinates $W_{\varepsilon_1, \dots, \varepsilon_m} = U_{\varepsilon_1} \times \cdots \times U_{\varepsilon_m} \times \mathbf{C}^{n-m}$ where $\varepsilon_j = 0$ or 1. For example, $W_{1,0,\dots,0}$ has the coordinates (as a real analytic manifold) $(s_1, \tilde{y}_1, \tilde{x}_2, t_2, \dots, \tilde{x}_m, t_m, u_{m+1}, \dots, u_n)$ so that the projection to the coordinate chart $\mathbf{u} \in W$ is given by

$$u_1 = \tilde{y}_1(s_1 + i), \quad u_2 = \tilde{x}_2(1 + it_2), \dots, u_m = \tilde{x}_m(1 + it_m).$$

In this coordinate chart, the exceptional real divisor $\tilde{E}_j$, $j \leq m$ is defined by $\tilde{E}_1 = \{\tilde{y}_1 = 0\}$ and $\tilde{E}_j = \{\tilde{x}_j = 0\}$ for $2 \leq j \leq m$.

4.4.2. *Normal polar blowing up*. We can also consider the composite of the polar blowing ups along exceptional divisors, which we denote as $\omega_p : \mathscr{P}X \to X$. In the same coordinate chart $(W, \mathbf{u})$, $\mathbf{u} = (u_1, \dots, u_n)$ as in the previous discussion, $\omega_p^{-1}(W)$ is written as

$$\omega_p^{-1}(W) = (\mathbf{R}^+ \times S^1) \times \cdots \times (\mathbf{R}^+ \times S^1) \times \mathbf{C}^{n-m}$$

with coordinates $(r_1, \exp(i\theta_1), \dots, r_m, \exp(i\theta_m), u_{m+1}, \dots, u_n)$ and the projection is given by

$$(r_1, \exp(i\theta_1), \dots, r_m, \exp(i\theta_m), u_{m+1}, \dots, u_n) \mapsto (u_1, \dots, u_n),$$
$$u_j = r_j \exp(i\theta_j), \quad j = 1, \dots, m.$$

Note that $\mathscr{P}X$ is a manifold with boundary and $\omega_p^{-1}(E_j)$ is the boundary component which is given by $\{r_j = 0\}$.

4.5. A resolution of a real type and a resolution of a polar type. Now we can state our main result for the resolution of non-degenerate mixed singularities. Assume that $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu, \mu} c_{\nu,\mu} \mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ is a non-degenerate convenient mixed function and consider the mixed hypersurface $V = f^{-1}(0)$. Let $\Gamma(f)$ be the Newton boundary and let $\Gamma^*(f)$ be the dual Newton diagram. Take a regular simplicial cone subdivision in the sense of [16] and let $\hat{\pi} : X \to \mathbf{C}^n$ be the associated toric modification. Let $\mathscr{V}^+$ be the set of strictly positive vertices of Σ^* and let $\hat{E}(P)$, $P \in \mathscr{V}^+$ be the exceptional divisors. We may assume that the vertices which are not strictly positive are the canonical bases $\{E_1, \dots, E_n\}$ of N by the convenience assumption where $E_j = {}^t(0, \dots, 1, \dots, 0)$. Put $\hat{E} = \bigcup_{P \in \mathscr{P}} \hat{E}(P)$. Then we take the normal real blowing-ups $\omega_{\mathbf{R}} : \mathscr{R}X \to X$ along the exceptional divisors of $\hat{E}$. Then we consider the composite

$$\Phi := \hat{\pi} \circ \omega_{\mathbf{R}} : \mathscr{R}X \xrightarrow{\omega_{\mathbf{R}}} X \xrightarrow{\hat{\pi}} \mathbf{C}^n, \quad \xi \mapsto \hat{\pi}(\omega_{\mathbf{R}}(\xi)).$$

Put $\tilde{E}(P) := \omega_{\mathbf{R}}^{-1}(\hat{E}(P))$ with $P \in \mathscr{V}^+$.

THEOREM 24. *$\Phi : \mathscr{R}X \to \mathbf{C}^n$ gives a good resolution of a real type of f at the origin and the exceptional divisors are $\tilde{E}(P)$ for $P \in \mathscr{V}^+$. The multiplicity of $\tilde{E}(P)$ of the function $\Phi^* f$ along $\tilde{E}(P)$ is $d(P; f)$.*

Let $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$ be the decomposition of f into the real and the imaginary part. Then the above assertion for the multiplicity is equivalent to: the mutiplicities of $\Phi^* g$, $\Phi^* h$ along $\tilde{E}(P)$ are the same and equal to $d(P; f)$.

Proof. We use the same notations as those in [14, 15, 16]. Let $\tilde{V}$, $\hat{V}$ be the strict transforms of V into $\mathscr{R}X$ and X respectively.

$$\begin{array}{ccccc} \Phi : \mathscr{R}X & \xrightarrow{\omega_{\mathbf{R}}} & X & \xrightarrow{\hat{\pi}} & \mathbf{C}^n \\ \cup & & \cup & & \cup \\ \tilde{V} & \xrightarrow{\omega_{\mathbf{R}}} & \hat{V} & \xrightarrow{\hat{\pi}} & V \end{array}$$

Take any point $\tilde{\xi} \in \tilde{V} \cap \Phi^{-1}(O)$ and consider $\hat{\xi} = \Phi(\tilde{\xi}) \in \hat{V}$. Assume that $\hat{\xi}$ is in a toric coordinate chart $\mathbf{C}^n_\sigma$ with $\sigma = (P_1, P_2, \dots, P_n)$ which is a unimodular matrix. Assume that $\hat{\xi} \in \hat{E}(P_1, P_2, \dots, P_s)^*$ where $\hat{E}(P_1, P_2, \dots, P_s)^* = \bigcap_{i=1}^{s} \hat{E}(P_i) \backslash \bigcup_{j>s} \hat{E}(P_j)$. For simplicity, we assume that $s = 1$ and $\hat{\xi} \in \hat{E}(P_1)^*$, leaving the other cases to the reader, as the argument is exactly the same. We denote the coordinates in this chart as $(u_{\sigma 1}, \dots, u_{\sigma n})$ and $u_{\sigma, j} = x_{\sigma j} + i y_{\sigma j}$. For simplicity, we write simply u_j, x_j, y_j for $u_{\sigma j}$, $x_{\sigma j}$, $y_{\sigma j}$ respectively. By the assumption, $\hat{\xi} = (0, \xi_2, \dots, \xi_n)$ with $\xi_j \neq 0$, $j \geq 2$ in the toric coordinate space $\mathbf{C}^n_\sigma$. We may assume that $\tilde{\xi} \in (\mathbf{C}^n_\sigma)_0$. The coordinates of $(\mathbf{C}^n_\sigma)_0$ are given by $(\tilde{x}_1, t_1, u_2, \dots, u_n)$. The divisor $\tilde{E}(P_1)$ is given by $\{\tilde{x}_1 = 0\}$ and the projection $\omega_{\mathbf{R}} \mid (\mathbf{C}^n_\sigma)_0 \to \mathbf{C}^n_\sigma$ is given as

$$(\tilde{x}_1, t_1, u_2, \dots, u_n) \mapsto (u_1, \dots, u_n), \quad u_1 = \tilde{x}_1(1 + it_1).$$

Let $\Delta = \Delta(P_1)$. Take an arbitrary monomial $\mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. Then we observe that

$$\pi_\sigma^*(\mathbf{z}^\nu \bar{\mathbf{z}}^\mu) = u_1^{P_1(\nu)} \cdots u_n^{P_n(\nu)} \times \bar{u}_1^{P_1(\mu)} \cdots \bar{u}_n^{P_n(\mu)} \quad \text{and}$$

$$\omega_{\mathbf{R}}^* \pi_\sigma^*(\mathbf{z}^\nu \bar{\mathbf{z}}^\mu) = \tilde{x}_1^{P_1(\nu+\mu)} (1 + it_1)^{P_1(\nu)} (1 - it_1)^{P_1(\mu)} \prod_{j=2}^{n} u_j^{P_j(\nu)} \bar{u}_j^{P_j(\mu)}.$$

Here we recall that $P_1(\nu) = \sum_{j=1}^{n} p_{j1} \nu_j$. By the definition of $d(P_1)$, for any monomial $\mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ which appears in $f(\mathbf{z}, \bar{\mathbf{z}})$, we have

$$P_1(\nu) + P_1(\mu) \geq d(P_1), \quad \text{and}$$

$$P_1(\nu) + P_1(\mu) = d(P_1) \quad \Leftrightarrow \quad \nu + \mu \in \Delta(P_1).$$

Thus we can write the pull-back function as

$$\Phi^* f(\tilde{x}_1, t_1, \mathbf{u}'_\sigma) = \tilde{x}_1^{d(P_1)} \times \hat{f}_\sigma(\tilde{x}_1, t_1, \mathbf{u}'_\sigma)$$
$$\Phi^* f_\Delta(\tilde{x}_1, t_1, \mathbf{u}'_\sigma) = \tilde{x}_1^{d(P_1)} \times \hat{f}_{\Delta,\sigma}(t_1, \mathbf{u}'_\sigma)$$
$$\hat{f}_\sigma(\tilde{x}_1, t_1, \mathbf{u}'_\sigma) \equiv \hat{f}_{\Delta,\sigma}(t_1, \mathbf{u}'_\sigma) \text{ modulo } (\tilde{x}_1).$$

where $\mathbf{u}'_\sigma = (u_2, \dots, u_n)$. The important point here is that $\hat{f}_{\Delta,\sigma}$ does not contain the variable $\tilde{x}_1$. In the above notation, the strict transform $\tilde{V}$ is defined by $\hat{f}_\sigma(\tilde{x}_1, t_1, \mathbf{u}'_\sigma) = 0$ in $(\mathbf{C}^n_\sigma)_0$. Let $\tilde{\xi} = (0, \tau_1, \xi_2, \dots, \xi_n)$ in the coordinates $(\tilde{x}_1, t_1, \mathbf{u}'_\sigma)$. Using the expression $f(\mathbf{z}, \bar{\mathbf{z}}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$ and $f_\Delta(\mathbf{z}, \bar{\mathbf{z}}) = g_\Delta(\mathbf{x}, \mathbf{y}) + ih_\Delta(\mathbf{x}, \mathbf{y})$, we write these functions $\hat{f}_\sigma$, $\hat{f}_{\Delta,\sigma}$ as the sum of real-valued functions:

$$\hat{f}_\sigma(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma) = \hat{g}_\sigma(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma) + i\hat{h}_\sigma(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)$$
$$\hat{f}_{\Delta,\sigma}(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma) = \hat{g}_{\Delta,\sigma}(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma) + i\hat{h}_{\Delta,\sigma}(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma),$$
$$\text{where } \mathbf{x}'_\sigma = (x_2, \dots, x_n), \quad \mathbf{y}'_\sigma = (y_2, \dots, y_n).$$

The main assertion in Theorem 24 is that the rank of the Jacobian matrix of the functions $\tilde{x}_1$, $\hat{g}_\sigma$, $\hat{h}_\sigma$:

$$J := \frac{\partial(\tilde{x}_1, \hat{g}_\sigma, \hat{h}_\sigma)}{\partial(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}) = \begin{pmatrix} 1 & 0 \\ * & \dfrac{\partial(\hat{g}_\sigma, \hat{h}_\sigma)}{\partial(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}) \end{pmatrix}$$

is 3, which is equivalent to

$$\operatorname{rank}\left(\frac{\partial(\hat{g}_\sigma, \hat{h}_\sigma)}{\partial(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi})\right) = 2.$$

Note that $g_{\Delta,\sigma}(\tilde{\xi}) = h_{\Delta,\sigma}(\tilde{\xi}) = 0$ and

$$g_\sigma - g_{\Delta,\sigma} \equiv 0, \quad h_\sigma - h_{\Delta,\sigma} \equiv 0 \text{ modulo } (\tilde{x}_1)$$

therefore

$$\frac{\partial(g_\sigma, h_\sigma)}{\partial(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}) = \frac{\partial(g_{\Delta,\sigma}, h_{\Delta,\sigma})}{\partial(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}). \tag{2}$$

Now recall that $g_{\Delta,\sigma}$, $h_{\Delta,\sigma}$ does not contain the variable $\tilde{x}_1$. Define a modified point $\tilde{\xi}' \in (\mathbf{C}^n_\sigma)_0$ by $\tilde{\xi}' = (1, \tau_1, \xi_2, \dots, \xi_n)$ and put $\hat{\xi}' = \omega_{\mathbf{R}}(\tilde{\xi}') \in \mathbf{C}^{*n}_\sigma$ and $\mathbf{w}_0 = \pi_\sigma(\hat{\xi}') \in \mathbf{C}^{*n}$. Put $\mathbf{w}_0 = \mathbf{x}_0 + i\mathbf{y}_0$. (Recall that $\pi_\sigma : \mathbf{C}^n_\sigma \to \mathbf{C}^n$ is the projection of the toric modification in this chart.) Then as $g_{\Delta,\sigma}(\hat{\xi}') = g_{\Delta,\sigma}(\tilde{\xi}') = 0$, we have

$$\operatorname{rank}\left(\frac{\partial(g_{\Delta,\sigma}, h_{\Delta,\sigma})}{\partial(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi})\right) = \operatorname{rank}\left(\frac{\partial(g_{\Delta,\sigma}, h_{\Delta,\sigma})}{\partial(t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}')\right)$$
$$= \operatorname{rank}\left(\frac{\partial(g_{\Delta,\sigma}, h_{\Delta,\sigma})}{\partial(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}')\right)$$

Now we consider the hypersurface

$$\begin{aligned} V_\Delta^* &:= \{\mathbf{z} \in \mathbf{C}^{*n} \mid f_\Delta(\mathbf{z}) = 0\} \\ &= \{\mathbf{x} + i\mathbf{y} \in \mathbf{C}^{*n} \mid g_\Delta(\mathbf{x}, \mathbf{y}) = h_\Delta(\mathbf{x}, \mathbf{y}) = 0\} \end{aligned}$$

where $z_j = x_j + iy_j$, $j = 1, \dots, n$ and $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$. Note that $\mathbf{w}_0 \in V_\Delta^*$. As $f_\Delta(\mathbf{w}_0) = 0$ and $\Phi = \hat{\pi} \circ \omega_{\mathbf{R}} : \Phi^{-1}(\mathbf{C}^{*n}) \to \mathbf{C}^{*n}$ is a diffeomorphism, we see that

$$\begin{aligned} \operatorname{rank}\left(\frac{\partial(g_{\Delta,\sigma}, h_{\Delta,\sigma})}{\partial(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}')\right) &= \operatorname{rank}\left(\frac{\partial(\tilde{x}_1^{d(P_1)} g_{\Delta,\sigma}, \tilde{x}_1^{d(P_1)} h_{\Delta,\sigma})}{\partial(\tilde{x}_1, t_1, \mathbf{x}'_\sigma, \mathbf{y}'_\sigma)}(\tilde{\xi}')\right) \\ &= \operatorname{rank}\left(\frac{\partial(g_\Delta, h_\Delta)}{\partial(\mathbf{x}, \mathbf{y})}(\mathbf{x}_0, \mathbf{y}_0)\right) = 2 \end{aligned}$$

where $\mathbf{w}_0 = \mathbf{x}_0 + i\mathbf{y}_0$. The first equality is the result of $g_{\Delta\sigma}(\tilde{\xi}') = h_{\Delta\sigma}(\tilde{\xi}') = 0$. The last equality follows from the non-degeneracy condition which assumes that $f_\Delta : \mathbf{C}^{*n} \to \mathbf{C}$ has 0 as a regular value. □

We can also use the normal polar blowing-up $\omega_p : \mathscr{P}X \to X$ along $\hat{E}(P)$, $P \in \mathscr{V}^+$ and the composite $\Phi_p : \mathscr{P}X \to \mathbf{C}^n$. Put $\tilde{E}(P) := \Phi_p^{-1}(\hat{E}(P))$, $P \in \mathscr{V}^+$.

Theorem 25. *Under the same assumption as in Theorem 24, $\Phi_p : \mathscr{P}X \to X$ gives a good resolution of a polar type of $f(\mathbf{z}, \bar{\mathbf{z}})$ where Φ_p is the composite*

$$\Phi_p : \mathscr{P}X \xrightarrow{\omega_p} X \xrightarrow{\hat{\pi}} \mathbf{C}^n.$$

The multiplicity of $\tilde{E}(P)$ of the function $\Phi_p^* f$ along $\tilde{E}(P)$ is $d(P; f)$. There is a canonical factorization $\eta : \mathscr{P}X \to \mathscr{R}X$ so that $\omega_p = \omega_{\mathbf{R}} \circ \eta$ and $\Phi_p = \Phi \circ \eta$.

Proof. The proof is almost the same as that of Theorem 24. For an arbitrary monomial $\mathbf{z}^\nu \bar{\mathbf{z}}^\mu$. Then we observe that

$$\begin{aligned} \pi_\sigma^*(\mathbf{z}^\nu \bar{\mathbf{z}}^\mu) &= u_1^{P_1(\nu)} \cdots u_n^{P_n(\nu)} \times \bar{u}_1^{P_1(\mu)} \cdots \bar{u}_n^{P_n(\mu)} \quad \text{and} \\ \omega_p^* \pi_\sigma^*(\mathbf{z}^\nu \bar{\mathbf{z}}^\mu) &= r_1^{P_1(\nu+\mu)} \exp(P_1(\nu - \mu)\theta_1 i) \prod_{j=2}^n u_j^{P_j(\nu)} \bar{u}_j^{P_j(\mu)}. \end{aligned}$$

Thus we simply replace $(\tilde{x}_1, t_1, \mathbf{u}'_\sigma)$ by $(r_1, \theta_1, \mathbf{u}'_\sigma)$ with $u_{\sigma 1} = r_1 \exp(i\theta_1) = \tilde{x}_1(1 + it_1)$ in the previous calculation. The factorization follows from §4.3.1. □

Remark 26. The assertion of Theorem 24 and Theorem 25 says that the strict transform $\tilde{V}$ is a "Cartier divisor" in the sense that it is locally defined by a single complex-valued real analytic function in $\mathscr{R}X$, although $\hat{V}$ is not a Cartier divisor in X. Note also that the pull-back of g and h are real-valued functions which have the same multiplicity $d(P)$ along $\tilde{E}(P)$, $P \in \mathscr{V}^+$.

Example 27. We consider two modifications:

$$\hat{\pi}_1 : X_1 \to \mathbf{C}^2, \quad \hat{\pi}_2 : X_2 \to \mathbf{C}^2$$

where $\hat{\pi}_j : X_j \to \mathbf{C}^2$ is the toric modification associated with the regular fan Σ_j^* $(j = 1, 2)$ which are defined by the vertices as follows.

$$\Sigma_1^* = \left\{ E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\Sigma_2^* = \left\{ E_1, P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, R = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, E_2 \right\}$$

1. Let $V_1 = f(\mathbf{z}, \bar{\mathbf{z}}) = \bar{z}_1^2 - z_2^2 = 0$. This is a union of two smooth real planes $z_2 \pm \bar{z}_1 = 0$. In the toric coordinate chart $\mathbf{C}_\sigma^2$ with $\sigma = \mathrm{Cone}(P, E_2)$, the strict transform $\tilde{V}_1$ of V_1 is defined in $\mathbf{C}_\sigma^{*2}$ by

$$\hat{V}_1 : \bar{u}_1^2 - u_1^2 u_2^2 = 0.$$

We have seen that $\hat{V}_1 \cap \hat{E}(P) = \{u_1 = 0 \,|\, |u_2| = 1\}$. Now take the normal real blowing up along $\hat{E}(P)$, $\omega_{\mathbf{R}} : \mathscr{R}X \to X$. The strict transform is defined in $(\mathbf{C}_\sigma^2)_\varepsilon$ as

$$\begin{aligned} \hat{V}_1 &= \{(\tilde{x}_1, t_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \,|\, (1 - it_1)^2 - (1 + it_1)^2 u_2^2 = 0\} \\ &= \{(\tilde{y}_1, s_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \,|\, (s_1 - i)^2 - (s_1 + i)^2 u_2^2 = 0\} \end{aligned}$$

Note these equations give two smooth components L_ε, $\varepsilon = \pm 1$ which are disjoint:

$$\{(\tilde{x}_1, t_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \,|\, (1 - it_1) \pm (1 + it_1) u_2 = 0\}.$$

This expression shows that the strict transform is embedded in the cylinder $|u_2| = 1$. Let us see this in a normal polar modification $\omega_p : \mathscr{P}X \to X$. Now $\mathscr{P}X$ is locally diffeomorphic to the product of $S^1 \times \mathbf{R}^+ \times \mathbf{C}$ and the strict transform is now defined in a simple equation

$$\tilde{V}_1 = \{(r_1, \exp(\theta i), u_2) \,|\, u_2 = \mp \exp(-2\theta i)\}$$

and it has two link components. This shows that the strict transform is a product (it does not depend on r_1) and for a fixed r_1, they are parallel torus knots in $S^1 \times S^1 = S^1 \times \{|u_2| = 1\}$. Observe that the direction of twisting is opposite in the first and the second S^1's with respect to the canonical orientation of S^1.

2. Let us consider another mixed curve:

$$V_2 : \{z_1 \bar{z}_1 - z_2^2 = 0\}$$

Equivalently V_2 is defined by

$$\{(x_1, y_1, x_2, y_2) \in \mathbf{R}^4 \,|\, x_1^2 + y_1^2 = x_2^2 - y_2^2, x_2 y_2 = 0\}.$$

This can be defined as

$$V_2 = \{(x_1, y_1, x_2, y_2) \in \mathbf{R}^4 \,|\, y_2 = 0, x_2^2 = x_1^2 + y_1^2\}.$$

This curve is real analytically (or real algebraically) irreducible at the origin (see [2] for the definition) but we can see that $V_2 \setminus \{O\}$ has two connected components $z_2 = |z_1|$ and $z_2 = -|z_1|$. Thus for the geometrical study of real analytic varieties, especially for the study of real analytic curves, it is better to see the connected components of $f^{-1}(0) \setminus \{O\}$. We apply the same toric modification $\hat{\pi}_1$ and we consider its strict transform on the toric chart $\mathrm{Cone}(P, E_2)$ where we use the same notation as in Example 27.

$$\hat{V}_2 : \bar{u}_1 - u_1 u_2^2 = 0.$$

Again we see that $\hat{V}_2 \cap \hat{E}(P) = \{(0, u_2) \mid |u_2| = 1\}$. Take the normal real blowing up along $\hat{E}(P)$. The strict transform is defined in $(\mathbf{C}^2_\sigma)_\varepsilon$ as

$$\begin{aligned}\tilde{V}_2 &= \{(\tilde{x}_1, t_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \mid (1 - it_1) - (1 + it_1)u_2^2 = 0\} \\ &= \{(\tilde{y}_1, s_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \mid (s_1 - i) - (s_1 + i)u_2^2 = 0\}\end{aligned}$$

which is non-singular. They have two real analytic components:

$$\{(\tilde{x}_1, t_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \mid u_2 \pm (1 - it_1)/\sqrt{1 + t_1^2} = 0\} \quad \text{or}$$
$$\{(\tilde{y}_1, s_1, u_2) \in \mathbf{R}^2 \times \mathbf{C} \mid u_2 \pm (s_1 - i)/\sqrt{s_1^2 + 1} = 0\}$$

Note that $\sqrt{1 + t_1^2}$ is a real analytic function, although $\sqrt{x_1^2 + y_1^2}$ is not an analytic function at O. The above expression says that $\tilde{V}_2$ is a product

$$\{(t_1, u_2) \mid \sqrt{1 + t_1^2}\, u_2 \pm (1 - it_1) = 0\} \times \mathbf{R}$$

where the second factor is the line with coordinate $\tilde{x}_1$. Using the resolution of a polar type, $\tilde{V}_2$ is simply written as

$$\tilde{V}_2 = \{(r_1, \theta_1, u_2) \in \mathbf{R}^+ \times S_1 \times \mathbf{C} \mid u_2 \pm \exp(-\theta_1 i) = 0\}.$$

Again we observe that it is a product of torus knots and $\mathbf{R}^+$.

3. Next we consider $V_3 = \{z_2^2 + z_1^2 \bar{z}_1 = 0\}$. The Newton boundary is the same with that of the cusp singularity $z_2^2 + z_1^3 = 0$. Thus we use the toric modification $\hat{\pi}_2 : X_2 \to \mathbf{C}^2$. Let (u_1, u_2) be the toric coordinate of the chart $\sigma = (Q, R) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Then the pull back of f is defined in this coordinate chart as

$$\hat{f}(u_1, u_2) = (u_1^3 u_2^2)^2 + (u_1^2 u_2)^2 (\bar{u}_1^2 \bar{u}_2) = u_1^4 u_2^2 (u_1^2 u_2^2 + \bar{u}_1^2 \bar{u}_2)$$

Thus the strict transform can be written as $u_1^2 u_2^2 + \bar{u}_1^2 \bar{u}_2 = 0$. Thus again we see that $\hat{V}_3 \cap \hat{E}(Q) = \{(0, u_2) \mid |u_2| = 1\}$ and S^1 appears as the limit of $\hat{V}_3 \cap \hat{E}(Q)$ where $\hat{E}(Q)$ is the exceptional divisor corresponding to Q. We take a normal polar modification $\omega_p : \mathscr{P}X_2 \to X_2$ and consider this in the coordinate chart $\omega_p^{-1}(\mathbf{C}^2_\sigma)$ with coordinates $(r_1, \exp(\theta_1 i), r_2, \exp(\theta_2 i))$ with $u_1 = r_1 \exp(\theta_1 i)$, $u_2 = r_2 \exp(\theta_2 i)$. Then $\tilde{V}_3$ is defined by

$$\tilde{V}_3 = \{r_2 \exp(3\theta_2 i) + \exp(-4\theta_1 i) = 0\}$$

which implies that $r_2 = 1$ and $3\theta_2 \equiv -4\theta_1 \mod 2\pi$. We see that $\tilde{V}_3 \cap \tilde{E}(Q)$ is a torus knot but the orientations for θ_1 and θ_2 are reversed.

In the resolution of a real type, the equation is apparently a little complicated. In the chart $\mathbf{C}_{\sigma,0,0}$, $\tilde{V}_3 \cap \tilde{E}(Q)$ is given by $g(t_1, s_2, \tilde{x}_2) = h(t_1, s_2, \tilde{x}_2) = 0$ where

$$g(t_1, s_2, \tilde{x}_2) = \tilde{x}_2 - \tilde{x}_2 s_2^2 - 4\tilde{x}_2 t_1 s_2 - \tilde{x}_2 t_1^2 + \tilde{x}_2 t_1^2 s_2^2 + 1 - 2t_1 s_2 - t_1^2$$

$$h(t_1, s_2, \tilde{x}_2) = -2\tilde{x}_2 s_2 - 2\tilde{x}_2 t_1 + 2\tilde{x}_2 t_1 s_2^2 + 2\tilde{x}_2 t_1^2 s_2 + s_2 + 2t_1 - t_1^2 s_2.$$

Taking the resultant of $g(t_1, s_2, \tilde{x}_2)$ and $h(t_1, s_2, \tilde{x}_2)$ in t_1, we see that $s_2^2\tilde{x}_2^2 + s_2^2 = 1$ which corresponds to $r_2 = 1$ in the polar resolution.

4.5.1. *Pseudo weighted homogeneous hypersurface*. Suppose that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient non-degenerate mixed function, let $\hat{\pi} : X \to \mathbf{C}^n$ be an admissible toric modification and let $\omega_{\mathbf{R}} : \mathscr{R}X \to X$ be a real modification along the exceptional divisors as in Theorem 24. Suppose that for a strictly positive weight P, $f_P(\mathbf{z}, \mathbf{z})$ is a pseudo weighted homogeneous polynomial. Write it as $f_P(\mathbf{z}, \bar{\mathbf{z}}) = Mh(\mathbf{z})$ where M is a mixed monomial and $h(\mathbf{z})$ is a weighted homogeneous polynomial with $h^{-1}(0) \cap \mathbf{C}^{*n}$ being smooth. Take a toric coordinate chart $\sigma = (P_1, \dots, P_n)$ with $P = P_1$. Put $d_M = \mathrm{rdeg}_P\, M$ and $d_h = \mathrm{rdeg}_P\, h$. Then $\mathrm{rdeg}_P\, f = d_M + d_h$. Then the strict transform $\hat{V}$ in the toric coordinates $\mathbf{C}_\sigma^n$ is already non-singular. Using the same notation as in the proof of Theorem 24, we have

$$\hat{\pi}^* f_P(\mathbf{u}_\sigma, \bar{\mathbf{u}}_\sigma) = \hat{\pi}^*(M)\hat{\pi}^* h(\mathbf{u}_\sigma) = M' u_1^{d_h} \tilde{h}(\mathbf{u}'_\sigma)$$

where M' is a mixed monomial and $h(\mathbf{u}')$ is a polynomial of $u_{\sigma 2}, \dots, u_{\sigma n}$. Let $E(P_1) = \{\mathbf{u}' \in \mathbf{C}_\sigma^{n-1} \mid \tilde{h}(\mathbf{u}'_\sigma) = 0\}$ be the exceptional divisor. Then $\hat{V}$ is diffeomorphic to the product $\mathbf{C} \times E(P_1)$. Now we take the normal real modification. The defining equation of the strict transform $\tilde{V}$ in $(\mathbf{C}_\sigma)_0$ is given as $\hat{f}_\sigma(\tilde{x}_1, t_\sigma, \mathbf{u}'_\sigma) = 0$ where

$$\hat{f}_\sigma(\tilde{x}_1, t_\sigma, \mathbf{u}'_\sigma) \equiv \hat{f}_{P,\sigma}(\tilde{x}_1, t_\sigma, \mathbf{u}'_\sigma) \text{ modulo } (\tilde{x}_1)$$

$$\Phi^* f_P(\tilde{x}_1, t_\sigma, \mathbf{u}'_\sigma) = \tilde{x}_1^{d(P_1)} \hat{f}_{P,\sigma}(\tilde{x}_1, t_\sigma, \mathbf{u}'_\sigma)$$

$$\hat{f}_{P,\sigma}(\tilde{x}_1, t_\sigma, \mathbf{u}'_\sigma) = (1 + t_1 i)^a (1 - t_1 i)^b \tilde{h}(u_{\sigma 2}, \dots, u_{\sigma n}).$$

Thus we see that $\tilde{V}$ is a product $\mathscr{R}\mathbf{C} \times E(P_1)$. The modification $\omega_p : \mathscr{P}X \to X$ is simply the polar modification of the trivial factor $\mathbf{C}$.

5. Milnor fibration

In this section, we study the Milnor fibration, assuming that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate convenient mixed function. We have seen in Theorem 19 that there exists a positive number r_0 such that $V = f^{-1}(0)$ is mixed non-singular except at the origin in the ball $B_{r_0}^{2n}$ and the sphere S_r^{2n-1} intersects transversely with V for any $0 < r \le r_0$. The following is a key assertion for which we need the strong non-degeneracy.

LEMMA 28. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly non-degenerate convenient mixed function. For any fixed positive number r_1 with $r_1 \le r_0$, there exists positive numbers $\delta_0 \ll r_1$ such that for any $\eta \neq 0$, $|\eta| \le \delta_0$ and r with $r_1 \le r \le r_0$,* (a) *the fiber $V_\eta := f^{-1}(\eta)$ has no mixed singularity inside the ball $B_{r_0}^{2n}$ and* (b) *the intersection $V_\eta \cap S_r^{2n-1}$ is transverse and smooth.*

Proof. As the assertion (b) follows from the compactness argument, we show the assertion (a) by contradiction. We assume that (a) does not hold. Then using the Curve Selection Lemma ([12, 7]), we can find an analytic path $\mathbf{z}(t)$, $0 \le t \le 1$ such that $\mathbf{z}(0) = O$ and $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \neq 0$ for $t \neq 0$ and $\mathbf{z}(t)$ is a critical point of the function $f : \mathbf{C}^n \to \mathbf{C}$. The proof is similar to that of Theorem 19 as we will see below. Using Proposition 8, we can find a real analytic family $\lambda(t)$ in $S^1 \subset \mathbf{C}$ such that

$$\overline{df}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \lambda(t)\, \bar{d}f(\mathbf{z}(t), \bar{\mathbf{z}}(t)). \tag{3}$$

Put $I = \{j \mid z_j(t) \not\equiv 0\}$. We may assume that $I = \{1, \dots, m\}$ and we consider f^I. As $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = f^I(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \not\equiv 0$, we see that $f^I \neq 0$. Consider the Taylor expansions of $\mathbf{z}(t)$ and $\lambda(t)$:

$$\mathbf{z}_i(t) = b_i t^{a_i} + (\text{higher terms}), \quad b_i \neq 0 \quad i = 1, \dots, m$$
$$\lambda(t) = \lambda_0 + \lambda_1 t + (\text{higher terms}), \quad \lambda_0 \in S^1 \subset \mathbf{C}.$$

Let $A = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_m)$ and we consider the face function f_A^I of $f^I(\mathbf{z}, \bar{\mathbf{z}})$. Then we have

$$\frac{\partial f}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_A^I}{\partial z_j}(\mathbf{b}) t^{d-a_j} + (\text{higher terms}),$$
$$\frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_A^I}{\partial z_j}(\bar{\mathbf{b}}) t^{d-a_j} + (\text{higher terms}), \quad d = d(A; f^I).$$

Observe that we have the following equality for any j by the equality (3):

$$\operatorname{ord}_t \frac{\partial f^I}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \operatorname{ord}_t \frac{\partial f^I}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)).$$

Thus by (3), we get the equality: $\overline{df_A^I}(\mathbf{b}, \bar{\mathbf{b}}) = \lambda_0\, \bar{d}f_A^I(\mathbf{b}, \bar{\mathbf{b}})$ and $\mathbf{b} \in \mathbf{C}^{*n}$. This implies that $\mathbf{b}$ is a critical point of $f_A^I : \mathbf{C}^{*I} \to \mathbf{C}$, which is a contradiction to the strong non-degeneracy of $f^I(\mathbf{z}, \bar{\mathbf{z}})$. □

5.1. Milnor fibration, the second description. Put

$$D(\delta_0)^* = \{\eta \in \mathbf{C} \mid 0 < |\eta| \le \delta_0\}, \quad S_{\delta_0}^1 = \partial D(\delta_0)^* = \{\eta \in \mathbf{C} \mid |\eta| = \delta_0\}$$
$$E(r, \delta_0)^* = f^{-1}(D(\delta_0)^*) \cap B_r^{2n}, \quad \partial E(r, \delta_0)^* = f^{-1}(S_{\delta_0}^1) \cap B_r^{2n}.$$

By Lemma 28 and the theorem of Ehresman ([24]), we obtain the following description of the Milnor fibration of the second type ([8]).

THEOREM 29 (The second description of the Milnor fibration). *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a convenient, strongly non-degenerate mixed function. Take positive numbers r_0, r_1 and δ_0 such that $r \le r_0$ and $\delta_0 \ll r_1$ as in Lemma 28. Then $f : E(r,\delta_0)^* \to D(\delta_0)^*$ and $f : \partial E(r,\delta_0)^* \to S^1_{\delta_0}$ are locally trivial fibrations and the topological isomorphism class does not depend on the choice of δ_0 and r.*

5.2. Milnor fibration, the first description. We consider now the original Milnor fibration on the sphere, which is defined as follows:

$$\varphi : S_r^{2n-1}\backslash K_r \to S^1, \quad \mathbf{z} \mapsto \varphi(\mathbf{z}) = f(\mathbf{z},\bar{\mathbf{z}})/|f(\mathbf{z},\bar{\mathbf{z}})|$$

where $K_r = V \cap S_r^{2n-1}$. The fibrations of this type for mixed functions and related topics have been studied by many authors ([20, 21, 5, 22, 19, 3]). But most of the works treat rather special classes of functions. The mapping φ can be identified with $\varphi(\mathbf{z}) = -\Re(i \log f(\mathbf{z}))$, taking the argument θ as a local coordinate of the circle S^1. We use the basis $\left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \,\middle|\, j = 1,\dots,n \right\}$ of the tangent space $T_{\mathbf{z}}\mathbf{C}^n \otimes \mathbf{C}$. For a mixed function $g(\mathbf{z},\bar{\mathbf{z}})$, we use two complex "gradient vectors" defined by

$$dg = \left(\frac{\partial g}{\partial z_1}, \dots, \frac{\partial g}{\partial z_n} \right), \quad \bar{d}g = \left(\frac{\partial g}{\partial \bar{z}_1}, \dots, \frac{\partial g}{\partial \bar{z}_n} \right).$$

Take a smooth path $\mathbf{z}(t)$, $-1 \le t \le 1$ with $\mathbf{z}(0) = \mathbf{w} \in \mathbf{C}^n \backslash V$ and put $\mathbf{v} = \frac{d\mathbf{z}}{dt}(0) \in T_{\mathbf{w}}\mathbf{C}^n$. Then we have

$$\begin{aligned}
&-\frac{d}{dt}(\Re(i \log f(\mathbf{z}(t),\bar{\mathbf{z}}(t)))_{t=0} \\
&\quad = -\Re\left(\sum_{i=1}^n i \left\{ \frac{\partial f}{\partial z_j}(\mathbf{w},\bar{\mathbf{w}}) \frac{dz_j}{dt}(0) + \frac{\partial f}{\partial \bar{z}_j}(\mathbf{w},\bar{\mathbf{w}}) \frac{d\bar{z}_j}{dt}(0) \right\} \Big/ f(\mathbf{w},\bar{\mathbf{w}}) \right) \\
&\quad = \Re(\mathbf{v}, i\overline{d \log f}(\mathbf{w},\bar{\mathbf{w}})) + \Re(\bar{\mathbf{v}}, i\overline{\bar{d} \log f}(\mathbf{w},\bar{\mathbf{w}})) \\
&\quad = \Re(\mathbf{v}, i\overline{d \log f}(\mathbf{w},\bar{\mathbf{w}})) + \Re(\mathbf{v}, -i\bar{d} \log f(\mathbf{w},\bar{\mathbf{w}})) \\
&\quad = \Re(\mathbf{v}, i(\overline{d \log f} - \bar{d} \log f)(\mathbf{w},\bar{\mathbf{w}})).
\end{aligned}$$

Namely we have

$$-\frac{d}{dt}(\Re(i \log f(\mathbf{z}(t),\bar{\mathbf{z}}(t)))_{t=0} = \Re(\mathbf{v}, i(\overline{d \log f} - \bar{d} \log f)(\mathbf{w},\bar{\mathbf{w}})). \tag{4}$$

Thus by the same argument as in Milnor [12], we get

LEMMA 30. *A point* $\mathbf{z} \in S_r^{2n-1} \backslash K_r$ *is a critical point of* φ *if and only if the two complex vectors* $i(\overline{d \log f(\mathbf{z}, \bar{\mathbf{z}})} - \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}}))$ *and* $\mathbf{z}$ *are linearly dependent over* $\mathbf{R}$.

The key assertion is the following.

LEMMA 31. *Assume that* $f(\mathbf{z}, \bar{\mathbf{z}})$ *is a strongly non-degenerate mixed function. Then there exists a positive number* r_0 *such that the two complex vectors* $i(\overline{d \log f(\mathbf{z}, \bar{\mathbf{z}})} - \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}}))$ *and* $\mathbf{z} \in S_r \backslash K_r$ *are linearly independent over* $\mathbf{R}$ *for any* r *with* $0 < r \le r_0$.

Proof. We do not assume the convenience of $f(\mathbf{z}, \bar{\mathbf{z}})$ for this lemma. We proceed as the proof of Lemma 4.3 [12]. Assuming the contrary, we can find an analytic path $\mathbf{z}(t)$, $0 \le t \le 1$ such that

(a) $\mathbf{z}(0) = O$ and $\mathbf{z}(t) \in \mathbf{C}^n \backslash V$ for $t > 0$.

(b) $i(\overline{d \log f} - \bar{d} \log f)(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \lambda(t)\mathbf{z}(t)$ for some $\lambda(t)$ such that $\lambda(t)$ is a real number.

As $\overline{d \log f} - \bar{d} \log f$ does not vanish outside of $f^{-1}(0)$ near the origin by Lemma 28 and Proposition 8, we see that $\lambda(t) \not\equiv 0$. Consider the subset $I = \{j \mid z_j(t) \not\equiv 0\} \subset \{1, \dots, n\}$. For simplicity, we may assume that $I = \{1, \dots, m\}$. Consider the Taylor expansions:

$$z_j(t) = a_j t^{p_j} + (\text{higher terms}), \quad a_j \ne 0,\ p_j > 0,\ j \in I.$$

Put $P = {}^t(p_1, \dots, p_m)$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbf{C}^{*I}$ and $d = d(P; f^I)$. Then we consider the expansions:

$$f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = f^I(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \alpha t^q + (\text{higher terms}), \quad q \ge d,\ \alpha \ne 0$$

$$\frac{\partial f^I}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \frac{\partial f_P^I}{\partial z_j}(\mathbf{a}, \bar{\mathbf{a}}) t^{d-p_j} + (\text{higher terms}), \quad 1 \le j \le m$$

$$\frac{\partial f^I}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t))) = \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}}) t^{d-p_j} + (\text{higher terms}), \quad 1 \le j \le m$$

$$\lambda(t) = \lambda_0 t^s + (\text{higher terms}), \quad \lambda_0 \in \mathbf{R}^*.$$

The assumption (b) implies that for $1 \le j \le m$,

$$i\left(\overline{\frac{\partial f_P^I}{\partial z_j}}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} - \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha\right) = \begin{cases} 0 & d - p_j - q < s + p_j \\ \lambda_0 a_j, & d - p_j - q = s + p_j. \end{cases}$$

Define $J \subset \{1, \dots, m\}$ by $J := \{j \mid d - p_j - q = s + p_j\}$. Assume that $J = \emptyset$. Then we have the equality

$$\overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}}) = \frac{\bar{\alpha}}{\alpha} \times \bar{d} f^I(\mathbf{a}, \bar{\mathbf{a}}), \quad j \le m$$

which implies $f_P^I : \mathbf{C}^{*I} \to \mathbf{C}$ has a critical point at $\mathbf{z} = \mathbf{a}$ by Proposition 8. This is a contradiction to the strong non-degeneracy. Thus we have shown that

$J \neq \emptyset$. We consider the differential:

$$\frac{d}{dt} f^I(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \sum_{j=1}^{m} \frac{\partial f^I}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \frac{dz_j(t)}{dt} + \sum_{j=1}^{m} \frac{\partial f^I}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \frac{d\bar{z}_j(t)}{dt}$$
$$= q\alpha t^{q-1} + (\text{higher terms}).$$

The two terms of the right side of the first row can be written as follows.

$$\left(\frac{d\mathbf{z}(t)}{dt}, \overline{df^I}(\mathbf{z}(t), \bar{\mathbf{z}}(t))\right) = (P\mathbf{a}, \overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}}))t^{d-1} + (\text{higher terms}),$$
$$\left(\frac{d\bar{\mathbf{z}}(t)}{dt}, \overline{\bar{d}f^I}(\mathbf{z}(t), \bar{\mathbf{z}}(t)\right) = (P\bar{\mathbf{a}}, \overline{\bar{d}f_P^I}(\mathbf{a}, \bar{\mathbf{a}}))t^{d-1} + (\text{higher terms})$$

where $P\mathbf{a} = (p_1 a_1, \dots, p_m a_m)$ and $P\bar{\mathbf{a}} = (p_1 \bar{a}_1, \dots, p_m \bar{a}_m)$. Thus we get
$q\alpha t^{q-1} + (\text{higher terms}) = ((P\mathbf{a}, \overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}})) + (P\bar{\mathbf{a}}, \overline{\bar{d}f_P^I}(\mathbf{a}, \bar{\mathbf{a}})))t^{d-1} + (\text{higher terms})$.
Observe that

$$\Re\left(P\mathbf{a}, i\frac{\overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}})}{\bar{\alpha}}\right) + \Re\left(P\bar{\mathbf{a}}, i\frac{\overline{\bar{d}f_P^I}(\bar{\mathbf{a}}, \bar{\mathbf{a}})}{\bar{\alpha}}\right)$$
$$= \Re\left(P\mathbf{a}, i\frac{\overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}})}{\bar{\alpha}} - i\frac{\bar{d}f_P^I(\bar{\mathbf{a}}, \bar{\mathbf{a}})}{\alpha}\right)$$
$$= \Re\left(\sum_{j \in J} \lambda_0 |a_j|^2 p_j\right) = \lambda_0 \sum_{j \in J} |a_j|^2 p_j \neq 0$$

as $J \neq \emptyset$. Thus we see that

$$(P\mathbf{a}, \overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}})) + (P\bar{\mathbf{a}}, \overline{\bar{d}f_P^I}(\mathbf{a}, \bar{\mathbf{a}}))$$
$$= \alpha i\left(\left(P\mathbf{a}, i\frac{\overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}})}{\bar{\alpha}}\right) + \left(P\bar{\mathbf{a}}, i\frac{\overline{\bar{d}f_P^I}(P\bar{\mathbf{a}}, \bar{\mathbf{a}})}{\bar{\alpha}}\right)\right) \neq 0.$$

This implies that $q = d$ (namely $f_P^I(\mathbf{a}, \bar{\mathbf{a}}) \neq 0$) and

$$q\alpha = (P\mathbf{a}, \overline{df_P^I}(\mathbf{a}, \bar{\mathbf{a}})) + (P\bar{\mathbf{a}}, \overline{\bar{d}f_P^I}(\mathbf{a}, \bar{\mathbf{a}})), \quad \text{or}$$
$$qi = \left(P\mathbf{a}, i\frac{\overline{\bar{d}f_P^I}(\mathbf{a}, \bar{\mathbf{a}})}{\bar{\alpha}}\right) + \left(P\bar{\mathbf{a}}, i\frac{\overline{\bar{d}f_P^I}(\bar{\mathbf{a}}, \bar{\mathbf{a}})}{\bar{\alpha}}\right).$$

Taking the real part of the last equality, we get an obvious contradiction:

$$0 = \Re\left(\sum_{j \in J} \lambda_0 |a_j|^2 p_j\right) = \sum_{j \in J} \lambda_0 |a_j|^2 p_j \neq 0. \qquad \square$$

OBSERVATION 32. *Let* $\mathbf{w} \in f^{-1}(\eta)$, $\eta \neq 0$ *be a smooth point. Then the tangent space* $T_{\mathbf{w}} f^{-1}(\eta)$ *is the real subspace of* $\mathbf{C}^n$ *whose vectors are orthogonal in* $\mathbf{R}^{2n}$ *to the two vectors*

$$i(\overline{d \log f} - \bar{d} \log f)(\mathbf{w}, \overline{\mathbf{w}}), \quad (\overline{d \log f} + \bar{d} \log f)(\mathbf{w}, \overline{\mathbf{w}}).$$

Proof. Assume that $\mathbf{z}(t)$, $-\varepsilon \leq t \leq \varepsilon$ is a smooth curve in $f^{-1}(\eta)$ with $\mathbf{z}(0) = \mathbf{w}$ and $\mathscr{V}(\mathbf{w}) = \mathbf{v} \in T_{\mathbf{w}} f^{-1}(\eta)$. The assertion follows from the next calculation.

$$\begin{aligned} \frac{d}{dt} \log f(\mathbf{z}(t), \overline{\mathbf{z}}(t)) &= \left. \frac{\Re \log f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{dt} \right|_{t=0} - \frac{d}{dt} (\Re(i \log f(\mathbf{z}(t), \overline{\mathbf{z}}(t)))_{t=0} \\ &= \Re(\mathbf{v}, (\overline{d \log f} + \bar{d} \log f)(\mathbf{w}, \overline{\mathbf{w}})) \\ &\quad + \Re(\mathbf{v}, i(\overline{d \log f} - \bar{d} \log f)(\mathbf{w}, \overline{\mathbf{w}})). \end{aligned}$$

□

Now we are ready to prove the existence of the Milnor fibration of the first description.

THEOREM 33 (Milnor fibration, the first description). *Let* $f(\mathbf{z}, \overline{\mathbf{z}})$ *be a strongly non-degenerate convenient mixed function. There exists a positive number* r_0 *such that*

$$\varphi = f/|f| : S_r^{2n-1} \backslash K_r \to S^1$$

is a locally trivial fibration for any r *with* $0 < r \leq r_0$.

Proof. Taking r_0, r_1, δ_0 sufficiently small so that $f^{-1}(\eta)$ and S_r^{2n-1} intersect transversely for any $\eta \in \mathbf{C}^*$ with $|\eta| \leq \delta_0$ and $r_1 \leq r \leq r_0$ by Lemma 28. Combining with Observation 32, the transversality implies that the three vectors

$$\mathbf{z}, \ i(\overline{d \log f} - \bar{d} \log f)(\mathbf{z}, \overline{\mathbf{z}}), \quad (\overline{d \log f} + \bar{d} \log f)(\mathbf{z}, \overline{\mathbf{z}})$$

are linearly independent over $\mathbf{R}$ on $\{\mathbf{z} \in S_r \mid 0 < |f(\mathbf{z}, \overline{\mathbf{z}})| \leq \delta_0\}$. Therefore we can construct a horizontal vector field $\mathscr{V}$ for φ on $S_r^{2n-1} \backslash K_r$ so that

(1) $\Re(\mathscr{V}(\mathbf{z}), i(\overline{d \log f} - \bar{d} \log f)(\mathbf{z}, \overline{\mathbf{z}})) = 1$ and $\Re(\mathscr{V}(\mathbf{z}), \mathbf{z}) = 0$ for any $\mathbf{z} \in S_r^{2n-1} - K_r$, and moreover

(2) $\Re(\mathscr{V}(\mathbf{z}), (\overline{d \log f} + \bar{d} \log f)(\mathbf{z}, \overline{\mathbf{z}})) = 0$ for $\mathbf{z} \in S_r$ with $|0 < |f(\mathbf{z}, \overline{\mathbf{z}})| \leq \delta_0$.

We show that the integral curve of $\mathscr{V}$ does not approach to K_r. In fact, assume that $\mathbf{z}(t)$, $-\varepsilon \leq t \leq \varepsilon$ be an integral curve with $\mathbf{z}(0) = \mathbf{w}$, $\mathscr{V}(\mathbf{w}) = \mathbf{v}$. As we have seen in Observation 32,

$$\frac{d}{dt} \log |f(\mathbf{z}(t), \overline{\mathbf{z}}(t))| = \Re(\mathbf{v}, (\overline{d \log f} + \bar{d} \log f)(\mathbf{w}, \overline{\mathbf{w}})). \tag{5}$$

Therefore the condition (2) guarantees that $\mathscr{V}(\mathbf{z})$ is tangent to the level real hypersurface of real codimension 1, $|f|_{\mathbf{z}} := \{\mathbf{w} \in \mathbf{C}^n \mid |f(\mathbf{w})| = |f(\mathbf{z})|\}$. Thus it is obvious that $\mathscr{V}$ is integrable for any finite time interval and we get the local triviality by the integration of $\mathscr{V}$. $\square$

5.3. Equivalence of two Milnor fibrations. Take positive numbers r, δ_0 with $\delta_0 \ll r$ as in Theorem 29. We compare the two fibrations

$$f : \partial E(r, \delta_0) \to S^1_{\delta_0}, \qquad \varphi : S^{2n-1}_r \backslash K_r \to S^1$$

and we will show that they are isomorphic. However the proof is much more complicated compared with the case of holomorphic functions. The reason is that we have to take care of the two vectors

$$i(\overline{d \log f} - \bar{d} \log f), \quad \overline{d \log f} + \bar{d} \log f$$

which are not perpendicular. (In the holomorphic case, the proof is easy as the two vectors reduce to the perpendicular vectors $i\overline{d \log f}, \overline{d \log f}$.) Consider a smooth curve $\mathbf{z}(t)$, $-1 \le t \le 1$, with $\mathbf{z}(0) = \mathbf{w} \in B^{2n}_r \backslash V$ and $\mathbf{v} = \frac{d\mathbf{z}(t)}{dt}(0)$. Put $\mathbf{v} = (v_1, \dots, v_n)$. First from (4) and (5), we observe that

$$\begin{aligned} \left.\frac{\log f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt}\right|_{t=0} &= \sum_{j=1}^{n} \left(v_j \frac{\partial \log f}{\partial z_j}(\mathbf{w}, \bar{\mathbf{w}}) + \bar{v}_j \frac{\partial \log f}{\partial \bar{z}_j}(\mathbf{w}, \bar{\mathbf{w}}) \right) \\ &= \Re(\mathbf{v}, (\overline{d \log f} + \bar{d} \log f)(\mathbf{w}, \bar{\mathbf{w}})) \\ &\quad + i\Re(\mathbf{v}, i(\overline{d \log f} - \bar{d} \log f)(\mathbf{w}, \bar{\mathbf{w}})). \end{aligned}$$

Define two vectors on $\mathbf{C}^n - V$:

$$\begin{aligned} \mathbf{v}_1(\mathbf{z}, \bar{\mathbf{z}}) &= \overline{d \log f}(\mathbf{z}, \bar{\mathbf{z}}) + \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}}) \\ \mathbf{v}_2(\mathbf{z}, \bar{\mathbf{z}}) &= i(\overline{d \log f}(\mathbf{z}, \bar{\mathbf{z}}) - \bar{d} \log f(\mathbf{z}, \bar{\mathbf{z}})) \end{aligned}$$

The above equality is translated as

$$\left.\frac{\log f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt}\right|_{t=0} = \Re(\mathbf{v}, \mathbf{v}_1(\mathbf{w}, \bar{\mathbf{w}})) + i\Re(\mathbf{v}, \mathbf{v}_2(\mathbf{w}, \bar{\mathbf{w}})). \tag{6}$$

The following will play the key role for the equivalence of two fibrations:

Lemma 34. *Under the same assumption as in Theorem* 33, *there exists a positive number r_0 so that for any $\mathbf{z}$ with $\|\mathbf{z}\| \le r_0$ and $f(\mathbf{z}, \bar{\mathbf{z}}) \ne 0$, the three vectors*

$$\mathbf{z}, \quad \mathbf{v}_1(\mathbf{z}, \bar{\mathbf{z}}), \quad \mathbf{v}_2(\mathbf{z}, \bar{\mathbf{z}})$$

are either (i) *linearly independent over* $\mathbf{R}$ *or* (ii) *they are linearly dependent over* $\mathbf{R}$ *and the relation can be written as*

$$\mathbf{z} = a\mathbf{v}_1(\mathbf{z}, \bar{\mathbf{z}}) + b\mathbf{v}_2(\mathbf{z}, \bar{\mathbf{z}}), \quad a, b \in \mathbf{R}. \tag{7}$$

and the coefficient a is positive.

Proof. First observe that the pairs

$$\mathscr{P}_1 = \{\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}}), \mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})\}, \quad \mathscr{P}_2 = \{\mathbf{z}, \mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})\}$$

are respectively linearly independent over $\mathbf{R}$ by Lemma 28, Lemma 31 and the above equality. Assume that the assertion does not hold. Consider the real analytic variety W where the three vectors are linearly dependent over $\mathbf{R}$. Let us consider the open set $U = \mathbf{C}^n \backslash V$. Then $W \cap U$ has a finite number of connected components. The sign of the coefficient a in (7) is constant on each component, as long as they are near enough to the origin. This is the result of the linear independence of $\mathbf{z}$, $\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})$. We will show that this sign is positive. We use the Curve Selection Lemma ([12, 7]) to find an analytic curve $\mathbf{z}(t)$, $0 \le t \le 1$, such that $\mathbf{z}(0) = O$ and $\mathbf{z}(t) \notin V$ for $t \neq 0$ and there exist real valued functions $\lambda(t)$, $\mu(t)$ so that

$$\mathbf{z}(t) = \lambda(t)\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}}) + \mu(t)\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}}).$$

Let $I = \{j \mid z_j(t) \not\equiv 0\}$. We may assume that $I = \{1,\dots,m\}$ and we do the argument in $\mathbf{C}^I$. We consider the Taylor expansions of $\mathbf{z}(t)$ and $f(\mathbf{z}(t),\bar{\mathbf{z}}(t))$, and the Laurent expansions of $\lambda(t)$ and $\mu(t)$:

$$\begin{aligned}
&z_j(t) = a_j t^{p_j} + (\text{higher terms}), \quad a_j \in \mathbf{C}^*,\ p_j \in \mathbf{N},\ 1 \le j \le m,\\
&f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = \alpha t^{\ell} + (\text{higher terms}), \quad \alpha \in \mathbf{C}^*,\ \ell \in \mathbf{N}\\
&\lambda(t) = \lambda_0 t^{\nu_1} + (\text{higher terms}), \quad \lambda_0 \in \mathbf{R}^*,\ \nu_1 \in \mathbf{Z}\\
&\mu(t) = \mu_0 t^{\nu_2} + (\text{higher terms}), \quad \mu_0 \in \mathbf{R}^*,\ \nu_2 \in \mathbf{Z}.
\end{aligned}$$

First we consider the equality:

$$\begin{aligned}
(8) \qquad z_j(t) &= \lambda(t)\left(\overline{\frac{\partial f}{\partial z_j}}\Big/\bar{f} + \frac{\partial f}{\partial \bar{z}_j}\Big/f\right)(\mathbf{z}(t),\bar{\mathbf{z}}(t))\\
&\quad + \mu(t)i\left(\overline{\frac{\partial f}{\partial z_j}}\Big/\bar{f} - \frac{\partial f}{\partial \bar{z}_j}\Big/f\right)(\mathbf{z}(t),\bar{\mathbf{z}}(t)), \quad j = 1,\dots,m.
\end{aligned}$$

Put $P = (p_1,\dots,p_m)$, $\mathbf{a} = (a_1,\dots,a_m)$ and $d = d(P, f^I)$. Then we observe that

$$\begin{aligned}
&\overline{\frac{\partial f}{\partial z_j}}(\mathbf{z}(t),\bar{\mathbf{z}}(t))/\bar{f}(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = \left(\overline{\frac{\partial f_P^I}{\partial z_j}}(\mathbf{a},\bar{\mathbf{a}})/\bar{\alpha}\right)t^{d-p_j-\ell} + (\text{higher terms}),\\
&\frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}(t),\bar{\mathbf{z}}(t)))/f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = \left(\frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a},\bar{\mathbf{a}})/\alpha\right)t^{d-p_j-\ell} + (\text{higher terms})
\end{aligned}$$

Thus comparing the equality (8), we see that

$$p_j \ge \min\{\nu_1 + d - p_j - \ell, \nu_2 + d - p_j - \ell\}.$$

To avoid the repetition of the similar argument and to treat the cases $\nu_2 = \nu_1$, $\nu_2 < \nu_1$ and $\nu_2 > \nu_1$ simultaneously, we put $\nu_0 = \min(\nu_1, \nu_2)$ and we rewrite the expansions as $\lambda(t)$, $\mu(t)$ as

$$\lambda(t) = r_0 t^{\nu_0} + \cdots, \quad r_0 \in \mathbf{R}$$
$$\mu(t) = m_0 t^{\nu_1} + \cdots, \quad m_0 \in \mathbf{R}.$$

Here we have $r_0 = 0$ or $r_0 = \lambda_0$ (respectively $m_0 = 0$ or $m_0 = \mu_0$) according to $\nu_1 > \nu_0$ or $\nu_1 = \nu_0$ (resp. $\nu_2 > \nu_0$ or $\nu_2 = \nu_0$). By (8), we get

$$\lambda_0 \left(\overline{\frac{\partial f_P^I}{\partial z_j}}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} + \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha \right) + i m_0 \left(\overline{\frac{\partial f_P^I}{\partial z_j}}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} - \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha \right)$$
$$= \begin{cases} a_j & p_j = d - p_j + \nu_0 - \ell \\ 0 & p_j > d - p_j + \nu_0 - \ell. \end{cases}$$

More precisely we assert

Assertion 35. *Put $p_{min} = \min\{p_i \mid i \in I\}$ and $K = \{i \in I \mid p_i = p_{min}\}$. Then we have*

$$\lambda_0 \left(\overline{\frac{\partial f_P^I}{\partial z_j}}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} + \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha \right) + m_0 i \left(\overline{\frac{\partial f_P^I}{\partial z_j}}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} - \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha \right) \tag{9}$$
$$= \begin{cases} a_j & j \in K \\ 0 & j \notin K. \end{cases}$$

Proof. We examine the equality (8). The order of $\|\mathbf{z}(t)\|$ is p_{min}. On the other hand, the order of j-th component of the right side of (8) is greater than or equal to $d - p_j + \nu_0 - \ell$ and the coefficient of $t^{d-p_j+\nu_0-\ell}$ is given by the left side of (9). If there is an index $j \notin K$ such that this coefficient is non-zero, then the order of the right side of (8) is strictly smaller than $d - p_{min} + \nu_0 - \ell$ and the limit of the normalized vector of the right side has 0 coefficient on any $j \in K$ and we have the contradiction to (8). $\square$

Thus we have proved (9). Now we examine the next equality more carefully:

$$\frac{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt} = \sum_{j=1}^{n} \left(\frac{\partial f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{\partial z_j} \frac{dz_j(t)}{dt} + \frac{\partial f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{\partial \bar{z}_j} \frac{d\bar{z}_j(t)}{dt} \right). \tag{10}$$

The left hand side is simply

$$\frac{df(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{dt} = \alpha \ell t^{\ell-1} + (\text{higher terms}).$$

We introduce the complex vectors:

$$\begin{cases} \mathbf{v} = (v_1, \dots, v_m), v_j = \sqrt{p_j} f_j, \\ \mathbf{w} = (w_1, \dots, w_m), w_j = \sqrt{p_j} f_{\bar{j}} \end{cases}$$

$$\text{where } f_j = \frac{\partial f_P^I}{\partial z_j}(\mathbf{a}, \bar{\mathbf{a}}),\ f_{\bar{j}} = \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}}),\ 1 \le j \le m.$$

The order of the right hand side of (10) is greater than or equal to $d-1$. Let R be the coefficient of t^{d-1} of the right side. By an easy calculation, we have

$$\begin{aligned} R &= \sum_{j=1}^m a_j p_j f_j + \sum_{j=1}^m \bar{a}_j p_j f_{\bar{j}} \\ &= \sum_{j=1}^m p_j f_j \left\{ \lambda_0 \left(\frac{\overline{f_j}}{\bar{\alpha}} + \frac{f_{\bar{j}}}{\alpha} \right) + i m_0 \left(\frac{\overline{f_j}}{\bar{\alpha}} - \frac{f_{\bar{j}}}{\alpha} \right) \right\} \\ &\quad + \sum_{j=1}^m p_j f_{\bar{j}} \left\{ \lambda_0 \left(\frac{f_j}{\alpha} + \frac{\overline{f_{\bar{j}}}}{\bar{\alpha}} \right) - i m_0 \left(\frac{f_j}{\alpha} - \frac{\overline{f_{\bar{j}}}}{\bar{\alpha}} \right) \right\} \\ &= \alpha \sum_{j=1}^m (p_j |f_j|^2 + p_j |f_{\bar{j}}|^2) \left(\frac{\lambda_0}{|\alpha|^2} + \frac{i m_0}{|\alpha|^2} \right) + \alpha \sum_{j=1}^m 2 p_j f_j f_{\bar{j}} \left(\frac{\lambda_0}{\alpha^2} - \frac{i m_0}{\alpha^2} \right) \\ &= \alpha (\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2) \left(\frac{\lambda_0}{|\alpha|^2} + \frac{i m_0}{|\alpha|^2} \right) + 2\alpha (\mathbf{w}, \bar{\mathbf{v}}) \left(\frac{\lambda_0}{\alpha^2} - \frac{i m_0}{\alpha^2} \right) \end{aligned}$$

Consider two complex numbers:

$$\beta := (\|\bar{\mathbf{v}}\|^2 + \|\mathbf{w}\|^2) \left(\frac{\lambda_0}{|\alpha|^2} + i \frac{m_0}{|\alpha|^2} \right), \quad \gamma := 2(\mathbf{w}, \bar{\mathbf{v}}) \left(\frac{\lambda_0}{\alpha^2} - i \frac{m_0}{\alpha^2} \right)$$

Using the Schwartz inequality, we see that

$$|\gamma| = 2 \frac{\sqrt{\lambda_0^2 + m_0^2}}{|\alpha|^2} |(\mathbf{w}, \bar{\mathbf{v}})| \le 2 \frac{\sqrt{\lambda_0^2 + m_0^2}}{|\alpha|^2} \|\bar{\mathbf{v}}\| \, \|\mathbf{w}\|$$

and by comparing with $|\beta|$, we get

$$|\beta| - |\gamma| \ge \left. \frac{\sqrt{\lambda_0^2 + m_0^2}}{|\alpha|^2} \right| (\|\bar{\mathbf{v}}\| - \|\mathbf{w}\|)^2 \ge 0.$$

For the equality $|\beta| = |\gamma|$, it is necessary that $\|\bar{\mathbf{v}}\| = \|\mathbf{w}\|$ and $|(\mathbf{w}, \bar{\mathbf{v}})| = \|\bar{\mathbf{v}}\| \times \|\mathbf{w}\|$, or

$$\mathbf{w} = u \bar{\mathbf{v}}, \quad \exists u \in S^1 \subset \mathbf{C}.$$

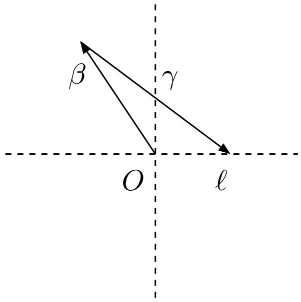

FIGURE 2. If $\lambda_0 \le 0$, $|\beta| < |\gamma|$

Note that this is equivalent to $(f_{\bar{1}}, \dots, f_{\bar{m}}) = u(\overline{f_1}, \dots, \overline{f_m})$ which implies $\bar{d}f(\mathbf{a}, \bar{\mathbf{a}}) = u\,\overline{df}(\mathbf{a}, \bar{\mathbf{a}})$. This is a contradiction to the non-degeneracy assumption for $f^I(\mathbf{z}, \bar{\mathbf{z}})$. Thus we conclude that $|\beta| > |\gamma|$ and $R \neq 0$.

Now the equality (10) says, $\ell - 1 = d - 1$ and

$$\ell\alpha = \alpha\beta + \alpha\gamma, \quad \text{or}$$

$$\ell = (\|\bar{\mathbf{v}}\|^2 + \|\mathbf{w}\|^2)\left(\frac{\lambda_0}{|\alpha|^2} + i\frac{m_0}{|\alpha|^2}\right) + 2(\mathbf{w}, \bar{\mathbf{v}})\left(\frac{\lambda_0}{\alpha^2} - i\frac{m_0}{\alpha^2}\right)$$

We now assert that $\lambda_0 > 0$. Assume $\lambda_0 \le 0$. Then $\Re(\beta) \le 0$ and to get the equality $\ell = \beta + \gamma$, we must have $|\gamma| > |\beta|$. This is impossible as we have seen that $|\beta| > |\gamma|$. See Figure 2. □

Now we are ready to prove the isomorphism theorem:

THEOREM 36. *Under the same assumption as in Theorem* 33, *the two fibrations*

$$f : \partial E(r, \delta_0) \to S^1_{\delta_0}, \quad \varphi : S_r^{2n-1} \backslash K_r \to S^1$$

are topologically isomorphic.

Proof. The proof is done as in the case of Milnor fibrations of a holomorphic function ([16]). We will construct a vector field $\mathscr{V}$ on

$$E^c(r, \delta_0) := B_r \backslash \mathrm{Int}(E(r, \delta_0)) = \{\mathbf{z} \in B_r \mid |f(\mathbf{z}, \bar{\mathbf{z}})| \ge \delta_0\}$$

so that

$$\begin{cases} \Re(\mathscr{V}(\mathbf{z}), \mathbf{v}_2(\mathbf{z}, \bar{\mathbf{z}})) = 0, \\ \Re(\mathscr{V}(\mathbf{z}), \mathbf{v}_1(\mathbf{z}, \bar{\mathbf{z}})) > 0, \\ \Re(\mathscr{V}(\mathbf{z}), \mathbf{z}) > 0. \end{cases} \tag{11}$$

Assume for a moment that we have constructed such a vector field. Along the integral curve $h(t, \mathbf{w})$ of $\mathscr{V}$ with $h(0, \mathbf{w}) = \mathbf{w}$, the argument of $f(h(t, \mathbf{w}), \bar{h}(t, \mathbf{w}))$ is constant and the absolute value $|f(h(t,\mathbf{w}))|$ and the norm $\|h(t,\mathbf{w})\|$ are monotone increasing. The integral curve is well-defined as long as $h(t,\mathbf{w})$ is inside $E^c(r,\delta_0)$. For each $\mathbf{w} \in E^c(r,\delta_0)$, there exists a unique $\tau(\mathbf{w})$ such that $\|h(\tau(\mathbf{w}))\| = r$. Thus this gives a topological isomorphism $\psi : \partial E(r,\delta_0) \to S_r^{2n-1} \backslash N_r$ which is defined by $\psi(\mathbf{w}) = h(\tau(\mathbf{w}),\mathbf{w})$

$$\begin{tikzcd}
\partial E(r,\delta_0) \arrow[r, "f"] \arrow[d, "\psi"] & S^1_{\delta_0} \arrow[d, "1/\delta_0"] \\
S_r^{2n-1}\backslash \mathrm{Int}\, N_r \arrow[r, "\varphi"] & S^1
\end{tikzcd}$$

where $N_r = S_r^{2n-1} \cap \{\mathbf{z} \,|\, |f(\mathbf{z})| \le \delta_0\}$. As $N_r \cong D(\delta_0)^* \times K_r$ with $D(\delta_0)^* = \{\eta \in \mathbf{C} \,|\, 0 < |\eta| \le \delta\}$, the restriction $\varphi : S_r^{2n-1}\backslash N_r \to S^1$ is isomorphic to the Milnor fibration $\varphi : S_r^{2n-1}\backslash K_r \to S^1$.

For the construction of $\mathscr{V}$, we use Lemma 34. Take a point $\mathbf{w} \in E^c(r,\delta_0)$. If the three vectors $\mathbf{v}_1(\mathbf{w},\overline{\mathbf{w}})$, $\mathbf{v}_2(\mathbf{w},\overline{\mathbf{w}})$, $\mathbf{w}$ are linearly independent over $\mathbf{R}$, it is also linearly independent over a small open neighborhood $U(\mathbf{w})$. It is easy to construct locally $\mathscr{V}$ on $U(\mathbf{w})$, satisfying the above property (11). If the three vectors are linearly dependent over $\mathbf{R}$, consider the expression:

$$\mathbf{w} = a\mathbf{v}_1(\mathbf{w},\overline{\mathbf{w}}) + b\mathbf{v}_2(\mathbf{w},\overline{\mathbf{w}}), \quad a,b \in \mathbf{R},$$

with $a > 0$, we construct $\mathscr{V}$ on a neighborhood $U(\mathbf{w})$ of $\mathbf{w}$ so that

$$\Re(\mathscr{V}(\mathbf{z}),\mathbf{v}_2(\mathbf{z},\overline{\mathbf{z}})) = 0, \quad \Re(\mathscr{V}(\mathbf{z}),\mathbf{v}_1(\mathbf{z},\overline{\mathbf{z}})) > 0.$$

on $U(\mathbf{w})$. Note that

$$\Re(\mathbf{w},\mathscr{V}(\mathbf{w})) = a\Re(\mathbf{v}_1(\mathbf{w},\overline{\mathbf{w}}),\mathscr{V}(\mathbf{w})) > 0$$

If $U(\mathbf{w})$ is sufficiently small, this inequality holds on $U(\mathbf{w})$. Consider the open covering $\mathscr{U} = \{U(\mathbf{w}) \,|\, \mathbf{w} \in E^c(r,\delta_0)\}$. Taking a locally finite refinement $\mathscr{U}'$ of this covering, we glue together vector fields constructed locally on each open set in $\mathscr{U}'$ using a partition of unity as usual. □

5.4. Polar weighted homogeneous polynomial and its Milnor fibration. Consider a mixed polynomial $f(\mathbf{z},\overline{\mathbf{z}})$ which is a radially weighted homogeneous polynomial of type $(q_1,\dots,q_n;d_r)$ and a polar weighted homogeneous polynomial of type $(p_1,\dots,p_n;d_p)$. Put $V = f^{-1}(0)$ as before. Then $f : \mathbf{C}^n\backslash V \to \mathbf{C}^*$ is a locally trivial fibration [17]. We call it *the global fibration*. On the other hand, the Milnor fibration of the first type:

$$\varphi := f/|f| : S_r\backslash K_r \to S^1, \quad K_r = f^{-1}(0) \cap S_r$$

always exists for any $r > 0$ and the isomorphism class does not depend on the choice of r. This can be shown easily, using the polar action. We simply use the polar action to show the local triviality:

$$\psi : \varphi^{-1}(\theta) \times (\theta - \pi, \theta + \pi) \to \varphi^{-1}((\theta - \pi, \theta + \pi))$$
$$\psi(\mathbf{z}, \theta + \eta) := (z_1 \exp(ip_1\eta/d_p), \dots, z_n \exp(ip_n\eta/d_p))$$

Now we have the following assertion which is a generalization of the same assertion for weighted homogeneous polynomials.

THEOREM 37. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a polar weighted polynomial as above. We assume that the radial weight vector ${}^t(q_1,\dots,q_n)$ is strictly positive. Then the two fibrations*

$$f : f^{-1}(S^1_\delta) \to S^1_\delta, \quad \varphi = f/|f| : S_r^{2n-1} - K_r \to S^1,$$

are isomorphic for any $r > 0$ and $\delta > 0$.

Proof. First, observe that the isomorphism class of the global fibration

$$f : f^{-1}(S^1_\delta) \to S^1_\delta$$

does not depend on $\delta > 0$. This follows from the commutative diagram:

$$\begin{array}{ccc} f^{-1}(1) & \xrightarrow{\phi_\delta} & f^{-1}(\delta) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ S^1 & \xrightarrow{\delta} & S^1_\delta \end{array}, \quad \phi_\delta : (\mathbf{z},\bar{\mathbf{z}}) \mapsto \sqrt[d]{\delta} \circ (\mathbf{z},\bar{\mathbf{z}})$$

where $d = \operatorname{rdeg} f$ and $\circ$ denotes the $\mathbf{R}^*$ action by the radial weights. Now the global fibration $f : f^{-1}(S^1_\delta) \to S^1_\delta$ is isomorphic to the second fibration $\varphi : S_r^{2n-1} \backslash K_r \to S^1$ as follows. For any $\mathbf{z} \in f^{-1}(S^1_\delta)$, consider the orbit of the radial action $\tau \mapsto \tau \circ \mathbf{z} = (\tau^{q_1} z_1, \dots, \tau^{q_n} z_n)$, $\tau > 0$. There exists a unique positive real number $\tau = \tau(\mathbf{z})$ so that $\|\tau(\mathbf{z}) \circ \mathbf{z}\| = r$ by the strict positivity assumption of Q. Put $\psi : f^{-1}(S^1_\delta) \to S_r^{2n-1} \backslash K_r$ by $\psi(\mathbf{z}) = \tau(\mathbf{z}) \circ \mathbf{z}$ and $\xi : S_\delta \to S_1$ by $\xi(\eta) = \eta/\delta$. Then we have a canonical commutative diagram which gives an isomorphism of two fibrations.

$$\begin{array}{ccc} f^{-1}(S^1_\delta) & \xrightarrow{f} & S^1_\delta \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \xi} \\ S_r^{2n-1} \backslash K_r & \xrightarrow{\varphi} & S^1 \end{array}$$

□

The following is an important criterion for the connectivity of the Milnor fiber of a polar weighted mixed polynomial.

PROPOSITION 38. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a polar weighted mixed polynomial of n variables $\mathbf{z} = (z_1,\dots,z_n)$. We assume that $f^{-1}(0)$ has at least one mixed smooth point. Then the fiber $F := f^{-1}(1) \subset \mathbf{C}^n$ is connected.*

Proof. Put $V = f^{-1}(0)$. Take two points $P, Q \in F$. Connect P, Q by a path ℓ in $\mathbf{C}^n \backslash V$. Then $f_\#(\ell)$ is a closed path in $\mathbf{C}^*$ based at $1 \in \mathbf{C}^*$ and let s be the rotation number of $f_\#(\ell)$ around the origin. Take a smooth point R in V

and take a small lasso ω around V in the normal plane at R. Connect ω to P in $\mathbf{C}^n \setminus V$ to get a closed path ω' at P. The image of ω' by f is a closed loop with the rotation number 1 around the origin. Take a new path $\ell' = \omega'^{-s} \circ \ell$. Then the image of ℓ' has 0 rotation number around the origin and thus it is homotopic to the constant loop at $1 \in \mathbf{C}^*$ by a homotopy $k_t : f \circ \omega' \simeq c_1$, where c_1 is the constant path at 1. Now lift this homotopy by the radial and the polar actions to get a path $\tilde{k}_1$ from P to Q. Obviously $f \circ \tilde{k}_1$ is the constant path c_1. Thus $\tilde{k}_1$ is a path in the fiber F which connects P and Q. (For a holomorphic case, this assertion follows from the Kato-Matsumoto theorem, [9]). $\square$

6. Curves defined by mixed functions

In this section, we focus our study to mixed plane curves ($n = 2$).

6.1. Holomorphic plane curves. Assume that C is a germ of a complex analytic curve defined by a convenient non-degenerate holomorphic function $f(z_1, z_2)$ and let Δ_j, $j = 1, \dots, r$ be the 1-dimensional faces and $M_0, M_1, \dots, M_{r-1}, M_r$ be the vertices of $\Gamma(f)$ such that $\Delta_j = \overline{M_{j-1}M_j}$ and M_0, M_r are on the coordinate axes. Then each face function f_{Δ_j} can be factorized as

$$f_{\Delta_j}(z_1, z_2) = c_j z_1^{a_j} z_2^{b_j} \prod_{i=1}^{\nu_j} (z_1^{p_j} + \alpha_{j,i} z_2^{q_j}), \quad \gcd(p_j, q_j) = 1$$

where $\alpha_{j,1}, \dots, \alpha_{j,\nu_j}$ are mutually distinct. Then any toric modification with respect to a regular simplicial cone subdivision Σ^* of the dual Newton diagram $\Gamma^*(f)$ gives a good resolution of $f : (\mathbf{C}^2, O) \to (\mathbf{C}, 0)$. Let P_j be the weight vector of the face Δ_j. Each vertex P of Σ^* gives an exceptional divisor $\hat{E}(P)$ and the strict transform $\tilde{C}$ intersects with $\hat{E}(P)$ if and only if $P = P_j$ for some $j = 1, \dots, r$. In the case $P = P_j$, $\hat{E}(P_j) \cap \tilde{C}$ is ν_j point which corresponds to irreducible components associated with f_{Δ_j}. The vertices $M_1, \dots, M_{r-1}$ do not contribute to the irreducible components. The number of irreducible components of (C, O) is given by $\sum_{i=1}^r \nu_i$. Note that $1 + \sum_{i=1}^r \nu_i$ is the number of integral points on $\Gamma(f)$ ([16]). The situation for mixed polynomials is more complicated as we will see later.

6.2. Mixed curves. Now we consider curves defined by a mixed function with the same Newton boundary as in the previous subsection. Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a non-degenerate convenient mixed function with two variables $\mathbf{z} = (z_1, z_2)$ and let $C = f^{-1}(0)$. Let

$$\varphi : Y \xrightarrow{\omega} X \xrightarrow{\hat{\pi}} \mathbf{C}^2$$

($Y = \mathscr{R}X$, $\omega = \omega_{\mathbf{R}}$ or $\mathscr{P}X$ and $\omega = \omega_p$) be the resolution map, described in Theorem 24 and Theorem 25. Let $\tilde{E}(P) = \omega^{-1}(\hat{E}(P))$ for a vertex P of Σ^*.

6.2.1. *Simple vertices.* A vertex $M = (a, b) \in \Gamma(f)$ is called *simple* if f_M contains only a single monomial $z_1^{a_1} z_2^{b_1} \bar{z}_1^{a_2} \bar{z}_2^{b_2}$ such that $a = a_1 + a_2$, $b = b_1 + b_2$. Otherwise we say M is a *multiple vertex* of $\Gamma(f)$.

Example 39. Let $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3 + t z_1^2 \bar{z}_1 + z_2^2$. Then $\Gamma(f)$ has one face with edge vertices $M_1 = (3, 0)$ and $M_2 = (0, 2)$. $f(\mathbf{z}, \bar{\mathbf{z}})$ is a radially weighted homogeneous polynomial of type $(2, 3; 6)$. The vertex M_1 is a multiple vertex as $f_{M_1}(\mathbf{z}, \bar{\mathbf{z}}) = z_1^3 + t z_1^2 \bar{z}_1$.

Lemma 40. *Suppose $M = (n, 0)$ and let $f_M(z_1, \bar{z}_1) = \sum_{j=0}^{n} c_j z_1^j \bar{z}_1^{n-j}$. Consider the factorization $f_M(z_1, \bar{z}_1) = c \prod_{j=1}^{n} (z_1 - \alpha_j \bar{z}_1)$. Then $V^* := \{z_1 \in \mathbf{C}^* \mid f_M(z_1, \bar{z}_1) = 0\}$ is empty if and only if $|\alpha_j| \neq 1$ for any $j = 1, \dots, n$.*

Proof. Let $V_j^* := \{z_1 \in \mathbf{C}^* \mid z_1 = \alpha_j \bar{z}_1\}$. Then $V^* = \bigcup_{j=1}^{n} V_j^*$. It is easy to see that V_j^* is not empty if and only if $|\alpha_j| = 1$. □

Note that $f_M(z_1, \bar{z}_1)$ is non-degenerate if and only if $V^* = \emptyset$. For an inside vertex M_j (namely, M_j is not on the axis), the criterion for non-degeneracy of the function $f_{M_j}(\mathbf{z}, \bar{\mathbf{z}})$ is not so simple.

Example 41. Consider

$$C := \{\mathbf{z} \in \mathbf{C}^2 \mid f_M(\mathbf{z}, \bar{\mathbf{z}}) = t z_1 z_2 + z_1 \bar{z}_2 + \bar{z}_1 z_2\}.$$

We assert that

Assertion 42. *$f_M^{-1}(0) \subset \mathbf{C}^{*2}$ is non-empty if and only if $|t| \leq 2$. f_M is non-degenerate if and only if $|t| > 2$ or $0 < |t| < 2$.*

Proof. Put

$$z_1 = \rho_1 \exp(\theta i), \quad z_2 = \rho_2 \exp(\eta i), \quad t = \xi \exp(\alpha i).$$

Then we see that C is radially homogeneous and it is defined by

$$C : 2 \cos(-\theta + \eta) + \xi e^{i(a + \theta + \eta)} = 0.$$

For the existence of non-trivial solutions, we need to have:

$$\xi = |t| \leq 2, \quad \begin{cases} \alpha + \theta + \eta = m\pi, & \exists m \in \mathbf{Z} \\ 2 \cos(-\theta + \eta) + \xi(-1)^m = 0 \end{cases} \tag{12}$$

$$\text{or} \quad \xi = 0, \quad \cos(-\theta + \eta) = 0 \tag{13}$$

Assume that $t \neq 0$, $|t| < 2$. The equation (12) has 8 solutions with $0 \leq \theta, \eta < 2\pi$ for $\xi < 2$ and 4 solutions for $\xi = 2$ or 0. We can show that $V = f_M^{-1}(0)$ is non-singular for $0 \neq \xi < 2$, using Proposition 8. In fact, assume that $\overline{df}(\mathbf{z}) = \lambda\, \bar{d}f(\mathbf{z})$ with $|\lambda| = 1$. Then we have

$$\bar{t}\bar{z}_2 + z_2 = \lambda z_2, \quad \bar{t}\bar{z}_1 + z_1 = \lambda z_1, \quad tz_1z_2 + (z_1\bar{z}_2 + \bar{z}_1 z_2) = 0$$

This implies that $\lambda^2 = \pm 1$ and $|t| = 0$ or $|t| = 2$. □

6.2.2. *Link components.* Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a mixed function with two variables $\mathbf{z} = (z_1, z_2)$ and let $C = f^{-1}(0)$. The *link components* at the origin are the components of $S_\varepsilon^3 \cap C$ for a sufficiently small ε. We are interested in finding out how to compute the number of the link components of C at the origin. Let us denote this number by $\mathrm{lkn}(C, O)$ and we call $\mathrm{lkn}(C, O)$ *the link component number*. Let us denote the number of components which are not the coordinate axes $z_1 = 0$ or $z_2 = 0$ by $\mathrm{lkn}^*(C, 0)$. In the case of f being a holomorphic function, $\mathrm{lkn}(C, O)$ is equal to the number of irreducible components of (C, O), which is a combinatorial invariant, provided f is Newton non-degenerate, as we have seen in the previous section §6.1. However for a generic mixed function, $\mathrm{lkn}(C, O)$ might be strictly greater than the number of irreducible components (see Example 27 for example).

Theorem 43. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient non-degenerate mixed polynomial of two variables $\mathbf{z} = (z_1, z_2)$ and let $C = f^{-1}(0)$. Let $\mathscr{F}$ be the set of 1-faces of $\Gamma(f)$. Assume that the vertices of $\Gamma(f)$ are simple. Then the number of the link components $\mathrm{lkn}(C, O)$ is given be the formula*:

$$\mathrm{lkn}(C, O) = \sum_{\Delta \in \mathscr{F}} \mathrm{lkn}^*(f_\Delta^{-1}(0), O).$$

Proof. Let $\Phi : \mathscr{R}X \to \mathbf{C}^2$ be the resolution of f by the composite of a toric modification $\hat{\pi} : X \to \mathbf{C}^2$ and the normal real blowing-up $\omega : \mathscr{R}X \to X$. The simplicity of the vertices implies that $\Phi^{-1}(\hat{E}(P)) \cap \tilde{C} = \emptyset$ for any P for which $\Delta(P)$ a vertex of $\Gamma(f)$. Thus by Theorem 24, it is immediate that there is one link component of (C, O) for every connected component of $\tilde{E}(P) = \Phi^{-1}(\hat{E}(P)) \cap \tilde{C}$ with $\Delta(P) \in \mathscr{F}$. The assertion follows from this observation. □

Now our interest is finding out how we can compute $\mathrm{lkn}^*(f_\Delta^{-1}(0), O)$. In general, it is not so easy to compute this number but there is a class for which the link number is easily computed.

6.2.3. *Good Newton polar boundary.* Suppose that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a mixed function of two variables and let Δ be a face of the Newton boundary. Suppose that $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ is also a polar weighted homogeneous polynomial. Let $Q = {}^t(q_1, q_2)$ and $P = {}^t(p_1, p_2)$ be the radial and the polar weight vectors and d_r, d_p be the respective degree. In general, the mixed face $\hat{\Delta}(Q)$ is two-dimensional as the possible monomial $z_1^{\nu_1} z_2^{\nu_2} \bar{z}_1^{\mu_1} \bar{z}_2^{\mu_2}$ satisfies two linear equations

$$(\nu_1 + \mu_1)q_1 + (\nu_2 + \mu_2)q_2 = d_r, \quad (\nu_1 - \mu_1)p_1 + (\nu_2 - \mu_2)p_2 = d_p.$$

We say that $f_\Delta(\mathbf{z},\bar{\mathbf{z}})$ is *a good polar weighted polynomial* if $\dim\hat{\Delta}=1$ and $f_\Delta(\mathbf{z},\bar{\mathbf{z}})$ factors as

$$f_\Delta(\mathbf{z},\bar{\mathbf{z}}) = c\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{n}}\prod_{j=1}^{k}(z_2^a\bar{z}_2^{a'}-\lambda_j z_1^b\bar{z}_1^{b'})^{\mu_j} \tag{14}$$

with $a\neq a'$, $b\neq b'$ and $\gcd(a,a',b,b')=1$. Note that in this case, $p_1(b-b')=p_2(a-a')$ and non-zero. We say that $f(\mathbf{z},\bar{\mathbf{z}})$ has *a good Newton polar boundary* if for every face Δ of $\Gamma(f)$, $f_\Delta(\mathbf{z},\bar{\mathbf{z}})$ is a good polar weighted polynomial.

LEMMA 44. *Assume that $f_\Delta(\mathbf{z},\bar{\mathbf{z}})$ is a good polar weighted polynomial and assume that a factorization of $f_\Delta(\mathbf{z},\bar{\mathbf{z}})$ is given as* (14). *Then $f_\Delta(\mathbf{z},\bar{\mathbf{z}})$ is non-degenerate if and only if $\mu_1=\cdots=\mu_k=1$.*

Proof. Assume that $\mu_j\geq 2$ for some j. Then it is easy to see that $df(\mathbf{z},\bar{\mathbf{z}})=\bar{d}f(\mathbf{z},\bar{\mathbf{z}})=0$ on $z_1^b\bar{z}_1^{b'}-\lambda_j z_2^a\bar{z}_2^{a'}=0$. Thus it is degenerate. Assume that $\mu_j=1$ for any j. As f_Δ is polar weighted, we only need to show that $f_\Delta^{-1}(0)\cap\mathbf{C}^{*2}$ is mixed non-singular. Take a point $\mathbf{w}\in f_\Delta^{-1}(0)\cap\mathbf{C}^{*2}$ such that $w_2^a\bar{w}_2^{a'}-\lambda_1 w_1^b\bar{w}_1^{b'}=0$ for example. Then we have

$$df(\mathbf{w},\bar{\mathbf{w}}) = c\mathbf{w}^{\nu}\bar{\mathbf{w}}^{\mu}\prod_{j=2}^{k}(w_2^a\bar{w}_2^{a'}-\lambda_j w_1^b\bar{w}_1^{b'})\times(-b\lambda_1 w_1^{b-1}\bar{w}_1^{b'},\,a w_2^{a-1}\bar{w}_2^{a'})$$

$$\bar{d}f(\mathbf{w},\bar{\mathbf{w}}) = c\mathbf{w}^{\nu}\bar{\mathbf{w}}^{\mu}\prod_{j=2}^{k}(w_2^a\bar{w}_2^{a'}-\lambda_j w_1^b\bar{w}_1^{b'})\times(-b'\lambda_1 w_1^{b}\bar{w}_1^{b'-1},\,a' w_2^{a}\bar{w}_2^{a'-1})$$

Suppose that $\overline{df}(\mathbf{w},\bar{\mathbf{w}})=u\,\bar{d}f(\mathbf{w},\bar{\mathbf{w}})$ for some u with $|u|=1$. This implies that $b=b'$, $a=a'$. This does not happen as we have assumed that $a\neq a'$, $b\neq b'$. □

Example 45. Let $f(\mathbf{z},\bar{\mathbf{z}})=z_1^5+\bar{z}_1z_2(z_1^2-\bar{z}_2^2)+\bar{z}_2^5$. Then $\Gamma(f)$ has three 1-faces and the corresponding face functions are

$$z_1^2\bar{z}_1(z_1^3\bar{z}_1^{-1}+z_2),\quad \bar{z}_1z_2(z_1^2-\bar{z}_2^2),\quad z_2\bar{z}_2^2(-\bar{z}_1+z_2^{-1}\bar{z}_2^3).$$

Thus f has a good Newton polar boundary.

6.2.4. *Good binomial polar weighted polynomial.* A polynomial $f(\mathbf{z},\bar{\mathbf{z}})=z_2^a\bar{z}_2^{a'}-\lambda z_1^b\bar{z}_1^{b'}$ with $a\neq a'$, $b\neq b'$, $\lambda\neq 0$ and $\gcd(a,a',b,b')=1$ is called an *irreducible binomial polar weighted homogeneous polynomial.* It is irreducible as a mixed polynomial. By Lemma 44, this is a basic polar weighted polynomial for our purpose. Put $c_1=b-b'$ and $c_2=a-a'$. Then the associated Laurent polynomial in the sense of [17] is

$$g(z_1,z_2)=z_2^{c_2}-\lambda z_1^{c_1}.$$

Let $C = \{f = 0\}$ and $C' = \{g = 0\}$. Note that $c_1, c_2 \neq 0$ by the polar weightedness.

LEMMA 46. *We have the equality*:

$$\operatorname{lkn}^*(C, O) = \gcd(c_1, c_2) = \#(C')$$

where $\#(C')$ *is the number if irreducible components of* C'.

Proof. It is easy to see that the number of irreducible components of C' in $\mathbf{C}^{*2}$ is $\gcd(c_1, c_2)$. We know that $C \cap \mathbf{C}^{*2}$ and $C' \cap \mathbf{C}^{*2}$ are homeomorphic by the same argument as in [17]. We will show that $\operatorname{lkn}^*(C, 0) = \gcd(c_1, c_2)$ without using this isomorphism. We consider components of C in $\mathbf{C}^{*2}$. For this purpose, we use the polar modification. So we put $z_1 = r_1 \exp(\theta_1 i)$ and $z_2 = r_2 \exp(\theta_2 i)$. Considering the conjugation diffeomorphism, we may assume that $c_1, c_2 > 0$. For brevity we put $r_1 = s_1^{c_2}$ and $\lambda = \rho^{c_2} \exp(\eta i)$ for some s_1, $\rho > 0$. Thus

$$\begin{aligned} f(\mathbf{z}, \bar{\mathbf{z}}) &= r_2^{c_2} \exp(c_2\theta_2 i) - \lambda r_1^{c_1} \exp(c_1\theta_1 i) \\ &= r_2^{c_2} \exp(c_2\theta_2 i) - \rho^{c_2}(s_1^{c_1})^{c_2} \exp((c_1\theta_1 + \eta)i) \end{aligned}$$

Thus we have $r_2 = \rho s_1^{c_1}$ and

$$\exp(c_2\theta_2 i) - \exp((c_1\theta_1 + \eta)i) = 0.$$

Put $c_0 = \gcd(c_1, c_2)$ and write $c_i = c_0 c_i'$ for $i = 1, 2$. The above equation is solved as follows.

$$r_2 = s_1^{c_1}\rho, \quad c_2\theta_2 \equiv c_1\theta_1 + \eta \text{ modulo } 2\pi.$$

The last equality can be solved so that the component C_j of C is given as

$$C_j := \{(s_1^{c_1}\rho \exp(\theta_1 i), s_1^{c_2 c_1} \exp(\theta_2 i)) \mid \theta_2 = \phi_k(\theta_1), 0 \leq \theta_1 \leq 2c_2'\pi\}$$

where $\phi_k(\theta_1) := c_1'\theta_1/c_2' - \eta/c_2 + 2k\pi/c_2$ for $k = 0, 1, \dots, c_2 - 1$. For $k \geq c_0$, write $k = c_0 k_1 + k_0$, $0 \leq k_0 < c_0$. Then $\phi_k(\theta_1) = \phi_{k_0}(\theta_1 + 2k_1\pi)$ and $C_k = C_{k_0}$ as we have

$$\begin{aligned} C_k &= \{(r_1 \exp(\theta_1 i), r_2 \exp(\phi_k(\theta_1)i) \mid 0 \leq \theta_1 \leq 2c_2'\pi\} \\ &= \{(r_1 \exp(\theta_1 i), r_2 \exp(\phi_k(\theta_1)i)) \mid 2k_1\pi \leq \theta_1 \leq 2c_2'\pi + 2k_1\pi\} \\ &= \{(r_1 \exp((\theta_1 + 2k_0\pi)i), r_2 \exp(\phi_{k_0}(\theta_1 + 2k_0\pi)i) \mid 0 \leq \theta_2 \leq 2c_2'\pi\} \\ &= C_{k_0}. \end{aligned}$$

Thus we get $\operatorname{lkn}^*(C, O) = c_0$. □

COROLLARY 47. *Let* $f_\Delta(\mathbf{z}, \bar{\mathbf{z}})$ *be a good polar weighted polynomial which is factored as*

$$f_\Delta(\mathbf{z}, \bar{\mathbf{z}}) = c\mathbf{z}^\nu \bar{\mathbf{z}}^\mu \prod_{j=1}^{k} (z_2^a \bar{z}_2^{a'} - \lambda_j z_1^b \bar{z}_1^{b'})$$

with $\gcd(a,a',b,b')=1$, $a\neq a'$, $b\neq b'$ *as in Lemma* 44 *and let* $C=f_\Delta^{-1}(0)$. *Then* $\mathrm{lkn}^*(C)=k\gcd(a-a',b-b')$.

6.2.5. *Newton pseudo conjugate weighted homogeneous function.* Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a non-degenerate Newton pseudo conjugate weighted homogeneous function. Then for any face Δ, we can write

$$f_\Delta(\mathbf{z},\bar{\mathbf{z}})=Mh(\mathbf{z},\bar{\mathbf{z}})$$

where M is a mixed monomial and h is a J-conjugate weighted homogeneous polynomial for some $J\subset\{1,2\}$. Thus we can factorize h as

$$\iota_J^*h(\mathbf{z},\bar{\mathbf{z}})=c\prod_{j=1}^{k}(z_2^{p_1}-\lambda_j z_1^{p_2}),\quad c\neq 0$$

with $\gcd(p_1,p_2)=1$. In this case, it is easy to see that

$$\mathrm{lkn}^*(f_\Delta^{-1}(0))=k.$$

Thus we obtain a similar formula:

Proposition 48. *Assume that* $f(\mathbf{z},\bar{\mathbf{z}})$ *is a non-degenerate convenient Newton pseudo conjugate weighted homogeneous function. Then*

$$\mathrm{lkn}(f^{-1}(0))+1=\textit{number of integral points on }\Gamma(f).$$

6.2.6. *Example of a radially weighted homogeneous polynomial with a non-simple vertex.* The link number for a radially weighted homogeneous polynomial with a non-simple vertex is more complicated, as is seen by the next example. Consider the radially weighted homogeneous polynomial

$$f(\mathbf{z},\bar{\mathbf{z}})=z_1^3+cz_1\bar{z}_1^2-z_2^3$$

and put $C=f^{-1}(0)$. Then $\Gamma(f)$ consists of a single face with vertices $(3,0)$, $(0,3)$. It is easy to see that f is non-degenerate if and only if $|c|\neq 1$. The vertex $(3,0)$ is not simple. For $|c|<1$, we have

$$z_2=z_1\omega^j(1+c\exp(-4\theta i))^{1/3},\quad j=0,1,2$$

where $\omega=\exp(2\pi i/3)$, $z_1=r\exp(\theta i)$ and $\mathrm{lkn}(C,O)=3$. The function $(1+c\exp(-4\theta i))^{1/3}$ is a well-defined single-valued function of c, z_1 with $|c|<1$ so that it takes value 1 for $c=0$. Considering the family $f(\mathbf{z},\bar{\mathbf{z}},t)=z_1^3+ctz_1\bar{z}_1^2-z_2^3$ for $0\le t\le 1$, we see that this curve is topologically the same as $z_1^3+z_2^3=0$.

Assume that $|c|>1$. Then $(1+c\exp(-4\theta i))^{1/3}$ is not a single valued function as a function of $0\le\theta\le 2\pi$. However we have a better expression. Put $z_1=r\exp(\theta i)$ and $c=s\exp(\eta i)$.

$$z_2=s^{1/3}r\omega^j\exp\left(i\frac{-\theta+\eta}{3}\right)\left(1+\frac{\exp(4\theta i)}{c}\right)^{1/3},\quad j=1,2,3$$

where $0 \le \theta \le 2\pi$. Note that $f^{-1}(0)\setminus\{O\}$ is a 3-sheeted covering over $\{z_1 \neq 0\}$ and three points over $\theta = 0$ are cyclically permuted by the monodromy $\theta : 0 \to 2\pi$. Thus this expression shows that $\mathrm{lkn}(C, O) = 1$. It is also easy to see that this knot is topologically the same with $z_1|z_1|^2 - z_2^3 = 0$. Thus we observe that *the topology of a mixed singularities is not a combinatorial invariant of* $\Gamma(f)$.

7. Milnor fibration for mixed polynomials with non-isolated singularities

We consider a true strongly non-degenerate mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ which is not necessarily convenient. Take a positive weight vector $P = {}^t(p_1, \dots, p_j) \in N^+$ which is not strictly positive and we put

$$I(P) = \{j \mid p_j = 0\}, \quad J(P) = \{j \mid p_j \neq 0\}.$$

We consider the face function $f_P(\mathbf{z}, \bar{\mathbf{z}})$ as a mixed polynomial in variables $\{z_j \mid j \in J(P)\}$ and we consider the other variables $\{z_i \mid i \in I(P)\}$ are fixed non-zero complex numbers. Thus it defines a family of mixed polynomial functions parameterized by $\mathbf{z}_{I(P)} = (z_i)_{i \in I(P)}$:

$$f_P : \mathbf{C}^{J(P)}(\mathbf{w}_{I(P)}) \to \mathbf{C}, \quad \mathbf{z}_{J(P)} \mapsto f_P(\mathbf{z}, \bar{\mathbf{z}}).$$

Here

$$\mathbf{C}^{*J(P)}(\mathbf{w}_{I(P)}) = \{\mathbf{z} \in \mathbf{C}^{*n} \mid \mathbf{z}_{I(P)} = \mathbf{w}_{I(P)} : \text{fixed}\} \cong \mathbf{C}^{*J(P)}.$$

Thus we are considering f_P as a family of mixed polynomials in $\mathbf{z}_{J(P)}$ with coefficients in $\mathbf{C}\{\mathbf{z}_{I(P)}, \bar{\mathbf{z}}_{I(P)}\}$. If $d(P, f) = 0$, then $f_P \in \mathbf{C}\{\mathbf{z}_{I(P)}, \bar{\mathbf{z}}_{I(P)}\}$.

Definition 49. We say that f is *super strongly non-degenerate* if the following condition (SSND) is satisfied.

(SSND): for any subset $P \in N^+$, either

(a) $d(P, f) = 0$ i.e., $f_P \in \mathbf{C}\{\mathbf{z}_{I(P)}, \bar{\mathbf{z}}_{I(P)}\}$ or

(b) $d(P, f) > 0$ and $f_P : \mathbf{C}^{*J(P)}(\mathbf{w}_{I(P)}) \to \mathbf{C}^*$ has no critical points for any $\mathbf{w}_{I(P)} \in \mathbf{C}^{*I(P)}$.

The following is an immediate consequence of the definition.

Proposition 50. (1) *If $f(\mathbf{z}, \bar{\mathbf{z}})$ is a convenient strongly non-degenerate mixed function, then $f(\mathbf{z}, \bar{\mathbf{z}})$ is super strongly non-degenerate.*

(2) *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is super strongly non-degenerate and $I \in \mathcal{NV}(f)$. Then f^I is also super strongly non-degenerate.*

The assertion (2) can be proved in the exact same way as the proof of Proposition 7. □

The following key lemma is a mixed polynomial version of Lemma (2.1.4) of Hamm-Lê [8].

LEMMA 51. *Assume that $f(\mathbf{z},\bar{\mathbf{z}})$ is a true super strongly non-degenerate mixed function and consider the mixed hypersurface V and its open subset $V^{\#}$. Take a positive number r_0 so that $V^{\#}\cap B_{r_0}$ is mixed non-singular and any sphere S_r intersects transversely with $V^{\#}$ for any $0<r\le r_0$. Then for any fixed $0<r\le r_0$, there exists a sufficiently small positive number δ such that for any $\eta\in\mathbf{C}$ with $0<|\eta|\le\delta$, the fiber $f^{-1}(\eta)\cap B_{r_0}$ is smooth and any sphere S_s intersects transversely with $f^{-1}(\eta)$ for any $r\le s\le r_0$ and η with $0<|\eta|\le\delta$.*

Proof. Assume that the assertion is not true. Using the Curve Selection Lemma ([12, 7]), we can find a real analytic curve $\mathbf{z}(t)$, $0\le t\le 1$ such that

$$r\le\|\mathbf{z}(t)\|\le r_0,\quad f(\mathbf{z}(t),\bar{\mathbf{z}}(t))\not\equiv 0,\quad \mathbf{z}(0)\in f^{-1}(0)$$

and the fiber $f^{-1}(\alpha(t))$ and the sphere of radius $\|\mathbf{z}(t)\|$ is not transverse at $\mathbf{z}(t)$ where $\alpha(t)=f(\mathbf{z}(t),\bar{\mathbf{z}}(t))$. Recall that we have defined two special vectors:

$$\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}})=\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}})+\bar{d}\log f(\mathbf{z},\bar{\mathbf{z}})$$

$$\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})=i(\overline{d\log f}(\mathbf{z},\bar{\mathbf{z}})-\bar{d}\log f(\mathbf{z},\bar{\mathbf{z}}))$$

Recall that the tangent space of the fiber $T_{\mathbf{z}}f^{-1}(\eta)$ is spanned by the vectors which are perpendicular to $\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}})$ and $\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})$. Thus under the assumption there exist real-valued analytic functions $\lambda(t)$, $\mu(t)$ so that

$$\mathbf{z}(t)=\lambda(t)\mathbf{v}_1(\mathbf{z},\bar{\mathbf{z}}))+\mu(t)\mathbf{v}_2(\mathbf{z},\bar{\mathbf{z}})),$$

as in the proof of Lemma 34. Let $I=\{j\,|\,z_j(t)\not\equiv 0\}$. Then $I\in\mathcal{NV}(f)$. We may assume that $I=\{1,\dots,m\}$ and we do the same argument in $\mathbf{C}^I$ as in the proof of Lemma 34. We consider the Taylor expansions of $\mathbf{z}(t)$, $f(\mathbf{z}(t),\bar{\mathbf{z}}(t))$ and the Laurent expansions of $\lambda(t)$ and $\mu(t)$:

$$z_j(t)=a_jt^{p_j}+(\text{higher terms}),\quad a_j\in\mathbf{C}^*,\ p_j\ge 0,\ 1\le j\le m,$$

$$f(\mathbf{z}(t),\bar{\mathbf{z}}(t))=\alpha t^{\ell}+(\text{higher terms}),\quad \alpha\in\mathbf{C}^*,\ \ell\in\mathbf{N}$$

$$\lambda(t)=\lambda_0t^{\nu_1}+(\text{higher terms}),\quad \lambda_0\in\mathbf{R}^*,\ \nu_1\in\mathbf{Z}$$

$$\mu(t)=\mu_0t^{\nu_2}+(\text{higher terms}),\quad \mu_0\in\mathbf{R}.$$

Here we understand $\nu_1=\infty$ or $\nu_2=\infty$ if $\lambda(t)\equiv 0$ or $\mu(t)\equiv 0$ respectively. We put $\nu_0=\min\{\nu_1,\mu_1\}$ and we write for simplicity as follows.

$$\lambda(t)=\hat{\lambda}_0t^{\nu_0}+(\text{higher terms}),\quad \lambda_0\in\mathbf{R}^*,\ \nu_1\in\mathbf{Z}$$

$$\mu(t)=\hat{\mu}_0t^{\nu_0}+(\text{higher terms}),\quad m_0\in\mathbf{R}$$

$$\text{where }\hat{\lambda}_0=\begin{cases}\lambda_0 & \text{if } \nu_1=\nu_0\\ 0 & \text{if } \nu_1>\nu_0\end{cases},$$

$$\hat{\mu}_0=\begin{cases}\mu_0 & \text{if } \nu_2=\nu_0\\ 0 & \text{if } \nu_2>\nu_0\end{cases}$$

Note that $\nu_0 < \infty$ and $(\hat{\lambda}_0, \hat{\mu}_0) \neq (0,0)$ anyway. Consider the equality:

$$z_j(t) = \lambda(t)\left(\frac{\overline{\partial f}}{\partial z_j}\Big/\bar{f} + \frac{\partial f}{\partial \bar{z}_j}\Big/f\right)(\mathbf{z}(t), \bar{\mathbf{z}}(t))$$

$$+ \mu(t)i\left(\frac{\overline{\partial f}}{\partial z_j}\Big/\bar{f} - \frac{\partial f}{\partial \bar{z}_j}\Big/f\right)(\mathbf{z}(t), \bar{\mathbf{z}}(t)), \quad j = 1, \dots, m.$$

Put $P = (p_1, \dots, p_m)$, $I(P) = \{j \mid p_j = 0\}$, $J(P) = \{j \mid p_j \neq 0\}$, $\mathbf{a} = (a_1, \dots, a_m)$ and $d = d(P, f^I)$. Note that $\mathbf{z}(0) \in \mathbf{C}^{*I(P)}$. Assume that $I(P) \in \mathcal{N}\mathcal{V}(f)$. Then $\mathbf{z}(0) \in V^{\#}$ and it is a smooth point. Thus by the assumption, the sphere $S_{\|\mathbf{z}(0)\|}$ intersects transversely with $V^{\#}$. Thus the same is true for $S_{\|\mathbf{z}(t)\|}$ and $f^{-1}(\alpha(t))$ for any sufficiently small t which is a contradiction to the assumption. Thus we may assume that $\mathbf{z}(0) \in V \setminus V^{\#}$ and therefore $I(P) \notin \mathcal{N}\mathcal{V}(f^I)$ $(\Leftrightarrow I(P) \cup I \notin \mathcal{N}\mathcal{V}(f))$. Then we observe that

$$\frac{\overline{\partial f}}{\partial z_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t))/\bar{f}(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \left(\frac{\overline{\partial f_P^I}}{\partial z_j}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha}\right)t^{d-p_j-\ell} + (\text{higher terms}),$$

$$\frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}(t), \bar{\mathbf{z}}(t))/f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \left(\frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha\right)t^{d-p_j-\ell} + (\text{higher terms})$$

By Assertion 35, we have

$$\hat{\lambda}_0\left(\frac{\overline{\partial f_P^I}}{\partial z_j}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} + \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha\right) + \hat{\mu}_0 i\left(\frac{\overline{\partial f_P^I}}{\partial z_j}(\mathbf{a}, \bar{\mathbf{a}})/\bar{\alpha} - \frac{\partial f_P^I}{\partial \bar{z}_j}(\mathbf{a}, \bar{\mathbf{a}})/\alpha\right) = 0, \quad j \in J(P).$$

This implies that $\mathbf{a}_{J(P)}$ is a critical point of the mixed polynomial $f_P^I : \mathbf{C}^{*J(P)}(\mathbf{a}_{I(P)}) \to \mathbf{C}$ and $f_P^I(\mathbf{a}, \bar{\mathbf{a}}) \neq 0$ with $\mathbf{z}_{I(P)} = \mathbf{a}_{I(P)}$ fixed. This is a contradiction to the super strong non-degeneracy of f^I. □

7.1. Milnor fibration for non-isolated singularities. Now, by Lemma 51 and Lemma 31, we have the following non-isolated version of the Milnor fibration. Note that $\varphi = f/|f| : S_r^{2n-1} \setminus K_r \to S^1$ is a fibration using a $|f|$-level preserving vector field near K_r by the transversality of $f^{-1}(\eta)$ and S_r for η, $|\eta| \ll \delta$.

THEOREM 52. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a super strongly non-degenerate mixed function. Then there exists a stable radius $r_0 > 0$ so that for any r with $0 < r \leq r_0$ and a sufficiently small number δ (compared with r), we have two equivalent fibrations:*

$$f : \partial E(r, \delta)^* \to S_\delta^1$$

$$\varphi = f/|f| : S_r^{2n-1} \setminus K_r \to S^1$$

where $K_r = f^{-1}(0) \cap S_r^{2n-1}$. Moreover, if f is a polar weighted polynomial, the global fibration $f : f^{-1}(S_\delta^1) \to S_\delta^1$ is also equivalent to the above fibration.

Example 53. I. A monomial $z_1^{\mu_1}\bar{z}_1^{\nu_1}z_2^{\mu_2}\bar{z}_2^{\nu_2}$ is called *an inside monomial* if $\mu_1+\nu_1, \mu_2+\nu_2>0$. An inside monomial $z_1^{\mu_1}\bar{z}_1^{\nu_1}z_2^{\mu_2}\bar{z}_2^{\nu_2}$ is called *polar admissible* if $\mu_1\neq\nu_1$ and $\mu_2\neq\nu_2$. Let $g(\mathbf{z},\bar{\mathbf{z}})$ be a strongly non-degenerate polar weighted mixed function of two variables $\mathbf{z}=(z_1,z_2)$ with two simple end vertices A, B of $\Gamma(g)$. We assume that

$$A=(m_1,n_1),\quad B=(m_2,n_2),\quad m_1<m_2,\ n_1>n_2$$

which come from the mixed monomials $z_1^{\mu_1}\bar{z}_1^{\nu_1}z_2^{\mu_2}\bar{z}_2^{\nu_2}$ and $z_1^{\mu_1'}\bar{z}_1^{\nu_1'}z_2^{\mu_2'}\bar{z}_2^{\nu_2'}$. Here

$$m_1=\mu_1+\nu_1,\quad n_1=\mu_2+\nu_2,\quad m_2=\mu_1'+\nu_1',\quad n_2=\mu_2'+\nu_2'.$$

Consider $P={}^t(1,0)$ for example. Then $g_P(\mathbf{z},\bar{\mathbf{z}})=c\mathbf{z}^\mu\bar{\mathbf{z}}^\nu$ with some non-zero constant c. Assume that $m_1>0$. To check if $g_P:\mathbf{C}^*\to\mathbf{C}^*$ has a critical point or not as a function of z_1 variable, we can use $\log g_P$ instead of g_P. Now we have

$$\overline{d_{z_1}\log g_P}(\mathbf{z},\bar{\mathbf{z}})=\frac{\mu_1}{\bar{z}_1},\quad \bar{d}_{z_1}\log g_P(\mathbf{z},\bar{\mathbf{z}})=\frac{\nu_1}{\bar{z}_1}.$$

If $z_1\in\mathbf{C}^*$ is a critical point of g_P for some fixed $z_2\in\mathbf{C}^*$, we must have $u\in S^1$ such that $\dfrac{\mu_1}{\bar{z}_1}=u\dfrac{\nu_1}{\bar{z}_1}$. This is only possible if $\nu_1=\mu_1$. By a similar discussion for $Q={}^t(0,1)$, we have shown the following.

Lemma 54. *Assume that $g(\mathbf{z},\bar{\mathbf{z}})$ is a non-degenerate polar weighted mixed polynomial whose two end monomials are $z_1^{\mu_1}\bar{z}_1^{\nu_1}z_2^{\mu_2}\bar{z}_2^{\nu_2}$ and $z_1^{\mu_1'}\bar{z}_1^{\nu_1'}z_2^{\mu_2'}\bar{z}_2^{\nu_2'}$ with $\mu_1+\nu_1<\mu_2+\nu_2$. Then $g(\mathbf{z},\bar{\mathbf{z}})$ is super strongly non-degenerate if and only if the following conditions are satisfied.*

(1) *Either $\mu_1=\nu_1=0$ or $z_1^{\mu_1}\bar{z}_1^{\nu_1}z_2^{\mu_2}\bar{z}_2^{\nu_2}$ is polar admissible.*

(2) *Either $\mu_2'=\nu_2'=0$ or $z_1^{\mu_1'}\bar{z}_1^{\nu_1'}z_2^{\mu_2'}\bar{z}_2^{\nu_2'}$ is polar admissible.*

II. Let $f(\mathbf{z},\bar{\mathbf{z}})=z_1^{a_1}\bar{z}_2^{b_1}+z_2^{a_2}\bar{z}_3^{b_2}+\cdots+z_n^{a_n}\bar{z}_1^{b_n}$ be a simplicial polar weighted homogeneous mixed polynomial. We assume that $a_j>b_{j-1}\geq 1$ for $j=1,\dots,n$ with $b_0=b_n$. We assert that f *is super strongly non-degenerate*.

Proof. Consider f_P^I for some $I\in\mathcal{NV}(f)$ and $P\in N^+$ and let $I(P)$, $J(P)$ be as in the proof of Lemma 51. We assume that $d(P,f)>0$. Suppose that $z_1^{a_1}\bar{z}_2^{b_1}$ is in f_P^I. Then $\{1,2\}\cap J(P)\neq\emptyset$. Assume that $2\in J(P)$ for example. Then $\dfrac{\partial f_P}{\partial\bar{z}_2}\neq 0$. If f_P^I has a critical point as a mapping $f_P^I:\mathbf{C}^{*J(P)}\to\mathbf{C}^*$, we need a non-zero $\dfrac{\partial f_P^I}{\partial z_2}$ by Proposition 7, which implies $z_2^{a_2}\bar{z}_3^{b_2}$ must be in f_P^I. As $a_2>b_1$ by the assumption and $p_1a_1+p_2b_1=p_2a_2+p_3b_2$, this implies that $1\in J(P)$ i.e., $p_1\neq 0$. This implies again that $\dfrac{\partial f_P^I}{\partial\bar{z}_1}\neq 0$ and therefore $z_n^{a_n}\bar{z}_1^{b_n}$ must be in f_P^I. By the same reasoning, $a_1>b_n$ implies that $p_n>0$ and $n\in J(P)$. Then we consider

$\frac{\partial f_P^I}{\partial z_n}$ and we see that $n-1 \in J(P)$ and $z_{n-1}^{a_{n-1}}\bar{z}_n^{b_{n-1}}$ is in f_P^I. Continuing the same discussion, we conclude $f_P^I = f$ i.e., $I = \{1,\dots,n\}$. However, $f(\mathbf{z},\bar{\mathbf{z}})$ is polar weighted and it has no critical point over $\mathbf{C}^*$. Thus f is super strongly non-degenerate. □

8. Resolution of a polar type and the zeta function

In this section, we will study the relation between a resolution of a polar type and the Milnor fibration of the second type. We expect a similar formula like the formula of A'Campo ([1]) or the formula of Varchenko [23]. We will restrict ourselves to the case of mixed curves.

8.1. Polar weighted case. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial of n variables $z_1,\dots,z_n$ and let $(q_1,\dots,q_n;d_r)$ and $(p_1,\dots,p_n;d_p)$ be the radial and polar weight types. We assume that $d_p > 0$. Then $f : \mathbf{C}^{*n} - f^{-1}(0) \to \mathbf{C}^*$ is a fibration. Put $F_s^* = f^{-1}(s)\cap \mathbf{C}^{*n}$ for $s \in \mathbf{C}^*$. Then the monodromy map $h : F_s^* \to F_s^*$ is given by the polar action as

$$h(z_1,\dots,z_n) = (z_1\omega^{p_1},\dots,z_n\omega^{p_n}), \quad \omega = \exp\left(\frac{2\pi i}{d_p}\right)$$

Put $F^* = F_1^*$ and let $\chi(F^*)$ be the Euler characteristic of F^*. Then the monodromy has the period d_p and the set of the fixed points of $h^j : F^* \to F^*$ is empty if $j \not\equiv 0$ modulo d_p, where $h^j = h\circ\cdots\circ h$ (j-times). Thus using the formula of the zeta function for a periodic mapping ([12]), we get

Lemma 55. *Under the above assumption, the zeta-function of $h : F^* \to F^*$ is given as*

$$\zeta(t) = (1-t^{d_p})^{-\chi(F^*)/d_p}.$$

The zeta function of the global fibration $f : \mathbf{C}^n \backslash f^{-1}(0) \to \mathbf{C}^*$ can be obtained by patching the data for each torus stratum.

Let us do this for curves ($n = 2$). Let $f(\mathbf{z})$ be a non-degenerate polar weighted homogeneous polynomial of type $(p_1,p_2;d_p)$. The signs of p_1, p_2 are chosen so that $d_p > 0$. Suppose that the two edge vertices of $\Gamma(f)$ are simple. Assume that the two end monomials are

$$z_1^{\mu_1}\bar{z}_1^{\nu_1}z_2^{\mu_2}\bar{z}_2^{\nu_2}, \quad z_1^{\mu_1'}\bar{z}_1^{\nu_1'}z_2^{\mu_2'}\bar{z}_2^{\nu_2'}$$

with $\mu_1+\nu_1 < \mu_1'+\nu_1'$ and $\mu_2+\nu_2 > \mu_2'+\nu_2'$.

Assume that $\mu_1 = \nu_1 = 0$ and $\mu_2' = \nu_2' = 0$ i.e., $f(\mathbf{z},\bar{\mathbf{z}})$ is convenient. In this case the two monomials reduces to $z_2^{\mu_2-\nu_2}|z_2|^{2\nu_2}$, $z_1^{\mu_1'-\nu_1'}|z_1|^{2\nu_1'}$. Let $F = f^{-1}(1) \subset \mathbf{C}^2$, $F_{z_1} = F\cap\{z_2 = 0\}$ and $F_{z_2} = F\cap\{z_1 = 0\}$. Note that

$$F_{z_1} = \{(z_1,0)\,|\,z_1^{\mu_1'-\nu_1'} = 1\}, \quad F_{z_2} = \{(0,z_2)\,|\,z_2^{\mu_2-\nu_2} = 1\}.$$

The monodromy map is defined by

$$h: F \to F, \quad (z_1, z_2) \mapsto (z_1\omega^{p_1}, z_2\omega^{p_2}), \quad \omega = \exp\left(\frac{2\pi i}{d_p}\right)$$

Note that $p_1(\mu_1' - \nu_1') = p_2(\mu_2 - \nu_2) = d_p$. Therefore the fixed points set $\mathrm{Fix}(h^j)$ of h^j is non-empty only for $j = |\mu_1' - \nu_1'|$, $|\mu_2 - \nu_2|$, or d_p and their multiples. Thus using the calculation through $\exp \zeta(t)$ as in [12], we get

Lemma 56. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted convenient polynomial as above. Let $z_1^{\mu_1'}\bar{z}_1^{\nu_1'}$, $z_2^{\mu_2}\bar{z}_2^{\nu_2}$ be the end monomials and let d_p be the polar degree. Then the Euler-Poincaré characteristic $\chi(F)$ and the zeta function of the monodromy $h: F \to F$ are given as*

$$\chi(F) = \chi(F^*) + |\mu_1' - \nu_1'| + |\mu_2 - \nu_2|, \quad \mu = 1 - \chi(F)$$

$$\zeta(t) = \frac{(1 - t^{d_p})^{-\chi(F^*)/d_p}}{(1 - t^{|\mu_1' - \nu_1'|})(1 - t^{|\mu_2 - \nu_2|})}$$

Remark 57. By a similar consideration, if $f(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted polynomial which is not convenient, the assertion is true under the following modification. Put $\varepsilon_1 = 1$ or 0 according to $\mu_2' + \nu_2' = 0$ or $\mu_2' + \nu_2' > 0$. Similarly $\varepsilon_2 = 1$ or 0 according to $\mu_1 + \nu_1 = 0$ or $\mu_1 + \nu_1 > 0$. Then

$$\chi(F) = \chi(F^*) + \varepsilon_1|\mu_1' - \nu_1'| + \varepsilon_2|\mu_2 - \nu_2|, \quad \mu = 1 - \chi(F)$$

$$\zeta(t) = \frac{(1 - t^{d_p})^{-\chi(F^*)/d_p}}{(1 - t^{|\mu_1' - \nu_1'|})^{\varepsilon_1}(1 - t^{|\mu_2 - \nu_2|})^{\varepsilon_2}}$$

8.1.1. *Simplicial polar weighted polynomial.* Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^m c_j \mathbf{z}^{\mu_j}\bar{\mathbf{z}}^{\nu_j}$. The associated Laurent polynomial $g(\mathbf{z})$ is defined by

$$g(\mathbf{z}) = \sum_{j=1}^m c_j \mathbf{z}^{\mu_j - \nu_j}.$$

Recall that $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *simplicial polar weighted homogeneous* if $m = n$ and the two matrices have a non-zero determinant [17]:

$$M = \begin{pmatrix} \mu_{11} + \nu_{11} & \cdots & \mu_{1n} + \nu_{1n} \\ \vdots & \dots & \vdots \\ \mu_{n1} + \nu_{n1} & \cdots & \mu_{nn} + \nu_{nn} \end{pmatrix}, \quad N = \begin{pmatrix} \mu_{11} - \nu_{11} & \cdots & \mu_{1n} - \nu_{1n} \\ \vdots & \dots & \vdots \\ \mu_{n1} - \nu_{n1} & \cdots & \mu_{nn} - \nu_{nn} \end{pmatrix}$$

where $\mu_j = (\mu_{j1}, \dots, \mu_{jn})$ and $\nu_j = (\nu_{j1}, \dots, \nu_{jn})$, $j = 1, \dots, n$ respectively. If f is a simplicial polar weighted homogeneous polynomial, we have shown that the two fibrations defined by $f(\mathbf{z}, \bar{\mathbf{z}})$ and $g(\mathbf{w})$:

$$f: \mathbf{C}^{*n} \backslash f^{-1}(0) \to \mathbf{C}^*, \quad g: \mathbf{C}^{*n} \backslash g^{-1}(0) \to \mathbf{C}^*$$

are equivalent ([17]). Thus the topology of the Milnor fibration is determined by the mixed face $\hat{\Delta}$ where Δ is the unique face of $\Gamma(f)$. In particular, the zeta function of $h : F^* \to F^*$ is given as $\zeta(t) = (1 - t^{d_p})^{(-1)^n d/d_p}$ where $d = |\det(N)|$ ([17]). On the other hand, if f is not simplicial, the topology is not even a combinatorial invariant of $\hat{\Delta}$ (§6.2.6). Therefore there does not exist any direct connection with the topology of the associated Laurent polynomial $g(\mathbf{z})$. However here is a useful lemma.

Lemma 58. *Suppose that $f_t(\mathbf{z}, \bar{\mathbf{z}})$, $0 \le t \le 1$ is a family of convenient, non-degenerate polar weighted homogeneous polynomials with the same radial and the polar weights, and assume that $\Gamma(f_t)$ is constant. Then the Milnor fibration $f_t : \mathbf{C}^n \backslash f_t^{-1}(0) \to \mathbf{C}^*$ and its restriction $\mathbf{C}^{*n} \backslash f_t^{-1}(0)) \to \mathbf{C}^*$ are homotopically equivalent to $f_0 : \mathbf{C}^n \backslash f_0^{-1}(0) \to \mathbf{C}^*$ and $f_0 : \mathbf{C}^{*n} \backslash f_0^{-1}(0) \to \mathbf{C}^*$ respectively.*

Proof. Consider the unit sphere $S^{2n-1} = S_1^{2n-1}$. For each $I \subset \{1, \dots, n\}$, $|I| \ne \emptyset$, the intersection $(f_t^I)^{-1}(0) \cap S^I$ is transverse and smooth for any t where $S^I = \{\mathbf{z}^I \in \mathbf{C}^I \mid \|\mathbf{z}\| = 1\}$. Thus by the compactness argument, there exists a common positive number δ such that the intersection $(f_t^I)^{-1}(\eta) \cap S^I$, is transverse and smooth for any t, $0 \le t \le 1$ and η with $|\eta| \le \delta$. This implies by the Ehresmann fibration theorem ([24]) that the fibrations

$$f_t^I : E_t^I(1, \delta)^* \to D(\delta)^*$$

are equivalent for each t, where

$$E_t^I(1, \delta) = (f_t^I)^{-1}(D(\delta)^*) \cap B^I, \quad B^I = \{\mathbf{z}^I \in \mathbf{C}^I \mid \|\mathbf{z}^I\| \le 1\}.$$

Thus we can construct characteristic diffeomorphisms

$$h_\theta : f_t^{-1}(\delta) \cap B^{2n} \to f_t^{-1}(\delta \exp(\theta i)) \cap B^{2n}$$

for $0 \le \theta \le 2\pi$ which preserve the stratification $f^{-1}(\delta) \cap B^I$, $I \subset \{1, \dots, n\}$. Now the assertion follows from Theorem 37. □

Example 59. Consider the family of polar weighted mixed polynomials in two variables:

$$f_t(\mathbf{z}, \bar{\mathbf{z}}) = -2z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + tz_1^2\bar{z}_2, \quad t \in \mathbf{C}$$

and let $C_t = f_t^{-1}(0)$. The radial and polar weight types are $(1, 1; 3)$ and $(1, 1; 1)$ respectively. Thus the critical points of $f_t : \mathbf{C}^2 \to \mathbf{C}$ are the solutions of

$$|\alpha| = 1, \quad \begin{cases} -4z_1\bar{z}_1 + 2\bar{t}\bar{z}_1 z_2 = -2\alpha z_1^2 \\ 2z_2\bar{z}_2 = \alpha(z_2^2 + tz_1^2) \\ -2z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + tz_1^2\bar{z}_2 = 0. \end{cases} \tag{15}$$

First it is easy to see that for a solution $(\mathbf{z}, \alpha)$ of (16), either $\mathbf{z} = (0, 0)$ or $\mathbf{z} \in \mathbf{C}^{*2}$. Secondly the equations are homogeneous in z_1, z_2. Thus we may assume that $|z_2| = 1$. By (15), we get $2z_2\bar{z}_2^2 = 2\alpha z_1^2\bar{z}_1$. Thus $|z_1| = 1$. Put $z_1/z_2 = \exp(\theta i)$. Then we can solve as

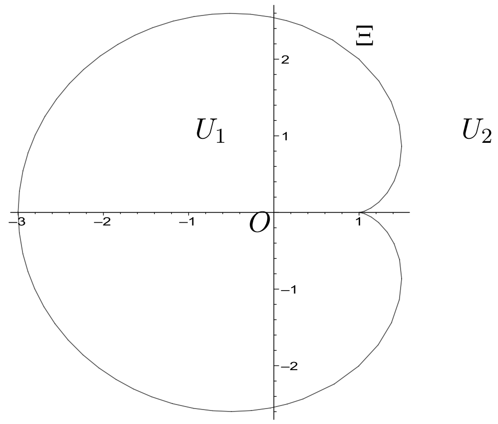

FIGURE 3. Degeneration locus Ξ

$$t = -\exp(-2\theta i) + 2\exp(-\theta i), \quad z_1 = z_2 \exp(\theta i), \quad \alpha = \frac{2}{z_2^2 + tz_1^2}.$$

Put $\Xi := \{-\exp(-2\theta i) + 2\exp(-\theta i) \mid 0 \le \theta \le 2\pi\}$. Ξ is the locus where f_t is degenerate. The complement $\mathbf{C} \backslash \Xi$ has two components, U_1, U_2 where U_1 is the bounded region with boundary Ξ. See Figure 3. By further calculation, we can see that $\operatorname{lkn}(C_t) = 1$, $\chi(F) = 1$, $\chi(F^*) = -1$ for $t \in U_1$ and $\operatorname{lkn}(C_t) = 3$, $\chi(F) = -1$, $\chi(F^*) = -3$ for $t \in U_2$. (See Appendix for the calculation.) The associated Laurent polynomial is $g_t(\mathbf{z}) = -2z_1 + z_2 + tz_1^2 z_2^{-1}$ which is non-degenerate for $t \neq 1, 0$. Thus we see that $\chi(G_t^*) = -2$ for $t \neq 0, 1$ where $G_t^* = g_t^{-1}(1) \cap \mathbf{C}^{*2}$ (see [16]). This example shows that Theorem 10 of [17] does not hold for non-simplicial polar weighted polynomials.

8.2. Zeta function of non-degenerate mixed curves. Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a convenient non-degenerate mixed polynomial and let $\Delta_1, \dots, \Delta_s$ be the faces of $\Gamma(f)$. Let $Q_j = {}^t(q_{j1}, q_{j2})$ be the weight vector of Δ_j for $j = 1, \dots, s$. Assume that each face function f_{Δ_j} is also polar weighted and the inside monomials corresponding to the vertices $M_j = \Delta_j \cap \Delta_{j+1}$, $j = 1, \dots, s-1$ are polar admissible. Let $(a_1 + 2b_1, 0)$, $(0, a_2 + 2b_2)$ be the vertices of $\Gamma(f)$ on the coordinate axes which come from the monomials $z_1^{a_1}|z_1|^{2b_1}$ and $z_2^{a_2}|z_2|^{2b_2}$ respectively. We call a_1, a_2 *the polar sections of* $\Gamma(f)$ on the respective coordinate axes $z_2 = 0$ and $z_1 = 0$. Let $f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}})$ be the face function of Δ_i and assume that $(p_{i1}, p_{i2}; m_i)$ is the polar weight type of $f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}})$. Let $F_i^* = \{\mathbf{z} \in \mathbf{C}^{*2} \mid f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}}) = 1\}$. Then we have the following.

THEOREM 60. *Assume that* $f(\mathbf{z}, \bar{\mathbf{z}})$ *is a non-degenerate convenient mixed polynomial such that its face functions* $f_{\Delta_j}(\mathbf{z}, \bar{\mathbf{z}})$, $j = 1, \dots, s$ *are polar weighted* *polynomials. Then the Euler-Poincaré characteristic of the Milnor fiber F of f and the zeta function of the monodromy $h : F \to F$ are given as follows.*

$$\chi(F) = \sum_{i=1}^{s} \chi(F_i^*) + |a_1| + |a_2|$$

$$\zeta(t) = \frac{\prod_{i=1}^{s}(1 - t^{m_i})^{-\chi(F_i^*)/m_i}}{(1 - t^{|a_1|})(1 - t^{|a_2|})}$$

where a_1, a_2 are the respective polar sections and m_j is the polar degree of the face function $f_{\Delta_i}(\mathbf{z}, \bar{\mathbf{z}})$, $j = 1, \dots, s$ as above $(m_j > 0)$.

Remark 61. The assertion is true for non-degenerate mixed polynomials with polar weighted face functions in two variables which may not be convenient. For example, if $\Gamma(f) \cap \{z_2 = 0\} = \emptyset$, we eliminate $|a_1|$ and $(1 - t^{|a_1|})$ from the formula.

The proof occupies the rest of the section. For the proof, we use the following multiplicative property of the zeta function. Consider an excision pair $\{A, B\}$ in the Milnor fiber F. We say $\{A, B\}$ is *stable* for the monodromy map h if $h(A) \subset A$ and $h(B) \subset B$.

Proposition 62 (Proposition 2.8, [16]). *Suppose that F decomposes into h stable excision couple A, B so that $F = A \cup B$. Put $C = A \cap B$. Then let $\zeta(t)$, $\zeta_A(t)$, $\zeta_B(t)$ and $\zeta_C(t)$ be the zeta functions of $h : F \to F$ and $h_A := h|_A : A \to A$, $h_B := h|_B : B \to B$ and $h_C := h|_C : C \to C$ respectively. Then*

$$\zeta(t) = \frac{\zeta_A(t)\zeta_B(t)}{\zeta_C(t)}.$$

8.2.1. *Resolution of a polar type and the Milnor fibration.* Let us consider an admissible toric modification $\hat{\pi} : X \to \mathbf{C}^2$ with respect to the regular fan Σ^* with vertices $\{P_0, P_1, \dots, P_{\ell+1}\}$ and we assume that $Q_j = P_{v_j}$, $j = 1, \dots, s$ and $P_0 = E_1 = {}^t(1, 0)$ and $P_{\ell+1} = E_2 = {}^t(0, 1)$. Then we take the polar modification $\omega_p : \mathscr{P}X \to X$ along $\hat{E}(P_1), \dots, \hat{E}(P_\ell)$. Put $\Phi_p : \mathscr{P}X \to \mathbf{C}^2$ be the composite with $\hat{\pi} : X \to \mathbf{C}^2$. Consider the second Milnor fibration

$$f \circ \Phi_p : \Phi_p^{-1}(E(r, \delta)^*) \to D(\delta)^*$$

on the resolution space $\mathscr{P}X$. Take P_j for $1 \le j \le \ell$. There are two toric coordinate charts of X which contain the vertex P_j:

$\sigma_{j-1} = \mathrm{Cone}(P_{j-1}, P_j)$ gives the coordinate chart $(U_{j-1}, (u_{j-1}, v_{j-1}))$

$\sigma_j = \mathrm{Cone}(P_j, P_{j+1})$ gives the coordinate chart $(U_j, (u_j, v_j))$.

Put $M = (P_j, P_{j+1})^{-1}(P_{j-1}, P_j)$. It takes the form:

$$M = \begin{pmatrix} \gamma_j & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the two coordinate systems are connected by the relation

$$u_j = u_{j-1}^{\gamma_j} v_{j-1}, \quad v_j = u_{j-1}^{-1}. \tag{16}$$

Put $P_j = {}^t(c_j, d_j)$, $j = 1, \dots, \ell$. The inverse image $\tilde{U}_j := \omega_p^{-1}(U_j)$ has the polar coordinates $(r_j, \theta_j, s_j, \eta_j)$ which corresponds to (u_j, v_j) with $u_j = r_j \exp(i\theta_j)$ and $v_j = s_j \exp(i\eta_j)$. The relation (16) says that

$$s_j = r_{j-1}^{-1}, \quad \eta_j = -\theta_{j-1}. \tag{17}$$

We do not take a normal polar modification along the two non-compact divisors $u_0 = 0$ and $v_\ell = 0$. Thus the coordinates of $\tilde{U}_0$ and $\tilde{U}_\ell$ are (u_0, s_0, η_0) and $(r_\ell, \theta_\ell, v_\ell)$ respectively. Recall that the exceptional divisor $\tilde{E}(P_j)$ is defined by $r_j = 0$ in $\tilde{U}_j$ and by $s_{j-1} = 0$ in $\tilde{U}_{j-1}$ for $1 \le j \le \ell$. Note that $u_0 = 0$ in U_0 corresponds bijectively to the axis $z_1 = 0$ in the base space $\mathbf{C}^2$ and

$$(P_0, P_1) = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \quad d_1 = 1, \quad z_1 = u_0 v_0^{c_1}, \quad z_2 = v_0.$$

Similarly on $\tilde{U}_\ell$, $v_\ell = 0$ corresponds to $z_2 = 0$ and

$$z_1 = u_\ell, \quad z_2 = u_\ell^{d_\ell} v_\ell, \quad c_\ell = 1.$$

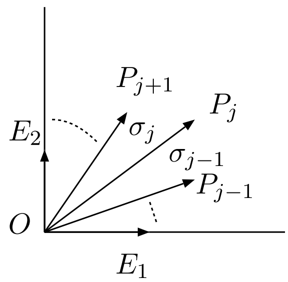

FIGURE 4. Regular fan Σ^*

8.3. Decomposition of the fiber. Recall that

$$E(r,\delta)^* = \{(z_1, z_2) \mid 0 < |f(z_1, z_2, \bar{z}_1, \bar{z}_2)| \le \delta, \|(z_1, z_2)\| \le r\}$$

$$\phi(\mathbf{z}) := \sqrt{|z_1|^2 + |z_2|^2}, \quad \tilde{B}_r = \phi^{-1}(B_r)$$

$$F_\delta = \{(z_1, z_2) \mid f(z_1, z_2, \bar{z}_1, \bar{z}_2) = \delta, (z_1, z_2) \in B_r\} : \text{Milnor fiber.}$$

We denote the pull-back of a function h on $\mathbf{C}^2$ to $\mathscr{P}X$ by $\tilde{h}$ for simplicity. On $\mathscr{P}X$, we consider the subsets

$$W_j(r, \rho) = \{\tilde{\mathbf{x}} = (r_j, \theta_j, s_j, \eta_j) \in \tilde{U}_j \mid 1/\rho \ge s_j \ge \rho\}$$

$$T_{j-1}(\rho) = \{(r_{j-1}, \theta_{j-1}, s_{j-1}, \eta_{j-1}) \in \tilde{U}_{j-1} \mid r_{j-1} \le \rho, s_{j-1} \le \rho\}$$

$$WT_j(\rho) = \{(r_j, \theta_j, s_j, \eta_j) \in \tilde{U}_j \mid s_j = \rho, r_j \le \rho\}$$

$$TW_j(\rho) = \{(r_{j-1}, \theta_{j-1}, s_{j-1}, \eta_{j-1}) \in \tilde{U}_{j-1} \mid r_{j-1} = \rho, s_{j-1} \le \rho\}$$

and

$$T_0(\rho) := \{(u_0, s_0, \eta_0) \in \tilde{U}_0 \mid |u_0| \le \rho, s_0 \le \rho\}$$
$$W_0(r, \rho) := \{(u_0, s_0, \eta_0) \in \tilde{U}_0 \mid \tilde{\phi}(u_0, s_0, \eta_0) \le r, |u_0| \ge \rho, s_0 \ge \rho\}$$
$$T_\ell(\rho) := \{(r_\ell, \theta_\ell, v_\ell) \in \tilde{U}_\ell \mid r_\ell \le \rho, |v_\ell| \le \rho\}$$
$$W_\ell(r, \rho) := \{(r_\ell, \theta_\ell, v_\ell) \in \tilde{U}_\ell \mid r_\ell \ge \rho, |v_\ell| \ge \rho, \tilde{\phi}(r_\ell, \theta_\ell, v_\ell) \le r\}$$

Note that

$$\tilde{\phi}(u_0, s_0, \eta_0) = s_0\sqrt{1 + |u_0|^2 s_0^{2c_1-2}} = s_0 + o(s_0)$$
$$\tilde{\phi}(r_\ell, \theta_\ell, v_\ell) = r_\ell\sqrt{1 + |v_\ell| r_\ell^{2d_\ell-2}} = r_\ell + o(r_\ell)$$

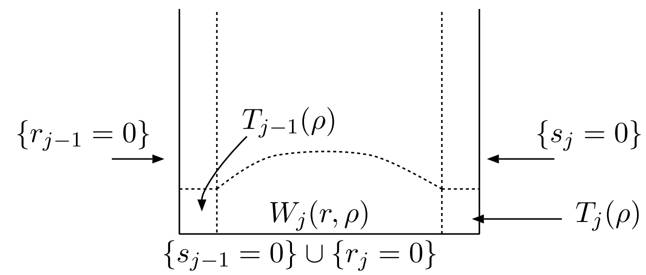

FIGURE 5. Decomposition of $\mathscr{P}X$

Here $o(s_0)$ implies $o(s_0)/s_0 \to 0$ when $s_0 \to 0$. Put

$$A(r, \rho) = \bigcup_{j=0}^{\ell+1} W_j(r, \rho) \cup \bigcup_{j=0}^{\ell} T_j(\rho).$$

Put $\tilde{E}(r, \delta)^* = \Phi_p^{-1}(E(r, \delta)^*)$ with $\delta \ll r$ and $A(r, \rho, \delta)^* = A(r, \rho) \cap \tilde{f}^{-1}(D_\delta^*)$ with $\delta \ll r, \rho$. It is easy to see that $A(r, \rho, \delta)^* = \tilde{E}(r, \delta)^*$ as long as $\rho \ll r$ and $\delta \ll \rho, r$. We see that the choice of ρ does not give any effect on $A(r, \rho, \delta)^*$, as long as $\delta \ll \rho \ll r$. Thus we can use $A(r, \rho, \delta)^*$ as the total space of the Milnor fibration: $\tilde{f} : A(r, \rho, \delta)^* \to D_\delta^*$. We decompose $A(r, \rho, \delta)^*$ into monodromy invariant subspaces as follows.

$$A(r, \rho, \delta)^* \cap W_j(r, \rho), \quad A(r, \rho, \delta)^* \cap T_j(\rho)$$
$$A(r, \rho, \delta)^* \cap TW_j(\rho), \quad A(r, \rho, \delta)^* \cap WT_j(\rho), \quad j = 0, \dots, \ell.$$

8.3.1. *Transversality*. Assume that $\Delta(P_j) = \Delta_t \cap \Delta_{t+1} = \{M_t\}$ and that M_t comes from the monomial $z_1^{\alpha_{t1}}|z_1|^{2\beta_{t1}} z_2^{\alpha_{t2}}|z_2|^{2\beta_{t2}}$. By the definition we can write

$$\tilde{f}(r_j, \theta_j, s_j, \eta_j) \equiv r_j^{d(P_j)} s_j^{d(P_{j+1})} \exp((\alpha_{t1} c_j + \alpha_{t2} d_j)\theta_j i) \\ \times \exp((\alpha_{t1} c_{j+1} + \alpha_{t2} d_{j+1})\eta_j i) + O(r_j^{d(P_j)+1}).$$

Thus it is easy to see that $\tilde{f}^{-1}(\xi)$, $|\xi| = \delta$ intersects transversely with $WT_j(\rho)$ if δ is sufficiently small and $\delta \ll r, \rho$. Similarly $\tilde{f}^{-1}(\xi)$ intersects transversely with $TW_j(\rho)$ under the same assumptions.

Fix such r, δ, ρ. Under the above decomposition of $A(r, \rho, \delta)^*$, the Milnor fiber $\tilde{F}_\delta := \tilde{f}^{-1}(\delta) \cap \tilde{B}$ decomposes into the following strata:

$$\tilde{F}_\delta \cap W_j(r, \rho), \quad \tilde{F}_\delta \cap T_j(\rho), \quad \tilde{F}_\delta \cap WT_j(\rho), \quad \tilde{F}_\delta \cap TW_j(\rho), \quad j = 0, \dots, \ell.$$

By the above transversality, we see that (after choosing a suitable vector field to define the characteristic diffeomorphisms) $\tilde{F}_\delta \cap W_j(r, \rho)$, $\tilde{F}_\delta \cap T_j(\rho)$, $\tilde{F}_\delta \cap TW_j(\rho)$ and $\tilde{F}_\delta \cap WT_j(\rho)$ are invariant by the monodromy $h : \tilde{F}_\delta \to \tilde{F}_\delta$. Now the proof of Theorem 60 follows from the following observations.

(1) The zeta functions of h restricted on $\tilde{F}_\delta \cap T_j(\rho)$ are trivial for $1 \le j \le \ell - 1$.
(2) The zeta functions of h restricted on $\tilde{F}_\delta \cap W_j(r, \rho)$ with $j \ne \nu_1, \dots, \nu_s$ are trivial.
(3) The zeta functions of h restricted on $\tilde{F}_\delta \cap WT_j(\rho)$ and $\tilde{F}_\delta \cap TW_j(\rho)$ are trivial.
(4) The zeta functions of h on $\tilde{F}_\delta \cap T_0(\rho)$ and $\tilde{F}_\delta \cap T_\ell(\rho)$ are respectively given by

$$\frac{1}{(1 - t^{|a_2|})}, \quad \frac{1}{(1 - t^{|a_1|})}.$$

(5) (Face contribution) The zeta function of $h : \tilde{F}_\delta \cap W_{\nu_j}(\rho)$ is $(1 - t^{m_j})^{-\chi(F_j^*)/m_j}$ where $F_j^* = f_{\Delta_j}^{-1}(1) \cap \mathbf{C}^{*2}$ and m_j is the polar degree of f_{Δ_j}.

8.4. Outline of the proof of the assertions (1) to (5).

(1) Consider $\tilde{F}_\delta \cap T_j(\rho)$. Assume that $\Delta(P_j) = \Delta_t \cap \Delta_{t+1} = \{M_j\}$ and that M_j comes from the monomial $z_1^{\alpha_{t1}} |z_1|^{2\beta_{t1}} z_2^{\alpha_{t2}} |z_2|^{2\beta_{t2}}$ as above. Then

$$\tilde{F}_\delta \cap T_j(\rho) = \{(r_j, \theta_j, s_j, \eta_j) \mid r_j, s_j \le \rho, \tilde{f}(r_j, \theta_j, s_j, \eta_j) = \delta\}.$$

$\tilde{f}(r_j, \theta_j, s_j, \eta_j)$ takes the form

$$\tilde{f}(r_j, \theta_j, s_j, \eta_j) \equiv c_{M_t} r_j^{d(P_j)} s_j^{d(P_{j+1})} \exp((\alpha_{t1} c_j + \alpha_{t2} d_j)\theta_j i) \\ \times \exp((\alpha_{t1} c_{j+1} + \alpha_{t2} d_{j+1})\eta_j i) + O(r_j^{d(P_j)+1} s_j^{d(P_{j+1})+1})$$

(c_{M_t} is a non-zero constant) and the homotopy type of this part of the Milnor fiber is given by

$$\{(\theta_j, \eta_j) \in S^1 \times S^1 \mid c_{M_t} \exp(((\alpha_{t1} c_j + \alpha_{t2} d_j)\theta_j + (\alpha_{t1} c_{j+1} \alpha_{t2} d_{j+1})\eta_j) i) = 1\}$$

which is a finite union of copies of S^1 by the following.

OBSERVATION 63. *Let a, b be integers with $(a,b) \neq (0,0)$ and let*

$$F = \{(\exp(\theta i), \exp(\eta i)) \in S^1 \times S^1 \mid \exp((a\theta + b\eta)i) = 1\}.$$

Then F is a disjoint union of copies of S^1 and the number of S^1 is $\gcd(a,b)$.

The monodromy acts as the permutation of the components and we see that the characteristic polynomials on the 0-th homology and the 1-th homology is the same. Thus the zeta function is trivial. The assertion (3) can be shown in the same way.

Let us see the assertion (2). By the same argument,

$$\tilde{F}_\delta \cap W_j(\rho) = \{\tilde{f}(r_j, \theta_j, s_j, \eta_j) = \delta, 1/\rho \geq s_j \geq \rho\}$$

and by throwing away higher terms, we may consider that $\tilde{f}$ is again homotopically defined by

$$c_{M_t} \exp((\alpha_{t1} c_j + \alpha_{t2} d_j)\theta_j i + (\alpha_{t1} c_{j+1} + \alpha_{t2} d_{j+1})\eta_j i)$$

Again we see that the Milnor fiber is fibered over the interval $\{\rho \leq s_j \leq 1/\rho\} = [\rho, 1/\rho]$ with fiber being a finite union of S^1's. Thus the zeta function is again trivial. (Recall that $r_{j-1} = 1/s_j$.)

(4) Let us consider the fibration restricted on $T_0(\rho)$. The situation is different from that of (3). Let $M_0 = \Gamma(f) \cap \{z_1 = 0\}$ and assume it comes from the monomial $z_2^{a_2}|z_2|^{2b_2}$. The pull back function takes the form:

$$\tilde{f}(u_0, s_0, \eta_0) = c_{M_0} s_0^{a_2 + 2b_2} \exp(a_2 \eta_0 i) + O(s_0^{a_2 + 2b_2 + 1})$$

and throwing away the higher term and putting $c_{M_0} = \tau_0 \exp(\xi i)$, we see that $\tilde{F}_\delta \cap T_0(\rho)$ consists of a_2-contractible components:

$$\tilde{F}_\delta \cap T_0(\rho) = \{(u_0, s_0, \eta_0) \mid \tau_0 s_0^{a_2 + 2b_2} = \delta, \xi + a_2 \eta_0 \equiv 0 \text{ modulo } 2\pi\}.$$

More precisely, 'throwing away' implies the following standard discussion. Consider the family of functions

$$\tilde{f}_\tau(u_0, s_0, \eta_0) := c_{M_0} s_0^{a_2 + 2b_2} \exp(a_2 \eta_0 i) + \tau O(s_0^{a_2 + 2b_2 + 1}), \quad 0 \leq \tau \leq 1.$$

In the level of the original function f, this corresponds to the family $f_\tau = c_{M_0} \mathbf{z}^{M_0} + \tau(f(\mathbf{z}, \bar{\mathbf{z}}) - c_{M_0} \mathbf{z}^{M_0})$. Consider the strata of the respective Milnor fibers restricted in this neighborhood $T_0(\rho)$ and their union:

$$\tilde{F}_{\delta,\tau} = \{(u_0, s_0, \eta_0) \mid \tilde{f}_\tau(u_0, s_0, \eta_0) = \delta, (u_0, s_0, \eta_0) \in T_0(\rho)\}$$

$$\tilde{\mathscr{F}}_\delta = \{(u_0, s_0, \eta_0, \tau) \mid \tilde{f}_\tau(u_0, s_0, \eta_0) = \delta, (u_0, s_0, \eta_0, \tau) \in T_0(\rho) \times [0,1]\}.$$

Taking δ sufficiently small, we may assume that $\tilde{F}_{\delta,\tau}$ is smooth and intersects transversely with the boundary of $T_0(\rho)$ for any $0 \leq \tau \leq 1$. Now we apply the Ehresmann fibering theorem ([24]) to the projection $\pi : \tilde{\mathscr{F}}_\delta \to [0,1]$ and we conclude that the Milnor fibers $\tilde{F}_{\delta,\tau}$, $0 \leq \tau \leq 1$ are diffeomorphic to $\tilde{F}_{\delta,0}$. (We apply this argument to each case (1) to (5).)

Thus using the Milnor fiber $\tilde{F}_{\delta,0}$, we see that each component is homeomorphic to a disk $\{u_0 \mid |u_0| \le \rho\}$, as the above equation has a_2 solutions for η_0. The monodromy is acting cyclically among these components. Thus the zeta function of this restriction is $1/(1-t^{|a_2|})$.

We see also that $\tilde{F}_\delta \cap W_0(\rho) = \emptyset$ if $\delta \ll \rho$.

The other edge $T_\ell(\rho)$ gives the term $1/(1-t^{|a_1|})$.

(5) Now we consider the restriction of $W_{\nu_j}(\rho)$. Then the principal part takes the form

$$\tilde{f}(r_j,\theta_j,s_j,\eta_j) = \tilde{f}_{\Delta_j}(r_j,\theta_j,s_j,\eta_j) + O(r^{d(P_{\nu_j})+1})$$

where $\mathrm{ord}_{r_j} f_{\Delta_j}(r_j,\theta_j,s_j,\eta_j) = d(P_{\nu_j})$ and the Milnor fibering restricted on this stratum $W_{\nu_j}(\rho)$ is determined by the principal part $\tilde{f}_{\Delta_j}(r_j,\theta_j,s_j,\eta_j)$. The last work for us is to determine this contribution.

Consider the curve $C_j = \{f_{\Delta_j}(\mathbf{z},\bar{\mathbf{z}}) = 0\}$ and its polar type resolution by the same mapping $\Phi_p : \mathscr{P}X \to \mathbf{C}^2$. By the polar admissibility assumption of the inside vertices, the Milnor fibration of the second description exists and it is equivalent to the Milnor fibration of the first description by Theorem 52. Then combining the assertions (1) to (4) applied for C_j, we see that the above contribution is nothing but the zeta function of the monodromy of $f_{\Delta_j} : \mathbf{C}^{*2} \backslash f_{\Delta_j}^{-1}(0) \to \mathbf{C}^*$, which is given by $(1-t^{m_j})^{-\chi(F_j^*)/m_j}$ as we have seen in Lemma 55 and Theorem 52. Note that $\tilde{F}_\delta \cap W_0(\rho) = \emptyset$ and $\tilde{F}_\delta \cap W_{\ell+1}(\rho) = \emptyset$. This completes the proof of Theorem 60.

8.5. Topology of a polar weighted polynomial and Kouchnirenko type formula. We consider a non-degenerate polar weighted mixed polynomial $f_\Delta(z_1,z_2,\bar{z}_1,\bar{z}_2)$ with $\Delta = \overline{AB}$ where A, B are polar admissible simple vertices. Let $(p_1,p_2;m_\Delta)$ be the polar weight type. Let $F_\Delta = f_\Delta^{-1}(1)$ be the fiber of the global fibration, $F_\Delta^* = F_\Delta \cap \mathbf{C}^{*2}$ and let $K_\Delta = f_\Delta^{-1}(0) \cap S^3$. Note that F_Δ is diffeomorphic to the fiber of the Milnor fibration $f_\Delta/|f_\Delta| : S^3 \backslash K_\Delta \to S^1$ or $f_\Delta : \partial E(r,\delta)^* \to S_\delta^1$, as f_Δ is super strongly non-degenerate by Theorem 52. The Milnor fiber is connected by Proposition 38. Let $P_i(t)$ be the characteristic polynomial of the monodromy at the i-th homology for $i = 0,1$. Then $P_1(t) = \zeta(t)(1-t)$ as $P_0(t) = (1-t)$. We consider the Wang sequence of the Milnor fibration:

$$0 \longrightarrow H_2(S^3 - K_\Delta) \longrightarrow H_1(F_\Delta) \xrightarrow{h_* - \mathrm{id}} H_1(F_\Delta) \longrightarrow H_1(S^3 - K_\Delta) \longrightarrow \mathbf{Z} \longrightarrow 0.$$

Put $r_\Delta^* = \mathrm{lkn}^*(f_\Delta^{-1}(0))$. Thus $H_0(K_\Delta) = \mathbf{Z}^{r_\Delta^* + \varepsilon(\Delta)}$ where $\varepsilon(\Delta)$ is the number of coordinate axes which are a subset of $f_\Delta^{-1}(0)$. Thus $\varepsilon(\Delta) = 0,1,2$ according to the two vertices A, B are either on the axis or not. Let μ_Δ and μ'_Δ be the multiplicities of the factor $(t-1)$ in $P_1(t)$ and $\zeta(t)$ respectively. Then by the equality $P_1(t) = \zeta(t)(1-t)$ and Lemma 56 and Remark 57,

$$\mu_\Delta = \mu'_\Delta + 1, \quad \mu'_\Delta = -\chi(F_\Delta^*)/m_\Delta - 2 + \varepsilon(\Delta).$$

On the other hand by the Alexander duality, we have the isomorphism:

$$H_2(S^3 - K_\Delta) \cong H^1(S^3, K_\Delta) \cong \tilde{H}^0(K_\Delta).$$

As the monodromy map h_* is periodic, we have

$$r_\Delta^* + \varepsilon(\Delta) - 1 = \dim \operatorname{Ker}\{h_* - \mathrm{id} : H_1(F_\Delta) \to H_1(F_\Delta)\} = \mu_\Delta.$$

Thus we obtain

LEMMA 64. *The Euler-Poincaré characteristic and the link component number satisfy the following equality*:

$$r_\Delta^* = -\chi(F_\Delta^*)/m_\Delta.$$

Usually it is easier to compute r_Δ^* and we can compute $\chi(F_\Delta^*)$ by Lemma 64. Now we can state our Kouchnirenko type formula:

THEOREM 65. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a non-degenerate convenient mixed polynomial as in Theorem 60. Let $\Delta_1, \dots, \Delta_s$ be faces of $\Gamma(f)$ and we assume that $f_{\Delta_j}(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial with polar degree m_j. Let $r_j = \mathrm{lkn}^*(f_{\Delta_j}^{-1}(0))$ for $j = 1, \dots, s$. Then the Milnor number $\mu(F) = b_1(F)$ is given by the formula*:

$$\mu(F) = \sum_{j=1}^{s} r_j m_j - |a_1| - |a_2| + 1.$$

Here m_j is the polar degree of f_{Δ_j} and we assume that $m_j > 0$. a_1, a_2 are the polar sections of $\Gamma(f)$ on the respective coordinate axes.

As a special case, the following is a formula for a good polar weighted mixed polynomial (see §6.2.3 for the definition) which corresponds to the Orlik-Milnor formula [13] for a weighted homogeneous isolated singularity.

COROLLARY 66. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a good polar weighted polynomial which is factored as*

$$f(\mathbf{z}, \bar{\mathbf{z}}) = c \prod_{j=1}^{k} (z_2^a |z_2|^{2a'} - \lambda_j z_1^b |z_1|^{2b'}), \quad c \neq 0 \tag{18}$$

with $a \neq 0$, $b \neq 0$. Let $r = \gcd(|a|, |b|)$. The polar weight is given by $P = {}^t(p_1\varepsilon_1, p_2\varepsilon_2)$ where $p_1 = |a|/r$, $p_2 = |b|/r$, $\varepsilon_1 = b/|b|$, $\varepsilon_2 = a/|a|$ and the polar degree d_p is given as $d_p = |a|\,|b|k/r$, $\mathrm{lkn}(f^{-1}(0)) = rk$ and

$$\mu = |a|\,|b|k^2 - k(|a| + |b|) + 1 = (k|a| - 1)(k|b| - 1) \quad \textit{and}$$

$$\zeta(t) = \frac{(1 - t^{d_p})^{rk}}{(1 - t^{|a|})(1 - t^{|b|})}.$$

8.6. Appendix: Calculation of Example 8.1.2. We give the detail of the calculation for Example 8.1.2. Let

$$f_t(\mathbf{z},\bar{\mathbf{z}}) = -2z_1^2\bar{z}_1 + z_2^2\bar{z}_2 + tz_1^2\bar{z}_2, \quad t \in \mathbf{C}$$
$$V_t^* := \{(z_1, z_2) \in \mathbf{C}^{*2} \mid f_t(\mathbf{z},\bar{\mathbf{z}}) = 0\}$$
$$F_t^* := \{(z_1, z_2) \in \mathbf{C}^{*2} \mid f_t(\mathbf{z},\bar{\mathbf{z}}) = 1\}.$$

and we compute link components. As f_t is radially weighted, we may assume that $|z_2| = 1$. Thus we compute the section with $|z_2| = 1$. We put

$$z_1 = x_1 + y_1 i, \quad z_2 = x_2 + y_2 i, \quad x_2 = \cos(a), \quad y_2 = \sin(\theta),$$

Then $f_t(\mathbf{z},\bar{\mathbf{z}}) = 0$ can be rewritten as $f_1 = f_2 = 0$ where

$$f_1 = -2x_1^3 - 2x_1y_1^2 + (\cos(a))^3 + \cos(a)(\sin(a))^2 + tx_1^2\cos(a)$$
$$+ 2tx_1y_1\sin(a) - ty_1^2\cos(a)$$
$$f_2 = -2x_1^2y_1 - 2y_1^3 + (\cos(a))^2\sin(a) + (\sin(a))^3 - tx_1^2\sin(a)$$
$$+ 2tx_1y_1\cos(a) + ty_1^2\sin(a)$$

The resultant R of f_1 and f_2 in y_1 takes the form $R = g_1g_2$ where

$$g_1 = 2x_1^3 - (\cos(a))^3 - tx_1^2\cos(a)$$
$$g_2 = t^4x_1^2 - t^3(\sin(a))^2 - 2\cos(a)x_1t^2 + (\cos(a))^2 + (\sin(a))^2$$

U_1: Assume that $t = 0$. Then $g_2 \equiv 1$. The equation $g_1 = 0$, $f_1 = f_2 = 0$ has a unique solution

$$\begin{cases} x_1 = \frac{1}{2}2^{2/3}\cos(a) \\ y_1 = \frac{1}{2}2^{2/3}\sin(a) \\ x_2 = \cos(a),\ y_2 = \sin(a), \end{cases} \qquad 0 \le a \le 2\pi.$$

This can be also observed by [20].

U_2: Consider the case $t = 3$ as a model of V_t, $t \in U_3$. First, g_1, g_2 takes the following form.

$$g_1 = 2x_1^3 - (\cos(a))^3 - 3x_1^2\cos(a)$$
$$g_2 = 81x_1^2 - 26(\sin(a))^2 - 18\cos(a)x_1 + (\cos(a))^2$$

Over $g_1 = 0$, we have one component parametrized as

$$x_1 = \left(\frac{1}{2}\sqrt[3]{3+2\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt[3]{3+2\sqrt{2}}} + \frac{1}{2}\right)\cos(a)$$

$$y_1 = -\frac{\sin(a)}{(3+2\sqrt{2})^{2/3} - (3+2\sqrt{2})^{2/3}\sqrt{2} - \sqrt[3]{3+2\sqrt{2}} + \sqrt[3]{3+2\sqrt{2}}\sqrt{2}}$$

$0 \le a \le 2\pi$.

Over $g_2 = 0$, we have two components parametrized as

$$x_1 = \frac{1}{9}\cos(a) \pm \frac{1}{9}\sqrt{26}\sin(a)$$

$$y_1 = \frac{1}{234}(\sqrt{26}\sin(a) \pm 26\cos(a))\sqrt{26}, \quad 0 \le a \le 2\pi.$$

Thus we have shown that $\mathrm{lkn}(V_0) = 1$ and $\mathrm{lkn}(V_3) = 3$.

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