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Non-Deterministic Distance Semantics for Handling Incomplete and Inconsistent Data

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Abstract. We introduce a modular framework for formalizing reasoning with incomplete and inconsistent information. This framework is composed of non-deterministic semantic structures and distance-based considerations. The combination of these two principles leads to a variety of entailment relations that can be used for reasoning about non-deterministic phenomena and are inconsistency-tolerant. We investigate the basic properties of these entailments and demonstrate their usefulness in the context of model-based diagnostic systems.

1   Introduction

In this paper, we propose a general framework for representing and reasoning with uncertain information and demonstrate this in the context of model-based diagnostic systems. Our framework consists of two main ingredients:

• *Semantic structures for describing incompleteness:* The principle of truth functionality, according to which the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas, is in an obvious conflict with non-deterministic phenomena and other unpredictable situations in everyday life. To handle this, Avron and Lev [6] introduced *non-deterministic matrices* (Nmatrices), where the value of a complex formula can be chosen *non-deterministically* out of a certain nonempty set of options. This idea turns out to be very useful for providing semantics to logics that handle uncertainty (see [4]). In this paper, we incorporate this idea and consider some additional types of (non-determinisitic) semantic structures for describing incompleteness.

• *Distance-based considerations for handling inconsistency:* Logics induced by Nmatrices are inconsistency-intolerant: whenever a theory has no models in a structure, everything follows from it, and so it becomes useless. To cope with this, we incorporate *distance-based reasoning*, a common technique for reflecting the principle of minimal change in different scenarios where information is dynamically evolving, such as belief revision, data-source mediators, and decision making in the context of social choice theory. Unlike 'standard' semantics, in which conclusions are drawn according to the models of the premises, reasoning

in distance-based semantics is based on the valuations that are *'as close as possible'* to the premises, according to a pre-defined metric. As this set of valuations is never empty, reasoning with inconsistent set of premises is not trivialized.

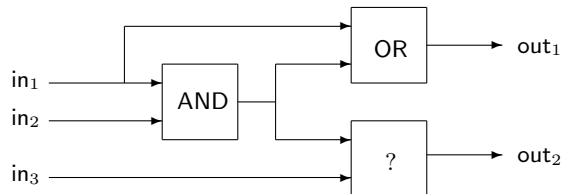*Example 1.* Consider the circuit that is represented in Figure 1.

Fig. 1. The circuit of Example 1.

Here, partial information (e.g., when it is unknown whether the ?-gate is an AND or an OR gate) may handled by non-deterministic semantics (see Example 7), and conflicting evidences (e.g., that the input line in_1 and the output line out_1 always have opposite values) can be handled by the incorporation of distance-based considerations (see Example 12).

In [2] Nmatrices were first combined with distance considerations and some properties of the resulting framework were investigated. This paper generalizes these results in two aspects: First, we incorporate new types of structures into the framework and study the relations among them. Secondly, we define new methods of constructing distance functions, tailored specifically for non-deterministic semantics, some of them are a conservative extension of well-known distances used in the classical case. The robustness of what is obtained for reasoning with uncertainty is demonstrated in the context of model-based diagnosis.

2 Semantic Structures for Incomplete Data

2.1 Preliminaries

Below, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{W}_{\mathcal{L}} = \{\psi, \phi, \ldots\}$ of well-formed formulas. $\mathsf{Atoms} = \{p, q, r \ldots\}$ are the atomic formulas in $\mathcal{W}_{\mathcal{L}}$. A theory Γ is a finite set of formulas in $\mathcal{W}_{\mathcal{L}}$. $\mathsf{Atoms}(\Gamma)$ and $\mathsf{SF}(\Gamma)$ denote, respectively, the atoms appearing in the formulas of Γ, and the subformulas of Γ.

Given a propositional language $\mathcal{L}$, a *propositional logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a consequence relation for $\mathcal{L}$, as defined below:

Definition 1. A (Tarskian) *consequence relation* for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas in $\mathcal{W}_{\mathcal{L}}$ and formulas in $\mathcal{W}_{\mathcal{L}}$, satisfying:

Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

2.2 Matrices, Nmatrices and Their Families

We start with the simplest semantic structures used for defining logics: *many-valued (deterministic) matrices* (see, e.g., [12] and [14]).

Definition 2. A (deterministic) *matrix* for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$.

A matrix $\mathcal{M}$ induces the usual semantic notions: An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that for each n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote by $\Lambda^{\mathsf{s}}_{\mathcal{M}}$ the set of all the $\mathcal{M}$-valuations of $\mathcal{L}$.[3] A valuation $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of ψ (or $\mathcal{M}$-*satisfies* ψ), if it belongs to $mod^{\mathsf{s}}_{\mathcal{M}}(\psi) = \{\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. A formula ψ is $\mathcal{M}$-*satisfiable* if $mod^{\mathsf{s}}_{\mathcal{M}}(\psi) \neq \emptyset$ and it is an $\mathcal{M}$-*tautology* if $mod^{\mathsf{s}}_{\mathcal{M}}(\psi) = \Lambda^{\mathsf{s}}_{\mathcal{M}}$. The $\mathcal{M}$-models of a theory Γ are the elements of the set $mod^{\mathsf{s}}_{\mathcal{M}}(\Gamma) = \cap_{\psi \in \Gamma} mod^{\mathsf{s}}_{\mathcal{M}}(\psi)$.

Definition 3. The relation $\vdash^{\mathsf{s}}_{\mathcal{M}}$ that is induced by a matrix $\mathcal{M}$ is defined for every theory Γ and formula $\psi \in \mathcal{W}_{\mathcal{L}}$ by $\Gamma \vdash^{\mathsf{s}}_{\mathcal{M}} \psi$ if $mod^{\mathsf{s}}_{\mathcal{M}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathcal{M}}(\psi)$.

It is well-known that $\vdash^{\mathsf{s}}_{\mathcal{M}}$ is a consequence relation in the sense of Definition 1.

Deterministic matrices do not always faithfully represent incompleteness. This brings us to the second type of structures, called *non-deterministic matrices*, where the truth-value of a complex formula is chosen non-deterministically out of a set of options.

Definition 4. [6] A *non-deterministic matrix* (Nmatrix) for $\mathcal{L}$ is a tuple $\mathcal{N} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{N}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$.

Example 2. Consider an AND-gate, $\diamond_1$, that operates correctly when its inputs have the same value and is unpredictable otherwise, and another gate, $\diamond_2$, that operates correctly, but it is not known whether its is an OR or a XOR gate. These gates may described by the following non-deterministic truth-tables:

$\widetilde{\diamond}_1$	t	f
t	{t}	{t, f}
f	{t, f}	{f}

$\widetilde{\diamond}_2$	t	f
t	{t, f}	{t}
f	{t}	{f}

Non-determinism can be incorporated into the truth-tables of the connectives by either a *dynamic* [6] or a *static* [5] approach, as defined below.

Definition 5. Let $\mathcal{N}$ be an Nmatrix for $\mathcal{L}$.

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[3] The '**s**', standing for 'static' semantics, is for uniformity with later notations.

– A *dynamic* $\mathcal{N}$*-valuation* is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$:

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}_{\mathcal{N}}(\nu(\psi_1), \ldots, \nu(\psi_n)). \qquad (1)$$

– A *static* $\mathcal{N}$*-valuation* is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ that satisfies condition (1) and the following compositionality principle: for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_n \in \mathcal{W}_{\mathcal{L}}$,

$$\text{if } \forall\, 1 \le i \le n \ \nu(\psi_i) = \nu(\phi_i), \text{ then } \nu(\diamond(\psi_1, \ldots, \psi_n)) = \nu(\diamond(\phi_1, \ldots \phi_n)). \quad (2)$$

We denote by $\Lambda_{\mathcal{N}}^{\mathsf{d}}$ the space of the dynamic $\mathcal{N}$-valuations and by $\Lambda_{\mathcal{N}}^{\mathsf{s}}$ the static $\mathcal{N}$-valuations. Clearly, $\Lambda_{\mathcal{N}}^{\mathsf{s}} \subseteq \Lambda_{\mathcal{N}}^{\mathsf{d}}$.

In both of the semantics considered above, the truth-value $\nu(\diamond(\psi_1, \ldots, \psi_n))$ assigned to the formula $\diamond(\psi_1, \ldots, \psi_n)$ is selected non-deterministically from a set of possible truth-values $\widetilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. In the dynamic approach this selection is made separately, independently for each tuple $\langle \psi_1, \ldots, \psi_n \rangle$, and $\nu(\psi_1), \ldots, \nu(\psi_n)$ do not uniquely determine $\nu(\diamond(\psi_1, \ldots, \psi_n))$. In the static semantics this choice is made globally, and so the interpretation of $\diamond$ is a function.

Note 1. In ordinary (deterministic) matrices each $\widetilde{\diamond}$ is a function having singleton values only (thus it can be treated as a function $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$). In this case the sets of static and dynamic valuations coincide, as we have full determinism.

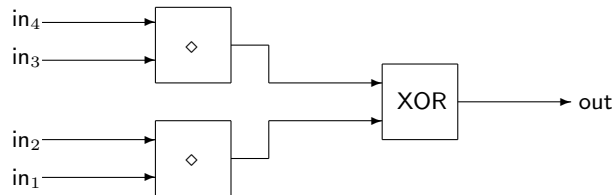*Example 3.* Consider the circuit of Figure 2.

Fig. 2. A circuit of Example 3.

If both of the $\diamond$ components implement the same Boolean function, which is unknown to the reasoner, the static approach would be more appropriate. In this case, for instance, whenever the inputs of these components are the same (that is, $\mathsf{in}_1 = \mathsf{in}_3$ and $\mathsf{in}_2 = \mathsf{in}_4$), the outputs will be the same as well, and so the output line (out) of the circuit will be turned off.

If, in addition, each one of these components has its own unpredictable behaviour, the dynamic semantics would be more appropriate. In this case, for instance, the outputs of the $\diamond$-components need not be the same for the same inputs, and so the value of the circuit's output line cannot be predicted either.[4]

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[4] Also, in Example 2, the situation represented by $\tilde{\diamond}_1$ is more suitable for dynamic semantics, while the one represented by $\tilde{\diamond}_2$ is more adequate for the static semantics.

Definition 6. Let $\mathcal{N}$ be an Nmatrix for $\mathcal{L}$.

- The *dynamic models* of ψ and Γ are defined, respectively, by:
  $mod_{\mathcal{N}}^{\mathsf{d}}(\psi) = \{\nu \in \Lambda_{\mathcal{N}}^{\mathsf{d}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod_{\mathcal{N}}^{\mathsf{d}}(\Gamma) = \cap_{\psi \in \Gamma} mod_{\mathcal{N}}^{\mathsf{d}}(\psi)$.
- The consequence relation induced by the dynamic semantics of $\mathcal{N}$ is
  $\Gamma \vdash_{\mathcal{N}}^{\mathsf{d}} \psi$ if $mod_{\mathcal{N}}^{\mathsf{d}}(\Gamma) \subseteq mod_{\mathcal{N}}^{\mathsf{d}}(\psi)$.
- The corresponding definitions for the static semantics are defined similarly, replacing d in the previous items by s.

Again, it is easily verified that $\vdash_{\mathcal{N}}^{\mathsf{d}}$ and $\vdash_{\mathcal{N}}^{\mathsf{s}}$ are consequence relations for $\mathcal{L}$.

Note 2. It is important to observe that by Note 1, if $\mathcal{N}$ is a deterministic Nmatrix and $\mathcal{M}$ is its corresponding (standard) matrix, it holds that $\vdash_{\mathcal{N}}^{\mathsf{d}} = \vdash_{\mathcal{N}}^{\mathsf{s}} = \vdash_{\mathcal{M}}^{\mathsf{s}}$.

Example 4. Consider again the circuit of Figure 2. The theory below represents this circuit and the assumption that both of the $\diamond$-gates have the same input:

$$\Gamma = \left\{ \, \mathsf{out} \leftrightarrow (\mathsf{in}_1 \diamond \mathsf{in}_2) \oplus (\mathsf{in}_3 \diamond \mathsf{in}_4), \ \mathsf{in}_1 \leftrightarrow \mathsf{in}_3, \ \mathsf{in}_2 \leftrightarrow \mathsf{in}_4 \, \right\}.$$

Suppose now that $\mathcal{N}$ is a two-valued non-deterministic matrix in which $\leftrightarrow$ and $\oplus$ have the standard interpretations for double-arrow and xor, and $\diamond$ has the truth-table of $\diamond_2$ in Example 2. Denote by $\mathbf{t}$ and $\mathbf{f}$ the propositional constants that are always assigned the truth-values t and f, respectively. Then $\Gamma \vdash_{\mathcal{N}}^{\mathsf{s}} \mathsf{out} \leftrightarrow \mathbf{f}$, while $\Gamma \nvdash_{\mathcal{N}}^{\mathsf{d}} \mathsf{out} \leftrightarrow \mathbf{f}$ (consider a valuation $\nu \in \Lambda_{\mathcal{N}}^{\mathsf{d}}$ such that $\nu(\mathsf{out}) = \nu(\mathsf{in}_i) = \mathsf{t}$ for $1 \leq i \leq 4$, and $\nu(\mathsf{in}_1 \diamond \mathsf{in}_2) = \mathsf{t}$ but $\nu(\mathsf{in}_3 \diamond \mathsf{in}_4) = \mathsf{f}$; see also Example 3).

A natural question to ask at this stage is whether logics induced by non-deterministic matrices are representable by (finite) deterministic matrices. The answer is negative for dynamic semantics (Proposition 1) and is positive for static semantics (Proposition 2). To show this, we use yet another type of semantic structures, which is a simplification of the notion of a *family of matrices* of [14].

Definition 7. A *family of matrices* is a finite set of deterministic matrices $\mathcal{F} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$, where $\mathcal{M}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle$ for all $1 \leq i \leq k$. An $\mathcal{F}$-valuation is any $\mathcal{M}_i$-valuation for $i \in \{1, \ldots, k\}$. We denote $\Lambda_{\mathcal{F}}^{\mathsf{s}} = \cup_{1 \leq i \leq k} \Lambda_{\mathcal{M}_i}^{\mathsf{s}}$. The relation $\vdash_{\mathcal{F}}^{\mathsf{s}}$ that is induced by $\mathcal{F}$ is defined by: $\Gamma \vdash_{\mathcal{F}}^{\mathsf{s}} \psi$ if $\Gamma \vdash_{\mathcal{M}}^{\mathsf{s}} \psi$ for every $\mathcal{M} \in \mathcal{F}$.

Example 5. The circuit of Figure 1 may be represented as follows:

$$\Gamma = \{\mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1 \, , \ \mathsf{out}_2 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \diamond \mathsf{in}_3\}.$$

Suppose that the connectives in Γ are interpreted by a family $\mathcal{F}$ of matrices with the standard meanings of $\wedge$, $\vee$, and $\leftrightarrow$, and the following interpretations for $\diamond$:

$\tilde{\diamond}_1$	t	f		$\tilde{\diamond}_2$	t	f		$\tilde{\diamond}_3$	t	f		$\tilde{\diamond}_4$	t	f
t	t	t		t	t	f		t	t	t		t	t	f
f	t	f		f	f	f		f	f	f		f	t	f

In this case we have, for instance, that $\Gamma \vdash_{\mathcal{F}}^{\mathsf{s}} \mathsf{out}_1 \leftrightarrow \mathsf{in}_1$, but $\Gamma \nvdash_{\mathcal{F}}^{\mathsf{s}} \mathsf{out}_2 \leftrightarrow \mathsf{in}_2$ (a counter-model assigns f to in_2, t to in_3, t to out_2, and interprets $\diamond$ by $\tilde{\diamond}_1$).

Lemma 1. *For a family $\mathcal{F}$ of matrices, denote $mod_{\mathcal{F}}^{\mathsf{s}}(\psi) = \{\nu \in \Lambda_{\mathcal{F}}^{\mathsf{s}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod_{\mathcal{F}}^{\mathsf{s}}(\Gamma) = \cap_{\psi \in \Gamma} mod_{\mathcal{F}}^{\mathsf{s}}(\psi)$. Then $\Gamma \vdash_{\mathcal{F}}^{\mathsf{s}} \psi$ iff $mod_{\mathcal{F}}^{\mathsf{s}}(\Gamma) \subseteq mod_{\mathcal{F}}^{\mathsf{s}}(\psi)$.* [5]

Corollary 1. *For a family $\mathcal{F}$ of matrices, $\vdash_{\mathcal{F}}^{\mathsf{s}}$ is a consequence relation for $\mathcal{L}$.*

The next proposition, generalizing [6, Theorem 3.4], shows that dynamic Nmatrices characterize logics that are not characterizable by ordinary matrices.

Proposition 1. *Let $\mathcal{N}$ be a two-valued Nmatrix with at least one proper non-deterministic operation. Then there is no family of matrices $\mathcal{F}$ such that $\vdash_{\mathcal{N}}^{\mathsf{d}} = \vdash_{\mathcal{F}}^{\mathsf{s}}$.*

In static semantics the situation is different, as reasoning with $\vdash_{\mathcal{N}}^{\mathsf{s}}$ can be simulated by a family of ordinary matrices. To show this, we need the following:

Definition 8. [4] Let $\mathcal{N}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{N}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$. $\mathcal{N}_1$ is called a *simple refinement* of $\mathcal{N}_2$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$, and $\tilde{\diamond}_{\mathcal{N}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{N}_2}(\overline{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every tuple $\overline{x} \in \mathcal{V}^n$.

Intuitively, an Nmatrix refines another Nmatrix if the former is more restricted than the latter in the non-deterministic choices of its operators.

Definition 9. For an Nmatrix $\mathcal{N}$, the family of matrices $\looparrowright\!\mathcal{N}$ is the set of all the deterministic matrices that are simple refinements of $\mathcal{N}$. A family of matrices $\mathcal{F}$ for $\mathcal{L}$ is called *Cartesian*, if there is some Nmatrix $\mathcal{N}$ for $\mathcal{L}$, such that $\mathcal{F} = \looparrowright\!\mathcal{N}$.

Example 6. Consider the Nmatrix $\mathcal{N}$ for describing $\diamond_1$ in Example 2. Then $\looparrowright\!\mathcal{N}$ is the (Cartesian) family of the four deterministic matrices in Example 5.

Proposition 2. *For every Nmatrix $\mathcal{N}$ it holds that $\vdash_{\mathcal{N}}^{\mathsf{s}} = \vdash_{\looparrowright\!\mathcal{N}}^{\mathsf{s}}$.*

Proposition 2 shows that Nmatrices are representable by Cartesian families of deterministic matrices. Yet, there are useful families that are not Cartesian:

Example 7. Suppose that a gate $\diamond$ is either an AND or an OR gate, but it is not known which one. This situation *cannot* be represented by truth table of $\tilde{\diamond}_1$ in Example 2, as in both static and dynamic semantics the two choices for $\tilde{\diamond}_1(\mathsf{t}, \mathsf{f})$ are completely independent of the choices for $\tilde{\diamond}_1(\mathsf{f}, \mathsf{t})$. What we need is a more precise representation that makes choices between two *deterministic matrices*, each one of which represents a possible behaviour of the unknown gate. Thus, among the four matrices of Example 5, only the first *two* faithfully describe $\diamond$:

$$\mathcal{F} = \left\{ \begin{array}{c|c|c} \tilde{\diamond}_1 & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{t} & \mathsf{t} & \mathsf{t} \\ \hline \mathsf{f} & \mathsf{t} & \mathsf{f} \end{array} \quad , \quad \begin{array}{c|c|c} \tilde{\diamond}_2 & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{t} & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{f} & \mathsf{f} & \mathsf{f} \end{array} \right\}$$

We now combine the concepts of Nmatrices and of their families.

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[5] Due to a lack of space proofs are omitted. For full proofs see the longer version of the paper in `http://www2.mta.ac.il/~oarieli/`, or ask the first author.

Definition 10. A *family of Nmatrices* is a finite set $\mathcal{G} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ of Nmatrices, where $\mathcal{N}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle$ for all $1 \leq i \leq k$.[6] A $\mathcal{G}$-valuation is any $\mathcal{N}_i$-valuation for $i \in \{1, \ldots, k\}$. For $\mathbf{x} \in \{\mathbf{d}, \mathbf{s}\}$, we denote $\Lambda_\mathcal{G}^{\mathbf{x}} = \cup_{1 \leq i \leq n} \Lambda_{\mathcal{N}_i}^{\mathbf{x}}$, and define: $\Gamma \vdash_\mathcal{G}^{\mathbf{x}} \psi$ if $\Gamma \vdash_\mathcal{N}^{\mathbf{x}} \psi$ for every $\mathcal{N} \in \mathcal{G}$.

Lemma 2. *Let $\mathcal{G} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ be a family of Nmatrices. For $\mathbf{x} \in \{\mathbf{d}, \mathbf{s}\}$, denote $mod_\mathcal{G}^{\mathbf{x}}(\psi) = \{\nu \in \Lambda_\mathcal{G}^{\mathbf{x}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod_\mathcal{G}^{\mathbf{x}}(\Gamma) = \cap_{\psi \in \Gamma} mod_\mathcal{G}^{\mathbf{x}}(\psi)$. Then $\Gamma \vdash_\mathcal{G}^{\mathbf{x}} \psi$ iff $mod_\mathcal{G}^{\mathbf{x}}(\Gamma) \subseteq mod_\mathcal{G}^{\mathbf{x}}(\psi)$.*

Corollary 2. *Both of $\vdash_\mathcal{G}^{\mathbf{d}}$ and $\vdash_\mathcal{G}^{\mathbf{s}}$ are consequence relations for $\mathcal{L}$.*

Concerning the simulation of $\vdash_\mathcal{G}^{\mathbf{x}}$ by other consequence relations, note that:

(a) In the dynamic case we have already seen that even logics induced by a single Nmatrix cannot be simulated by a family of ordinary matrices.

(b) In the static case, logics induced by a family of Nmatrices can be simulated using a family of ordinary matrices:

Proposition 3. *For every family of Nmatrices $\mathcal{G}$ there is a family of matrices $\mathcal{F}$ such that $\vdash_\mathcal{G}^{\mathbf{s}} = \vdash_\mathcal{F}^{\mathbf{s}}$.*

2.3 Hierarchy of the Two-Valued Semantic Structures

In the rest of the paper we focus on the two-valued case, using a language $\mathcal{L}$ that includes the propositional constants $\mathbf{t}$ and $\mathbf{f}$. We shall also use a meta-variable $\mathfrak{M}$ that ranges over the two-valued structures defined above, and the metavariable $\mathbf{x}$ that ranges over $\{\mathbf{s}, \mathbf{d}\}$, denoting the restriction on valuations. Accordingly, $\Lambda_\mathfrak{M}^{\mathbf{x}}$ and $mod_\mathfrak{M}^{\mathbf{x}}(\psi)$ denote, respectively, the relevant space of valuations and the models of ψ. The following conventions will be useful in what follows:

- An M-logic is a logic that is induced by a (standard) two-valued matrix. The class of M-logics is denoted by **M**.
- An SN-logic (resp., a DN-logic) is a logic based on a static (resp., a dynamic) two-valued Nmatrix. The class of SN-logics (DN-logics) is denoted **SN** (**DN**).
- An F-logic is a logic that is induced by a family of two-valued matrices. The corresponding class of F-logics is denoted by **F**.
- An SG-logic (DG-logic) is a logic based on a family of static (dynamic) two-valued Nmatrices. The class of SG-logics (DG-logics) is denoted **SG** (**DG**).

For relating the classes of logics above, we need the following proposition.

Proposition 4. *Let $\mathcal{F}$ be a family of matrices for $\mathcal{L}$ with standard negation and conjunction. Then $\mathbf{L} = \langle \mathcal{L}, \vdash_\mathcal{F}^{\mathbf{s}} \rangle$ is an SN-logic iff $\mathcal{F}$ is Cartesian.*

Example 8. The family of matrices $\mathcal{F}$ in Example 7 (enriched with classical negation and conjunction) is not Cartesian and so, by Propositions 1 and 4, it is not representable by a (finite) non-deterministic matrix.

Theorem 1. *In the notations above, we have that:* (a) $\mathbf{M} = \mathbf{DN} \cap \mathbf{SN}$, (b) $\mathbf{SN} \subsetneq \mathbf{F}$, (c) $\mathbf{F} \not\subset \mathbf{DN}$, (d) $\mathbf{SG} = \mathbf{F}$, *and* (e) $\mathbf{DN} \subsetneq \mathbf{DG}$

A graphic representation of Theorem 1 is given in Figure 3.

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[6] To the best of our knowledge, these structures have not been considered yet.
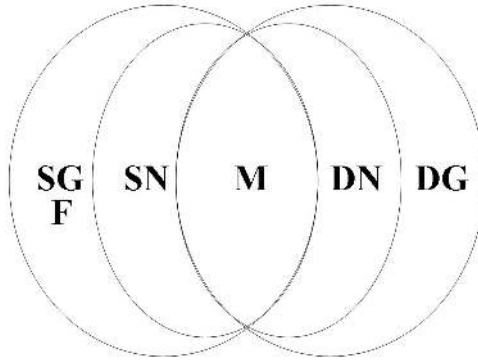
Fig. 3. Relations among the different classes of logics

3 Distance Semantics for Inconsistent Data

A major drawback of the logics considered above is that they do not tolerate inconsistency properly. Indeed, if Γ is not $\mathfrak{M}$-consistent, then $\Gamma \vdash_{\mathfrak{M}}^{\mathsf{x}} \psi$ for every ψ. To overcome this, we incorporate distance-based considerations. The idea is simply to define a distance-like measurement between valuations and theories, and for drawing conclusions, to consider the valuations that are 'closest' to the premises. This intuition is formalized in [2] for deterministic matrices and for Nmatrices under two-valued dynamic semantics only. It can also be viewed as a kind of a *preferential semantics* [13]. Below, we extend this method to all the semantic structures of Section 2. We also introduce a new method for constructing distances, which allows us to define a wide range of distance-based entailments.

3.1 Distances Between Valuations

Definition 11. A *pseudo-distance* on a set S is a total function $d : S \times S \to \mathbb{R}^+$ that is symmetric ($\forall \nu, \mu \in S \ d(\nu, \mu) = d(\mu, \nu)$) and preserves identity ($\forall \nu, \mu \in S \ d(\nu, \mu) = 0$ iff $\nu = \mu$). A pseudo-distance d is a *distance* (*metric*) on S if it also satisfies the has the triangular inequality ($\forall \nu, \mu, \sigma \in S \ d(\nu, \sigma) \leq d(\nu, \mu) + d(\mu, \sigma)$).

Example 9. The following functions are two common distances on the space of the two-valued valuations.

- *The drastic distance:* $d_U(\nu, \mu) = 0$ if $\nu = \mu$, otherwise $d_U(\nu, \mu) = 1$.
- *The Hamming distance:* $d_H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$.[7]

These distances can be applied on *any* space of static valuations (see also Note 3 below).

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[7] Note that this definition assumes a finite number of atomic formulas in the language.

In the context of non-deterministic semantics, one needs to be more cautious in defining distances, as two dynamic valuations can agree on all the atoms of a complex formula, but still assign two different values to that formula. Therefore, complex formulas should also be taken into account in the distance definitions, but there are infinitely many of them to consider. To handle this, we restrict the distance computations to some *context*, i.e., to a certain set of relevant formulas.[8]

Definition 12. A *context* C is a finite set of formulas closed under subformulas. The *restriction to C* of $\nu \in \Lambda_{\mathfrak{M}}^{\mathtt{x}}$ is a valuation $\nu^{\downarrow \mathsf{C}}$ on C, such that $\nu^{\downarrow \mathsf{C}}(\psi) = \nu(\psi)$ for every ψ in C. The restriction to C of $\Lambda_{\mathfrak{M}}^{\mathtt{x}}$ is the set $\Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}} = \{\nu^{\downarrow \mathsf{C}} \mid \nu \in \Lambda_{\mathfrak{M}}^{\mathtt{x}}\}$.

Distances between valuations are now defined as follows:

Definition 13. Let $\mathfrak{M}$ be a semantic structure, $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, and d a function on $\bigcup_{\{\mathsf{C}=\mathsf{SF}(\Gamma) \mid \Gamma \subseteq \mathcal{W}_{\mathcal{L}}\}} \Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}} \times \Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}}$.

- The *restriction* of d to C is a function $d^{\downarrow \mathsf{C}}$ s.t. $\forall \nu, \mu \in \Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}}$, $d^{\downarrow \mathsf{C}}(\nu, \mu) = d(\nu, \mu)$.
- d is a *generic (pseudo) distance* on $\Lambda_{\mathfrak{M}}^{\mathtt{x}}$ if for every context C, $d^{\downarrow \mathsf{C}}$ is a (pseudo) distance on $\Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}}$.

General Constructions of Generic Distances

We now introduce a general method of constructing generic distances. These constructions include the functions of Example 9 as particular cases of generic distances, restricted to the context $\mathsf{C} = \mathsf{Atoms}$ (see Note 3 and Proposition 6).

Definition 14. A *numeric aggregation function* is a complete mapping f from multisets of real numbers to real numbers, such that: (a) f is non-decreasing in the values of the elements of its argument, (b) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$, and (c) $f(\{x\}) = x$ for every $x \in \mathbb{R}$.

As we aggregate non-negative (distance) values, functions that meet the conditions in Definition 14 are, e.g., summation, average, and the maximum.

Definition 15. Let $\mathfrak{M}$ be a (two-valued) structure, C a context, and $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$. For every $\psi \in \mathsf{C}$, define the function $\bowtie^{\psi} \colon \Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}} \times \Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}} \to \{0, 1\}$ as follows:

- for $v_1, v_2 \in \{\mathtt{t}, \mathtt{f}\}$, let $\nabla(v_1, v_2) = 0$ if $v_1 = v_2$, otherwise $\nabla(v_1, v_2) = 1$.
- for an atomic formula p, let $\bowtie^p(\nu, \mu) = \nabla(\nu(p), \mu(p))$
- for a formula $\psi = \diamond(\psi_1, \ldots, \psi_n)$, define

$$\bowtie^{\psi}(\nu, \mu) = \begin{cases} 1 & \text{if } \nu(\psi) \neq \mu(\psi) \text{ but } \forall i \ \nu(\psi_i) = \mu(\psi_i), \\ 0 & \text{otherwise.} \end{cases}$$

For an aggregation g, define the following functions from $\Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}} \times \Lambda_{\mathfrak{M}}^{\mathtt{x} \downarrow \mathsf{C}}$ to $\mathbb{R}^+$:

- $d_{\nabla, g}^{\downarrow \mathsf{C}}(\nu, \mu) = g(\{\nabla(\nu(\psi), \mu(\psi)) \mid \psi \in \mathsf{C}\})$,
- $d_{\bowtie, g}^{\downarrow \mathsf{C}}(\nu, \mu) = g(\{\bowtie^{\psi}(\nu, \mu) \mid \psi \in \mathsf{C}\})$.

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[8] Thus, unlike [1, 8] and other frameworks that use distances as those of Example 9, we will not need the rather restricting assumption that the number of atoms in the language is finite.

Proposition 5. *Both of $d_{\nabla,g}^{\downarrow\mathsf{C}}$ and $d_{\bowtie,g}^{\downarrow\mathsf{C}}$ are pseudo-distances on $\Lambda_{\mathfrak{M}}^{\mathtt{x}\downarrow\mathsf{C}}$.*

The difference between $d_{\nabla,g}^{\downarrow\mathsf{C}}$ and $d_{\bowtie,g}^{\downarrow\mathsf{C}}$ is that while $d_{\nabla,g}$ compares *truth assignments*, $d_{\bowtie,g}$ compares (non-deterministic) *choices* (see also Example 10).

Note 3. The pseudo distances defined above generalize those of Example 9:

• Both $d_{\nabla,\max}$ and $d_{\bowtie,\max}$ are natural generalizations of d_U. Moreover, for any $\nu, \mu \in \Lambda_{\mathfrak{M}}^{\mathsf{s}}$ and finite set Atoms, $\ d_U(\nu,\mu) = d_{\nabla,\max}^{\downarrow\mathsf{Atoms}}(\nu,\mu) = d_{\bowtie,\max}^{\downarrow\mathsf{Atoms}}(\nu,\mu)$.

• Both $d_{\nabla,\Sigma}$ and $d_{\bowtie,\Sigma}$ are natural generalizations of d_H. Moreover, for any $\nu, \mu \in \Lambda_{\mathfrak{M}}^{\mathsf{s}}$ and finite set Atoms, $d_H(\nu,\mu) = d_{\nabla,\Sigma}^{\downarrow\mathsf{Atoms}}(\nu,\mu) = d_{\bowtie,\Sigma}^{\downarrow\mathsf{Atoms}}(\nu,\mu)$.

Proposition 6. *If $g(\{x_1,\ldots,x_n,0\}) = g(\{x_1,\ldots,x_n\})$ for all $x_1,\ldots,x_n \in \{0,1\}$, then $d_{\bowtie,g}^{\downarrow\mathsf{C}}(\nu,\mu) = d_{\bowtie,g}^{\downarrow\mathsf{Atoms}(\mathsf{C})}(\nu,\mu)$.*

By Proposition 5, generic pseudo distances may be constructed as follows:

Proposition 7. *For an aggregation function g, define the following functions:*

$$d_{\nabla,g}(\nu,\mu) = g\big(\{\nabla(\nu(\psi),\mu(\psi)) \mid \psi \in \mathsf{C}\}\big), \tag{3}$$

$$d_{\bowtie,g}(\nu,\mu) = g\big(\{\bowtie^{\psi}(\nu,\mu) \mid \psi \in \mathsf{C}\}\big). \tag{4}$$

Then $d_{\nabla,g}$ and $d_{\bowtie,g}$ are generic pseudo distances on $\Lambda_{\mathfrak{M}}^{\mathtt{x}}$.

3.2 Distance-based Entailments

We now use the distances between valuations for defining entailments relations.

Definition 16. A (semantical) *setting* for $\mathcal{L}$ is a tuple $\mathcal{S} = \langle \mathfrak{M},(d,\mathtt{x}),f \rangle$, where $\mathfrak{M}$ is a structure, d is a generic pseudo distance on $\Lambda_{\mathfrak{M}}^{\mathtt{x}}$ for some $\mathtt{x} \in \{\mathtt{d},\mathtt{s}\}$, and f is an aggregation function.

A setting identifies the underlying semantics, and can be used for measuring the correspondence between valuations and theories.

Definition 17. Given a setting $\mathcal{S} = \langle \mathfrak{M},(d,\mathtt{x}),f \rangle$ define

$$- \ d^{\downarrow\mathsf{C}}(\nu,\psi_i) = \begin{cases} \min\{d^{\downarrow\mathsf{C}}(\nu^{\downarrow\mathsf{C}},\mu^{\downarrow\mathsf{C}}) \mid \mu \in mod_{\mathfrak{M}}^{\mathtt{x}}(\psi_i)\} & \text{if } mod_{\mathfrak{M}}^{\mathtt{x}}(\psi_i) \neq \emptyset, \\ 1 + \max\{d^{\downarrow\mathsf{C}}(\mu_1^{\downarrow\mathsf{C}},\mu_2^{\downarrow\mathsf{C}}) \mid \mu_1,\mu_2 \in \Lambda_{\mathfrak{M}}^{x}\} & \text{otherwise.} \end{cases}$$

$$- \ \delta_{d,f}^{\downarrow\mathsf{C}}(\nu,\Gamma) = f(\{d^{\downarrow\mathsf{C}}(\nu,\psi_1),\ldots,d^{\downarrow\mathsf{C}}(\nu,\psi_n)\}).$$

The intuition here is to measure how 'close' a valuation is to satisfying a formula and a theory. To be faithful to this intuition, we are interested only in contexts where the distance between a formula and its model is zero, and is strictly positive otherwise.

Proposition 8. *Let $\mathfrak{M}$ be a semantic structure, C a context, and $\mathtt{x} \in \{\mathtt{d},\mathtt{s}\}$.*

• *If $\mathsf{Atoms}(\psi) \subseteq \mathsf{C}$, then $d^{\downarrow\mathsf{C}}(\nu,\psi) = 0$ iff $\nu \in mod_{\mathfrak{M}}^{\mathsf{s}}(\psi)$.*

• *If $\mathsf{SF}(\psi) \subseteq \mathsf{C}$, then $d^{\downarrow\mathsf{C}}(\nu,\psi) = 0$ iff $\nu \in mod_{\mathfrak{M}}^{\mathtt{d}}(\psi)$.*

It follows that the most appropriate contexts to use are the following:

Definition 18. Given a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), f \rangle$, denote:

$$\mathsf{C}_{\mathsf{x}}(\Gamma) = \begin{cases} \mathsf{Atoms}(\Gamma) & \text{if } \mathsf{x} = \mathsf{s}, \\ \mathsf{SF}(\Gamma) & \text{if } \mathsf{x} = \mathsf{d}. \end{cases}$$

Definition 19. The *most plausible valuations* of Γ with respect to a semantic setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), f \rangle$ are the elements of the following set:

$$\Delta_{\mathcal{S}}(\Gamma) = \begin{cases} \{\nu \in \Lambda_{\mathfrak{M}}^{\mathsf{x}} \mid \forall \mu \in \Lambda_{\mathfrak{M}}^{\mathsf{x}} \; \delta_{d,f}^{\downarrow\mathsf{C}_{\mathsf{x}}(\Gamma)}(\nu, \Gamma) \le \delta_{d,f}^{\downarrow\mathsf{C}_{\mathsf{x}}(\Gamma)}(\mu, \Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda_{\mathfrak{M}}^{\mathsf{x}} & \text{otherwise.} \end{cases}$$

Example 10. Let $\mathcal{N}$ be an Nmatrix that interprets negation in the standard way, and $\diamond$ according to $\diamond_1$ in Example 2. Then $\Gamma = \{p, q, \neg(p \diamond q)\}$ is not $\mathcal{N}$-satisfiable, and $mod_{\mathcal{N}}^{\mathsf{d}}(\Gamma) = \emptyset$. Consider now the settings $\mathcal{S}_1 = \langle \mathcal{N}, (d_{\nabla, \Sigma}, \mathsf{d}), \Sigma \rangle$ and $\mathcal{S}_2 = \langle \mathcal{N}, (d_{\bowtie, \Sigma}, \mathsf{d}), \Sigma \rangle$, where $d_{\nabla, \Sigma}$ and $d_{\bowtie, \Sigma}$ are, respectively, the generic distances defined in (3) and (4). Then:

	p	q	$p \diamond q$	$\neg(p \diamond q)$	$\delta_{\mathcal{S}_1}^{\downarrow\mathsf{C}_{\mathsf{d}}}(\nu_i, \Gamma)$	$\delta_{\mathcal{S}_2}^{\downarrow\mathsf{C}_{\mathsf{d}}}(\nu_i, \Gamma)$
ν_1	t	t	t	f	3	1
ν_2	t	f	t	f	3	2
ν_3	t	f	f	t	1	1
ν_4	f	t	t	f	3	2
ν_5	f	t	f	t	1	1
ν_6	f	f	f	t	2	2

and so $\Delta_{\mathcal{S}_1}(\Gamma) = \{\nu_3, \nu_5\}$ and $\Delta_{\mathcal{S}_2}(\Gamma) = \{\nu_1, \nu_3, \nu_5\}$.

Proposition 9. *For every $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), f \rangle$ and Γ, $\Delta_{\mathcal{S}}(\Gamma)$ is nonempty. If Γ is $\mathfrak{M}$-satisfiable, $\Delta_{\mathcal{S}}(\Gamma) = mod_{\mathfrak{M}}^{\mathsf{x}}(\Gamma)$.*

Next, we formalize the idea that, according distance-based entailments, conclusions should follow from all of the most plausible valuations of the premises.

Definition 20. For $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), f \rangle$, denote: $\Gamma \hspace{1pt}\vdash\hspace{-7pt}\sim_{\mathcal{S}} \psi$ if $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathfrak{M}}^{\mathsf{x}}(\psi)$ or $\Gamma = \{\psi\}$.

Example 11. In Example 10, under the standard interpretation of disjunction, $\Gamma \hspace{1pt}\vdash\hspace{-7pt}\sim_{\mathcal{S}_1} \neg p \vee \neg q$ while $\Gamma \hspace{1pt}\not\vdash\hspace{-7pt}\sim_{\mathcal{S}_2} \neg p \vee \neg q$.

Example 12. Consider the $\mathcal{F}$-consistent theory Γ of Example 5 that represents the circuit of Figure 1. Learning that lines in_1 and out_1 always have opposite values, the revised theory, $\Gamma' = \Gamma \cup \{\mathsf{out}_1 \leftrightarrow \neg\mathsf{in}_1\}$, is not $\mathcal{F}$-satisfiable anymore, so $\vdash_{\mathcal{F}}$ is useless for making plausible conclusions from Γ'. However, using the setting $\mathcal{S} = \langle \mathcal{F}, (d_{\nabla, \Sigma}, \mathsf{s}), \Sigma \rangle$, or $\mathcal{S} = \langle \mathcal{F}, (d_{\bowtie, \Sigma}, \mathsf{s}), \Sigma \rangle$, it can be verified that:

• The assertion $\mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1$ is falsified by some most plausible valuations of Γ', and so, while $\Gamma \vdash_{\mathcal{F}} \mathsf{out}_1 \leftrightarrow \mathsf{in}_1$, we have $\Gamma' \hspace{1pt}\not\vdash\hspace{-7pt}\sim_{\mathcal{S}} \mathsf{out}_1 \leftrightarrow \mathsf{in}_1$.
• The assertion $\mathsf{out}_2 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \diamond \mathsf{in}_3$ is validated by all the most plausible valuations of Γ', and so, despite the $\mathcal{F}$-inconsistency of Γ', the information about the relation between out_2 and in_1, in_2 may be retained.

The distance-based entailments defined above generalize the usual methods for distance-based reasoning in the context of deterministic matrices. This includes, among others, the operators in [8, 10, 11] and the distance-based entailments for deterministic matrices in [1, 3]. The entailment $\hspace{-0.2em}\sim_{\mathcal{S}}$ for Nmatrices and dynamic valuations is studied in [2]. To the best of our knowledge, distance entailments for Nmatrices and static valuations, and entailments based on families of matrices and (static or dynamic) Nmatrices, have not been considered before.

Theorem 2. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$. For every $\mathfrak{M}$-consistent theory Γ, it holds that $\Gamma \hspace{-0.2em}\sim_{\mathcal{S}} \psi$ iff $\Gamma \vdash^{\mathtt{x}}_{\mathfrak{M}} \psi$.*

Theorem 3. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a setting in which f is hereditary.[9] Then $\hspace{-0.2em}\sim_{\mathcal{S}}$ is a cautious consequence relation, i.e., it has the following properties:*

Cautious Reflexivity:	$\psi \hspace{-0.2em}\sim_{\mathcal{S}} \psi$
Cautious Monotonicity [9]:	*if $\Gamma \hspace{-0.2em}\sim_{\mathcal{S}} \psi$ and $\Gamma \hspace{-0.2em}\sim_{\mathcal{S}} \phi$, then $\Gamma, \psi \hspace{-0.2em}\sim_{\mathcal{S}} \phi$.*
Cautious Transitivity [7]:	*if $\Gamma \hspace{-0.2em}\sim_{\mathcal{S}} \psi$ and $\Gamma, \psi \hspace{-0.2em}\sim_{\mathcal{S}} \phi$, then $\Gamma \hspace{-0.2em}\sim_{\mathcal{S}} \phi$.*

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[9] An aggregation function f is *hereditary*, if $f(\{x_1, \ldots, x_n\}) < f(\{y_1, \ldots, y_n\})$ implies that $f(\{x_1, \ldots, x_n, z\}) < f(\{y_1, \ldots, y_n, z\})$.