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NON-DETERMINISTIC SEMANTICS FOR LOGICAL SYSTEMS

1 INTRODUCTION

1.1 The Key Idea

The principle of truth-functionality (or compositionality) is a basic principle in many-valued logic in general, and in classical logic in particular. According to this principle, the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. However, real-world information is inescapably incomplete, uncertain, vague, imprecise or inconsistent, and these phenomena are in an obvious conflict with the principle of truth-functionality. One possible solution to this problem is to relax this principle by borrowing from automata and computability theory the idea of non-deterministic computations, and apply it in evaluations of truth-values of formulas. This leads to the introduction of *non-deterministic matrices* (Nmatrices) — a natural generalization of ordinary multi-valued matrices, in which the truth-value of a complex formula can be chosen *non-deterministically* out of some non-empty set of options. There are many natural motivations for introducing non-determinism into the truth-tables of logical connectives. We discuss some of them below. They give rise to two different ways in which non-determinism can be incorporated: the *dynamic* and the *static*[1]. In both the value $v(\diamond(\psi_1, \ldots, \psi_n))$ assigned to the formula $\diamond(\psi_1, ..., \psi_n)$ is selected from a set $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ (where $\widetilde{\diamond}$ is the interpretation of $\diamond$). In the dynamic approach this selection is made separately and independently for each tuple $\langle\psi_1, \ldots, \psi_n\rangle$. Thus the choice of one of the possible values is made at the lowest possible (local) level of computation, or on-line, and $v(\psi_1), \ldots, v(\psi_n)$ do not uniquely determine $v(\diamond(\psi_1, \ldots, \psi_n))$. In contrast, in the static semantics this choice is made globally, system-wide, and the interpretation of $\diamond$ is a function, which is selected before any computation begins. This function is a "determinisation" of the non-deterministic interpretation $\widetilde{\diamond}$, to be applied in computing the value of any formula under the given valuation. This limits non-determinism, but still leaves the freedom of choosing the above function among all those that are compatible with the non-deterministic interpretation $\widetilde{\diamond}$ of $\diamond$.

[1]The dynamic approach was introduced together with the concept of Nmatrices. The static approach was later introduced in [Avron and Konikowska, 2005]

D. Gabbay and F. Guenthner (eds.),
Handbook of Philosophical Logic, Volume ??, 1–73.

1.2 Some Intuitive Motivations

We start by presenting some cases in which the need for non-deterministic semantics naturally arises.

Syntactic "underspecification":
Consider the standard Gentzen-type system LK for propositional classical logic (see e.g. [Troelstra and Schwichtenberg, 2000]). Its introduction rules for $\neg$ and $\vee$ are usually formulated as follows:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}\ (\Rightarrow \neg)$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}\ (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}\ (\Rightarrow \vee)$$

The corresponding semantics is given by the following classical truth-tables:

	$\neg$
t	**f**
f	**t**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**f**	**t**
f	**f**	**f**

Note that each syntactic rule of LK dictates some semantic condition on the connective it introduces: $(\neg \Rightarrow)$ corresponds to the condition $\tilde{\neg}(t) = f$, while $(\Rightarrow \neg)$ corresponds to the condition $\tilde{\neg}(f) = t$, thus completely determining the truth-table for negation. Similarly, $(\vee \Rightarrow)$ dictates the last line of the truth-table for $\vee$, i.e $\tilde{\vee}(f, f) = f$, while $(\Rightarrow \vee)$ dictates the other three lines. Now suppose we want to reject the *law of excluded middle* (LEM), in the spirit of intuitionistic logic. This can most simply be done by discarding the rule $(\Rightarrow \neg)$, which corresponds to LEM, while keeping the rest of the rules unchanged. What is the semantics of the resulting system? Intuitively, by discarding $(\Rightarrow \neg)$, we lose the information concerning the second line of the truth-table for $\neg$. Accordingly, we are left with a problem of *underspecification*. This can be modelled using Nmatrices in a very natural way: in case of underspecification, all possible truth-values are allowed. The corresponding semantics in the case we consider would be as follows (we use sets of possible truth-values instead of truth-values):

	$\neg$
t	{**f**}
f	{**t**,**f**}

		$\vee$
t	**t**	{**t**}
t	**f**	{**t**}
f	**f**	{**t**}
f	**f**	{**f**}

Linguistic ambiguity:
In many natural languages the meaning of the words "either ... or" is ambiguous. Thus the Oxford English Dictionary explains the meaning of this phrase as follows:

> The primary function of either, etc., is to emphasize the indifference of the two (or more) things or courses, ..., but a secondary function is to emphasize the mutual exclusiveness (i.e. either of the two, but not both).

Following this kind of common-sense intuition about "or", it follows that in many natural languages the word "or" has both an "inclusive" and an "exclusive" sense. For instance, when some mathematician promises: *"I shall either attack problem A or attack problem B"*, then in many cases he might at the end solve the two problems, but there are certainly situations in which what he means is "but do not expect me to attack them both". In the first case the meaning of "or" is inclusive, while in the latter case it is exclusive. Now in many cases one is uncertain whether the meaning of a speaker's "or" is inclusive or exclusive. However, even in cases like this one would still like to be able to make some certain inferences from what has been said. This situation can be captured by dynamic semantics based on the following non-deterministic truth-table for $\vee$:

		$\vee$
$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{t}, \mathbf{f}\}$
$\mathbf{t}$	$\mathbf{f}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{t}\}$
$\mathbf{f}$	$\mathbf{f}$	$\{\mathbf{f}\}$

Note that the static semantics is less appropriate here, since the meaning of a speaker's "or" is not predetermined, and he might use both meanings of "or" in two different sentences within the same discourse.

Inherent non-deterministic behavior of circuits:
Nmatrices can be applied to model non-deterministic behavior of various elements of electrical circuits. An ideal logic gate performing operations on boolean variables is an abstraction of a physical gate operating with a continuous range of electrical quantity. This electrical quantity is turned into a discrete variable by associating a whole range of electrical voltages with the logical values 1 and 0 (see [Rabaey et. al, 2003] for further details). There are a number of reasons, due to which the measured behavior of a circuit may deviate from the expected behavior. One reason can be the variations in the manufacturing process: the dimension and device parameters may vary, affecting the electrical behavior of the circuit. The presence of disturbing noise sources, temperature and other conditions are another

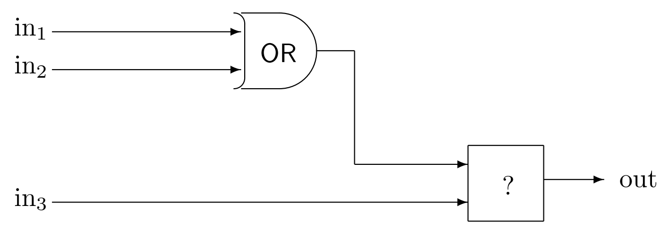

Figure 1. The circuit C

source of deviations in the circuit response. The exact mathematical form of the relation between input and output in a given logical gate is not always known, and so it can be approximated by a non-deterministic truth-table. For instance, suppose that the circuit C given in Figure 1 consists of a standard OR gate and a faulty AND gate, which responds correctly if the inputs are similar, and unpredictably otherwise. The behavior of the gate can be described by the following truth-table, equipped with the dynamic semantics:

		AND
t	**t**	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{f}, \mathbf{t}\}$
f	**t**	$\{\mathbf{f}, \mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

Computation with unknown functions:
Let us return to Figure 1, and suppose that this time it represents a circuit about which only some partial information is known. Namely, it is known that the gate labelled with "?" is either an XOR gate or an OR gate, but it is not known which one. Thus the function describing the second gate is deterministic, but unknown to us. This situation can be represented by using the non-deterministic truth-table for $\vee$ given in the "linguistic ambiguity" example, equipped with the static semantics.

Verification with unknown evaluation models:
There are two well-known three-valued logics for describing different types of computational models. The first, which captures parallel evaluation, was described in the context of computational mathematics by Kleene ([Kleene, 1938]); the second, programming oriented method, in which evaluation proceeds sequentially, was proposed by McCarthy ([McCarthy, 1963]). Below

are the corresponding truth-tables for $\vee$:

(Kleene)

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**t**
t	**t**	**t**	**t**

(McCarthy)

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**e**
t	**t**	**t**	**t**

Now suppose we are sending an expression $\psi \vee \varphi$ for evaluation to some distant computer, for which it is not known whether it performs parallel or sequential computations. Hence we know that $\psi \vee \varphi$ will be evaluated using a deterministic function $\tilde{\vee}$, defined by either Kleene's or McCarthy's truth-table for $\vee$, but we have no information which of the two. Again this can be captured by using a static interpretation of the following "truth-table":

$\widetilde{\vee}$	**f**	**e**	**t**
f	$\{\mathbf{f}\}$	$\{\mathbf{e}\}$	$\{\mathbf{t}\}$
e	$\{\mathbf{e}\}$	$\{\mathbf{e}\}$	$\{\mathbf{e},\mathbf{t}\}$
t	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

According to this static interpretation, the function $f_\vee : \{\mathbf{t},\mathbf{f},\mathbf{e}\}^2 \to \{\mathbf{t},\mathbf{f},\mathbf{e}\}$ used by the computer satisfies either $f_\vee(\mathbf{t},\mathbf{e}) = \mathbf{t}$ (in case the computation is parallel) or $f_\vee(\mathbf{t},\mathbf{e}) = \mathbf{e}$ (in case it is sequential). However, it is not known which of these two conditions is satisfied.

Incompleteness and inconsistency:
This example is taken from [Avron et. al., 2006; Avron et. al., 2008]. Suppose we have a framework for information collecting and processing, which consists of a set S of information sources and a processor P. The sources provide information about formulas over $\{\neg, \vee\}$, and we assume that for each such formula ψ a source $s \in S$ can say that ψ is true (i.e., assigned the truth-value 1), ψ is false (i.e., assigned the truth-value 0), or that it has no knowledge about ψ. In turn, the processor collects information from the sources, combines it according to some strategy and defines the resulting combined valuation of formulas. Thus for every formula ψ the processor can encounter one of the four possible situations: (a) it has information that ψ is true, but no information that ψ is false, (b) it has information that ψ is false, but no information that ψ is true, (c) it has both information that ψ is true and information that it is false, and (d) it has no information on ψ at all. In view of this, it was suggested by Belnap in [Belnap, 1977] (following works and ideas of Dunn, e.g. [Dunn, 1976]) to account for incomplete and contradictory information by using the following four logical truth values:

$$\mathbf{t} = \{1\}, \mathbf{f} = \{0\}, \top = \{0,1\}, \bot = \emptyset$$

Here 1 and 0 represent "true" and "false" respectively, and so $\top$ represents inconsistent information, while $\bot$ represents absence of information.

The above scenario has many ramifications, corresponding to various assumptions regarding the kind of information provided by the sources and the strategy used by the processor to combine it. We assume that the processor respects at least the deterministic consequences (in both ways) of each of the classical truth tables. This assumption means that the values assigned by the processor to complex formulas and those it assigns to their immediate subformulas are interrelated according to the following principles derived from the classical truth-tables of $\neg$ and $\vee$:

1. The processor ascribes 1 to $\neg\varphi$ iff it ascribes 0 to φ.
2. The processor ascribes 0 to $\neg\varphi$ iff it ascribes 1 to φ.
3. If the processor ascribes 1 to either φ or ψ, then it ascribes 1 to $\varphi \vee \psi$.
4. The processor ascribes 0 to $\varphi \vee \psi$ iff it ascribes 0 to both φ and ψ.

Here the statement "the processor ascribes 0 to ψ" means that 0 is included in the subset of $\{0, 1\}$ which is assigned by the processor to ψ (recall that the truth-values used by the processor correspond to subsets of $\{0, 1\}$). It is crucial to note that the converse of (3) does *not* hold, since some source might inform the processor that $\varphi \vee \psi$ is true, without providing information about the truth/falsehood of either φ or ψ. Under the above assumptions, there can be a number of possible scenarios concerning the type of formulas evaluated by the sources. The case when the sources provide information only about atomic formulas has been considered in [Belnap, 1977]. This case is deterministic, and leads to the famous Dunn-Belnap four-valued logic. Now consider the case when the sources provide information about arbitrary formulas (also complex ones), but not necessarily all of them. In this case the assumptions above are reflected in the following *non-deterministic* truth-tables:

$\widetilde{\vee}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}, \top\}$	$\{\mathbf{t}, \bot\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\bot$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$	$\{\mathbf{t}\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

	$\widetilde{\neg}$
$\mathbf{f}$	$\{\mathbf{f}\}$
$\bot$	$\{\bot\}$
$\top$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$

Note that the table for negation reflects the principles 1 and 2, while the table for disjunction reflects the principles 3 and 4. To see this, let us examine one of the most peculiar cases: the entry $\mathbf{f}\widetilde{\vee}\mathbf{f} = \{\mathbf{f}, \top\}$. Suppose that ψ and φ are both assigned the truth-value $\mathbf{f} = \{0\}$. Then by principle 4 above, the truth-value of $\psi \vee \varphi$ (which is a subset of $\{0, 1\}$) must include 0. If in addition one of the sources assigned 1 to $\psi \vee \varphi$, then the processor ascribes 1 to $\psi \vee \varphi$ too, and so the truth-value it assigned to $\psi \vee \varphi$ is in this case $\top$. Otherwise it is **f**. This justifies the two options in the truth-table. The rest of the entries can be explained in a similar way.

1.3 Things To Come

The rest of this survey is divided into two parts. Part I describes the propositional framework of Nmatrices. We begin with some preliminaries and a review of many-valued matrices in Section 2. The basic definitions of the framework of Nmatrices are presented in Section 3. In Section 4 we introduce *canonical signed calculi*, a natural family of proof systems manipulating sets of signed formulae (Gentzen-type systems can be thought of as a specific instance of such calculi). The relation between Nmatrices and canonical calculi is then explored in two complementary directions. In Section 4.1 we provide a general proof theory for Nmatrices using canonical calculi. In Section 4.2 modular non-deterministic semantics is provided for every canonical calculus (satisfying a simple syntactic condition). We then proceed to describe further applications of Nmatrices. In Section 5 we extend the modular approach to two non-canonical families of Gentzen-type calculi: those that are obtained from the positive fragments of classical logic and intuitionistic logic by adding various natural Gentzen-type rules for negation. In Section 6 Nmatrices are used for yet another family of non-classical logics: *paraconsistent logics*, designed for reasoning in the presence of contradictions. In Part II we handle the extension of the framework of Nmatrices to the first-order level and beyond. In Section 7 we briefly review the two standard approaches to interpreting unary quantifiers in many-valued logics. In Section 8 we extend the propositional framework of Nmatrices to languages with such quantifiers and discuss the problems that this move reveals (and were not evident on the propositional level). Section 9 is devoted to the particular case of the usual first-order quantifiers. An application of this case is presented in Section 10, where we extend the results from Section 6, and provide semantics for a large family of first-order paraconsistent logics. Section 11 further generalizes the framework of Nmatrices to *multi-ary quantifiers* and extends the relation between Nmatrices and canonical signed calculi to languages with such quantifiers.

Due to lack of space, we omit in what follows most of the proofs, providing instead pointers to the relevant papers. Those of the proofs we do include are intended to give the reader a better insight into the nature of Nmatrices, and a flavour of the (mostly new) methods that can be employed in handling and applying them.

PART I: THE PROPOSITIONAL CASE

2 PRELIMINARIES

In what follows, $\mathcal{L}$ is a propositional language and $Frm_{\mathcal{L}}$ is its set of wffs. The metavariables ψ, φ range over $\mathcal{L}$-formulas, and Γ, Δ over sets of $\mathcal{L}$-formulas. For an $\mathcal{L}$-formula ψ, we denote by $\mathsf{Atoms}(\psi)$ the set of atomic formulas in ψ. We denote by $SF(\Gamma)$ the set of all subformulas of Γ.

2.1 Logics, Consequence Relations and Abstract Rules

DEFINITION 1.

1. A *Scott consequence relation* (*scr* for short) for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following three conditions:

   strong reflexivity: if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.
   monotonicity: if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.
   Transitivity (*cut*): if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

2. A Tarskian consequence relation (tcr) $\vdash^1$ for a language $\mathcal{L}$ is a binary relation between sets of L-formulas and L-formulas, that satisfies the following conditions:

   strong reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash^1 \psi$.
   monotonicity: if $\Gamma \vdash^1 \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash^1 \psi$.
   Transitivity (*cut*): if $\Gamma \vdash^1 \psi$ and $\Gamma', \psi \vdash^1 \varphi$ then $\Gamma, \Gamma' \vdash^1 \varphi$.

3. A tcr $\vdash$ for $\mathcal{L}$ is *structural* if for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ, if $\Gamma \vdash \psi$ then $\sigma(\Gamma) \vdash \sigma(\psi)$. $\vdash$ is *finitary* if whenever $\Gamma \vdash \psi$, there exists some finite $\Gamma' \subseteq \Gamma$, such that $\Gamma' \vdash \psi$. $\vdash$ is *consistent* (or *non-trivial*) if there exist some non-empty Γ and some ψ s.t. $\Gamma \not\vdash \psi$. $\vdash$ is *uniform* if $\Gamma \vdash \psi$ whenever $\Gamma, \Delta \vdash \psi$, $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\Delta) = \emptyset$, and Δ is consistent (i.e. there exists φ such that $\Gamma \not\vdash \varphi$). Similar properties can be defined for an scr.

4. *A Tarskian propositional logic* (*propositional logic*) is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural and consistent tcr (scr) for $\mathcal{L}$. The logic $\langle \mathcal{L}, \vdash \rangle$ is finitary if $\vdash$ is finitary.

For the rest of this section, we focus on scrs. However, the properties below can be formulated in the context of tcrs as well.

There are several ways of defining consequence relations for a language $\mathcal{L}$. The two most common ones are the proof-theoretical and the model-theoretical approaches. In the former, the definition of a consequence relation is based on some notion of a *proof* in some formal calculus. In the latter approach, the definition is based on a notion of a *semantics* for $\mathcal{L}$. The general notion of an abstract semantics is rather opaque. One usually starts by defining a notion of a *valuation* as a certain type of partial functions from $Frm_{\mathcal{L}}$ to some set. Then ones defines what it means for a valuation to *satisfy* a formula (or to be a *model* of a formula). A semantics is then some set S of valuations, and the consequence relation induced by S is defined as follows: $\Gamma \vdash_{\mathsf{S}} \Delta$ if every *total* valuation in S which satisfies all the formulas in Γ, satisfies some formula in Δ as well (note that this always defines an scr). We say that a semantics S is *analytic*[2] if every partial valuation in S, whose domain is closed under subformulas, can be extended to a full (i.e. total) valuation in S. This implies that the exact identity of the language $\mathcal{L}$ is not important, since analycity allows us to focus on some subset of its connectives. (See Remark 12 below for another important consequence of analycity.) We shall shortly see that both ordinary many-valued semantics and non-deterministic semantics based on propositional Nmatrices are always analytic. However this is not necessarily the case in general [3].

DEFINITION 2.

1. A *pure (abstract) rule* in a propositional language $\mathcal{L}$ is any ordered pair $\langle \Gamma, \Delta \rangle$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ (We shall usually denote such a rule by $\Gamma \Rightarrow \Delta$ rather than by $\langle \Gamma, \Delta \rangle$).

2. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_1 \rangle$ be a propositional logic, and let S be a set of rules in a propositional language $\mathcal{L}'$. The *extension* $\mathbf{L}[S]$ of $\langle \mathcal{L}, \vdash_1 \rangle$ by S is[4] the logic $\langle \mathcal{L}^*, \vdash^* \rangle$, where $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}'$, and $\vdash^*$ is the least *structural* scr $\vdash$ such that $\Gamma \vdash \Delta$ whenever $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$.

REMARK 3. It is easy to see that $\vdash^*$ is the closure under cuts and weakenings of the set of all pairs $\langle \sigma(\Gamma), \sigma(\Delta) \rangle$, where σ is a uniform substitution in $\mathcal{L}^*$, and either $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$. This in turn implies that an extension of a finitary logic by a set of pure rules is again finitary.

[2] The term ‘effective’ was used in [Avron, 2007a; Avron and Zamansky, 2007d; Avron and Zamansky, 2007a] instead of ‘analytic’.

[3] For instance, in the bivaluations semantics and the possible translations semantics described in [Carnielli, 1998; Carnielli and Marcos, 2002; Carnielli and Marcos, 2007] no general theorem of analycity is available. Hence analycity should be proved from scratch for every useful instance of these types of semantics.

[4] Obviously, the extension of $\langle \mathcal{L}, \vdash_1 \rangle$ by S is well-defined (i.e. a logic) only if $\vdash^*$ is consistent. In all the cases we consider below this will easily be guaranteed by the semantics we provide (and so we shall not even mention it).

CONVENTION 4. To emphasize the fact that the presence of a rule in a system means the presence of all its instances, we shall usually describe a rule using the metavariables φ, ψ, θ rather than the atomic formulas $p_1, p_2, \ldots$. Thus although formally $(\supset\Rightarrow)$ is the rule $p_1, p_1 \supset p_2 \Rightarrow p_2$, we shall write it as $\varphi, \varphi \supset \psi \Rightarrow \psi$.

REMARK 5. Suppose that the formula θ occurs in a pure rule of a logic $\mathcal{L}$, and we decide to select θ as the "principal formula" of that rule. Assume e.g. that the rule is of the form $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k, \theta$ (the consideration in the other case is similar). Suppose further that $\Gamma_i \vdash \Delta_i, \varphi_i$ for $i = 1, \ldots, n$ and $\psi_j, \Gamma_j \vdash \Delta_j$ for $j = 1, \ldots, k$. Then $\Gamma_1, \ldots, \Gamma_n \vdash \Delta_1, \ldots, \Delta_k, \theta$ (by n+k cuts). It follows that $\mathcal{L}$ is closed in this case under the Gentzen-type rule:

$$\frac{\Gamma_i \Rightarrow \Delta_i, \varphi_i \;\; (i = 1, \ldots, n) \qquad \psi_j, \Gamma_j \Rightarrow \Delta_j \;\; (j = 1, \ldots, k)}{\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_k, \theta}$$

Conversely, if $\mathcal{L}$ is closed under this Gentzen-type rule then by applying it to the reflexivity axioms $\varphi_i \vdash \varphi_i$ $(i = 1, \ldots, n)$ and $\psi_j \vdash \psi_j$ $(j = 1, \ldots, k)$ we get $\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_k, \theta$. It follows that every pure rule in the sense of Definition 2 is equivalent to some *multiplicative* (in the terminology of [Girard, 1987]) or *pure* (in the terminology of [Avron, 1991]) Gentzen-type rule. Moreover: it is easy to see that most standard rules used in Gentzen-type systems are equivalent to finite sets of pure rules in the sense of Definition 2. For example: the usual $(\supset\Rightarrow)$ rule of classical logic is equivalent by what we have just shown to the pure rule $\varphi, \varphi \supset \psi \Rightarrow \psi$. The classical $(\Rightarrow\supset)$, in turn, can be split into the following two rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi \supset \psi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

Hence $(\Rightarrow\supset)$ is equivalent to the set $\{\psi \Rightarrow \varphi \supset \psi, \;\; \Rightarrow \varphi, \varphi \supset \psi\}$. [5]

2.2 Many-valued Matrices

The most standard general method for defining propositional logics is by using many-valued (deterministic) matrices ([Rosser and Turquette, 1952; Bolc and Borowik, 1992; Malinowski, 1993; Gottwald, 2001; Hähnle, 2001; Urquhart, 2001]):

DEFINITION 6.

1. A matrix for $\mathcal{L}$ is a tuple $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:
   - $\mathcal{V}$ is a non-empty set of truth values.
   - $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.

[5] Recall that formally we should have written here $\{p_2 \Rightarrow p_1 \supset p_2, \;\; \Rightarrow p_1, p_1 \supset p_2\}$.

- For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$.

We say that $\mathcal{P}$ is (in)finite if so is $\mathcal{V}$.

2. A partial valuation in $\mathcal{P}$ is a function v to $\mathcal{V}$ from some subset of $Frm_{\mathcal{L}}$ which is closed under subformulas, such that for each n-ary connective $\diamond$ of $\mathcal{L}$, the following holds for all $\psi_1, \ldots, \psi_n \in Frm_{\mathcal{L}}$:

$$v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$$

A partial valuation in $\mathcal{P}$ is a (full) *valuation* if its domain is $Frm_{\mathcal{L}}$. A partial valuation v in $\mathcal{P}$ satisfies a formula ψ ($v \models \psi$) if $v(\psi) \in \mathcal{D}$.

3. Let $\mathcal{P}$ be a matrix. We say that $\Gamma \vdash_{\mathcal{P}} \Delta$ if whenever a valuation in $\mathcal{P}$ satisfies all the formulas of Γ, it satisfies also at least one of the formulas of Δ. We say that $\Gamma \vdash^1_{\mathcal{P}} \psi$ if $\Gamma \vdash_{\mathcal{P}} \{\psi\}$. For a family of matrices F, we say that $\Gamma \vdash_F \Delta$ if $\Gamma \vdash_{\mathcal{P}} \Delta$ for every $\mathcal{P}$ in F.

4. A logic $\mathbf{L}$ is sound for a matrix $\mathcal{P}$ if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{P}}$. $\mathbf{L}$ is complete for a matrix $\mathcal{P}$ if $\vdash_{\mathcal{P}} \subseteq \vdash_{\mathbf{L}}$. $\mathcal{P}$ is a characteristic matrix for a logic $\mathbf{L}$ if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{P}}$. F is a characteristic set of matrices for $\mathbf{L}$ if $\vdash_{\mathbf{L}} = \vdash_F$.

The following well-known theorem can easily be proved:

THEOREM 7. *For every matrix $\mathcal{P}$ for $\mathcal{L}$, $\vdash_{\mathcal{P}}$ is a uniform propositional logic, and $\vdash^1_{\mathcal{P}}$ is a uniform Tarskian propositional logic.*

The converse of this theorem also holds (see [Urquhart, 2001]):

THEOREM 8. *Every (Tarskian) uniform structural logic has a characteristic matrix.*

REMARK 9. Although every Tarskian uniform structural logic has a characteristic matrix, it is often the case that this matrix is infinite, and is hard to find and use. We will shortly see that finite characteristic Nmatrices exist for many logics which have only infinite characteristic matrices (see Theorem 24).

THEOREM 10. **(Compactness)** ([Shoesmith, 1971]) *If $\mathcal{P}$ is a finite matrix then $\vdash_{\mathcal{P}}$ and $\vdash^1_{\mathcal{P}}$ are finitary.*

The next important result is again very easy to prove:

PROPOSITION 11. **(Analycity)** *Any partial valuation in a matrix $\mathcal{P}$ for $\mathcal{L}$, which is defined on a set of $\mathcal{L}$-formulas closed under subformulas, can be extended to a full valuation in $\mathcal{P}$.*

REMARK 12. At this point the importance of analycity should again be stressed. Because of this property $\vdash_{\mathcal{S}}$ is decidable whenever $\mathcal{S}$ is a finite matrix. Moreover, analycity guarantees semi-decidability of non-theoremhood even if a matrix $\mathcal{P}$ is infinite, provided that $\mathcal{P}$ is effective (i.e, the set of truth-values is countable, the interpretation functions of the connectives are computable, and the set of designated truth-values is decidable). Note that this implies decidability in case $\vdash_{\mathcal{S}}$ also has a corresponding sound and complete proof system.

REMARK 13. One of the main shortcomings of matrix-based semantics is its lack of modularity with respect to proof systems. To use this type of semantics, the rules and axioms of a system which are related to a given connective should be considered as a whole, and there is no method for separately determining the semantic effects of each rule alone. Take for example the standard Gentzen-type rules for negation:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}\ (\Rightarrow \neg)$$

The corresponding truth-table is the classical one:

	$\neg$
t	**f**
f	**t**

However, if one of the negation rules is discarded, the resulting system has no finite characteristic matrix (this is a special case of Theorem 24 below). It follows that in the framework of (ordinary) matrices the semantic effects of each of the above two rules of negation cannot be analyzed separately. We will shortly see that in contrast, the semantics of non-deterministic matrices does allow a high degree of modularity: In many cases the effect of each syntactic rule or axiom alone can easily be determined, and the semantics of a proof system can then be constructed by straightforwardly combining the semantics of its various rules and axioms.

3 INTRODUCING NMATRICES

Nmatrices were introduced in [Avron and Lev, 2001; Avron and Lev, 2005; Avron and Konikowska, 2005]. The definitions below are taken from there.

DEFINITION 14. A *non-deterministic matrix (Nmatrix)* for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.
- For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$.

DEFINITION 15. Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix for $\mathcal{L}$.

1. A partial *dynamic valuation* in $\mathcal{M}$ (or an $\mathcal{M}$-legal partial dynamic valuation) is a function v from some subset of $Frm_{\mathcal{L}}$ to $\mathcal{V}$, which is closed under subformulas, such that for each n-ary connective $\diamond$ of $\mathcal{L}$, the following holds for all $\psi_1, \ldots, \psi_n \in Frm_{\mathcal{L}}$:

$$\text{(SLC)} \qquad v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

   A partial valuation in $\mathcal{M}$ is called a *valuation* if its domain is $Frm_{\mathcal{L}}$.

2. A (partial) *static valuation* in $\mathcal{M}$ (or an $\mathcal{M}$-legal (partial) static valuation) is a (partial) dynamic valuation which satisfies also the following compositionality (or functionality) principle (CMP): for each $\diamond$ of $\mathcal{L}$ and for every $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n \in Frm_{\mathcal{L}}$,

$$v(\diamond(\psi_1, \ldots, \psi_n)) = v(\diamond(\varphi_1, \ldots, \varphi_n)) \text{ if } v(\psi_i) = v(\varphi_i) \ (i = 1 \ldots n)$$

REMARK 16. Ordinary (deterministic) matrices correspond to the case when each $\widetilde{\diamond}$ is a function taking singleton values only (then it can be treated as a function $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$). In this case there is no difference between static and dynamic valuations, and we have full determinism.

REMARK 17. Like in usual multi-valued semantics, the principle here is that each formula has a definite logical value. This is why we exclude $\emptyset$ from being a value of $\widetilde{\diamond}$. However, the absence of any logical value for a formula can still be simulated in our formalism by introducing a special logical value $\perp$ representing exactly this case (which is a well-known procedure in the framework of partial logics ([Blamey, 1986])).

To understand the difference between ordinary matrices and Nmatrices, recall that in the deterministic case (see Defn. 6), the truth-value assigned by a valuation v to a complex formula is defined as follows: $v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$. Thus the truth-value assigned to $\diamond(\psi_1, ..., \psi_n)$ is uniquely determined by the truth-values of its subformulas: $v(\psi_1), ..., v(\psi_n)$. This, however, is not the case in dynamic valuations in Nmatrices: in general the truth-values assigned to $\psi_1, ..., \psi_n$ do not uniquely determine the truth-value assigned to $\diamond(\psi_1, ..., \psi_n)$ because v makes a non-deterministic choice out of the set of options $\tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$. Therefore the non-deterministic semantics is non-truth-functional, as opposed to the deterministic one.

DEFINITION 18.

1. A (partial) valuation v in $\mathcal{M}$ *satisfies* a formula ψ ($v \models \psi$) if ($v(\psi)$ is defined and) $v(\psi) \in \mathcal{D}$. It is a *model of* Γ ($v \models \Gamma$) if it satisfies every formula in Γ.

2. We say that ψ is dynamically (statically) valid in $\mathcal{M}$, in symbols $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$), if $v \models \psi$ for each dynamic (static) valuation v in $\mathcal{M}$.

3. A logic **L** is *dynamically (statically) weakly sound for an Nmatrix* $\mathcal{M}$ if $\vdash_{\mathbf{L}} \psi$ implies $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$). A logic **L** is *dynamically (statically) weakly complete for* $\mathcal{M}$ if $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$) implies $\vdash_{\mathbf{L}} \psi$. $\mathcal{M}$ is a *dynamically (statically) weakly characteristic* for **L** if **L** is dynamically (statically) both weakly sound and weakly complete for $\mathcal{M}$.

4. $\vdash^d_{\mathcal{M}}$ ($\vdash^s_{\mathcal{M}}$), the *dynamic (static) consequence relation induced by* $\mathcal{M}$, is defined as follows: $\Gamma \vdash^d_{\mathcal{M}} \Delta$ ($\Gamma \vdash^s_{\mathcal{M}} \Delta$), if every dynamic (static) model v in $\mathcal{M}$ of Γ satisfies some $\psi \in \Delta$.

5. A logic $\mathbf{L} = \langle \vdash_{\mathbf{L}}, \mathcal{L} \rangle$ is dynamically (statically) sound for an Nmatrix $\mathcal{M}$ for $\mathcal{L}$ if $\vdash_{\mathbf{L}} \subseteq \vdash^d_{\mathcal{M}}$ ($\vdash_{\mathbf{L}} \subseteq \vdash^s_{\mathcal{M}}$). **L** is dynamically (statically) complete for $\mathcal{M}$ if $\vdash^d_{\mathcal{M}} \subseteq \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} \subseteq \vdash_{\mathbf{L}}$). $\mathcal{M}$ is dynamically (statically) characteristic for **L** if $\vdash^d_{\mathcal{M}} = \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} = \vdash_{\mathbf{L}}$).

REMARK 19. Obviously, the static consequence relation includes the dynamic one, i.e. $\vdash^s_{\mathcal{M}} \supseteq \vdash^d_{\mathcal{M}}$. Also, for ordinary matrices $\vdash^s_{\mathcal{M}} = \vdash^d_{\mathcal{M}}$.

CONVENTION 20. We shall denote $\mathcal{F} = \mathcal{V} \setminus \mathcal{D}$, and shall usually identify singletons of truth-values with the truth-values themselves.

EXAMPLE 21. Assume that $\mathcal{L}$ has binary connectives $\vee$, $\wedge$, and $\supset$ interpreted classically, and a unary connective $\neg$, for which the law of contradiction obtains, but not necessarily the law of excluded middle. This leads to the Nmatrix $\mathcal{M}^2 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ for $\mathcal{L}$, where Let $\mathcal{V} = \{\mathbf{f}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$, and $\mathcal{O}$ is given by:

		$\widetilde{\vee}$	$\widetilde{\wedge}$	$\widetilde{\supset}$
t	**t**	**t**	**t**	**t**
t	**f**	**t**	**f**	**f**
f	**t**	**t**	**f**	**t**
f	**f**	**f**	**f**	**t**

	$\widetilde{\neg}$
t	**f**
f	$\{\mathbf{t}, \mathbf{f}\}$

Note that classical negation can be defined in $\mathcal{M}^2$ by: $\sim\psi = \psi \supset \neg\psi$ (this is a semantic counterpart of the observation made in [Béziau, 1999]).

EXAMPLE 22. Consider the following two 3-valued Nmatrices $\mathcal{M}^3_L, \mathcal{M}^3_S$. In both we have $\mathcal{V} = \{\mathbf{f}, \top, \mathbf{t}\}, \mathcal{D} = \{\top, \mathbf{t}\}$. Also the interpretations of disjunction, conjunction and implication are the same in both of them, and correspond to those in positive classical logic:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

However, negation is interpreted differently: more liberally in $\mathcal{M}_L^3$, and more strictly in $\mathcal{M}_S^3$:

$\mathcal{M}_L^3$:

	$\tilde{\neg}$
t	**f**
$\top$	$\mathcal{V}$
f	**t**

$\mathcal{M}_S^3$:

	$\tilde{\neg}$
t	**f**
$\top$	$\mathcal{D}$
f	**t**

EXAMPLE 23. After considering 2-valued Nmatrices and 3-valued Nmatrices, our last example is the 4-valued Nmatrix $\mathcal{M}_4 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V} = \{\mathbf{f}, \bot, \top, \mathbf{t}\}$, $\mathcal{D} = \{\top, \mathbf{t}\}$, $\wedge, \vee, \supset$ are defined by the general rules given in Example 22 (applied, however, to the sets $\mathcal{D}$ and $\mathcal{F} = \mathcal{V} \setminus \mathcal{D}$ appearing in the current example), while $\neg$ is the negation of the bilattice $\mathcal{FOUR}$ ([Belnap, 1977; Ginsberg, 1988; Fitting, 1994; Arieli and Avron, 1996]):

	$\tilde{\neg}$
t	**f**
$\top$	$\top$
$\bot$	$\bot$
f	**t**

At this point it is natural to ask whether finite Nmatrices can be used for characterizing logics that cannot be characterized by finite ordinary matrices. The next theorem provides a positive answer to this question:

THEOREM 24. *Let $\mathcal{M}$ be a two-valued Nmatrix which has at least one proper non-deterministic operation. Then there is no finite family of finite ordinary matrices F, such that $\vdash^d_{\mathcal{M}} = \vdash_F$. If in addition $\mathcal{M}$ includes the classical implication, then there is no finite family of ordinary matrices F, such that $\vdash^d_{\mathcal{M}} \psi$ iff $\vdash_F \psi$.*

Proof: a straightforward modification of the proof of Theorem 3.4 in [Avron and Lev, 2005].

As the next easy theorem shows, things are different in the case of the static semantics:

THEOREM 25. *For every (finite) Nmatrix $\mathcal{M}$, there is a (finite) family of ordinary matrices, such that $\vdash^s_{\mathcal{M}} = \vdash_F$.*

Thus only the expressive power of the dynamic semantics based on Nmatrices is stronger than that of ordinary matrices. For this reason (after providing general proof theory for both kinds of semantics in the next subsection) our main focus will be on this semantics and what it induces. Accordingly, *we shall usually write simply* $\vdash_{\mathcal{M}}$ *instead of* $\vdash^{d}_{\mathcal{M}}$.

The following theorem from [Avron and Lev, 2005] is a generalization of Theorem 10 to the case of Nmatrices:

THEOREM 26. **(Compactness)** $\vdash_{\mathcal{M}}$ *is finitary for any finite Nmatrix* $\mathcal{M}$.

Later we shall prove a stronger version of this theorem (see Theorem 53). The proof of the next important result is as easy for Nmatrices as it is for ordinary matrices:

PROPOSITION 27. **(Analycity)** *Let* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *be an Nmatrix for* $\mathcal{L}$, *and let* v' *be a partial valuation in* $\mathcal{M}$. *Then* v' *can be extended to a (full) valuation in* $\mathcal{M}$.

It is easy to show that like in the case of ordinary matrices (see Remark 12), Proposition 27 implies the following Theorem:

THEOREM 28. *Non-theoremhood of a logic which has an effective characteristic Nmatrix* $\mathcal{M}$ *is semi-decidable. If* $\mathcal{M}$ *is finite, or* L *also has a sound and complete formal proof system, then* L *is decidable.*

The following is an easy analogue for Nmatrices of Theorem 7:

PROPOSITION 29. *For any Nmatrix* $\mathcal{M}$, $\vdash_{\mathcal{M}}$ *is uniform.*

We end this subsection by introducing the notion of a *refinement*:

DEFINITION 30. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$.

1. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$ such that:
   (a) For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$.
   (b) $F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n, y \in \mathcal{V}_1$ such that $y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$.
2. $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.

THEOREM 31. *If* $\mathcal{M}_1$ *is a refinement of* $\mathcal{M}_2$ *then* $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

REMARK 32. An important case in which $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is when $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\widetilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{V}_1^n$. It is easy to see that the identity function on $\mathcal{V}_1$ is in this case a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. A refinement of this sort will be called *simple*.

4 CANONICAL DEDUCTION SYSTEMS AND NMATRICES

The idea of "canonical" systems implicitly underlies a long tradition in the philosophy of logic, established by G. Gentzen in his classical paper [Gentzen, 1969]. According to this tradition, the meaning of a connective is determined by the introduction and the elimination rules which are associated with it (see, e.g., [Zucker, 1978a; Zucker, 1978b]). The supporters of this thesis usually have in mind Natural Deduction systems of an ideal type. In this type of "canonical systems" each connective $\diamond$ has its own introduction and elimination rules, in each of which $\diamond$ is mentioned exactly once, and no other connective is involved. The rules should also be pure in the sense of [Avron, 1991]. Unfortunately, already the handling of negation requires rules which are not canonical in this sense. This problem was solved by Gentzen himself by moving to what is now known as (multiple-conclusion) Gentzen-type calculi, which instead of introduction and elimination rules use left and right introduction rules. The intuitive notion of a "canonical rule" can be adapted to such systems in a straightforward way, and it is well-known that the usual classical connectives can indeed be fully characterized in this framework by such rules. Moreover, the cut-elimination theorem obtains in all the known Gentzen-type calculi for propositional classical logic (or some fragment of it) which employ only rules of this type. These facts were generalized in [Avron and Lev, 2005], where the notion of a canonical propositional Gentzen-type system has been introduced. This notion was further generalized in [Avron and Konikowska, 2005; Avron and Zamansky, 2008b] to *canonical signed calculi*. These calculi and their intimate connections with finite Nmatrices are the subject of the present section.

Signed calculi consist of rules operating on finite sets of *signed formulas*, and axioms being sets of such formulas. The deduction formalism we use here for them is similar to the Rasiowa-Sikorski (R-S) systems ([Rasiowa and Sikorski, 1963; Konikowska, 2002]), known also as dual tableaux ([Baaz et. al., 1993; Hähnle, 1999]).

Henceforth (until the end of Section 4) $\mathcal{V}$ *denotes some finite set of signs.*

DEFINITION 33. A *signed formula* for $(\mathcal{L}, \mathcal{V})$ is an expression of the form $s : \psi$, where $s \in \mathcal{V}$ and $\psi \in Frm_{\mathcal{L}}$. A signed formula $s : \psi$ is atomic if ψ is an atomic formula. A *sequent* for $(\mathcal{L}, \mathcal{V})$ is a finite set of signed formulas for $(\mathcal{L}, \mathcal{V})$. A *clause* is a sequent consisting of atomic signed formulas.

REMARK 34. The usual (two-sided) sequent notation $\Gamma \Rightarrow \Delta$ can be interpreted as $\{f : \Gamma\} \cup \{t : \Delta\}$, i.e. a sequent in the sense of Defn. 33 over the two signs $\{t, f\}$.

DEFINITION 35. Let v be a function from the set of formulas of $\mathcal{L}$ to $\mathcal{V}$.

1. v satisfies a signed formula $\gamma = (l : \psi)$, denoted by $v \models (l : \psi)$, if $v(\psi) = l$.
2. v satisfies a set of signed formulas Υ, denoted by $v \models \Upsilon$, if there is some $\gamma \in \Upsilon$, such that $v \models \gamma$.

CONVENTION 36. Formulas will be denoted by φ, ψ, signed formulas - by $\alpha, \beta, \gamma, \delta$, sets of signed formulas - by Υ, Λ, sequents - by Ω, Σ, Π, sets of sets of signed formulas - by Φ, Ψ and sets of sequents - by Θ, Ξ. We write $S : \psi$ instead of $\{s : \psi \mid s \in S\}$, and $S : \Delta$ instead of $\{s : \psi \mid s \in S, \psi \in \Delta\}$.

DEFINITION 37. A *signed canonical (propositional) rule of arity n* for $(\mathcal{L}, \mathcal{V})$ is an expression of the form $[\Theta/S : \diamond(p_1, \dots, p_n)]$, where S is a non-empty subset of $\mathcal{V}$, $\diamond$ is an n-ary connective of $\mathcal{L}$ and $\Theta = \{\Sigma_1, ..., \Sigma_m\}$, where $m \geq 0$ and for every $1 \leq j \leq m$, Σ_j are clauses (see Definition 33) consisting of signed formulas of the form $a : p_k$, where $a \in \mathcal{V}$ and $1 \leq k \leq n$. An *application* of a rule $[\{\Sigma_1, ..., \Sigma_m\}/S : \diamond(p_1, \dots, p_n)]$ is any inference of the form:

$$\frac{\Omega \cup \Sigma_1^* \quad ... \quad \Omega \cup \Sigma_m^*}{\Omega \cup S : \diamond(\psi_1, ..., \psi_n)}$$

where $\psi_1, ..., \psi_n$ are $\mathcal{L}$-formulas, Ω is a sequent, and for all $1 \leq i \leq m$: Σ_i^* is obtained from Σ_i by replacing p_j by ψ_j for every $1 \leq j \leq n$.

EXAMPLE 38.

1. The standard Gentzen-style introduction rules for the classical conjunction are usually defined as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

   Using the notation in Remark 34, we can write $\{f : \Gamma\} \cup \{t : \Delta\}$ (that is, ψ occurs with a sign 't' if $\psi \in \Gamma$ and with a sign 'f' if $\psi \in \Delta$), thus the canonical representation of the rules above is as follows:

$$[\{\{f : p_1, f : p_2\}\}/\{f\} : p_1 \wedge p_2] \qquad [\{\{t : p_1\}, \{t : p_2\}\}/\{t\} : p_1 \wedge p_2]$$

   Applications of these rules have the forms:

$$\frac{\Omega \cup \{f : \psi_1, f : \psi_2\}}{\Omega \cup \{f : \psi_1 \wedge \psi_2\}} \qquad \frac{\Omega \cup \{t : \psi_1\} \quad \Omega \cup \{t : \psi_2\}}{\Omega \cup \{t : \psi_1 \wedge \psi_2\}}$$

2. Consider a calculus over $\mathcal{V} = \{a, b, c\}$ with the following introduction rules for a ternary connective $\circ$:

$$[\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\}/\{a, c\} : \circ(p_1, p_2, p_3)]$$

$$[\{\{c : p_2\}, \{a : p_3, b : p_3\}, \{c : p_1\}\}/\{b, c\} : \circ(p_1, p_2, p_3)]$$

Their applications are of the forms:

$$\frac{\Omega \cup \{a : \psi_1, c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_2\}}{\Omega \cup \{a : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

$$\frac{\Omega \cup \{c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_3\} \quad \Omega \cup \{c : \psi_1\}}{\Omega \cup \{b : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

DEFINITION 39. Let $\mathcal{V}$ be a finite set of signs.

1. A logical axiom for $\mathcal{V}$ is a sequent of the form: $\{l : \psi \mid l \in \mathcal{V}\}$.

2. The cut and weakening rules for $\mathcal{V}$ are defined as follows:

$$\frac{\Omega \cup \{l : \psi \mid l \in L_1\} \quad \Omega \cup \{l : \psi \mid l \in L_2\}}{\Omega \cup \{l : \psi \mid l \in L_1 \cap L_2\}}\ CUT$$

$$\frac{\Omega}{\Omega, l : \psi}\ WEAK$$

where $L_1, L_2 \subseteq \mathcal{V}$ and $l \in \mathcal{V}$.

The following proposition follows from the completeness of many-valued resolution (see [Baaz et. al., 1995]):

PROPOSITION 40. *Let Θ be a set of clauses and Ω - a clause. Then Ω follows from Θ (in the sense that every v which satisfies Θ also satisfies Ω) iff there is some $\Omega' \subseteq \Omega$, such that Ω' is derivable from Θ by cuts.*

COROLLARY 41. *Let Θ be a set of clauses. The empty sequent is derivable from Θ by cuts iff Θ is not satisfiable.*

Now we are ready to define "canonical signed calculi" in precise terms:

DEFINITION 42. A signed calculus over a language $\mathcal{L}$ and a finite set of signs $\mathcal{V}$ is *canonical* if it consists of:

1. All logical axioms for $\mathcal{V}$.

2. The rules of cut and weakening from Defn. 39.

3. Any number of signed canonical inference rules.

Not all canonical calculi are useful. Of interest are only those of them which "define" the semantic meaning of the logical connectives they introduce. It turned out that this property can be captured syntactically by a simple syntactic criterion called *coherence*, introduced in [Avron and Lev, 2005] for canonical Gentzen-type systems, and extended in [Avron and Zamansky, 2008b] to signed calculi.

DEFINITION 43. A canonical calculus G is *coherent* if $\Theta_1 \cup ... \cup \Theta_m$ is unsatisfiable whenever $\{[\Theta_1/S_1 : \psi], ..., [\Theta_m/S_m : \psi]\}$ is a set of rules of G such that $S_1 \cap ... \cap S_m = \emptyset$ (here $\psi = \diamond(p_1, \ldots, p_n)$ for some n-ary $\diamond$).

Obviously, coherence is a decidable property of canonical calculi. Note also that by Corollary 41, a canonical calculus G is *coherent* if whenever $\{[\Theta_1/S_1 : \psi], ..., [\Theta_m/S_m : \psi]\}$ is a set of rules of G, and $S_1 \cap ... \cap S_m = \emptyset$, we have that $\Theta_1 \cup ... \cup \Theta_m$ is inconsistent (i.e. the empty sequent can be derived from it using cuts).

EXAMPLE 44.

1. Consider the canonical calculus G_1 over $\mathcal{L} = \{\wedge\}$ and $\mathcal{V} = \{t, f\}$, the canonical rules of which are the two rules for $\wedge$ from Example 38. We can derive the empty sequent from $\{\{t : p_1\}, \{t : p_2\}, \{f : p_1, f : p_2\}\}$ as follows:

$$\dfrac{\dfrac{\{t : p_1\} \quad \{f : p_1, f : p_2\}}{\{f : p_2\}}\ CUT \qquad \{t : p_2\}}{\emptyset}\ CUT$$

   Thus G_1 is coherent.

2. Consider the canonical calculus G_2 over $\mathcal{V} = \{a, b, c\}$ with the following introduction rules for the ternary connective $\circ$:

$$[\{\{a : p_1\}, \{b : p_2\}\}/\{a, b\} : \circ(p_1, p_2, p_3)]$$

$$[\{\{a : p_2, c : p_3\}\}/\{c\} : \circ(p_1, p_2, p_3)]$$

   Clearly, the set $\{\{a : p_1\}, \{b : p_2\}, \{a : p_2, c : p_3\}\}$ is satisfiable, thus G_2 is not coherent.

REMARK 45. [Ciabattoni and Terui, 2006a] investigates a general class of single-conclusion two-sided (sequent) calculi called *simple calculi.* These calculi may include any set of structural rules, and so the two-sided canonical calculi are a particular instance of simple calculi which include all of the standard structural rules. The reductivity condition of [Ciabattoni and Terui, 2006a] can be shown to be equivalent to our coherence criterion in the context of two-sided canonical systems.

Next we define some notions of cut-elimination[6] in canonical calculi:

DEFINITION 46. Let G be a canonical signed calculus and let Θ be some set of sequents.

1. A cut is called a Θ-*cut* if the cut formula occurs in Θ. We say that a proof is Θ-cut-free if the only cuts in it are Θ-cuts.

2. A cut is called Θ-*analytic* if the cut formula is a subformula of some formula occurring in Θ. A proof is called Θ-analytic[7] if all cuts in it are Θ-analytic.

3. A canonical calculus G admits *(standard) cut-elimination* if whenever $\vdash_G \Omega$, Ω has a cut-free proof in G. G admits *strong cut-elimination*[8] if whenever $\Theta \vdash_G \Omega$, Ω has in G a Θ-cut-free proof from Θ.

4. G admits *strong analytic cut-elimination* if whenever $\Theta \vdash_G \Omega$, Ω has in G a $\Theta \cup \{\Omega\}$-analytic proof from Θ. G admits *analytic cut-elimination* if whenever $\vdash_G \Omega$, Ω has in G a $\{\Omega\}$-analytic proof.

EXAMPLE 47. Consider the following calculus G' for a language with a binary connective $\circ$ and $\mathcal{V} = \{a, b, c\}$. The rules of G' are as follows:

$$R_1 = \{\{a : p_1\}\}/\{a, b\} : p_1 \circ p_2\} \qquad R_2 = \{\{a : p_1\}\}/\{b, c\} : p_1 \circ p_2\}$$

In the following proof in G', the cut in the final step is analytic:

$$\dfrac{\dfrac{a : p_1, b : p_1, c : p_1}{b : p_1, c : p_1, b : (p_1 \circ p_2), c : (p_1 \circ p_2)} \qquad \dfrac{a : p_1, b : p_1, c : p_1}{b : p_1, c : p_1, a : (p_1 \circ p_2), b : (p_1 \circ p_2)}}{b : p_1, c : p_1, b : (p_1 \circ p_2)}$$

4.1 Canonical Calculi for Nmatrices

There are numerous works on proof theory for logics based on finite *ordinary* matrices, mainly using many-placed sequent calculi or tableaux systems with truth values as signs (cf. [Baaz et. al., 1993; Borowik, 1986; Carnielli, 1991; Rousseau, 1967; Takahashi, 1967; Hähnle, 1999; Baaz et. al., 2000]). In this section we present analogous canonical signed calculi

[6]We note that by 'cut-elimination' we mean here just the *existence* of proofs without (certain forms of) cuts, rather than an algorithm to transform a given proof to a cut-free one (for the assumptions-free case the term "cut-admissibility" is sometimes used, but this notion is too weak for our purposes).

[7]This is a generalization of the notion of analytic cut (see e.g. [Baaz et. al., 2001]).

[8]The notion of strong cut-elimination from [Avron, 1993] was studied in the context of canonical Gentzen-type systems in [Avron and Zamansky, 2007c].

for logics based on finite Nmatrices (developed in [Avron and Konikowska, 2005]).

DEFINITION 48. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$.

1. $\Phi \vdash^d_{\mathcal{M}} \Upsilon$ ($\Phi \vdash^s_{\mathcal{M}} \Upsilon$) if $v \models \Upsilon$ for every $\mathcal{M}$-legal dynamic (static) valuation v which satisfies all the sets in Φ.

2. Let G be a deduction system based on sequents. G is *dynamically (statically) strongly sound for* $\mathcal{M}$ if $\Theta \vdash_G \Omega$ implies that $\Theta \vdash^d_{\mathcal{M}} \Omega$ ($\Theta \vdash^s_{\mathcal{M}} \Omega$). G is *dynamically (statically) strongly complete for* $\mathcal{M}$ if $\Theta \vdash^d_{\mathcal{M}} \Omega$ ($\Theta \vdash^s_{\mathcal{M}} \Omega$) implies $\Theta \vdash_G \Omega$. $\mathcal{M}$ is a *dynamically (statically) strongly characteristic Nmatrix for* G if G is dynamically (statically) strongly sound and strongly complete for $\mathcal{M}$. (The notions of soundness, completeness and a characteristic Nmatrix are defined similarly by setting $\Theta = \emptyset$.)

It should be noted that the set of designated values $\mathcal{D}$ in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ has not been used in the semantic definitions above. This is because the consequence relations defined above are between sets of sequents and sequents. Recall, however, that the set $\mathcal{D}$ *is* used in Definition 18, where the consequence relations between *sets of formulas* are defined. The following easy observations are the key for using proof systems based on sets of signed formulas for characterizing logics induced by Nmatrices:

PROPOSITION 49. *Let* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *be an Nmatrix for* $\mathcal{L}$. *Then:*

$$\Gamma \vdash^d_{\mathcal{M}} \Delta \textit{ iff } \{\mathcal{D} : \psi \mid \psi \in \Gamma\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta\} \vdash^d_{\mathcal{M}} \emptyset \textit{ iff } \vdash^d_{\mathcal{M}} \mathcal{F} : \Gamma \cup \mathcal{D} : \Delta$$

$$\Gamma \vdash^s_{\mathcal{M}} \Delta \textit{ iff } \{\mathcal{D} : \psi \mid \psi \in \Gamma\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta\} \vdash^s_{\mathcal{M}} \emptyset \textit{ iff } \vdash^s_{\mathcal{M}} \mathcal{F} : \Gamma \cup \mathcal{D} : \Delta$$

DEFINITION 50. The proof system $SF^d_{\mathcal{M}}$ for the dynamic semantics of a finite-valued Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is the canonical signed calculus for $(\mathcal{L}, \mathcal{V})$ which for every n-ary $\diamond$, and every $a_1, \ldots, a_m, b_1, \ldots, b_k \in \mathcal{V}$ such that $\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$, includes the rule:

$$[\{a_1 : p_1\}, \ldots, \{a_m : p_m\} / \{b_1, \ldots, b_k\} : \diamond(p_1, \ldots, p_n)]$$

or in a more conventional formulation:

$$(\diamond\text{-D}) \qquad \frac{\Omega \cup \{a_1 : \varphi_1\} \quad \ldots \quad \Omega \cup \{a_m : \varphi_m\}}{\Omega \cup \{b_1 : \diamond(\varphi_1, \ldots, \varphi_m), \ldots, b_k : \diamond(\varphi_1, \ldots, \varphi_m)\}}$$

The following theorem is a generalization of a result first shown in [Avron and Konikowska, 2005]. Its proof requires just a straightforward extension of the argument given there:

THEOREM 51. *$SF^d_{\mathcal{M}}$ is dynamically strongly characteristic for $\mathcal{M}$.*

The following corollary follows from Prop. 49 and the above theorem:

COROLLARY 52. *$\Gamma \vdash^d_{\mathcal{M}} \Delta$ iff $\{\mathcal{D} : \psi \mid \psi \in \Gamma\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta\} \vdash_{SF^d_{\mathcal{M}}} \emptyset$. Moreover, if Γ and Δ are finite then $\Gamma \vdash^d_{\mathcal{M}} \Delta$ iff $\vdash_{SF^d_{\mathcal{M}}} \mathcal{F} : \Gamma \cup \mathcal{D} : \Delta$.*

Now we are ready to prove the general compactness theorem mentioned immediately after Theorem 26:

THEOREM 53. **(Compactness)**

1. *Let Θ be a set of sequents and Ω a sequent. If $\Theta \vdash^d_{\mathcal{M}} \Omega$, then there is some finite $\Theta' \subseteq \Theta$, such that $\Theta' \vdash^d_{\mathcal{M}} \Omega$.*
2. *Let Γ, Δ be two sets of $\mathcal{L}$-formulas. If $\Gamma \vdash^d_{\mathcal{M}} \Delta$, then there are some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, such that $\Gamma' \vdash^d_{\mathcal{M}} \Delta'$.*

Proof. For the first part, assume that $\Theta \vdash^d_{\mathcal{M}} \Omega$. Then $\Theta \vdash_{SF^d_{\mathcal{M}}} \Omega$ by Theorem 51, and so there is some finite $\Theta' \subseteq \Theta$, such that $\Theta' \vdash_{SF^d_{\mathcal{M}}} \Omega$. Hence (again by Theorem 51) $\Theta' \vdash^d_{\mathcal{M}} \Omega$. For the second part, suppose that $\Gamma \vdash^d_{\mathcal{M}} \Delta$. Then by Proposition 49, $\{\mathcal{D} : \psi \mid \psi \in \Gamma\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta\} \vdash^d_{\mathcal{M}} \emptyset$. By the first part, there are some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, such that $\{\mathcal{D} : \psi \mid \psi \in \Gamma'\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta'\} \vdash^d_{\mathcal{M}} \emptyset$. By Proposition 49, $\Gamma' \vdash^d_{\mathcal{M}} \Delta'$. ■

DEFINITION 54. The proof system $SF^s_{\mathcal{M}}$ for the static semantics of $\mathcal{M}$ is obtained from the system $SF^d_{\mathcal{M}}$ by adding, for any m-ary connective $\diamond$ of $\mathcal{L}$ and any $a_1, \ldots, a_m, b \in \mathcal{V}$ such that $b \in \widetilde{\diamond}(a_1, \ldots, a_m)$, the rule ($\diamond$-S):

$$\frac{\{\Omega \cup \{a_j : \varphi_j\}\}_{1 \leq j \leq m} \quad \{\Omega \cup \{a_j : \psi_j\}\}_{1 \leq j \leq m} \quad \Omega \cup \{b : \diamond(\psi_1, \ldots, \psi_m)\}}{\Omega \cup \{b : \diamond(\varphi_1, \ldots, \varphi_m)\}}$$

Obviously, these $(2m+1)$-premise inference rules are not very convenient. More importantly: they are not analytic[9]. However, they can be simplified at the price of extending the language with a constant $\underline{a}$ for every $a \in \mathcal{V}$. In that case we can also resign from repeating the inference rules from the dynamic semantics, adding instead equivalent axioms for the constants:

DEFINITION 55. The proof system $SF^{sc}_{\mathcal{M}}$ for the static semantics of the language featuring constants consists of:

[9]By an analytic rule we mean a rule which has some kind of a subformula property (see, e.g. [Baaz et. al., 2000]). This should not be confused with analycity of semantics (see Remark 12).

- **Axioms**: Each set of signed formulas containing either:
    1. $\{a : \varphi \mid a \in \mathcal{V}\}$, where φ is any formula in $\mathcal{W}$; or
    2. $\{a : \underline{a}\}$, for any $a \in \mathcal{V}$; or
    3. $\{b_1 : \diamond(\underline{a_1}, \ldots, \underline{a_m}), \ldots, b_k : \diamond(\underline{a_1}, \ldots, \underline{a_m})\}$ for any m-ary connective $\diamond$ of $\mathcal{L}$ and any $a_1, \ldots, a_m,\ b_1, \ldots, b_k \in \mathcal{V}$ such that $\widetilde{\diamond}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$.
- **Inference rules:** For any $a_1, \ldots, a_m, b \in \mathcal{V}$ and any m-ary connective $\diamond$ such that $b \in \widetilde{\diamond}(a_1, \ldots, a_m)$, the rule ($\diamond$-SC):

$$\frac{\Omega \cup \{a_1 : \varphi_1\} \quad \ldots \quad \Omega \cup \{a_m : \varphi_m\} \quad \Omega \cup \{b : \diamond(\underline{a_1}, \ldots, \underline{a_m})\}}{\Omega \cup \{b : \diamond(\varphi_1, \ldots, \varphi_m)\}}$$

REMARK 56. Examining the generic deduction systems given above, we can easily observe that the inference rules of the static semantics really differ from those of the dynamic semantics only in case of truly non-deterministic values of the connectives. Indeed, if the value of the connective is a singleton, i.e. $\widetilde{\diamond}(\underline{a_1}, \ldots, \underline{a_m}) = \{b\}$, the rule ($\diamond$-S) is just a weaker version of ($\diamond$-D), and so need not be included in $SF^{s}_{\mathcal{M}}$. As for $SF^{sc}_{\mathcal{M}}$, the last premise of rule ($\diamond$-SC) is derivable in the system by virtue of the singleton set $\{b :\ \diamond(\underline{a_1}, \ldots, \underline{a_m})\}$ being an axiom — hence it can be skipped. As the other premises of the "static" and "dynamic" rules coincide, and so do the conclusions in such a "singleton" case, the rules can be considered identical. in this case the "static" Axiom 3 corresponding to such a singleton value of the connective can be deleted too, since it is derivable from rule ($\diamond$-D) and the basic axioms for the constants ("static" Axiom 2).

REMARK 57. It can easily be proved that the weakening rule is admissible in $SF^{sc}_{\mathcal{M}}$. This is the reason why it is not necessary to officially include it among the rules of this system.

The following generalizes a theorem from [Avron and Konikowska, 2005]:

THEOREM 58. *$SF^{sc}_{\mathcal{M}}$ is statically strongly characteristic for $\mathcal{M}$.*

COROLLARY 59. $\Gamma \vdash^{s}_{\mathcal{M}} \Delta$ *iff* $\{\mathcal{D} : \psi \mid \psi \in \Gamma\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta\} \vdash_{SF^{s}_{\mathcal{M}}} \emptyset$. *Moreover, if* Γ *and* Δ *are finite, then* $\Gamma \vdash^{s}_{\mathcal{M}} \Delta$ *iff* $\vdash_{SF^{sc}_{\mathcal{M}}} \mathcal{F} : \Gamma \cup \mathcal{D} : \Delta$.

4.2 Nmatrices for Canonical Calculi

In this subsection we provide, in a modular way, finite non-deterministic semantics for signed canonical calculi. Moreover, we show that there is an exact correspondence between the coherence of a canonical calculus G, the existence of a strongly characteristic Nmatrix for G, and analytic cut-elimination (Definition 46) in G. Then we focus on stronger notions of cut-elimination and show that coherence is not a sufficient condition for them.

Therefore we define a stronger criterion of *density* which is a necessary and sufficient condition for strong cut-elimination in canonical calculi. Finally, we focus on the special case of Gentzen-type (two-signed) canonical calculi and show how the correspondence theorem can be used to provide a solution to the well-known "Tonk" problem of Prior ([Prior, 1960]).

Modular Semantics for Signed Canonical Calculi [10]

We start by defining semantics for the simplest canonical calculus: the one without any canonical rules.

DEFINITION 60. $G_0^{(\mathcal{L},\mathcal{V})}$ is the canonical calculus over a language $\mathcal{L}$ and a set of signs $\mathcal{V}$, whose set of canonical rules is empty.

In the rest of this section we assume that our language $\mathcal{L}$, the set of signs $\mathcal{V}$, and the set of designated signs $\mathcal{D}$, are fixed. Accordingly, we shall write G_0 instead of $G_0^{(\mathcal{L},\mathcal{V})}$. It is easy to see that G_0 is (trivially) coherent. We now define a strongly characteristic Nmatrix for G_0. It has the maximal degree of non-determinism in interpreting the connectives of $\mathcal{L}$.

DEFINITION 61. $\mathcal{M}_0 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is the Nmatrix in which $\tilde{\diamond}(a_1, ..., a_n) = \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and $a_1, ..., a_n \in \mathcal{V}$.

THEOREM 62. *$\mathcal{M}_0$ is (dynamically) strongly characteristic for G_0.*

Now to the modular effects of canonical rules. The idea is that each rule which is added to G_0 imposes a certain semantic condition on refinements of $\mathcal{M}_0$, while coherence guarantees that these semantic conditions are not contradictory. This can be formalized as follows:

DEFINITION 63. For $\langle a_1, ..., a_n \rangle \in \mathcal{V}^n$, the set of clauses $\mathsf{C}_{\langle a_1,...,a_n \rangle}$ is defined as follows:

$$\mathsf{C}_{\langle a_1,...,a_n \rangle} = \{\{a_1 : p_1\}, \{a_2 : p_2\}, ..., \{a_n : p_n\}\}$$

DEFINITION 64. Let R be a canonical rule of the form $[\Theta/S : \diamond]$. $\mathsf{C}(\mathsf{R})$, the refining condition induced by R, is defined as follows:

$\mathsf{C}(\mathsf{R})$: For $a_1, ..., a_n \in \mathcal{V}$, if $\mathsf{C}_{\langle a_1,...,a_n \rangle} \cup \Theta$ is consistent, then $\tilde{\diamond}(a_1, ..., a_n) \subseteq S$.

Intuitively, a rule $[\Theta/S : \diamond]$ leads to the deletion from $\tilde{\diamond}(a_1, ..., a_n)$ of all the truth-values which are not in S. If some rules $[\Theta_1/S_1 : \diamond], ..., [\Theta_m/S_2 : \diamond]$ "overlap", their overall effect leads to $S_1 \cap ... \cap S_m$ (the coherence of a calculus guarantees that $S_1 \cap ... \cap S_m$ is not empty in such a case).

[10]This subsection is based on [Avron and Zamansky, 2008b], where all proofs can be found.

DEFINITION 65. Let G be a canonical calculus for $(\mathcal{L}, \mathcal{V})$.

1. Define an application of a rule $[\Theta/S : \diamond]$ of G for some n-ary connective $\diamond$ on $a_1, ..., a_n \in \mathcal{V}$ as follows:

$$[\Theta/S : \diamond](a_1, ..., a_n) = \begin{cases} S & \text{if } \Theta \cup \mathsf{C}_{\langle a_1,...,a_n \rangle} \text{ is consistent} \\ \mathcal{V} & \textit{otherwise} \end{cases}$$

2. $\mathcal{M}_G = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is any Nmatrix, such that for every n-ary connective $\diamond$ for $\mathcal{L}$ and every $a_1, ..., a_n \in \mathcal{V}$:

$$\tilde{\diamond}_{\mathcal{M}_G}(a_1, ..., a_n) = \bigcap \{[\Theta/S : \diamond](a_1, ..., a_n) \mid [\Theta/S : \diamond] \in G\}$$

PROPOSITION 66. *If G is coherent, then $\mathcal{M}_G$ is well-defined.*

REMARK 67. It is easy to see that for a coherent calculus G, $\mathcal{M}_G$ is the weakest refinement of $\mathcal{M}_0$, in which all the conditions induced by the rules of G are satisfied. Thus if G' is a coherent calculus obtained from G by adding a new canonical rule, $\mathcal{M}'_G$ can be straightforwardly obtained from $\mathcal{M}_G$ by some deletions of options as dictated by the condition which corresponds to the new rule.

THEOREM 68. *For every coherent canonical calculus G, $\mathcal{M}_G$ is a (dynamically) strongly characteristic Nmatrix for G.*

REMARK 69. The last theorem provides the converse of Theorem 51.

The next theorem is the most important result of this subsection. It establishes a quadruple correspondence between coherence of canonical calculi, non-deterministic matrices and analytic cut-elimination.

THEOREM 70. *Let G be a canonical calculus. The following statements concerning G are equivalent.*

1. *G is coherent.*
2. *G has a strongly characteristic Nmatrix.*
3. *G admits strong analytic cut-elimination.*
4. *G admits analytic cut-elimination.*

What about full (strong) cut-elimination? The next example shows that coherence is *not* a sufficient condition for it. Therefore a stronger condition is provided in the definition that follows that example.

EXAMPLE 71. Consider the calculus G' from Example 47. G' is obviously coherent. A proof of the sequent $\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}$ is given in that example. However, this sequent clearly has no cut-free proof in G'.

DEFINITION 72. A canonical calculus G is *dense* if for every $a_1, ..., a_n \in \mathcal{V}$ and every two rules of G of the forms $[\Theta_1/S_1 : \diamond]$ and $[\Theta_2/S_2 : \diamond]$, such that $\Theta_1 \cup \Theta_2 \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ is consistent, there is some rule $[\Theta/S : \diamond]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ is consistent and $S \subseteq S_1 \cap S_2$.

LEMMA 73. *Every dense canonical calculus is coherent.*

THEOREM 74. *Let G be a canonical calculus. Then the following statements concerning G are equivalent:*

1. *G is dense.*
2. *G admits cut-elimination.*
3. *G admits strong cut-elimination.*

Canonical Gentzen-type Calculi and Tonk

A very important class of canonical signed calculi is the class of canonical *ordinary* Gentzen-type calculi ([Avron and Lev, 2001; Avron and Lev, 2005]), i.e. calculi employing ordinary sequents of the form $\Gamma \Rightarrow \Delta$. As noted above, such calculi can be thought of as a special case of canonical signed calculi in which the set $\mathcal{V}$ of signs is $\{t, f\}$, and $\mathcal{D}$ is $\{t\}$. For this particular class, the criteria of coherence and density can be simplified, because the next proposition can easily be proved:

PROPOSITION 75. *A canonical ordinary Gentzen-type calculus G is* coherent *iff for every two canonical rules of G of the form $\Theta_1/\{t\} : \diamond$ and $\Theta_2/\{f\} : \diamond$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent (and so the empty sequent can be derived from it using cuts). Moreover, such a calculus is dense iff it is coherent.*

The following characterization theorem[11] easily follows from Theorems 70, 74, and Proposition 75:

THEOREM 76. *Let G be a canonical calculus with the set of signs $\mathcal{V} = \{t, f\}$. Then the following statements concerning G are equivalent:*

1. *G is coherent.*
2. *G is non-trivial.*
3. *G has a characteristic two-valued Nmatrix.*
4. *G admits cut-elimination.*
5. *G admits strong cut-elimination.*

[11]With the exception of the last item (concerning *strong* cut-elimination), this theorem was originally proved in [Avron and Lev, 2005].

Theorem 76 was used in [Avron and Lev, 2001; Avron and Lev, 2005] to provide a complete solution for the old "Tonk" problem of Prior in the multiple conclusion framework (the single conclusion case is handled in [Avron, 2008b]). In [Prior, 1960] Prior strongly challenged the above mentioned Gentzen's thesis that the semantic meaning of a connective is determined by its introduction and elimination rules. He did that by introducing his famous binary "connective" Tonk (denoted below by T), which has two rules of the "ideal" type. The introduction rule allows to infer $\varphi \mathsf{T} \psi$ from φ. The elimination rule allows to infer ψ from $\varphi \mathsf{T} \psi$. In the presence of Tonk, every formula can be derived from any other formula, making trivial the "logic" that is "defined" by any system which includes this "connective". Prior's paper made it clear that not every combination of "ideal" introduction and elimination rules can be used for defining the semantic meaning of a connective, and some constraints should be imposed on the set of rules. Such a constraint was indeed suggested by Belnap in [Belnap, 1962]: the rules for a connective $\diamond$ should be *conservative*, in the sense that if $T \vdash \psi$ is derivable using them, and $\diamond$ does not occur in $T \cup \{\psi\}$, then $T \vdash \psi$ can also be derived without using the rules for $\diamond$. However, Belnap did not provide any effective necessary and sufficient criterion for checking whether a given set of rules is conservative in the above sense. Moreover: he formulated the condition of conservativity only with respect to the basic deduction framework, in which no connectives are assumed. Accordingly, nothing in what he wrote excludes the possibility of a system G having two connectives, each of them "defined" by a set of rules which is conservative over the basic system, while G itself is not conservative over it. To prevent this situation one should demand a much stronger conservativity condition than Belnap's, and it might not even be clear how it should be formulated. Later attempts of solutions of the Tonk problem insisted on closer connections between the introduction and the elimination rules for a given connective than those implicit in Belnap's condition of conservativity. Usually it is demanded that the introduction and elimination rules should precisely "match" (see, e.g., [Sundholm, 2002; Hodges, 2001]) in the sense that the elimination rules could be derived from the introduction rules by some syntactic procedure. From Theorem 76 it follows that this condition is too strong. What should be required from the set of rules is only *coherence*, which is an absolute (and minimal) condition for non-triviality. Tonk's rules indeed do not meet this condition: in the framework of canonical Gentzen-type systems its rules are translated into the following pair of rules: $\{\{f : p_1\}\}/\{f\} : \mathsf{T}$ and $\{\{t : p_2\}\}/\{t\} : \mathsf{T}$. This pair is not coherent, since the set $\{\{f : p_1\}, \{t : p_2\}\}$ is classically consistent. It is no wonder therefore that the resulting calculus is inconsistent. On the other hand every coherent set of canonical rules does indeed define a *unique* non-deterministic connective over $\{t, f\}$. This proves Gentzen's thesis at least in the multiple-conclusion canonical case. For further discussion and generalizations, we refer the reader to [Avron, 2008b].

5 USING NMATRICES FOR NON-CANONICAL SYSTEMS

In the previous section we have applied finite Nmatrices in a modular way to characterize canonical calculi. The goal of this section is to show that the modular approach can be further extended and fruitfully applied (at least in many important cases) also to *non-canonical* Gentzen-type calculi. As our example we take the most common type of non-canonical rules that can be found in the literature: those which involve a combination of negation with other connectives. We investigate the semantic effects of rules of this type in the context of two major families of non-canonical Gentzen-type calculi: those that are obtained from the positive fragments of classical logic and intuitionistic logic by adding various natural Gentzen-type rules for negation. Not surprisingly, while Nmatrices suffice for providing adequate semantics for the first family, for the second one we need a combination of Nmatrices with intuitionistic Kripke frames. We demonstrate the power of this semantic tool by using it for solving the following important problem: given a system from the second family, determine whether or not it is a conservative extension of the positive fragment of intuitionistic logic.

The material of this section is based on [Avron, 2007b; Avron, 2005a].

5.1 Extensions of Classical Logic

In this section $\mathcal{L}$ denotes the propositional language $\{\wedge, \vee, \supset, \neg\}$, while $\mathcal{L}_{\mathbf{ff}}$ is the language obtained from $\mathcal{L}$ by adding the constant $\mathbf{ff}$. LK^+ denotes positive classical logic taken over $\mathcal{L}$, and LK denotes positive classical logic taken over $\mathcal{L}_{\mathbf{ff}}$. $\mathbf{G}[LK^+]$, the standard Gentzen-type (canonical) for LK^+, is given in Figure 2. $\mathbf{G}[LK]$, the Gentzen-type system for LK, is obtained from $\mathbf{G}[LK^+]$ by adding the sequent $\mathbf{ff} \Rightarrow$ as an additional axiom.

The table in Figure 3 lists the most common and natural logical rules for formulas involving negation and its combinations with other connectives (along with corresponding Hilbert-style axioms and Gentzen-style rules). Note that only the first two rules in this table are canonical.

DEFINITION 77. Denote by NIR the set of rules in Figure 3. For a logic $\mathbf{L}$ and $S \subseteq NIR$, let $\mathbf{L}[S]$ be the extension of $\mathbf{L}$ by S.

CONVENTION 78. For a rule R, denote by $\mathbf{H_R}$ its corresponding Hilbert-style axiom, and by $\mathbf{G_R}$ its corresponding Gentzen-style rule.

REMARK 79. It is easy to see that for every $S \subseteq NIR$ and $\mathbf{L} \in \{LK^+, LK\}$, a sound and complete Hilbert-style axiomatization for $\mathbf{L}[S]$ can be obtained by adding to some axiomatization of $\mathbf{L}$ the set of axioms $\{\mathbf{H_R} \mid \mathbf{R} \in S\}$, and similarly for Gentzen-type axiomatizations. We denote the resulting systems by $\mathbf{HL}[S]$ and $\mathbf{GL}[S]$, respectively.

Axioms:

$$A \Rightarrow A$$

Structural Rules:

Cut, Weakening

Logical Rules:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}\ (\supset\Rightarrow) \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}\ (\Rightarrow\supset)$$

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}\ (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \wedge \varphi, \Delta}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \psi, \varphi, \Delta}{\Gamma \Rightarrow \psi \vee \varphi, \Delta}\ (\Rightarrow \vee)$$

Figure 2. The system LK

Now we provide semantics for the systems introduced in Defn. 77. The basic idea is to let the value assigned to a sentence φ provide information not only about the truth/falsity of φ, but also about the truth/falsity of its negation. This leads to the use of elements from $\{0, 1\}^2$ as our truth-values, where the intended intuitive meaning of $v(\varphi) = \langle x, y \rangle$ is the following:

- $x = 1$ iff φ is "true" (i.e. $v(\varphi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).

This interpretation of the truth-values dictates the following constraint on any valuation v (where $P_1(\langle x_1, x_2 \rangle) = x_1$, and $P_2(\langle x_1, x_2 \rangle) = x_2$):

$$P_1(v(\neg\varphi)) = P_2(v(\varphi))$$

In terms of Nmatrices this constraint translates into the condition:

$$\text{(NEG)} \quad \tilde{\neg} a \subseteq \{y \mid P_1(y) = P_2(a)\}$$

We start our semantic investigation of NIR with the weakest Nmatrix which satisfies Condition (NEG) and has the standard interpretation of $\supset, \wedge$, and $\vee$ (since the standard rules for these connectives are in LK^+).

Rule	Abstract form	Hilbert-style axiom	Gentzen-style rule
$(\neg \Rightarrow)$	$\neg\varphi, \varphi \vdash$	$\neg\varphi \supset (\varphi \supset \psi)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta}$
$(\Rightarrow \neg)$	$\vdash \neg\varphi, \varphi$	$\neg\varphi \vee \varphi$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$
$(\neg\neg \Rightarrow)$	$\neg\neg\varphi \vdash \varphi$	$\neg\neg\varphi \supset \varphi$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$
$(\Rightarrow \neg\neg)$	$\varphi \vdash \neg\neg\varphi$	$\varphi \supset \neg\neg\varphi$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
$(\neg \supset\Rightarrow)_1$	$\neg(\varphi \supset \psi) \vdash \varphi$	$\neg(\varphi \supset \psi) \supset \varphi$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\neg \supset\Rightarrow)_2$	$\neg(\varphi \supset \psi) \vdash \neg\psi$	$\neg(\varphi \supset \psi) \supset \neg\psi$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$
$(\Rightarrow \neg \supset)$	$\varphi, \neg\psi \vdash \neg(\varphi \supset \psi)$	$\varphi \supset (\neg\psi \supset \neg(\varphi \supset \psi))$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \supset \psi)}$
$(\neg\vee \Rightarrow)_1$	$\neg(\varphi \vee \psi) \vdash \neg\varphi$	$\neg(\varphi \vee \psi) \supset \neg\varphi$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\neg\vee \Rightarrow)_2$	$\neg(\varphi \vee \psi) \vdash \neg\psi$	$\neg(\varphi \vee \psi) \supset \neg\psi$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$
$(\Rightarrow \neg\vee)$	$\neg\varphi, \neg\psi \vdash \neg(\psi \vee \varphi)$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$
$(\neg\wedge \Rightarrow)$	$\neg(\varphi \wedge \psi) \vdash \neg\varphi, \neg\psi$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$
$(\Rightarrow \neg\wedge)_1$	$\neg\varphi \vdash \neg(\varphi \wedge \psi)$	$\neg\varphi \supset \neg(\varphi \wedge \psi)$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$
$(\Rightarrow \neg\wedge)_2$	$\neg\psi \vdash \neg(\varphi \wedge \psi)$	$\neg\psi \supset \neg(\varphi \wedge \psi)$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$

Figure 3. The set of rules NIR

DEFINITION 80. Let $\mathcal{M}_4^B = \langle \mathcal{V}_4, \mathcal{D}_4, \mathcal{O}_4 \rangle$ be the following Nmatrix for $\mathcal{L}$:

- $\mathcal{V}_4 = \{t, \top, \bot, f\}$[12] where:

$$\begin{array}{rcl} t & = & \langle 1,0 \rangle \\ \top & = & \langle 1,1 \rangle \\ \bot & = & \langle 0,0 \rangle \\ f & = & \langle 0,1 \rangle \end{array}$$

- $\mathcal{D}_4 = \{a \in \mathcal{V}_4 \mid P_1(a) = 1\} = \{t, \top\}$
- Let $\mathcal{V} = \mathcal{V}_4$, $\mathcal{D} = \mathcal{D}_4$, $\mathcal{F} = \mathcal{V}_4 - \mathcal{D}$. The operations in $\mathcal{O}_4$ are:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \quad (\text{i.e. } a \in \{f, \top\}) \\ \mathcal{F} & \text{if } P_2(a) = 0 \quad (\text{i.e. } a \in \{t, \bot\}) \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$\mathcal{M}_4^{B\mathbf{ff}}$ (for $\mathcal{L}_{\mathbf{ff}}$) is obtained from $\mathcal{M}_4^B$ by adding the condition: $\tilde{\mathbf{ff}} \in \mathcal{F}$.

THEOREM 81. *$\mathcal{M}_4^B$ ($\mathcal{M}_4^{B\mathbf{ff}}$) is a (dynamically) characteristic Nmatrix for LK^+ (LK).*

Now each rule of NIR induces a semantic condition, and $\mathbf{L}[S]$ ($\mathbf{L} \in \{LK^+, LK\}$) is characterized by the simple refinement (Remark 32) of $\mathcal{M}_4^B/\mathcal{M}_4^{B\mathbf{ff}}$, induced by the conditions that correspond to the rules in S.

DEFINITION 82. *The refining conditions induced by the rules in NIR:*

C($\neg \Rightarrow$) : If $P_1(a) = 1$ then $P_2(a) = 0$

C($\Rightarrow \neg$) : If $P_1(a) = 0$ then $P_2(a) = 1$

C($\Rightarrow \neg\neg$) : If $P_1(a) = 1$ then $\tilde{\neg} a \subseteq \{x \mid P_2(x) = 1\}$

C($\neg\neg \Rightarrow$) : If $P_1(a) = 0$ then $\tilde{\neg} a \subseteq \{x \mid P_2(x) = 0\}$

C($\neg \supset \Rightarrow$)$_1$: If $P_1(a) = 0$ then $a \tilde{\supset} b \subseteq \{x \mid P_2(x) = 0\}$

C($\neg \supset \Rightarrow$)$_2$: If $P_2(b) = 0$ then $a \tilde{\supset} b \subseteq \{x \mid P_2(x) = 0\}$

[12] The intuition behind these four truth-values is like in Dunn-Belnap's logic, see the end of the Introduction.

C($\Rightarrow \neg \supset$) : If $P_1(a) = 1$ and $P_2(b) = 1$ then $a \tilde{\supset} b \subseteq \{x \mid P_2(x) = 1\}$

C($\neg\vee \Rightarrow$)$_1$: If $P_2(a) = 0$ then $a \tilde{\vee} b \subseteq \{x \mid P_2(x) = 0\}$

C($\neg\vee \Rightarrow$)$_2$: If $P_2(b) = 0$ then $a \tilde{\vee} b \subseteq \{x \mid P_2(x) = 0\}$

C($\Rightarrow \neg\vee$) : If $P_2(a) = 1$ and $P_2(b) = 1$ then $a \tilde{\vee} b \subseteq \{x \mid P_2(x) = 1\}$

C($\Rightarrow \neg\wedge$)$_1$: If $P_2(a) = 1$ then $a \tilde{\wedge} b \subseteq \{x \mid P_2(x) = 1\}$

C($\Rightarrow \neg\wedge$)$_2$: If $P_2(b) = 1$ then $a \tilde{\wedge} b \subseteq \{x \mid P_2(x) = 1\}$

C($\neg\wedge \Rightarrow$) : If $P_2(a) = 0$ and $P_2(b) = 0$ then $a \tilde{\vee} b \subseteq \{x \mid P_2(x) = 0\}$

As an example how these conditions have been derived, take $(\neg \supset \Rightarrow)_2$. This rule is valid if $\neg(a \supset b)$ is in $\mathcal{F}$ whenever $\neg b$ is in $\mathcal{F}$ (where $x \supset y$ denotes some element in $x \tilde{\supset} y$, and $\neg x$ denotes some element in $\tilde{\neg} x$). This is equivalent to: if $P_2(b) = 0$ then $P_2(a \supset b) = 0$, which is exactly C($\neg \supset\Rightarrow$)$_2$.

REMARK 83. With the obvious extensions of P_1 and P_2, The above formulation of the conditions in $C(NIR)$ can be applied whenever the truth-values are finite sequences of 0's and 1's, the designated elements are those for which the first component is 1, and condition (NEG) is satisfied. However, these conditions can be simplified in case exactly $\{t, f, \top, \bot\}$ are used. Thus the conditions involving $\neg$ and $\supset$ can be reformulated as follows:

C($\neg \Rightarrow$) : Use only t, f and $\bot$

C($\Rightarrow \neg$) : Use only t, f and $\top$

C($\Rightarrow \neg\neg$) : $\tilde{\neg} t = \{f\}$, $\tilde{\neg} \top = \{\top\}$

C($\neg\neg \Rightarrow$) : $\tilde{\neg} f = \{t\}$, $\tilde{\neg} \bot = \{\bot\}$

C($\neg \supset\Rightarrow$)$_1$: If $a \in \mathcal{F}$ then $a \tilde{\supset} b \subseteq \{t, \bot\}$

C($\neg \supset\Rightarrow$)$_2$: If $b \in \{t, \bot\}$ then $a \tilde{\supset} b \subseteq \{t, \bot\}$

C($\Rightarrow \neg \supset$) : If $a \in \mathcal{D}$ and $b \in \{\top, f\}$ then $a \tilde{\supset} b \subseteq \{\top, f\}$

Moreover, if we consider only simple refinements of $\mathcal{M}_4^B$, then the three last conditions can be further transformed into more specific ones:

C($\neg \supset\Rightarrow$)$_1$: If $a \in \mathcal{F}$ then $a \tilde{\supset} b = \{t\}$

C($\neg \supset\Rightarrow$)$_2$: If $b = t$ then $a \tilde{\supset} b = \{t\}$
If $b = \bot$ and $a \in \mathcal{F}$ then $a \tilde{\supset} b = \{t\}$
If $b = \bot$ and $a \in \mathcal{D}$ then $a \tilde{\supset} b = \{\bot\}$

C($\Rightarrow \neg \supset$) : If $a \in \mathcal{D}$ and $b \in \{\top, f\}$ then $a \tilde{\supset} b = \{b\}$

DEFINITION 84.

1. For $S \subseteq NIR$, let $C(S) = \{\mathrm{C}r \mid r \in S\}$
2. For $S \subseteq NIR$, let $\mathcal{M}_S$ ($\mathcal{M}_S^{\mathbf{ff}}$) be the weakest simple refinement of $\mathcal{M}_4^B$ ($\mathcal{M}_4^{B\mathbf{ff}}$) in which the conditions in $C(S)$ are all satisfied. In other words: $\mathcal{M}_S = \langle \mathcal{V}_S, \mathcal{D}_S, \mathcal{O}_S \rangle$, where $\mathcal{V}_S$ is the set of values from $\mathcal{V}_4$ which are not rejected by any condition in S, $\mathcal{D}_S = \mathcal{D}_4 \cap \mathcal{V}_S$, and for any connective $\diamond$ and $\vec{x} \in \mathcal{V}_S^n$ (where n is the arity of $\diamond$), the interpretation in $\mathcal{O}$ of $\diamond$ assigns to $\vec{x}$ the set of all the values in $\tilde{\diamond}(\vec{x})$ which are not forbidden by any condition in $C(S)$ (it is easy to check that for $S \subseteq NIR$ this set is never empty. The same is true for $\mathcal{D}_S$).

EXAMPLE 85.

1. Let $C_{min} = LK^+[\{(\Rightarrow \neg), (\neg\neg \Rightarrow)\}]$. Then $\mathcal{M}_{C_{min}}$ is the three-valued Nmatrix[13] $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:
   - $\mathcal{V} = \{t, \top, f\}$ (the rule $(\Rightarrow \neg)$ causes the deletion of $\bot$)
   - $\mathcal{D} = \{t, \top\}$
   - The operations in $\mathcal{O}$ are:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } a = \top \\ \{f\} & \text{if } a = t \\ \{t\} & \text{if } a = f \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a = f \text{ or } b \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$$

2. Let $\mathcal{FOUR} = NIR - \{(\neg \Rightarrow), (\Rightarrow \neg)\}$. Then $\mathcal{M}_{\mathcal{FOUR}}$ is a 4-valued *deterministic* Nmatrix (i.e. an ordinary *matrix*). The operations in this matrix are defined as follows (where $a \leq_t \bot, \top \leq_t t$): [14]

$$\tilde{\neg} t = f \quad \tilde{\neg} f = t \quad \tilde{\neg} \top = \top \quad \tilde{\neg} \bot = \bot$$

[13] C_{min} is studied in [Carnielli and Marcos, 1999]. The 3-valued Nmatrix for this logic described here was first introduced in [Avron and Lev, 2005].

[14] Without $\tilde{\supset}$, $\mathcal{M}_{\mathcal{FOUR}}$ is the famous 4-valued matrix of Dunn and Belnap ([Dunn, 1976; Belnap, 1977]). The connective $\tilde{\supset}$ of $\mathcal{O}_4$ was introduced in [Arieli and Avron, 1996]. The soundness and completeness of the logic $\mathcal{FOUR}$ for $\mathcal{M}_{\mathcal{FOUR}}$ was also first stated and proved there.

$$a \tilde{\supset} b = \begin{cases} t & \text{if } a \notin \mathcal{D} \\ b & \text{otherwise} \end{cases}$$

$$a \tilde{\wedge} b = inf_{\leq_t}\{a, b\} \qquad a \tilde{\vee} b = sup_{\leq_t}\{a, b\}$$

Now the modular character of the semantics of Nmatrices allows us to formulate and prove together soundness and completeness theorems for 2^{13} systems (most of which define different logics):

THEOREM 86. *For $S \subseteq NIR$, $\mathcal{M}_S$ ($\mathcal{M}_S^{\mathbf{ff}}$) is a (dynamically) characteristic Nmatrix for $LK^+[S]$ ($LK[S]$).*

Proof. We give an outline of the completeness part. So let $\mathbf{L} \in \{LK^+, LK\}$, and assume that $\mathcal{T} \not\vdash_{\mathbf{L}[S]} \psi_0$. We construct a model of $\mathcal{T}$ in $\mathcal{M}_S$ which is not a model of ψ_0. For this extend $\mathcal{T}$ to a maximal set $\mathcal{T}^*$ of formulas such that $\mathcal{T}^* \not\vdash_{\mathbf{L}[S]} \psi_0$. Then $\varphi \notin \mathcal{T}^*$ iff $\mathcal{T}^*, \varphi \vdash_{\mathbf{L}[S]} \psi_0$. Define now a valuation v by $v(\varphi) = \langle x(\varphi), y(\varphi)\rangle$, where:

$$x(\varphi) = \begin{cases} 1 & \varphi \in \mathcal{T}^* \\ 0 & \varphi \notin \mathcal{T}^* \end{cases} \qquad y(\varphi) = \begin{cases} 1 & \neg\varphi \in \mathcal{T}^* \\ 0 & \neg\varphi \notin \mathcal{T}^* \end{cases}$$

It is not difficult to see that v is a legal valuation in $\mathcal{M}_4^B$. To show that it is also a legal valuation in $\mathcal{M}_S$, we need to check that it respects the conditions in $C(S)$ (as formulated in Remark 83). We do some of the cases, leaving the rest for the reader:

C($\neg \Rightarrow$) : Assume $(\neg \Rightarrow) \in S$. Then there can be no sentence φ such that $\{\varphi, \neg\varphi\} \subseteq \mathcal{T}^*$. Hence $v(\varphi) \neq \top$ for all φ.

C($\Rightarrow \neg$) : Assume $(\Rightarrow \neg) \in S$, but $v(\varphi) = \perp$ for some φ. Then $\varphi \notin \mathcal{T}^*$ and $\neg\varphi \notin \mathcal{T}^*$. It follows that $\mathcal{T}^*, \varphi \vdash_{\mathbf{L}[S]} \psi_0$ and $\mathcal{T}^*, \neg\varphi \vdash_{\mathbf{L}[S]} \psi_0$. Hence $\mathcal{T}^* \vdash_{\mathbf{L}[S]} \varphi \supset \psi_0$, and $\mathcal{T}^* \vdash_{\mathbf{L}[S]} \neg\varphi \supset \psi_0$. This contradicts the fact that $\mathcal{T}^* \not\vdash_{\mathbf{L}[S]} \psi_0$, since $\varphi \supset \psi_0, \neg\varphi \supset \psi_0 \vdash_{\mathbf{L}[S]} \psi_0$ in case $(\neg \Rightarrow) \in S$.

C($\Rightarrow \neg\neg$) : Assume $(\Rightarrow \neg\neg) \in S$.

- Suppose $v(\varphi) = t$. Then $\varphi \in \mathcal{T}^*$ and $\neg\varphi \notin \mathcal{T}^*$. By $(\Rightarrow \neg\neg)$, also $\neg\neg\varphi \in \mathcal{T}^*$. Hence $v(\neg\varphi) = f$ by definition of v.
- Suppose $v(\varphi) = \top$. Then $\varphi \in \mathcal{T}^*$ and $\neg\varphi \in \mathcal{T}^*$. By $(\Rightarrow \neg\neg)$, also $\neg\neg\varphi \in \mathcal{T}^*$. Hence $v(\neg\varphi) = \top$ by definition of v.

C($\neg \supset\Rightarrow)_1$: Assume $(\neg \supset\Rightarrow)_1 \in S$. Suppose that $v(\varphi) \notin \mathcal{D}$. Then $\varphi \notin \mathcal{T}^*$, and so also $\neg(\varphi \supset \psi) \notin \mathcal{T}^*$. It follows that $v(\varphi \supset \psi) \in \{t, \perp\}$.

Obviously, v is a model of $\mathcal{T}$ in $\mathcal{M}_S$ which is not a model of ψ_0. ■

The following corollary is implied by the above theorem and the analycity of Nmatrices:

COROLLARY 87. *$LK^+[S]$ and $LK[S]$ are decidable for every $S \subseteq NIR$.*

5.2 Extensions of Intuitionistic Logic

Let LJ denote propositional intuitionistic logic (over $\{\wedge, \vee, \supset, \mathbf{ff}\}$), and let LJ^+ be its positive fragment (i.e. its $\{\wedge, \vee, \supset\}$-fragment). Next we investigate extensions of LJ^+ and LJ by a negation connective $\neg$.[15] Now, it is well known that it is impossible to conservatively add to LJ^+ or LJ a connective $\neg$ which is both explosive (i.e.: $\neg A, A \vdash B$ for all A, B) and satisfies the law of excluded middle LEM. With such an addition we get classical logic. The intuitionists indeed reject LEM, retaining the explosive nature of negation (which is usually defined by $\sim \varphi =^{Def} \varphi \supset \mathbf{ff}$). In this subsection we show that this is not the only possible choice. The main problem we shall solve is: Which of the logics $LJ^+[S]$ ($S \subseteq NIR$) is conservative over LJ^+ (and similarly for LJ)? We believe that each such logic is entitled to be called "a logic with a constructive negation".

REMARK 88. $\mathbf{G}[LJ^+]$ ($\mathbf{G}[LJ]$), a multiple-conclusioned Gentzen-type system for LJ^+ (LJ), is obtained from $\mathbf{G}[LK^+]$ ($\mathbf{G}[LK]$) by replacing the ($\Rightarrow\supset$) rule with the following impure (single-conclusion) rule:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ (\Rightarrow\supset)$$

It is again easy to see that for every $S \subseteq NIR$ and $\mathbf{L} \in \{LJ^+, LJ\}$, a Gentzen-type system $\mathbf{GL}[S]$ which is sound and complete for $\mathbf{L}[S]$ can be obtained by adding to $\mathbf{GL}$ the Gentzen-type versions of the rules in S. In what follows we identify $\mathbf{L}[S]$ and $\mathbf{GL}[S]$.

Like in the classical case, we start by generalizing the standard, two-valued semantics of LJ^+ (or LJ). Recall that this semantics is usually provided by the class of all Kripke frames of the form $\mathcal{W} = \langle W, \leq, v\rangle$ [16], where $\langle W, \leq\rangle$ is a nonempty partially ordered set (of "worlds"), and v is a function from $W \times Frm_{\mathcal{L}}$ to $\mathcal{V}$ that satisfies the following conditions:

1. If $y \geq x$ and $v(x, \varphi) = t$ then $v(y, \varphi) = t$.[17]

[15] Positive intuitionistic logic might be a better starting point for investigating negations than positive classical logic (especially *constructive* negations), because its valid sentences are all intuitively correct. LK^+, in contrast, includes counterintuitive tautologies like $(A \wedge B \supset C) \supset (A \supset C) \vee (B \supset C)$ or $A \vee (A \supset B)$. Moreover: the classical natural deduction rules for the positive connectives ($\wedge, \vee$ and $\supset$) define LJ^+, not LK^+. It is only with the aid of the classical rules for (the classical) negation that one can prove the counterintuitive positive tautologies mentioned above.

[16] In the literature by a "frame" one usually means just the pair $\langle W, \leq\rangle$. Here we have found it convenient to use this technical term differently, so that the valuation v is an integral part of it.

[17] For the language of LJ it suffices to demand this condition for atomic formulas only; then one can prove that every formula has this property. This is not the case for the nondeterministic generalizations with $\neg$ that we present below.

2.
   - $v(x, \varphi \wedge \psi) = t$ iff $v(x, \varphi) = t$ and $v(x, \psi) = t$
   - $v(x, \varphi \vee \psi) = t$ iff $v(x, \varphi) = t$ or $v(x, \psi) = t$
   - $v(x, \mathbf{ff}) = f$ (if $\mathbf{ff}$ is in the language).

3. $v(x, \mathbf{ff}) = f$ (if $\mathbf{ff}$ is in the language).

4. $v(x, \varphi \supset \psi) = t$ iff $v(y, \psi) = t$ for every $y \geq x$ such that $v(y, \varphi) = t$

Obviously, if $\mathcal{W} = \langle W, \leq, v \rangle$ is a frame, then for every $x \in W$ the function $\lambda\varphi.v(x, \varphi)$ behaves like an ordinary classical valuation with respect to all the connectives except $\supset$. The treatment of $\supset$ is indeed what distinguishes between classical logic and intuitionistic logic. This observation leads to the following nondeterministic generalization of intuitionistic Kripke frames:

DEFINITION 89. Let $\supset$ be one of the connectives of a propositional language $\mathcal{L}$, and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L} - \{\supset\}$. An $\mathcal{M}$-*frame* for $\mathcal{L}$ is a triple $\mathcal{W} = \langle W, \leq, v \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set

2. $v : W \times Frm_{\mathcal{L}} \rightarrow \mathcal{V}$ satisfies the following conditions:
   - *Persistence:* if $y \geq x$ and $v(x, \varphi) \in \mathcal{D}$ then $v(y, \varphi) \in \mathcal{D}$
   - For every $x \in W$, $\lambda\varphi.v(x, \varphi)$ is a legal $\mathcal{M}$-valuation.
   - $v(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v(y, \varphi) \in \mathcal{D}$

We say that a formula φ is *true* in a world $x \in W$ of a frame $\mathcal{W}$ if $v(x, \varphi) \in \mathcal{D}$. A sequent $\Gamma \Rightarrow \Delta$ is *valid* in $\mathcal{W}$ if for every $x \in W$ there is either $\varphi \in \Gamma$ such that φ is not true in x, or $\psi \in \Delta$ such that ψ is true in x.

Obviously, if $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$, then any $\mathcal{M}_1$-frame is also an $\mathcal{M}_2$-frame, and every sequent valid in $\mathcal{M}_2$ is also valid in $\mathcal{M}_1$.

DEFINITION 90.

1. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LJ^+. We say that $\mathcal{M}$ is *suitable* for LJ^+ if the following conditions are satisfied (where again $\mathcal{V} - \mathcal{D}$ is denoted by $\mathcal{F}$):
   - If $a \in \mathcal{D}$ and $b \in \mathcal{D}$ then $a \wedge b \subseteq \mathcal{D}$
   - If $a \notin \mathcal{D}$ then $a \wedge b \subseteq \mathcal{F}$
   - If $b \notin \mathcal{D}$ then $a \wedge b \subseteq \mathcal{F}$

   - If $a \in \mathcal{D}$ then $a \vee b \subseteq \mathcal{D}$
   - If $b \in \mathcal{D}$ then $a \vee b \subseteq \mathcal{D}$

- If $a \notin \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \vee b \subseteq \mathcal{F}$

- If $b \in \mathcal{D}$ then $a \supset b \subseteq \mathcal{D}$
- If $a \in \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \supset b \subseteq \mathcal{F}$

2. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LJ. We say that $\mathcal{M}$ is *suitable* for LJ if it is suitable for LJ^+, and the following condition is satisfied:

   - $\mathbf{ff} \subseteq \mathcal{F}$

THEOREM 91. *Assume $\mathcal{W}$ is an $\mathcal{M}$-frame, where $\mathcal{M}$ is suitable for LJ^+ (LJ). Then any sequent provable in LJ^+ (LJ) is valid in $\mathcal{W}$.*

Below we concentrate on the systems $LJ^+(S)$ for $S \subseteq NIR$ (obtaining similar results for $LJ(S)$ causes no further difficulties).

DEFINITION 92. Let $\mathcal{M}_4^{IB}$ be the following Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\mathcal{L}$:

- $\mathcal{V} = \{t, \top, f, \bot\}$
- $\mathcal{D} = \{t, \top\}$
- $a \supset b = \begin{cases} \mathcal{D} & b \in \mathcal{D} \\ \mathcal{F} & b \notin \mathcal{D}, a \in \mathcal{D} \\ \mathcal{V} & a, b \in \mathcal{F} \end{cases}$

  $a \vee b = \begin{cases} \mathcal{D} & a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$

  $a \wedge b = \begin{cases} \mathcal{D} & a, b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$

  $\neg t = \neg \bot = \mathcal{F} \quad \neg f = \neg \top = \mathcal{D}$

REMARK 93. The only difference between $\mathcal{M}_4^{IB}$ and $\mathcal{M}_4^{B}$ (recall Defn. 80) is that in $\mathcal{M}_4^{IB}$ we have $a \supset b = \mathcal{V}$ in case $a, b \in \mathcal{F} = \mathcal{V} - \mathcal{D}$, while in $\mathcal{M}_4^{B}$ $a \supset b = \mathcal{D}$ in this case.

PROPOSITION 94. *Let $\mathcal{M}$ be a refinement of $\mathcal{M}_4^{IB}$. Then LJ^+ is sound for every $\mathcal{M}$-frame.*

Now we turn to the effects of the various negation rules in the context of our semantics for LJ^+ and its extensions. The conditions we associated with these conditions in the previous subsection lead this time to refinements of $\mathcal{M}_4^{IB}$ on which the corresponding frames are based.

DEFINITION 95. For $S \subseteq NIR$, let $\mathcal{M}_S^I$ be the weakest refinement of $\mathcal{M}_4^{IB}$ in which the conditions in $C(S)$ are satisfied.[18]

THEOREM 96. *If $S \subseteq NIR$ then $LJ^+(S)$ is sound and strongly complete for $\mathcal{M}_S^I$-frames: $\Gamma \vdash_{LJ^+(S)} \psi$ iff for every $\mathcal{M}_S^I$-frame $\mathcal{W} = \langle W, \leq, v \rangle$, and every $x \in W$, if $v(x, \varphi) \in \mathcal{D}$ for every $\varphi \in \Gamma$ then also $v(x, \psi) \in \mathcal{D}$.*

EXAMPLE 97. The system C_ω of da Costa ([da Costa, 1974]) is identical to $LJ^+(\{(\Rightarrow \neg), (\neg\neg \Rightarrow)\})$. Theorem 96 provides illuminating semantics for it which is much simpler than the Kripke-type semantics given in [Baaz, 1986] and the bivaluations semantics of [Loparić, 1986] (and can be used for a decision procedure — see Corrolary 104 below). Here is a compact description of this semantics: A frame for C_ω is a triple $\langle W, \leq, v \rangle$ such that $\langle W, \leq \rangle$ is a nonempty partially ordered set, and $v : W \times \mathcal{F} \to \{t, f, \top\}$ is a valuation which satisfies the following conditions:

- If $x \leq y$ then $v(x, \varphi) \leq_k v(y, \varphi)$
- $v(x, \varphi \wedge \psi) = f$ iff $v(x, \varphi) = f$ or $v(x, \psi) = f$
- $v(x, \varphi \vee \psi) = f$ iff $v(x, \varphi) = f$ and $v(x, \psi) = f$
- $v(x, \varphi \supset \psi) = f$ iff for some $y \geq x$, $v(y, \varphi) \neq f$ while $v(y, \psi) = f$
- $v(x, \neg\varphi) = f$ iff $v(x, \varphi) = t$
- If $v(x, \varphi) = f$ then $v(x, \neg\varphi) = t$

A frame is a model of a formula φ if $v(x, \varphi) \neq f$ for every $x \in W$.

Theorem 96 does not have much value in itself. Indeed, it does not guarantee that $LJ^+(S)$ is conservative over LJ^+, and neither does it provide a decision procedure for $LJ^+(S)$. The reason for this is that the current semantic framework (of Nmatrices combined with intuitionistic frames) is not always analytic (recall Remark 12). Next we provide a definition of this notion which is suitable for the present context. For this we need first the following important observation.

PROPOSITION 98. *Let $\mathcal{M}$ be a refinement of $\mathcal{M}_4^{IB}$. Then the persistence condition for $\mathcal{M}$ is equivalent to the following* monotonicity condition*:*

- If $x \leq y$ then $v(x, \varphi) \leq_k v(y, \varphi)$, where the partial order $\leq_k$ on $\mathcal{V}_4$ is defined by: $\bot \leq_k t, f \leq_k \top$. [19]

[18] It is advisable here to read again the first part of Remark 83.

[19] $\leq_k$ had a crucial role already in [Belnap, 1977]. The structure obtained by equipping $\mathcal{V}_4$ with both $\leq_t$ and $\leq_k$ is nowadays known as the basic (distributive) *bilattice* (see [Ginsberg, 1988; Fitting, 1994; Arieli and Avron, 1996]).

Analycity for the semantics of frames can now be defined as follows:

DEFINITION 99. Let $\mathcal{M} = \mathcal{M}^I_S$ for some $S \subseteq NIR$.

1. An $\mathcal{M}$-*semiframe* is a triple $\mathcal{W} = \langle W, \leq, v' \rangle$ such that:
   (a) $\langle W, \leq \rangle$ is a nonempty partially ordered set.
   (b) $v' : W \times \mathcal{F}' \to \mathcal{V}$ is a partial valuation such that:
      - $\mathcal{F}'$ is a subset of $Frm_{\mathcal{L}}$ which is closed under subformulas.
      - v' satisfies the monotonicity condition: if $y \geq x$ and $\varphi \in \mathcal{F}'$, then $v'(x, \varphi) \leq_k v'(y, \varphi)$.
      - v' respects $\mathcal{M}$: If $\diamond(\psi_1, \dots, \psi_n) \in \mathcal{F}'$, then $v'(x, \diamond(\psi_1, \dots, \psi_n))$ is in $\tilde{\diamond}(v'(x, \psi_1), \dots, v'(x, \psi_n))$.
      - If $\varphi \supset \psi \in \mathcal{F}'$ then $v'(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v'(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v'(y, \varphi) \in \mathcal{D}$.
2. $\mathcal{M}^I_S$ is *analytic* if for any $\mathcal{M}^I_S$- semiframe $\langle W, \leq, v' \rangle$ there exists an $\mathcal{M}^I_S$-frame $\langle W, \leq, v \rangle$ such that v extends v'.

The next theorem provides the conditions under which $\mathcal{M}^I_S$ is analytic:

THEOREM 100. *$\mathcal{M}^I_S$ ($S \subseteq NIR$) is analytic iff either* $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S$ *or* $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \not\subseteq S$.

REMARK 101. In [Avron, 2005a] it is shown that Theorem 100 would have failed had the definition of a semiframe included the persistence condition rather than the monotonicity condition.

REMARK 102. The problem with the combination $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\}$ is that the condition imposed by $(\neg \supset\Rightarrow)_1$ is not consistent with the condition of monotonicity in case $\perp$ is not available.

As an immediate application, Theorem 100 can be used to determine for which $S \subseteq NIR$ the system $LJ^+(S)$ is conservative over LJ^+. This is done in the next theorem. The proof of this theorem nicely demonstrates how our semantic framework can be used, as well as the crucial role of the analycity property. Therefore we include here this proof.

THEOREM 103. *Let $S \subseteq NIR$. If neither* $\{(\Rightarrow \neg), (\neg \Rightarrow)\} \subseteq S$ *nor* $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \subseteq S$, *then $LJ^+(S)$ is a conservative extension of LJ^+. Otherwise $LJ^+(S) = LK^+(S)$.*

Proof. It is easy to see that the two conditions are necessary. Let $SN = NIR - \{(\Rightarrow \neg)\}$, $SP = NIR - \{(\neg \Rightarrow), (\neg \supset\Rightarrow)_1\}$. To show that the two conditions together are also sufficient, it suffices to show that both $LJ^+(SN)$ and $LJ^+(SP)$ are conservative over LJ^+. So let ψ be a sentence in the language of LJ^+ which is not provable in LJ^+. We show that ψ is provable in neither $LJ^+(SN)$ nor $LJ^+(SP)$. Since $\not\vdash_{LJ^+} \psi$, there is an *ordinary* two-valued Kripke frame $\langle W, \leq, u\rangle$ (where $u : W \times Frm_{\mathcal{L}} \rightarrow \{t, f\}$) in which ψ is not valid (i.e. $u(x_0, \psi) = f$ for some $x_0 \in W$). Now we define the corresponding semiframes for $LJ^+(SN)$ and $LJ^+(SP)$. Let $\mathcal{F}'$ be the set of formulas in the language of LJ^+.

$LJ^+(SN)$: Define v'_N on $W \times \mathcal{F}'$ by:

$$v'_N(x, \varphi) = \begin{cases} t & \text{if } u(x, \varphi) = t \\ \perp & \text{if } u(x, \varphi) = f \end{cases}$$

It is straightforward to check that $\langle W, \leq, v'_N\rangle$ is an $\mathcal{M}_{IP}[SN]$-semiframe (note that any condition concerning $\neg$ is vacuously satisfied, since there is no sentence of the form $\neg\varphi$ in $\mathcal{F}'$).

$LJ^+(SP)$: Define v'_P on $W \times \mathcal{F}'$ by:

$$v'_P(x, \varphi) = \begin{cases} \top & \text{if } u(x, \varphi) = t \\ f & \text{if } u(x, \varphi) = f \end{cases}$$

Again, it is easy to check that $\langle W, \leq, v'_P\rangle$ is an $\mathcal{M}_{IP}[SP]$-semiframe.

By Theorem 100, $\langle W, \leq, v'_N\rangle$ and $\langle W, \leq, v'_P\rangle$ can respectively be extended to an $\mathcal{M}_{IP}[SN]$-frame $\langle W, \leq, v_N\rangle$ and an $\mathcal{M}_{IP}[SP]$-frame $\langle W, \leq, v_P\rangle$. Since $v_N(x_0, \psi) = v'_N(x_0, \psi) = \perp$, ψ is not valid in $\langle W, \leq, v'_N\rangle$, and so it is not provable in $LJ^+(SN)$. Similarly, $v_P(x_0, \psi) = v'_P(x_0, \psi) = f$. Hence ψ is not valid in $\langle W, \leq, v'_P\rangle$, and so is not provable in $LJ^+(SP)$. ■

Theorems 100, 103 and Corollary 87 immediately entail:

COROLLARY 104. *$LJ^+(S)$ is decidable for every $S \subseteq NIR$.*

It follows from Theorem 103 that $LJ^+(SN)$ and $LJ^+(SP)$ are the two maximal logics in the family $\{LJ^+(S) \mid S \subseteq NIR\}$ which are conservative extensions of constructive positive logic. Now the first is the well-known system **N** of Nelson ([Almukdad and Nelson, 1984]) and von Kutschera ([von Kutschera, 1969]). The other, in contrast, is new. However, it is a very attractive system for constructive negation. First: it is paraconsistent (i.e.: a single contradiction does not imply everything in it). Second: LEM is valid in it. In fact, $LJ^+(SP)$ is obtained from **N** by replacing two of its axioms by LEM. Now, while LEM is very intuitive, the two axioms it replaces are not. Indeed, one of them, $\neg\varphi \supset (\varphi \supset \psi)$, intuitively means that if φ is false then it implies everything. The second, $\neg(\varphi \supset \psi) \supset \varphi$, intuitively means that if there is something that φ does not imply, then φ should be true (i.e.: it cannot be false). Obviously, these two principles are similar — and counterintuitive. It is no wonder that from a constructive point of view, each of them is inconsistent with LEM, and is rejected in

$LJ^+(SP)$. It is also worth noting that despite the paraconsistent nature of $LJ^+(SP)$, the basic intuitive law of contradiction $\neg(\varphi \wedge \neg\varphi)$ *is* valid in it.

Next we turn to another application of Theorems 96 and 100: eliminations of cuts. It was shown in [Avron, 2007b] that in general the cut-elimination theorem does not hold for the Gentzen-type systems presented in this subsection. Moreover: examples have been given there of a subset S of NIR and a sequent which is provable in $LJ^+(S)$, but any proof of it there should contain a non-analytic cut. This is perhaps not surprising, since our logical rules themselves do not have the strict subformula property: some of them involve negations of subformulas of their conclusion, even though those negations are not subformulas themselves. Therefore, it is reasonable to expect the same from cuts. This leads to the following theorem from [Avron, 2005a]:

THEOREM 105. *Assume that $S \subseteq NIR$, and $\{(\Rightarrow \neg), (\neg \supset\Rightarrow)_1\} \not\subseteq S$. Then for every sequent s in the language of LJ^+ there is either a* finite *$\mathcal{M}^I_S$-frame in which s is not valid, or a proof in $LJ^+(S)$ in which every cut is either on a subformula of s or on a negation of such a subformula.*

6 NMATRICES FOR LOGICS OF FORMAL INCONSISTENCY

In this section we apply the framework of Nmatrices to provide modular semantics for yet another family of non-classical logics: da Costa's paraconsistent logics. A paraconsistent logic is a logic which allows non-trivial inconsistent theories. One of the oldest and best known approaches to the problem of designing useful paraconsistent logics is da Costa's approach. This approach is based on two main ideas. The first is to limit the applicability of the classical (and intuitionistic) rule $\neg\varphi, \varphi \vdash \psi$ to the case where φ is "consistent". The second is to express this assumption of consistency of φ within the language. The easiest way to implement these ideas is to include in the language a special connective $\circ$, with the intended meaning of $\circ\varphi$ being "φ is consistent". Then one can explicitly add the assumption of the consistency of φ to the problematic (from a paraconsistent point of view) rule, getting the rule called (**b**) below. Other rules concerning $\neg$ and $\circ$ can then be added, leading to a large family of logics known as "Logics of Formal Inconsistency" (LFIs - see [da Costa, 1974; Carnielli and Marcos, 2002; Carnielli and Marcos, 2007]). In this chapter we investigate those that are obtained using the rules in NIR, as well as the main rules involving the consistency operator that have been studied in the literature on LFIs. The latter rules are listed in Figure 4 (in which $\diamond \in \{\wedge, \vee, \supset\}$). The material of this section is based on [Avron, 2007a]. Throughout it, we fix the language $\mathcal{L}_C = \{\neg, \circ, \supset, \wedge, \vee\}$. Again our basic system will be LK^+ (the positive fragment of classical logic).

Name of rule	Abstract form	Hilbert-style axiom
(**b**)	$\circ\varphi, \neg\varphi, \varphi \vdash$	$(\circ\varphi \wedge \neg\varphi \wedge \varphi) \supset \psi$
(**k1**)	$\vdash \circ\varphi, \varphi$	$\circ\varphi \vee \varphi$
(**k2**)	$\vdash \circ\varphi, \neg\varphi$	$\circ\varphi \vee \neg\varphi$
(**i1**)	$\neg\circ\varphi \vdash \varphi$	$\neg\circ\varphi \supset \varphi$
(**i2**)	$\neg\circ\varphi \vdash \neg\varphi$	$\neg\circ\varphi \supset \neg\varphi$
($\mathbf{a}_\neg$)	$\circ\varphi \vdash \circ\neg\varphi$	$\circ\varphi \supset \circ\neg\varphi$
($\mathbf{a}_\diamond$)	$\circ\varphi, \circ\psi \vdash \circ(\varphi \diamond \psi)$	$\circ\varphi \supset (\circ\psi \supset \circ(\varphi \diamond \psi))$
($\mathbf{o}^1_\diamond$)	$\circ\varphi \vdash \circ(\varphi \diamond \psi)$	$\circ\varphi \supset \circ(\varphi \diamond \psi)$
($\mathbf{o}^2_\diamond$)	$\circ\psi \vdash \circ(\varphi \diamond \psi)$	$\circ\psi \supset \circ(\varphi \diamond \psi)$
(**l**)	$\neg(\varphi \wedge \neg\varphi) \vdash \circ\varphi$	$\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

Figure 4. Schemata involving $\circ$

6.1 LFIs with Finite Characteristic Nmatrices

DEFINITION 106.

1. Let FCR be the set of all the rules in the table above *except the last one* (**l**). We shall write (**i**) instead of the combination of (**i1**) and (**i2**), (**a**) instead of $\{(\mathbf{a}_\diamond) \mid \diamond \in \{\wedge, \vee, \supset\}\}$ and similarly for (**o**).

2. Let $LFIR = NIR \cup FCR$. We denote by $HLFIR$ the set of Hilbert-style axioms corresponding to the rules in $LFIR$.

3. For $S \subseteq LFIR$ let $LK^+[S]$ be the extension of LK^+ by S.

The basic idea in providing semantics for $LK^+[S]$ (where $S \subseteq LFIR$) is this time to let the value assigned to a sentence φ provide information not only about the truth/falsity of φ and $\neg\varphi$, but also about the truth/falsity of $\circ\varphi$. This leads to the use of elements from $\{0,1\}^3$ as our truth-values, where the intended intuitive meaning of $v(\varphi) = \langle x, y, z\rangle$ is now:

- $x = 1$ iff φ is "true" (i.e. $v(\varphi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).
- $z = 1$ iff $\circ\varphi$ is "true" (i.e. $v(\circ\varphi) \in \mathcal{D}$).

In addition to (NEG), which remains unchanged, this interpretation dictates also the following condition:

$$\text{(CON)} \quad \widetilde{\circ}a \subseteq \{y \mid P_1(y) = P_3(a)\}$$

Accordingly, this time we start our semantic investigation of $LFIR$ with the weakest Nmatrix which satisfies both (NEG) and (CON). Then we show that every logic which is defined by some subset of $LFIR$ is characterized by some (easily computable) simple refinement of that Nmatrix.

DEFINITION 107. The Nmatrix $\mathcal{M}_8^B = \langle \mathcal{V}_8, \mathcal{D}_8, \mathcal{O}_8 \rangle$ is defined as follows:

- $\mathcal{V}_8 = \{0,1\}^3$
- $\mathcal{D}_8 = \{a \in \mathcal{V}_8 \mid P_1(a) = 1\}$
- Let $\mathcal{V} = \mathcal{V}_8$, $\mathcal{D} = \mathcal{D}_8$, $\mathcal{F} = \mathcal{V}_8 - \mathcal{D}$. The operations in $\mathcal{O}_8$ are:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \\ \mathcal{F} & \text{if } P_2(a) = 0 \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } P_3(a) = 1 \\ \mathcal{F} & \text{if } P_3(a) = 0 \end{cases}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

DEFINITION 108.

1. The general refining conditions induced by the conditions in NIR are identical to those given in Definition 82.
2. The general refining conditions induced by the conditions in FCR are:

   C(**b**): If $P_1(a) = 1$ and $P_2(a) = 1$ then $P_3(a) = 0$
   C(**k1**): If $P_1(a) = 0$ then $P_3(a) = 1$
   C(**k2**): If $P_2(a) = 0$ then $P_3(a) = 1$
   C(**i1**): If $P_1(a) = 0$ then $\tilde{\circ} a \subseteq \{x \mid P_2(x) = 0\}$
   C(**i2**): If $P_2(a) = 0$ then $\tilde{\circ} a \subseteq \{x \mid P_2(x) = 0\}$
   C($\mathbf{a}_\neg$): If $P_3(a) = 1$ then $\tilde{\neg} a \subseteq \{x \mid P_3(x) = 1\}$
   C($\mathbf{a}_\diamond$): If $P_3(a) = 1$ and $P_3(b) = 1$ then $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$
   C($\mathbf{o}^1_\diamond$): If $P_3(a) = 1$ then $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$
   C($\mathbf{o}^2_\diamond$): If $P_3(b) = 1$ then $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$

3. For $S \subseteq LFIR$, let $C(S) = \{Cr \mid r \in S\}$, and let $\mathcal{M}_S$ be the weakest simple refinement of $\mathcal{M}_8^B$ in which the conditions in $C(S)$ are all satisfied (again it is not difficult to check that this is well-defined for every $S \subseteq LFIR$).

THEOREM 109. *$\mathcal{M}_S$ ($S \subseteq LFIR$) is a characteristic Nmatrix for $LK^+[S]$.*

COROLLARY 110. *$LK^+[S]$ is decidable for every $S \subseteq LFIR$.*

EXAMPLE 111.

1. Let $\mathbf{B} = LK^+[\{(\Rightarrow \neg), (\mathbf{b})\}]$. This logic is the basic logic of formal inconsistency from [Carnielli and Marcos, 2002; Carnielli and Marcos, 2007] (where it is called *mbC*). By Theorem 109, the following Nmatrix $\mathcal{M}_5^B = \langle \mathcal{V}_5, \mathcal{D}_5, \mathcal{O}_5 \rangle$ is characteristic for it:

   - $\mathcal{V}_5 = \{t, t_I, I, f_I, f\}$ where:

   $$\begin{array}{lcl} t & = & \langle 1,0,1 \rangle \\ t_I & = & \langle 1,0,0 \rangle \\ I & = & \langle 1,1,0 \rangle \\ f & = & \langle 0,1,1 \rangle \\ f_I & = & \langle 0,1,0 \rangle \end{array}$$

   - $\mathcal{D}_5 = \{t, I, t_I\}$ $(= \{\langle x,y,z \rangle \in \mathcal{V}_5 \mid x = 1\})$.
   - Let $\mathcal{D} = \mathcal{D}_5$, $\mathcal{F} = \mathcal{V}_5 - \mathcal{D}$. The operations in $\mathcal{O}_5$ are defined by:

   $$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } a \in \{I, f, f_I\} \\ \mathcal{F} & \text{if } a \in \{t, t_I\} \end{cases}$$

   $$\tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{I, t_I, f_I\} \end{cases}$$

   The rest of the operations are defined like in Definition 107.

2. Let $S = \{(\Rightarrow \neg), (\mathbf{b}), (\Rightarrow \neg \supset), (\mathbf{i1}), (\mathbf{a}_\neg)\}$. Then $\mathcal{M}_S = \langle \mathcal{V}_S, \mathcal{D}_S, \mathcal{O}_S \rangle$, where:

   - $\mathcal{V}_S = \{t, t_I, I, f\}$
   - $\mathcal{D}_S = \{t, I, t_I\}$
   - $a \tilde{\supset} b = \begin{cases} \mathcal{D}_S & \text{if either } a = f \text{ or } b \in \{t, t_I\} \\ \{I\} & \text{if } a \in \mathcal{D}_S \text{ and } b = I \\ \{f\} & \text{if } a \in \mathcal{D}_S \text{ and } b = f \end{cases}$
   - $\tilde{\neg} t = \tilde{\neg} t_I = \{f\}$ $\quad \tilde{\neg} I = \mathcal{D}_S$ $\quad \tilde{\neg} f = \{t\}$
   - $\tilde{\circ} t = \mathcal{D}_S$ $\quad \tilde{\circ} t_I = \tilde{\circ} I = \{f\}$ $\quad \tilde{\circ} f = \{t, t_I\}$

3. Let $\mathbf{Cia} = \{(\Rightarrow \neg), (\mathbf{b}), (\neg\neg \Rightarrow), (\mathbf{i}), (\mathbf{a})\}$. $\mathcal{M}_{Cia} = \langle \mathcal{V}_{Cia}, \mathcal{D}_{Cia}, \mathcal{O}_{Cia} \rangle$, where:

   - $\mathcal{V}_{Cia} = \{t, I, f\}$
   - $\mathcal{D}_{Cia} = \{t, I\}$

- $a\tilde{\supset}b = \begin{cases} \{f\} & \text{if } a \in \{t, I\} \text{ and } b = f \\ \{t\} & \text{if either } a = f,\ b \in \{f, t\} \text{ or } a = t,\ b = t \\ \{t, I\} & \text{otherwise} \end{cases}$
- $a\tilde{\vee}b = \begin{cases} \{f\} & \text{if } a = f \text{ and } b = f \\ \{t\} & \text{if either } a = t,\ b \in \{f, t\} \text{ or } b = t,\ a \in \{f, t\} \\ \{t, I\} & \text{otherwise} \end{cases}$
- $a\tilde{\wedge}b = \begin{cases} \{f\} & \text{if } a = f \text{ or } b = f \\ \{t\} & \text{if } a = t \text{ and } b = t \\ \{t, I\} & \text{otherwise} \end{cases}$
- $\tilde{\neg}t = \{f\} \quad \tilde{\neg}I = \{I\} \quad \tilde{\neg}f = \{t\}$
- $\tilde{\circ}t = \tilde{\circ}f = \{t\} \quad \tilde{\circ}I = \{f\}$

6.2 LFIs with Infinite Characteristic Nmatrices

The family of LFIs for which we provided semantics in the previous subsection does not include the well-known da Costa's original logic C_1 from ([da Costa, 1974]). Now C_1 is just the $\circ$-free fragment of **Cila**, the logic which is obtained by adding the rule **(l)** from Figure 4 to the system **Cia** from Example 111. This rule is problematic, because of the following theorem:

THEOREM 112. *No system between* **Bl** *and* **Bl**$[(\Rightarrow \neg\neg), (\neg\neg \Rightarrow), (\mathbf{i}), (\mathbf{o})]$ *has a finite characteristic Nmatrix (and so none of them has a finite characteristic ordinary matrix).* [20]

It follows that the method used in the previous subsection cannot work for logics like **Cila**. As a reasonable useful substitute, in this subsection we present *infinite* (but still effective) characteristic Nmatrices for a family of such systems (which includes **Cila**). Then we show that these Nmatrices can still be used to provide decision procedures for the logics they characterize.

As usual, we start with the basic LFI which includes (**l**), and find first a characteristic Nmatrix for it.

DEFINITION 113. The system **Bl** is obtained from the basic system **B** (from Example 111) by adding (**l**) as an axiom.

Now the validity of (**l**) in an Nmatrix means that whenever $\circ\varphi$ is "false", so is $\neg(\varphi \wedge \neg\varphi)$. Accordingly, Nmatrices appropriate for **Bl** should be able to distinguish between conjunctions of an "inconsistent" formula with its negation from other types of conjunctions. Therefore such Nmatrices should enforce an intimate connection between the truth-value of an "inconsistent"

[20]It easily follows from this theorem that C_1 has no finite characteristic Nmatrix. Now it has been known before that C_1 and some other LFIs have no characteristic *ordinary* matrices (see e.g. [Carnielli and Marcos, 2002; Carnielli and Marcos, 2007]). However, the result of Theorem 112 is much stronger.

formula and the truth-value of its negation. This in turn requires a supply of infinitely many truth-values, corresponding to the potentially infinite number of "inconsistent" formulas. But from where will we take these truth-values, and how should we define the operations on them? A key observation in our path to solve these problems is that **(k1)** and **(k2)** are theorems of **Bl**. Hence **Bl** extends $\mathbf{B}[\{(\mathbf{b}), (\Rightarrow\neg), (\mathbf{k1}), (\mathbf{k2})\}]$. Accordingly, the Nmatrices which we will use for characterizing **Bl** and its extensions will be refinements (see Definition 30) of $\mathcal{M}_{\{(\mathbf{b}),(\Rightarrow\neg),(\mathbf{k1}),(\mathbf{k2})\}}$. The latter is an Nmatrix with three truth-values: those that were denoted above by t, f, and I. Now one of the most productive method of refining a given *Nmatrix* $\mathcal{M}$ (which is not available in the framework of ordinary matrices!) is to first *duplicate* its elements: we can construct an Nmatrix $\mathcal{M}'$ which is completely equivalent to $\mathcal{M}$ by replacing each element a by a nonempty set of "copies", and then defining the operations in $\mathcal{M}'$ to be "the same" as in the original $\mathcal{M}$, but without distinguishing between two copies of the same element of $\mathcal{M}$. In other words: if $b', a'_1, \ldots, a'_n$ are copies in $\mathcal{M}'$ of $b, a_1 \ldots, a_n$ (respectively), then $b' \in \widetilde{\diamond}(a'_1 \ldots, a'_n)$ in $\mathcal{M}'$ iff $b \in \widetilde{\diamond}(a_1 \ldots, a_n)$ in $\mathcal{M}$. [21] What we shall do in order to construct an Nmatrix for **Bl** is first to duplicate the elements of $\mathcal{M}_{\{(\mathbf{b}),(\Rightarrow\neg),(\mathbf{k1}),(\mathbf{k2})\}}$ (actually only t and I) infinitely many times. Then we shall refine the resulting Nmatrix in the way hinted above so that axiom (**l**) becomes valid.

DEFINITION 114. Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$. The Nmatrix $\mathcal{M}_{\mathbf{Bl}} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined as follows:

1. $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$ and $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$.

2. $\mathcal{O}$ is defined by:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

[21]Actually, we have already implicitly used this method above several times. Thus from the point of view of the positive classical connectives, $\mathcal{M}_8^B$ is just a duplication of the classical two-valued matrix: all elements of $\mathcal{D}$ are copies of "true", all elements of $\mathcal{F}$ are copies of "false".

$$\widetilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

THEOREM 115. *$\mathcal{M}_{\mathbf{Bl}}$ is a characteristic Nmatrix for* **Bl**.

Now we turn to the extensions of **Bl** with axioms.

DEFINITION 116.

1. Let $LFIR^l = LFIR - \{(\Rightarrow \neg), (\neg \Rightarrow), (\mathbf{b}), (\mathbf{k_1}), (\mathbf{k_2})\}$.
2. For $S \subseteq LFIR^l$, the system $\mathbf{Bl}[S]$ is obtained from **Bl** by adding the schemata in S.

Like in the previous subsection, each of the schemata in $LFIR^l$ corresponds to some easily computed semantic condition, this time on simple refinements of the basic Nmatrix $\mathcal{M}_{\mathbf{Bl}}$. These conditions are in fact identical to the conditions that correspond to these axioms in refinements of $\mathcal{M}_{\{(\mathbf{b}),(\Rightarrow\neg),(\mathbf{k1}),(\mathbf{k2})\}}$, but with t replaced by $\mathcal{T}$, and I replaced by $\mathcal{I}$.

DEFINITION 117. For $S \subseteq LFIR^l$, $\mathcal{M}_{\mathbf{Bl}[S]}$ is the weakest simple refinement of $\mathcal{M}_{\mathbf{Bl}}$ which satisfies the following conditions:

1. If $(\neg\neg \Rightarrow) \in S$ then $a \in \mathcal{F} \Rightarrow \widetilde{\neg}(a) \subseteq \mathcal{T}$
2. If $(\Rightarrow \neg\neg) \in S$ then $a \in \mathcal{I} \Rightarrow \widetilde{\neg}(a) \subseteq \mathcal{I}$
3. If $(\mathbf{i1}) \in S$ then $a \in \mathcal{T} \Rightarrow \widetilde{\circ}(a) \subseteq \mathcal{T}$
4. If $(\mathbf{i1}) \in S$ then $a \in \mathcal{F} \Rightarrow \widetilde{\circ}(a) \subseteq \mathcal{T}$
5. If $(\mathbf{a}_\neg) \in S$ then $a \in \mathcal{I} \Rightarrow \widetilde{\circ}a \subseteq \mathcal{I}$
6. If $(\mathbf{a}_\diamond) \in S$ then $a \in \mathcal{F} \cup \mathcal{I}, b \in \mathcal{F} \cup \mathcal{I} \Rightarrow a \widetilde{\diamond} b \subseteq \mathcal{F} \cup \mathcal{I}$
7. If $(\mathbf{o}^{\mathbf{1}}_\diamond) \in S$ then $a \in \mathcal{F} \cup \mathcal{T} \Rightarrow a \widetilde{\diamond} b \subseteq \mathcal{F} \cup \mathcal{T}$
8. If $(\mathbf{o}^{\mathbf{2}}_\diamond) \in S$ then $b \in \mathcal{F} \cup \mathcal{T} \Rightarrow a \widetilde{\diamond} b \subseteq \mathcal{F} \cup \mathcal{T}$
9. If $(\neg \supset\Rightarrow)_1 \in S$ then $a \in \mathcal{F} \Rightarrow (a \widetilde{\supset} b) \subseteq \mathcal{T}$
10. If $(\neg \supset\Rightarrow)_2 \in S$ then $b \in \mathcal{T} \Rightarrow (a \widetilde{\supset} b) \subseteq \mathcal{T}$
11. If $(\Rightarrow \neg \supset) \in S$ then $a \in \mathcal{D}, b \in \mathcal{F} \cup \mathcal{I} \Rightarrow a \widetilde{\supset} b \subseteq \mathcal{F} \cup \mathcal{I}$

THEOREM 118. *For $S \subseteq LFIR$, $\mathcal{M}_{\mathbf{Bl}[S]}$ is a (dynamically) characteristic Nmatrix for* $\mathbf{Bl}[S]$.

COROLLARY 119. *For every $S \subseteq LFIR^l$, the logic* $\mathbf{Bl}[S]$ *is decidable.*

Proof. The proof of Theorem 118 in [Avron, 2007a] implies that to check whether a given formula φ is provable in **L**, it suffices to check all legal partial valuations v in $\mathcal{M}_L$ which assign to subformulas of φ values in

$$\{f\} \cup \{t_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\} \cup \{I_i^j \mid 0 \leq i \leq n(\varphi), 0 \leq j \leq k(\varphi)\}$$

where $n(\varphi)$ is the number of subformulas of φ which do not begin with $\neg$, and $k(\varphi)$ is the maximal number of consecutive negation symbols occurring within φ. This is a finite process.

COROLLARY 120. *da Costa's system C_1 is decidable,* [22] *and it has a characteristic Nmatrix $\mathcal{M}_{C_1}$, in which the sets of truth-values and designated truth-values are like in $\mathcal{M}_{\mathbf{Bl}}$, and the interpretations of the connectives are defined as follows:*

$$a \tilde{\supset} b = \begin{cases} \mathcal{F} & a \in \mathcal{D},\, b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{F},\, b \notin \mathcal{I} \\ \mathcal{T} & b \in \mathcal{T},\, a \notin \mathcal{I} \\ \mathcal{D} & otherwise \end{cases} \qquad a \tilde{\wedge} b = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T},\, b \in \mathcal{T} \\ \mathcal{T} & a = I_i^j,\, b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & otherwise \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{T} & a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & a = I_i^j \end{cases} \qquad a \tilde{\vee} b = \begin{cases} \mathcal{F} & a \in \mathcal{F},\, b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T},\, b \notin \mathcal{I} \\ \mathcal{T} & b \in \mathcal{T},\, a \notin \mathcal{I} \\ \mathcal{D} & otherwise \end{cases}$$

PART II: THE FIRST-ORDER CASE AND BEYOND

In the first part we have described the semantic framework of Nmatrices on the propositional level and presented a number of applications of this framework. However, no semantic framework can be considered really useful unless it can be naturally extended at least to the first-order level. Accordingly, this part is devoted to extending the framework of Nmatrices to languages with quantifiers.

The simplest and most well-known quantifiers are of course the first-order quantifiers $\forall$ and $\exists$ (and we shall devote Section 9 to them). However, we start by exploring a more general notion of quantifiers. By a (unary) quantifier we mean a logical constant which (may) bind a variable when applied to a formula. In other words, if $\mathcal{Q}$ is a quantifier, x is a variable and ψ is a formula, then $\mathcal{Q}x\psi$ is a formula in which all occurrences of x are bound by $\mathcal{Q}$. It should be noted that this notion can be further generalized to *multi-ary quantifiers*, which are logical constants that can be applied to more than

[22]The decidability of C_1, as well as of most of the systems presented here is not new (see, e.g. [Carnielli and Marcos, 2002; Carnielli and Marcos, 2007]).

one formula. If $\mathcal{Q}$ is an n-ary quantifier, x is a variable and $\psi_1, ..., \psi_n$ are formulas, then $\mathcal{Q}x(\psi_1, ..., \psi_n)$ is a formula in which all occurrences of x are bound by $\mathcal{Q}$. In this context the ordinary quantifiers can be thought of as unary quantifiers, while the bounded universal and existential quantifiers $\overline{\forall}$ and $\overline{\exists}$ used in syllogistic reasoning are examples of binary quantifiers[23].

7 MANY-VALUED MATRICES WITH QUANTIFERS

We start with ordinary (unary) quantifiers and their treatment in the framework of standard many-valued matrices. In what follows, L is a language, which includes a set of propositional connectives, a set of quantifiers, a countable set of variables, and a signature, consisting of a non-empty set of predicate symbols, a set of function symbols, and a set of constants. Frm_L is the set of (standardly defined) wffs of L, and Frm_L^{cl} is its set of closed wffs. Trm_L is the set of terms of L, and Trm_L^{cl} is its set of closed terms. In ordinary (deterministic) many-valued matrices (unary) quantifiers are standardly interpreted using the notion of *distributions*. This notion is due to Mostowski ([Mostowski, 1961]; the term 'distribution' was later coined in [Carnielli, 1987].

DEFINITION 121. Given a set of truth values $\mathcal{V}$, a *distribution* of a quantifier $\mathcal{Q}$ is a function $\lambda_{\mathcal{Q}} : (2^{\mathcal{V}} \setminus \{\emptyset\}) \to \mathcal{V}$.

The following is a standard definition (see, e.g. [Urquhart, 2001]) of a deterministic matrix with distribution quantifiers:

DEFINITION 122. A matrix for L is a tuple $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth-values,
- $\mathcal{D}$ is a non-empty proper set of $\mathcal{V}$,
- $\mathcal{O}$ includes a function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective of L, and a function $\tilde{\mathcal{Q}} : 2^{\mathcal{V}} \setminus \{\emptyset\} \to \mathcal{V}$ for every quantifier of L.

EXAMPLE 123. Consider the matrix $\mathcal{P} = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O} \rangle$ for a first-order language L, where $\mathcal{O}$ contains the following (standard) interpretations of $\forall$ and $\exists$:

$\mathbf{H}$	$\tilde{\forall}(\mathbf{H})$	$\tilde{\exists}(\mathbf{H})$
$\{\mathbf{t}\}$	$\mathbf{t}$	$\mathbf{t}$
$\{\mathbf{t}, \mathbf{f}\}$	$\mathbf{f}$	$\mathbf{t}$
$\{\mathbf{f}\}$	$\mathbf{f}$	$\mathbf{f}$

[23] The respective meanings of $\overline{\forall}x(\psi_1, \psi_2)$ and $\overline{\exists}x(\psi_1, \psi_2)$ are $\forall x(\psi_1 \to \psi_2)$ and $\exists x(\psi_1 \wedge \psi_2)$.

The notion of a structure is defined standardly:

DEFINITION 124. Let $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for L. An L-structure S for $\mathcal{P}$ is a pair $\langle D, I \rangle$ where D is a (non-empty) domain and I is an interpretation of constants, predicate symbols and function symbols of L, which satisfies:

- For every constant c of L: $I(c) \in D$.
- For every n-ary predicate symbol p of L: $I(p) \in D^n \to \mathcal{V}$.
- For every n-ary function symbol f of L: $I(f) \in D^n \to D$.

There are two main approaches to interpreting quantified formulas: the objectual (referential) approach, which uses *assignments*, and the substitutional approach ([Leblanc, 2001]), which is based on *substitutions*. Below we shortly review these two approaches. In the better known objectual approach (used in most standard textbooks on classical first-order logic, like [Mendelson, 1964; Enderton, 1972; van Dalen, 1980]), a variable is thought of as ranging over a set of objects from the domain, and assignments map variables to elements of the domain. In the context of many-valued deterministic matrices this is usually formalized as follows (see e.g. [Urquhart, 2001; Hähnle, 1999]).

DEFINITION 125. Given an L-structure $S = \langle D, I \rangle$, an *assignment* G in S is any function mapping the variables of L to D. For any $a \in D$ we denote by $G[x := a]$ the assignment which is similar to G, except that it assigns a to x. G is extended to L-terms as follows: $G(c) = I(c)$ for every constant c of L and $G(f(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(f)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$ for every n-ary function symbol f of L and $\mathbf{t}_1, ..., \mathbf{t}_n \in Trm_L$.

DEFINITION 126. Let S be an L-structure for a matrix $\mathcal{P}$ and let G be an assignment in S. The valuation $v_{S,G} : Frm_L \to \mathcal{V}$ is defined as follows:

- $v_{S,G}(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$.
- $v_{S,G}(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v_{S,G}(\psi_1), ..., v_{S,G}(\psi_n))$.
- $v_{S,G}(\mathcal{Q}x\psi) = \tilde{\mathcal{Q}}(\{v_{S,G[x:=a]}(\psi) \mid a \in D\})$.

In the alternative substitutional approach to quantification (used e.g. for first-order classical logic in [Shoenfield, 1967]) a variable is thought of as ranging over syntactical (closed) terms rather than over elements of the domain. Accordingly, the key notion in this approach is that of a *substitution instance* (rather than an assignment):

DEFINITION 127. For any formula ψ, a *substitution L-instance of ψ* is a formula $\psi\{\mathbf{t}_1/x_1, ..., \mathbf{t}_n/x_n\}$, where for all $1 \leq i \leq n$, $\mathbf{t}_i$ is an L-term free for

x_i in ψ. A *substitution L-instance of* Γ is a set $\{\psi\{\mathbf{t}_1/x_1, ..., \mathbf{t}_n/x_n\} \mid \psi \in \Gamma\}$ for some $\mathbf{t}_1, ..., \mathbf{t}_n \in Trm_L$ which are free for $x_1, ..., x_n$ (respectively) in all the formulas of Γ.

The main idea of the substitutional approach is that a formula is interpreted in terms of its substitution instances. Thus a formula $\forall\psi x$ ($\exists x\psi$) is true if and only if each (at least one) of the closed substitution instances of ψ is true. To apply this approach, we need to assume that every element of the domain has a closed term referring to it. This condition can be satisfied by extending the language with individual constants:

DEFINITION 128. For an L-structure $S = \langle D, I\rangle$ for $\mathcal{P}$, $L(D)$ is the language obtained from L by adding to it the set of *individual constants* $\{\overline{a} \mid a \in D\}$. The $L(D)$-structure which is induced by S is $\langle D, I'\rangle$, where I' is the unique extension of I to $L(D)$ such that $I'(\overline{a}) = a$.

Henceforth we shall identify an L-structure S with the $L(D)$-structure which is induced by S.

Here is the substitutional counterpart of the notion of a valuation given in Definition 126:

DEFINITION 129. Let $S = \langle D, I\rangle$ be an L-structure for a matrix $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$. The *valuation* $v_S : Frm^{\mathsf{cl}}_{L(D)} \to \mathcal{V}$ is defined as follows:

- $v_S(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$
- $v_S(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$
- $v_S(\mathcal{Q}x\psi) = \tilde{\mathcal{Q}}(\{v_S(\psi\{\overline{a}/x\}) \mid a \in D\})$

For reasons that will become clear in the sequel, in what follows we shall use the substitutional approach to define the consequence relations we are interested in, and not the objectual one.

DEFINITION 130. Let $S = \langle D, I\rangle$ be an L-structure for a matrix $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$.

- The valuation v_S *satisfies* a sentence ψ (denoted by $v_S \models \psi$), if $v_S(\psi) \in \mathcal{D}$. v_S is a *model* of $\Gamma \subseteq Frm^{\mathsf{cl}}_{L(D)}$ (denoted by $v_S \models \Gamma$), if $v_S(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.
- v_S satisfies a formula $\varphi \in Frm_L$, denoted by $v_S \models \varphi$, if for every closed $L(D)$-instance φ' of φ, ($v_S(\varphi')$ is defined and) $v_S(\varphi') \in \mathcal{D}$. v_S satisfies a set of formulas $\Gamma \subseteq Frm_L$, denoted by $v_S \models \Gamma$, if for every closed $L(D)$-instance Γ' of Γ, $v_S \models \Gamma'$.

In analogy to the propositional case (recall Definition 1), a (Taskian) logic $\mathbf{L}$ is a pair $\langle L, \vdash\rangle$, where L is a language and $\vdash$ is a structural and consistent scr (tcr) for L.[24] However, unlike in the propositional case, when variables and quantifiers are involved, there is more than one natural way of defining consequence relations induced by a given matrix. Two such relations which are usually associated with first-order logic are the truth and the validity consequence relations ([Avron, 1991]). Using the substitutional approach they can be generalized to the context of many-valued matrices as follows:

DEFINITION 131.

- For sets of L-formulas Γ, Δ, we say that $\Gamma \vdash^t_{\mathcal{P}} \Delta$ if for every L-structure S and every closed $L(D)$-instance $\Gamma' \cup \Delta'$ of $\Gamma \cup \Delta$: $v_S \models \Gamma'$ implies that $v_S \models \psi$ for some $\psi \in \Delta'$.
- We say that $\Gamma \vdash^v_{\mathcal{P}} \Delta$ if for every L-structure S: $v_S \models \Gamma$ implies that $v_S \models \psi$ for some $\psi \in \Delta$.

To demonstrate the difference between the validity and the truth consequence relations, consider a matrix $\mathcal{P}$ for a first-order language L with the standard interpretations of the quantifiers $\forall$ and $\exists$ from Example 123. Then $p(x) \vdash^v_{\mathcal{P}} \forall x p(x)$, but $p(x) \not\vdash^t_{\mathcal{P}} \forall x p(x)$. On the other hand, the classical deduction theorem holds for $\vdash^t_{\mathcal{P}}$, but not for $\vdash^v_{\mathcal{P}}$. However, the two consequence relations are identical from the point of view of theoremhood (i.e., $\vdash^t_{\mathcal{P}} \psi$ iff $\vdash^v_{\mathcal{P}} \psi$). This is a special case of the second part of the following well-known proposition:

PROPOSITION 132. *Let $\mathcal{P}$ be a matrix for L.*

1. $\Gamma \vdash^t_{\mathcal{P}} \psi$ *implies* $\Gamma \vdash^v_{\mathcal{P}} \psi$.
2. *If* $\Gamma \subseteq Frm^{\mathsf{cl}}_L$ *(i.e, Γ contsists of* sentences*), then* $\Gamma \vdash^t_{\mathcal{P}} \psi$ *iff* $\Gamma \vdash^v_{\mathcal{P}} \psi$.

8 NMATRICES WITH QUANTIFIERS

The extension of Nmatrices to languages with quantifiers is a natural generalization of Definition 122:

DEFINITION 133. An Nmatrix for L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth-values,
- $\mathcal{D}$ is a non-empty proper set of $\mathcal{V}$,
- $\mathcal{O}$ includes a function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective of L, and a function $\tilde{\mathcal{Q}} : 2^{\mathcal{V}} \setminus \{\emptyset\} \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every quantifier of L.

[24] The extension of the notion of structurality to languages with quantifiers is not an immediate matter. We omit the technical details.

EXAMPLE 134. Consider the Nmatrix $\mathcal{M} = \langle\{\mathbf{t},\mathbf{f}\},\{\mathbf{t}\},\mathcal{O}\rangle$ for a first-order language L, where $\mathcal{O}$ contains the following (non-standard) interpretations of $\forall$ and $\exists$:

$\mathbf{H}$	$\tilde{\forall}(\mathbf{H})$	$\tilde{\exists}(\mathbf{H})$
$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$	$\{\mathbf{t}\}$
$\{\mathbf{t},\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$

L-structures for Nmatrices are defined like in Definition 124. However, it seems difficult to apply the objectual approach to quantification in the context of Nmatrices. The reason for this is that unlike the deterministic case (recall Definition 126), an L-structure S and an assignment G *do not uniquely determine* the valuation $v_{S,G}$ in a Nmatrix $\mathcal{M}$. Thus the expression $v_{S,G[x:=a]}$ (used in Definition 126) is not well-defined. The substitutional approach, in contrast, *is* suitable for the non-deterministic context.

DEFINITION 135. Let $S = \langle D, I\rangle$ be an L-structure.

1. A set of sentences $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ is *closed under subsentences with respect to* S if (i) for every n-ary connective $\diamond$ of L: $\psi_1,...,\psi_n \in W$ whenever $\diamond(\psi_1,...,\psi_n) \in W$, and (ii) for every quantifier $\mathcal{Q}$ of L and every $a \in D$: if $\mathcal{Q}x\psi \in W$, then $\psi\{\overline{a}/x\} \in W$.
2. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S. We say that a partial S-valuation $v : W \to \mathcal{V}$ is *semi-legal in* $\mathcal{M}$ if it satisfies the following conditions:
   - $v(p(\mathbf{t}_1,...,\mathbf{t}_n)) = I(p)(I(\mathbf{t}_1),...,I(\mathbf{t}_n))$
   - $v(\diamond(\psi_1,...,\psi_n)) \in \tilde{\diamond}_{\mathcal{M}}(v(\psi_1),...,v(\psi_n))$
   - $v(\mathcal{Q}x\psi) \in \tilde{\mathcal{Q}}(\{v(\psi\{\overline{a}/x\}) \mid a \in D\})$

   A partial S-valuation v in $\mathcal{M}$ is a (full) S-valuation if its domain is $Frm^{\mathsf{cl}}_{L(D)}$.

It is easy to see that the above notion of a valuation is now well-defined. This is due to the fact that the truth-value $v(\mathcal{Q}x\psi)$ depends on the truth-values assigned by v *itself* to the subsentences of $\mathcal{Q}x\psi$ (unlike in our previous attempt using objectual quantification, where $v_{S,G[x:=a]}$ was used in the definition of $v_{S,G}$).

REMARK 136. It is important to stress the difference between our use of notation in the above definition and the one used in Definition 129. Given a (deterministic) matrix $\mathcal{P}$ and an L-structure S, the valuation v_S is uniquely determined by S and $\mathcal{P}$. However, this is not the case for non-deterministic valuations in an Nmatrix $\mathcal{M}$ (although S does determine the truth-values of the atomic sentences), and so we write "an S-valuation v" (compare to "the valuation v_S").

DEFINITION 137. Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S, and let $v : W \to \mathcal{V}$ be a partial S-valuation.

- v *satisfies* a sentence $\psi \in W$ (denoted by $v \models \psi$), if $v(\psi) \in \mathcal{D}$. v is a *model* of $\Gamma \subseteq W$ (denoted by $v \models \Gamma$), if $v(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.
- v satisfies a formula $\varphi \in Frm_L$ (denoted by $v \models \varphi$), if for every closed $L(D)$-instance φ' of φ, ($v(\varphi')$ is defined and) $v(\varphi') \in \mathcal{D}$. v is a *model* of $\Gamma \subseteq Frm_L$ (denoted by $v \models \Gamma$), if for every closed $L(D)$-instance Γ' of Γ, $v \models \Gamma'$.

The following analycity property is analogous to that given in Proposition 27 for the propositional case:

PROPOSITION 138. *Let $\mathcal{M}$ be an Nmatrix for L and S an L-structure for $\mathcal{M}$. Any partial S-valuation v, which is semi-legal in $\mathcal{M}$ can be extended to a full S-valuation, which is semi-legal in $\mathcal{M}$.*

At this point we note two important problems concerning the above naive semantics, which do not arise on the propositional level. The first problem is related to the principle of α-equivalence, capturing the idea that the names of bound variables are immaterial. It is of course quite reasonable to expect that in any useful semantics two α-equivalent sentences are always assigned the same truth-value. However, this is not necessarily the case for valuations in Nmatrices as defined above. As an example, consider a language L_a with the unary connective $\neg$ and the quantifier $\forall$. Let $\mathcal{M}_a = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O}\rangle$ be the Nmatrix for L_a with the standard (deterministic) interpretation of $\forall$ and the non-deterministic interpretation of $\neg$ given in Example 21. Let $S_a = \langle \{a\}, I_a\rangle$ be the simple L_a-structure, such that $I_a(c_a) = a$ and $I_a(p)(a) = \mathbf{t}$. Clearly, there is a $\mathcal{M}_a$-semi-legal S_a-valuation v, such that $v(\neg\forall x p(x)) = \mathbf{t}$ and $v(\neg\forall y p(y)) = \mathbf{f}$. Hence two α-equivalent formulas are not necessarily assigned the same truth-value by a $\mathcal{M}_a$-semi-legal S_a-valuation![25] The second problem is related to the nature of identity and becomes really crucial if equality is added to the language. Suppose we have two terms, denoting the same object. It is again reasonable to expect that we should be able to use these terms interchangeably, or substitute one term for another in any context. Returning to our example, suppose we add another constant d_a to the language L_a and extend the structure S_a to interpret it: $I(d_a) = a$. Thus the constants d_a and c_a refer to the same

[25] Of course, two different occurrences of the same formula are still assigned the same truth-value, since a valuation is a mapping from *formulas* to truth-values.

element a, but there is a $\mathcal{M}_a$-legal valuation v, such that $v(\neg p(c_a)) = \mathbf{t}$ and $v(\neg p(d_a)) = \mathbf{f}$.

These problems are directly related to introducing a new level of freedom by the non-deterministic choice of truth-values for quantified formulas. In view of these issues, further limitations need to be imposed on this choice. This can be done by introducing the following congruence relation, capturing these principles.

DEFINITION 139. Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ between terms of $L(D)$ is defined as follows:

- $x \sim^S x$ for every variable x of L.
- If $\mathbf{t}, \mathbf{t}' \in Trm^{\mathsf{cl}}_{L(D)}$ and $I[\mathbf{t}] = I[\mathbf{t}']$, then $\mathbf{t} \sim^S \mathbf{t}'$.
- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $f(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S f(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.

The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:

- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, \mathbf{t}_2 \sim^S \mathbf{t}'_2, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $p(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S p(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.
- If $\psi_i \sim^S \varphi_i$ for all $1 \leq i \leq n$, then $\diamond(\psi_1, ..., \psi_n) \sim^S \diamond(\varphi_1, ..., \varphi_n)$ for every n-ary connective $\diamond$ of $\mathcal{L}$.
- If $\psi\{z/x\} \sim^S \varphi\{z/y\}$, where x, y are distinct variables and z is a new variable, then $\mathcal{Q}x\psi \sim^S \mathcal{Q}y\varphi$ for every quantifier $\mathcal{Q}$ of L.

The following lemma is easy to prove:

LEMMA 140. *Let S be an L-structure, and let $\boldsymbol{t}_1, \boldsymbol{t}_2$ be closed terms of $L(D)$ such that $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$. Let ψ_1, ψ_2 be $L(D)$-formulas such that $\psi_1 \sim^S \psi_2$. Then $\psi_1\{\boldsymbol{t}/x\} \sim^S \psi_2\{\boldsymbol{t}_2/x\}$.*

Using the above congruence relation, we can now modify Definition 135 as follows:

DEFINITION 141. Let S be an L-structure and $\mathcal{M}$ an Nmatrix for L. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S. A partial S-valuation $v : W \to \mathcal{V}$ is *$\sim^S$-legal in $\mathcal{M}$* if it is semi-legal in $\mathcal{M}$ and for every $\psi, \varphi \in W$: $\psi \sim^S \varphi$ implies $v(\psi) = v(\varphi)$.

Now we come to the definition of consequence relations induced by Nmatrices, analogous to Definition 131:

DEFINITION 142.

- For sets of L-formulas Γ, Δ, we say that $\Gamma \vdash^t_{\mathcal{M}} \Delta$ if for every L-structure S, every S-valuation v which is $\sim^S$-legal in $\mathcal{M}$, and every closed $L(D)$-instance $\Gamma' \cup \Delta'$ of $\Gamma \cup \Delta$: $v \models \Gamma'$ implies $v \models \psi$ for some $\psi \in \Delta'$.

- We say that $\Gamma \vdash^v_{\mathcal{M}} \Delta$ if for every L-structure S and S-valuation v which is $\sim^S$-legal in $\mathcal{M}$: $v \models \Gamma$ implies $v \models \psi$ for some $\psi \in \Delta$.

The following extension of Proposition 132 to the context of Nmatrices can be easily proved:

PROPOSITION 143. *Let $\mathcal{M}$ be an Nmatrix for L.*

1. $\Gamma \vdash^t_{\mathcal{M}} \psi$ *implies* $\Gamma \vdash^v_{\mathcal{M}} \psi$.
2. *If* $\Gamma \subseteq Frm^{\mathsf{cl}}_L$ *(i.e, Γ contains only closed formulas), then* $\Gamma \vdash^t_{\mathcal{M}} \psi$ *iff* $\Gamma \vdash^v_{\mathcal{M}} \psi$.

As for analycity, the following analogue of Proposition 138 can be proved (the presence of the $\sim^S$-relation makes its proof less trivial):

PROPOSITION 144. *Let $\mathcal{M}$ be an Nmatrix for L and S an L-structure. Then any partial S-valuation which is $\sim^S$-legal in $\mathcal{M}$ can be extended to a full S-valuation which is $\sim^S$-legal in $\mathcal{M}$.*

We end this section by generalizing the notions of reduction and refinement from Definition 30 to languages with quantifiers:

DEFINITION 145. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be two Nmatrices for L.

1. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$, such that:
   - For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$.
   - $F(y) \in \tilde{\diamond}_{\mathcal{M}_2}(F(x_1), ..., F(x_n))$ for every n-ary connective $\diamond$ of L and every $x_1, ..., x_n, y \in \mathcal{V}_1$, such that $y \in \tilde{\diamond}_{\mathcal{M}_1}(x_1, ..., x_n)$.
   - $F(y) \in \tilde{\mathcal{Q}}_{\mathcal{M}_2}(\{F(z) \mid z \in H\})$ for every quantifier $\mathcal{Q}$ of L, every $y \in \mathcal{V}_1$ and $H \in 2^{\mathcal{V}_1} \setminus \{\emptyset\}$, such that $y \in \tilde{\mathcal{Q}}_{\mathcal{M}_1}(H)$.
2. $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.

THEOREM 146. *Let $\mathcal{M}_1$ be a refinement of $\mathcal{M}_2$. Then* $\vdash^t_{\mathcal{M}_2} \subseteq \vdash^t_{\mathcal{M}_1}$ *and* $\vdash^v_{\mathcal{M}_2} \subseteq \vdash^v_{\mathcal{M}_1}$.

REMARK 147. Again an important case in which $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is when $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, $\widetilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every n-ary connective $\diamond$ of L and every $\vec{x} \in \mathcal{V}_1^n$, and $\tilde{\mathcal{Q}}_{\mathcal{M}_1}(H) \subseteq \tilde{\mathcal{Q}}_{\mathcal{M}_2}(H)$ for every quantifier $\mathcal{Q}$ of L and every $H \in 2^{\mathcal{V}_1} \setminus \{\emptyset\}$. It is easy to see that the identity function on $\mathcal{V}_1$ is in this case a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. We will refer to this kind of refinement as *simple*.

9 THE FIRST-ORDER CASE

Next we focus on the first-order quantifiers $\forall$ and $\exists$ with their natural interpretations. Throughout this section we assume that $\forall$ and $\exists$ are in L. In Example 123 we have seen the standard interpretation of these quantifiers in the two-valued case. This can be generalized to an arbitrary number of truth-values as follows:

DEFINITION 148. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for L. We say that a quantifier $\mathcal{Q}$ is *universally interpreted in* $\mathcal{M}$ if for all $H \in 2^{\mathcal{V}} \setminus \{\emptyset\}$:

$$\widetilde{\mathcal{Q}}(H) \subseteq \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

A quantifier $\mathcal{Q}$ is *existentially interpreted in* $\mathcal{M}$ if for all $H \in 2^{\mathcal{V}} \setminus \{\emptyset\}$:

$$\widetilde{\mathcal{Q}}(H) \subseteq \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

At this point we note a problem, the nature of which is very similar to the problems of the α-equivalence and identity principles which we handled in the previous section. Namely, in the context of universally and existentially interpreted quantifiers, one would expect the equivalence of two formulas, where one is obtained from the other by deletion or addition of *void* quantifiers (by a void quantifier we mean the case then a variable is bound vacuously). For instance, we expect $\neg\forall x p(c)$ and $\neg p(c)$ to be equivalent. This, however, is not always the case, again due to the degree of freedom introduced by the non-deterministic choice in our semantic framework. For an example, consider again the Nmatrix $\mathcal{M}_a = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O} \rangle$ discussed in the previous section, where $\neg$ is interpreted like in Example 21, and $\forall$ and $\exists$ have the universal and the existential interpretations in $\mathcal{M}_a$ (respectively). Then there exists an L-structure S and an S-valuation v legal in $\mathcal{M}_a$, such that $v(\neg\forall x p(c)) = \mathbf{t}$, but $v(\neg p(c)) = \mathbf{f}$.
The solution is similar to the one in the previous section: we extend the congruence relation $\sim^S$ to capture the principle of void quantification:

DEFINITION 149. Let L be a language which includes the quantifiers $\forall$ and $\exists$ and let $S = \langle D, I \rangle$ be an L-structure. $\sim^S_{\forall\exists}$ is the minimal congruence relation between $L(D)$-formulas, which satisfies: (i) $\sim^S \subseteq \sim^S_{\forall\exists}$, and (ii) If $\psi \sim^S_{\forall\exists} \psi'$ and x does not occur free in ψ, then $\mathcal{Q}x\psi \sim^S_{\forall\exists} \psi'$ for $\mathcal{Q} \in \{\forall, \exists\}$.

The following extension of Lemma 140 is again easy to prove:

LEMMA 150. *Let S be an L-structure, and let $\boldsymbol{t}_1, \boldsymbol{t}_2$ be closed terms of $L(D)$ such that $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$. Let ψ_1, ψ_2 be $L(D)$-formulas such that $\psi_1 \sim^S_{\forall\exists} \psi_2$. Then $\psi_1\{\boldsymbol{t}/x\} \sim^S_{\forall\exists} \psi_2\{\boldsymbol{t}_2/x\}$.*

DEFINITION 151. Let S be an L-structure and $\mathcal{M}$ an Nmatrix for L. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S. A partial S-valuation $v : W \to \mathcal{V}$ is $\sim^S_{\forall\exists}$*-legal in* $\mathcal{M}$ if it is semi-legal in $\mathcal{M}$ and for every $\psi, \varphi \in W$: $\psi \sim^S_{\forall\exists} \varphi$ implies $v(\psi) = v(\varphi)$.

Using the above definition, we can now modify the notions of truth- and validity-based consequence relations from Definition 142:

DEFINITION 152. The consequence relations $\vdash^t_{\mathcal{M},\forall\exists}$ and $\vdash^v_{\mathcal{M},\forall\exists}$ are defined like $\vdash^t_{\mathcal{M}}$ and $\vdash^v_{\mathcal{M}}$ (respectively), but using $\sim^S_{\forall\exists}$ rather than $\sim^S$.

PROPOSITION 153. *Let $\mathcal{M}$ be an Nmatrix for L.*

1. $\Gamma \vdash^t_{\mathcal{M},\forall\exists} \psi$ *implies* $\Gamma \vdash^v_{\mathcal{M},\forall\exists} \psi$.
2. *If* $\Gamma \subseteq Frm^{\mathsf{cl}}_L$ *(i.e, Γ contains only closed formulas), then* $\Gamma \vdash^t_{\mathcal{M},\forall\exists} \psi$ *iff* $\Gamma \vdash^v_{\mathcal{M},\forall\exists} \psi$.

It should be noted that analycity for $\sim^S_{\forall\exists}$ is *not* always guaranteed. Consider, for instance, an Nmatrix $\mathcal{M}_v = \langle\{\mathbf{t},\mathbf{f}\},\{\mathbf{t}\},\mathcal{O}\rangle$ for some first-order language L, with the following interpretation of $\forall$: $\tilde{\forall}[\{H\}] = \{\mathbf{t}\}$ for every $H \subseteq P^+(\{\mathbf{t},\mathbf{f}\})$. Let $S = \langle\{a\}, I\rangle$ be an L-structure, such that $I(c) = a$ and $I(p) = \emptyset$. Let $W = \{p(c)\}$. Then no partial valuation on W can be extended to a full $\mathcal{M}$-legal valuation v which respects $\sim^S_{\forall\exists}$. Next we characterize those Nmatrices in which this problem does not occur.

DEFINITION 154. Let L include propositional connectives and (at most) the quantifiers $\forall$ and $\exists$. An Nmatrix $\mathcal{M}$ for L is $\{\forall,\exists\}$*-analytic* if every L-structure S has the property that every partial S-valuation which is $\sim^S_{\forall\exists}$-legal in $\mathcal{M}$ can be extended to a full S-valuation which is $\sim^S_{\forall\exists}$-legal in $\mathcal{M}$.

THEOREM 155. *Let $\mathcal{M} = \langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$ be an Nmatrix for a language which in addition to propositional connectives includes (at most) the quantifiers $\forall$ and $\exists$. $\mathcal{M}$ is $\{\forall,\exists\}$-analytic iff for every $a \in \mathcal{V}$: $a \in \tilde{\mathcal{Q}}[\{a\}]$ for $\mathcal{Q} \in \{\forall,\exists\}$.*

Next we turn to the problem of extending a propositional formal system having a nondeterministic semantics to the first-order level. We take as an example $\mathbf{H}LK^+$, the Hilbert-type system which corresponds to the basic Nmatrix $\mathcal{M}^B_4$ from Definition 80 (see Remark 79).

DEFINITION 156. $\mathbf{QHL_0}$ is obtained by adding to $\mathbf{H}LK^+$ the following standard axioms and inference rules for $\forall$ and $\exists$:

$$\forall x\psi \supset \psi\{\mathbf{t}/x\} \qquad \psi\{\mathbf{t}/x\} \supset \exists x\psi$$

$$\frac{(\varphi \supset \theta)}{(\varphi \supset \forall x\theta)} \qquad \frac{(\theta \supset \varphi)}{(\exists x\theta \supset \varphi)}$$

where $\mathbf{t}$ is any term free for x in ψ, and x does not occur free in φ.

Unfortunately, $\mathbf{QHL_0}$ is not very useful. Due to the absence of axioms for negation, neither the α-equivalence principle, nor the void quantification principle, are derivable in it. For instance, $\nvdash_{\mathbf{QHL_0}} \neg\forall xp(x) \leftrightarrow \neg\forall yp(y)$, and $\nvdash_{\mathbf{QHL_0}} (\neg\forall xp(c)) \leftrightarrow \neg p(c)$. To handle this, we follow da Costa's approach from [da Costa, 1974]:

DEFINITION 157. $\sim^{dc}$ is the minimal congruence relation between formulas, which satisfies for $\mathcal{Q} \in \{\forall, \exists\}$:

- If $\psi\{z/x\} \sim^{dc} \psi'\{z/y\}$, where z is fresh, then $\mathcal{Q}x\psi \sim^{dc} \mathcal{Q}y\psi'$.
- If $\psi \sim^{dc} \psi'$ and x does not occur free in ψ, then $\mathcal{Q}x\psi \sim^{dc} \psi'$.

DEFINITION 158. Let **QHL** be the system obtained from $\mathbf{QHL_0}$ by adding the axiom **(DC)** $\psi \supset \psi'$ whenever $\psi \sim^{dc} \psi'$.

DEFINITION 159. Let the Nmatrix $\mathcal{QM}_4^B$ be the extension of the Nmatrix $\mathcal{M}_4^B$ (Definition 80) with the following interpretations of $\forall$ and $\exists$:

$$\widetilde{\forall}(H) = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\widetilde{\exists}(H) = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

PROPOSITION 160. *$\Gamma \vdash_{\mathbf{QHL}} \psi$ iff $\Gamma \vdash^v_{\mathcal{QM}_4^B,\forall\exists} \psi$.*

The proof is very similar to the proof of Theorem 163 below.

10 AN APPLICATION: NMATRICES FOR FIRST-ORDER LOGICS OF FORMAL INCONSISTENCY

In this section we further apply the framework of Nmatrices with first-order quantifiers to provide semantics for first-order LFIs (the propositional fragments of which were already handled in section 6). For simplicity of presentation, we formulate these logics in terms of Hilbert-style systems, rather than in terms of abstract consequence relations. The results of this section are mainly taken from [Avron and Zamansky, 2007d; Avron and Zamansky, 2007b]. Throughout it, we let $L_C = \{\vee, \wedge, \supset, \neg, \circ, \forall, \exists\}$.

Our starting point will be the basic paraconsistent system **QHB**, obtained from **QHL** (Definition 158) by the addition of the following schemata:

$$(\Rightarrow \neg)\ \varphi \vee \neg\varphi \qquad (\mathbf{b})\ (\circ\varphi \wedge \neg\varphi \wedge \varphi) \supset \psi$$

CONVENTION 161. **QHB** is the obvious first-order extension of the Hilbert-style axiomatization of the logic **B** from Example 111. Accordingly, in this section we shall refer to **QHB** simply as **B**.

We obtain a large family of first-order LFIs by extending **B** with various combinations of axioms from $HLFIR$ (Definition 106), to which we add the following quantifier-related versions of the axioms (see, e.g. [Carnielli et. al., 2000]) (**a**) and (**o**) which were considered in section 6: [26]

$$(\mathbf{a}_{\mathcal{Q}})\ \forall x \circ\varphi \supset (\circ(\mathcal{Q}x\varphi)) \qquad (\mathbf{o}_{\mathcal{Q}})\ \exists x \circ\varphi \supset (\circ(\mathcal{Q}x\varphi)) \qquad (\mathcal{Q} \in \{\forall, \exists\})$$

DEFINITION 162. Let $\mathcal{QR} = HLFIR \cup \{(\mathbf{a}_\forall), (\mathbf{a}_\exists), (\mathbf{o}_\forall), (\mathbf{o}_\exists)\}$. For a set $S \subseteq \mathcal{QR}$, $\mathbf{B}[S]$ is the system obtained by adding the axioms in S to **B**.

Our Nmatrix for **B** is a straightforward extentension of the Nmatrix $\mathcal{M}_5^B$ from Example 111:

THEOREM 163. *Let $\mathcal{QM}_5^B$ be the extension of $\mathcal{M}_5^B$ with the following interpretations of quantifiers:*

$$\widetilde{\forall}(H) = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \qquad \widetilde{\exists}(H) = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Then $\Gamma \vdash^v_{\mathcal{QM}_5^B, \forall\exists} \psi_0$ iff $\Gamma \vdash_{\mathbf{B}} \psi_0$.

Proof. The proof of soundness is not hard and is left to the reader. For completeness, we first note that by definition of the interpretation of $\forall$ in $\mathcal{QM}_5^B$, $\forall x\varphi \vdash_{\mathcal{QM}_5^B} \varphi$ and $\varphi \vdash_{\mathcal{QM}_5^B} \forall x\varphi$ for every formula φ and every variable x. Obviously the same relations hold between φ and $\forall x\varphi$ also in **B**. It follows that we may assume that all formulas in $\Gamma \cup \{\psi_0\}$ are sentences. It is also easy to see that we may restrict ourselves to sentences in σ_r, the signature consisting of all the constants, function, and predicate symbols occurring in $\Gamma \cup \{\psi_0\}$. Now suppose that $\Gamma \not\vdash_{\mathbf{B}} \psi_0$. We will construct an σ_r-structure S and a $\mathcal{QM}_5^B$-legal S-valuation v, such that $v \models \Gamma$, but $v \not\models \psi_0$. Let L' be the language obtained from σ_r by adding a countably infinite set of new constants. It is a standard matter to show (using a usual Henkin-type construction) that Γ can be extended to a maximal set Γ^* of sentences in L', such that: (i) $\Gamma^* \not\vdash_{\mathbf{B}} \psi_0$, (ii) $\Gamma \subseteq \Gamma^*$, (iii) For every L'-sentence $\exists x\psi \in \Gamma^*$ there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \in \Gamma^*$, and (iv) For every L'-sentence $\forall x\psi \notin \Gamma^*$, there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \notin \Gamma^*$. (The last property follows from property (iii), the deduction theorem for **B**, and the fact that for any $x \notin Fv(\varphi)$, $(\forall x\psi \supset \varphi) \supset \exists x(\psi \supset \varphi)$ is provable in **B**.) It is now easy to show that Γ^* has the following properties: (1) If

[26] See [Zamansky and Avron, 2006b; Avron and Zamansky, 2007d; Avron and Zamansky, 2007b] for other quantifier-related axioms treated in the context of Nmatrices.

$\psi \notin \Gamma^*$, then $\psi \supset \psi_0 \in \Gamma^*$, (2) $\psi \vee \varphi \in \Gamma^*$ iff either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$, (3) $\psi \wedge \varphi \in \Gamma^*$ iff both $\varphi \in \Gamma^*$ and $\psi \in \Gamma^*$, (4) $\varphi \supset \psi \in \Gamma^*$ iff either $\varphi \notin \Gamma^*$ or $\psi \in \Gamma^*$, (5) Either $\psi \in \Gamma^*$ or $\neg\psi \in \Gamma^*$, (6) If ψ and $\neg\psi$ are both in Γ^*, then $\circ\psi \notin \Gamma^*$, (7) If $\psi \in \Gamma^*$, then for every L'-sentence ψ' such that $\psi' \sim^{dc} \psi$: $\psi' \in \Gamma^*$, (8) If $\forall x\theta \in \Gamma^*$, then for every closed L'-term $\mathbf{t}$: $\theta\{\mathbf{t}/x\} \in \Gamma^*$. If $\forall x\theta \notin \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L', such that $\theta\{\mathbf{t}_\theta/x\} \notin \Gamma^*$, (9) If $\exists x\theta \in \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L, such that $\theta\{\mathbf{t}_\theta/x\} \in \Gamma^*$. If $\exists x\theta \notin \Gamma^*$, then for every closed term $\mathbf{t}$ of L': $\theta\{\mathbf{t}/x\} \notin \Gamma^*$.
The L'-structure $S = \langle D, I\rangle$ is defined as follows:

- D is the set of all the closed terms of L'.
- For every constant c of L': $I(c) = c$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I(f)(\mathbf{t}_1, ..., \mathbf{t}_n) = f(\mathbf{t}_1, ..., \mathbf{t}_n)$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I(p)(\mathbf{t}_1, ..., \mathbf{t}_n) = \langle x, y, z\rangle$, where $x, y, z \in \{0, 1\}$ and (i) $x = 1$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$, (ii) $y = 1$ iff $\neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$, (iii) $z = 1$ iff $\circ p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$.

Given an $L'(D)$-sentence ψ, let the sentence $\widetilde{\psi}$ be obtained by replacing all individual constants $\overline{\mathbf{t}}$ occurring in ψ by the respective (closed) term $\mathbf{t}$. Then the following lemma is easy to prove:

LEMMA 164. *For any* $\psi, \varphi \in Frm^{\mathsf{cl}}_{L'(D)}$*: if* $\psi \sim^S_{\forall\exists} \varphi$*, then* $\widetilde{\psi} \sim^{dc} \widetilde{\varphi}$*.*

The refuting S-valuation $v : Frm^{\mathsf{cl}}_{L'(D)} \to \mathcal{V}$ is defined as follows:

$$v(\psi) = \langle x_\psi, y_\psi, z_\psi\rangle$$

where $x_\psi, y_\psi, z_\psi \in \{0, 1\}$ and: (i) $x_\psi = 1$ iff $\widetilde{\psi} \in \Gamma^*$, (ii) $y_\psi = 1$ iff $\widetilde{\neg\psi} \in \Gamma^*$, (iii) $z_\psi = 1$ iff $\widetilde{\circ\psi} \in \Gamma^*$.

Let ψ, ψ' be two $L'(D)$-sentences, such that $\psi \sim^S_{\forall\exists} \psi'$. Then by lemma 164, $\widetilde{\psi} \sim^{dc} \widetilde{\psi'}$, and by property 7 of Γ^*, $\widetilde{\psi} \in \Gamma^*$ iff $\widetilde{\psi'} \in \Gamma^*$. Similarly, since $\neg\psi \sim^S_{\forall\exists} \neg\psi'$ and $\circ\psi \sim^S_{\forall\exists} \circ\psi'$, $\neg\widetilde{\psi} = \widetilde{\neg\psi} \sim^{dc} \widetilde{\neg\psi'} = \neg\widetilde{\psi'}$ and $\widetilde{\circ\psi} \sim^{dc} \widetilde{\circ\psi'}$. Thus $\widetilde{\neg\psi} \in \Gamma^*$ iff $\widetilde{\neg\psi'} \in \Gamma^*$ and $\widetilde{\circ\psi} \in \Gamma^*$ iff $\widetilde{\circ\psi'} \in \Gamma^*$. Hence $v(\psi) = v(\psi')$ and so v respects the $\sim^S_{\forall\exists}$ relation.
It remains to check that v respects the interpretations of the connectives and quantifiers in $\mathcal{QM}_5$. This is guaranteed by the properties of Γ^*. We prove this for the case of $\forall$:

- Let $\forall x\psi$ be an $L'(D)$-sentence, such that $\{v(\psi\{\overline{a}/x\}) \mid a \in D\} \subseteq \mathcal{D}$. Suppose by contradiction that $v(\forall x\psi) \notin \mathcal{D}$. Then $\widetilde{\forall x\psi} = \forall x\widetilde{\psi} \notin \Gamma^*$. By property 8 of Γ^*, there exists some closed L'-term $\mathbf{t}$, such that

$\widetilde{\psi}\{\mathbf{t}/x\} \notin \Gamma^*$. Then $v(\widetilde{\psi}\{\mathbf{t}/x\}) \notin \mathcal{D}$. Since $\psi \sim^S_{\forall\exists} \widetilde{\psi}$, by lemma 150 also $\psi\{\mathbf{t}/x\} \sim^S_{\forall\exists} \widetilde{\psi}\{\mathbf{t}/x\}$. We have already shown that v respects the $\sim^S_{\forall\exists}$ relation, and so $v(\psi\{\mathbf{t}/x\}) \notin \mathcal{D}$. By lemma 150 again, $\psi\{\mathbf{t}/x\} \sim^S_{\forall\exists} \psi\{\overline{\mathbf{t}}/x\}$, and so $v(\psi\{\overline{\mathbf{t}}/x\}) \notin \mathcal{D}$, in contradiction to our assumption.

- Let $\forall x\psi$ be an $L'(D)$-sentence, such that $\{v(\psi\{\overline{a}/x\}) \mid a \in D\} \cap \mathcal{F} \neq \emptyset$. Suppose by contradiction that $v(\forall x\psi) \notin \mathcal{F}$. Then $\forall x\widetilde{\psi} \in \Gamma^*$. By property 8 of Γ^*, for every closed L'-term $\mathbf{t}$: $\widetilde{\psi}\{\mathbf{t}/x\} \in \Gamma^*$. Then $v(\widetilde{\psi}\{\mathbf{t}/x\}) \in \mathcal{D}$. Similarly to the previous case, we get that $v(\psi\{\overline{a}/x\}) \in \mathcal{D}$ for every $a \in D$, in contradiction to our assumption.

Now for every L'-sentence ψ: $v(\psi) \in \mathcal{D}$ iff $\psi \in \Gamma^*$. So $v \models \Gamma$ (recall that $\Gamma \subseteq \Gamma^*$), but $v \not\models \psi_0$. ■

Like in the propositional case, the systems obtained by adding some set of axioms from $\mathcal{QR}$ to **B** can be characterized by the simple refinement of $\mathcal{QM}^B_5$ induced by the conditions corresponding to the axioms from $\mathcal{QR}$:

DEFINITION 165.

1. Let $\mathsf{Con} = \{\langle x, y, 1\rangle \mid x, y \in \{0, 1\}\}$.
    - For $r \in HLFIR$, $C(r)$ is defined like in Definition 82 (for NIR) or Definition 108 (for FCR).
    - C($\mathbf{a}_\mathcal{Q}$): If $H \subseteq \mathsf{Con}$, then $\tilde{\mathcal{Q}}(H) \subseteq \mathsf{Con}$
    - C($\mathbf{o}_\mathcal{Q}$): If $H \cap \mathsf{Con} \neq \emptyset$, then $\tilde{\mathcal{Q}}(H) \subseteq \mathsf{Con}$
2. For $S \subseteq \mathcal{QR}$, $\mathrm{C}(S) = \{\mathrm{C}r \mid r \in S\}$, and $\mathcal{QM}_S$ is the weakest simple refinement of $\mathcal{QM}^B_5$ in which the conditions in $C(S)$ are all satisfied.

EXAMPLE 166. Let $S_i = \{(\mathbf{i})\}$, $S_a = S_i \cup \{(\mathbf{a})\}$ and $S_o = S_i \cup \{(\mathbf{o})\}$. The interpretations of $\forall$ and $\exists$ are defined in $\mathcal{QM}_{S_i}$, $\mathcal{QM}_{S_a}$ and $\mathcal{QM}_{S_o}$ (respectively) as follows:[27]

$\mathcal{QM}_{S_i}$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t, I\}$	$\{t, I\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t, I\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t, I\}$

[27] Recall that by C($\mathbf{i_1}$) and C($\mathbf{i_2}$) the truth-values t_I and f_I are deleted and we are left with only three truth-values: t, f and I.

$\mathcal{QM}_{S_o}$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t\}$	$\{t\}$
$\{f, I\}$	$\{f\}$	$\{t\}$
$\{t, f, I\}$	$\{f\}$	$\{t\}$

$\mathcal{QM}_{S_a}$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t, I\}$

THEOREM 167. *For* $S \subseteq \mathcal{QR}$, $\Gamma \vdash^v_{\mathcal{QM}_S,\forall\exists} \psi$ *iff* $\Gamma \vdash_{\mathbf{B}[S]} \psi$.

And what about systems which include the problematic axiom (**l**) (see Figure 4 and Section 6.2)? It suffices to say that they can be handled in a way which is very similar to the systems discussed so far in this section. The only difference is that their semantics is based on the Nmatrix $\mathcal{M}_{\mathbf{Bl}}$ from Definition 114 rather than on $\mathcal{M}^B_5$.

EXAMPLE 168. da Costa's well-known first-order logic C^*_1 is the $\circ$-free fragment of $\mathbf{B}[\{(\mathbf{i}), (\mathbf{c}), (\mathbf{a})\}]$ (note that the axioms $(\mathbf{a}_\forall)$ and $(\mathbf{a}_\exists)$ are also included). Let $\mathcal{M}_{C^*_1}$ be the Nmatrix which extends $\mathcal{M}_{C_1}$ from Corollary 120 with the following interpretations of quantifiers:

$$\widetilde{\forall}(H) = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \\ \mathcal{D} & \text{if } H \subseteq \mathcal{D} \text{ and } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\widetilde{\exists}(H) = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \cup \mathcal{F} \text{ and } H \cap \mathcal{T} \neq \emptyset \\ \mathcal{D} & \text{if } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Then $\Gamma \vdash^v_{\mathcal{QM}_{C^*_1},\forall\exists} \psi$ iff $\Gamma \vdash_{C^*_1} \psi$.

11 CANONICAL DEDUCTION SYSTEMS AND NMATRICES WITH MORE GENERAL QUANTIFIERS

The main goal of this section is to extend the notion of coherent canonical calculi from the propositional case to the level of multi-ary quantifiers. After that we briefly summarize the main related results, omitting the (quite complicated) technical details, which can be found in [Avron and Zamansky, 2008b].

$\mathbf{H}$	$\tilde{\overline{\forall}}(\mathbf{H})$	$\tilde{\overline{\exists}}(\mathbf{H})$	$\tilde{\mathcal{Q}}_2(\mathbf{H})$
$\{\langle\mathbf{t},\mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{t},\mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle\mathbf{f},\mathbf{f}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{f},\mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle\mathbf{t},\mathbf{t}\rangle,\langle\mathbf{t},\mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{t},\mathbf{t}\rangle,\langle\mathbf{f},\mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{t},\mathbf{t}\rangle,\langle\mathbf{f},\mathbf{f}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{f},\mathbf{t}\rangle,\langle\mathbf{t},\mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle\mathbf{f},\mathbf{t}\rangle,\langle\mathbf{f},\mathbf{f}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle\mathbf{t},\mathbf{f}\rangle,\langle\mathbf{f},\mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle\mathbf{t},\mathbf{t}\rangle,\langle\mathbf{t},\mathbf{f}\rangle,\langle\mathbf{f},\mathbf{t}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\{\langle\mathbf{t},\mathbf{t}\rangle,\langle\mathbf{f},\mathbf{f}\rangle,\langle\mathbf{f},\mathbf{t}\rangle\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{f},\mathbf{t}\rangle,\langle\mathbf{t},\mathbf{f}\rangle,\langle\mathbf{f},\mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\{\langle\mathbf{f},\mathbf{f}\rangle,\langle\mathbf{t},\mathbf{f}\rangle,\langle\mathbf{f},\mathbf{t}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t},\mathbf{f}\}$
$\{\langle\mathbf{t},\mathbf{t}\rangle,\langle\mathbf{t},\mathbf{f}\rangle,\langle\mathbf{f},\mathbf{t}\rangle,\langle\mathbf{f},\mathbf{f}\rangle\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

Figure 5. The interpretation of the quantifiers in Example 170

Henceforth L is a language with multi-ary quantifiers[28]. Recall that the interpretation of a unary quantifier $\mathcal{Q}_1$ in an Nmatrix $\mathcal{M} = \langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$ for L is a function $\tilde{\mathcal{Q}}_1 : 2^{\mathcal{V}} \setminus \{\emptyset\} \to \mathcal{V}$. Similarly, an n-ary quantifier will be interpreted by a function $\tilde{\mathcal{Q}}_n : 2^{\mathcal{V}^n} \setminus \{\emptyset\} \to \mathcal{V}$. Thus the following is an extension of Definition 133 to the level of multi-ary quantifiers:

DEFINITION 169. An Nmatrix for L is a tuple $\mathcal{M} = \langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth-values,
- $\mathcal{D}$ is a non-empty proper set of $\mathcal{V}$,
- $\mathcal{O}$ includes a function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective, and a function $\tilde{\mathcal{Q}} : 2^{\mathcal{V}^n} \setminus \{\emptyset\} \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary quantifier.

EXAMPLE 170. Consider the Nmatrix $\mathcal{M} = \langle\{\mathbf{t},\mathbf{f}\},\{\mathbf{t}\},\mathcal{O}\rangle$ for a language with the standard bounded universal and existential (binary) quantifiers $\overline{\forall}$ and $\overline{\exists}$ used in syllogistic reasoning (see footnote 23). In addition, the language contains a binary quantifier $\mathcal{Q}_2$. The interpretations of the quantifiers in $\mathcal{M}$ are given in Figure 5.

The congruence relations $\sim^S$ and $\sim^S_{\forall\exists}$ (Definitions 139 and 149) are naturally extended to languages with multi-ary quantifiers. All is needed is to

[28]For simplicity of presentation, we assume that the language L does not include any propositional connectives. The latter can anyway be thought of as multi-ary quantifiers which bind no variables.

modify the third condition of the second part of Definition 135 as follows:

$$v(\mathcal{Q}x(\psi_1, ..., \psi_n)) \in \tilde{\mathcal{Q}}_{\mathcal{M}}(\langle\{v(\psi_1\{\overline{a}/x\}), ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\})$$

After this modification, Definitions 141 and 142 remain the same. Then to work with signed formulas, we need to extend also the semantic notions from Definitions 35 and 48 to languages with multi-ary quantifiers. This is done by replacing "$\mathcal{M}$-legal valuation v" by "a structure S and an S-valuation v which is $\sim^S$-legal in $\mathcal{M}$", and $\vdash^d_{\mathcal{M}}$ by $\vdash^t_{\mathcal{M}}$. We then have the following counterpart of Proposition 49:

PROPOSITION 171. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ be an Nmatrix for L. Then:*

$$\Gamma \vdash^t_{\mathcal{M}} \Delta \textit{ iff } \{\mathcal{D} : \psi \mid \psi \in \Gamma\} \cup \{\mathcal{F} : \psi \mid \psi \in \Delta\} \vdash^t_{\mathcal{M}} \emptyset \textit{ iff } \vdash^t_{\mathcal{M}} \mathcal{F} : \Gamma \cup \mathcal{D} : \Delta$$

In order to represent canonical rules with multi-ary quantifiers, we shall use a simplified language which abstracts over the internal structure of L-formulas. For a single canonical rule introducing some n-ary quantifier, this representation language includes the unary predicate symbols $p_1, ..., p_n$ and some finite sets of variables and constants: a constant signifies the case of a term variable, while a variable signifies an eigenvariable.

DEFINITION 172. For $n \geq 1$ and a set of constants Con, $QL_n(Con)$ is the first-order language with n unary predicate symbols $p_1, ..., p_n$ and the set of constants Con ($QL_n(Con)$ contains no quantifiers or logical connectives).

CONVENTION 173. In case the set Con is clear from context, we will write QL_n instead of $QL_n(Con)$.

DEFINITION 174. A *signed canonical quantifier rule of arity n* over a finite set of signs $\mathcal{V}$ is an expression of the form $[\Theta/\ S : \mathcal{Q}]$, where $\mathcal{Q}$ is an n-ary quantifier, $S \subseteq \mathcal{V}$, and $\Theta = \{\Sigma_1, ..., \Sigma_m\}$, where for all $1 \leq j \leq m$, Σ_j is a clause over QL_n (i.e. it consists of signed formulas of the form $s : p_i(x)$ or $s : p_i(c)$, where $s \in \mathcal{V}$ and $1 \leq i \leq n$).

EXAMPLE 175. Using the notation in Remark 34, applications of the standard Gentzen-type introduction rules for $\forall$ have the following forms:

$$\frac{\Omega, t : \psi\{z/w\}}{\Omega, t : \forall w\psi} \qquad \frac{\Omega, f : \psi\{\mathbf{t}/w\}}{\Omega, f : \forall w\psi}$$

where z and $\mathbf{t}$ are free for w in ψ and z does not occur free in $\Omega \cup \{\forall w\psi\}$. The canonical representation of these rules will be:

$$[\{\{t : p_1(x)\}\}/\ \{t\} : \forall] \qquad [\{\{f : p_1(c_1)\}\}/\{f\} : \forall]$$

This shows that for instantiating a canonical rule we need a context and some notion of a mapping from the terms and formulas of QL_n to the terms and formulas of L, which handles with care the choice of terms and variables of L, so that they satisfy the appropriate conditions.

DEFINITION 176. For a canonical rule $R = [\Theta/\ S : \mathcal{Q}]$ and a sequent Ω over L, an $\langle R, \Omega, z\rangle$-*mapping* is any function χ from the predicate symbols, terms and formulas of QL_n to formulas and terms of L, satisfying the following conditions:

- For every $1 \leq i \leq n$, $\chi(p_i)$ is an L-formula.
- $\chi(y)$ is a variable of L.
- $\chi(x) \neq \chi(y)$ for every two variables $x \neq y$ of QL_n.
- $\chi(c)$ is an L-term, such that $\chi(x)$ does not occur in $\chi(c)$ for any variable x occurring in Θ.
- For every $1 \leq i \leq n$, if $p_i(\mathbf{t})$ occurs in Θ, $\chi(\mathbf{t})$ is a term free for z in $\chi(p_i)$, and if $\mathbf{t}$ is a variable, then $\chi(\mathbf{t})$ does not occur free in $\Omega \cup \{\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))\}$.
- $\chi(p_i(\mathbf{t})) = \chi(p_i)\{\chi(\mathbf{t})/z\}$.

χ is extended to sequents as follows: $\chi(\Sigma) = \{a : \chi(\psi) \mid a : \psi \in \Sigma\}$.

DEFINITION 177. Let $\mathcal{Q}$ be an n-ary quantifier. An *application of a canonical quantifier rule* $R = [\{\Sigma_1, ..., \Sigma_m\}/\ S : \mathcal{Q}]$ is any inference step of the form:

$$\frac{\Omega \cup \chi(\Sigma_1) \quad \dots \quad \Omega \cup \chi(\Sigma_m)}{\Omega \cup S : \mathcal{Q}x(\chi(p_1), \dots, \chi(p_n))}$$

where Ω is a sequent and χ is some $\langle R, \Omega, x\rangle$-mapping.

EXAMPLE 178. The introduction rules for the bounded universal binary quantifier $\overline{\forall}$ over $\mathcal{V} = \langle t, \top, f, \bot\rangle$ can be formulated as follows (taking t and $\top$ as the designated truth-values, this is a natural generalization of its classical interpretation):

$$[\ \{\ \{f : p_1(x), \bot : p_1(x), t : p_2(x), \top : p_2(x)\}\ \}\ /\ \{t, \top\} : \overline{\forall}\ ]$$

$$[\ \{\ \{t : p_1(c_1), \top : p_1(c_1)\},\ \{f : p_2(c_1), \bot : p_2(c_1)\}\ \}\ /\ \{f, \bot\} : \overline{\forall}\ ]$$

Their applications have the forms:

$$\frac{\Omega \cup \{f : \psi_1\{y/z\}, \bot : \psi_1\{y/z\}, t : \psi_2\{y/z\}, \top : \psi_2\{y/z\}\}}{\Omega \cup \{t : \overline{\forall}z(\psi_1, \psi_2), \top : \overline{\forall}z(\psi_1, \psi_2)\}}$$

$$\frac{\Omega \cup \{t : \psi_1\{\mathbf{t}/z\}, \top : \psi_1\{\mathbf{t}/z\}\} \quad \Omega \cup \{f : \psi_2\{\mathbf{t}/z\}, \bot : \psi_2\{\mathbf{t}/z\}\}}{\Omega \cup \{f : \overline{\forall}z(\psi_1, \psi_2), \bot : \overline{\forall}z(\psi_1, \psi_2)\}}$$

On the level of quantifiers two new elements are added to canonical calculi: the axiom of α-equivalence and the rule of substitution.

DEFINITION 179. Let $\mathcal{V} = \{l_1, ..., l_n\}$ be a finite set of signs.

1. A *logical axiom*[29] for $\mathcal{V}$ is any sequent $\{l_1 : \psi_1, l_2 : \psi_2 ..., l_n : \psi_n\}$, where $\psi_1 \equiv_\alpha \psi_2 ... \equiv_\alpha \psi_n$.
2. The substitution rule for $\mathcal{V}$ is defined as follows:

$$\frac{\Omega}{\Omega'}\ \textbf{Sub}$$

where Ω' is obtained from Ω by legal substitutions of terms for free variables.

The following proposition follows from the completeness of many-valued resolution ([Baaz et. al., 1995]):

PROPOSITION 180. *Let Θ be a set of clauses. The empty sequent can be derived from Θ using cuts and substitutions iff Θ is not satisfiable.*

DEFINITION 181. We say that a signed calculus over $\mathcal{V}$ is *canonical* if it consists of: (i) All logical axioms for $\mathcal{V}$, (ii) The rules of cut, weakening and substitution, and (iii) A finite number of signed canonical quantifier rules.

Next we extend the propositional criterion of coherence to canonical calculi with multi-ary quantifiers.

DEFINITION 182. For sets of clauses $\Theta_1, ..., \Theta_m$, $\mathsf{Rnm}(\Theta_1 \cup ... \cup \Theta_m)$ is a set $\Theta'_1 \cup ... \cup \Theta'_m$, such that for all $1 \leq i \leq n$, Θ'_i is obtained from Θ_i by renaming the constants and variables which occur in Θ_i, and no constant or variable occur in both Θ'_i and Θ'_j in case $i \neq j$.

DEFINITION 183. A canonical calculus G is *coherent* if $\mathsf{Rnm}(\Theta_1 \cup ... \cup \Theta_m)$ is unsatisfiable whenever $[\Theta_1 / \ S_1 : \mathcal{Q}], ..., [\Theta_m / \ S_m : \mathcal{Q}]$ is a set of rules of G, such that $S_1 \cap ... \cap S_m = \emptyset$.

Note that by Proposition 180, the above definition of coherence can be translated into a purely syntactic one.

PROPOSITION 184. *The coherence of a canonical calculus is decidable.*

EXAMPLE 185. Consider a canonical calculus over $\mathcal{V} = \{t, \top, f\}$ with the following rules for a unary quantifier $\mathcal{Q}$:

$$R_1 \ = \ [\ \{\ \{t, \top\} : p_1(x)\ \}\ /\ \{t, \top\} : \mathcal{Q}\ ]$$

$$R_2 \ = \ [\ \{\ \{\top, f\} : p_1(y)\ \}\ /\ \{\top, f\} : \mathcal{Q}\ ]$$

$$R_3 \ = \ [\ \{\ \{t, f\} : p_1(c_1)\ \}\ /\ \{t, f\} : \mathcal{Q}\ ]$$

[29]This is an extension of the α-axiom from [Zamansky and Avron, 2006c]

Since $\{t, \top\} \cap \{\top, f\} \cap \{t, f\} = \emptyset$, we need to check whether the empty sequent is derivable (using cuts and substitutions) from the set of premises of these rules:

$$\dfrac{\dfrac{\{t,f\} : p_1(c_1) \qquad \dfrac{\{t,\top\} : p_1(x)}{\{t,\top\} : p_1(c_1)}\,\textbf{Sub}}{\{t\} : p_1(c_1)}\,\textbf{Cut} \qquad \dfrac{\{\top,f\} : p_1(y)}{\{\top,f\} : p_1(c_1)}\,\textbf{Sub}}{\emptyset}\,\textbf{Cut}$$

Thus this calculus is coherent (note that each pair of premises is consistent, but the three of them together are not).

Below we briefly review the main results related to the connection between canonical calculi and finite Nmatrices. What follows is an extension of the results for the propositional case from Sections 4.1 and 4.2.

The notions of standard, analytic and strong cut-elimination from Definition 46 can be naturally extended to calculi with multi-ary quantifiers.

THEOREM 186. *A coherent canonical calculus which admits strong cut-elimination can be constructed for every finite Nmatrix.*

As a corollary, we have the following extension of Theorem 53:

COROLLARY 187. **(Compactness)** *Let Θ be a set of sequents and Ω a sequent.*

1. *If $\Theta \vdash^t_{\mathcal{M}} \Omega$, then there is some finite $\Theta' \subseteq \Theta$, such that $\Theta' \vdash^t_{\mathcal{M}} \Omega$.*

2. *Let Γ, Δ be two sets of L-formulas. If $\Gamma \vdash^t_{\mathcal{M}} \Delta$, then there are some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, such that $\Gamma' \vdash^t_{\mathcal{M}} \Delta'$.*

In the converse direction, every coherent calculus has a corresponding finite Nmatrix. Moreover, there is a direct correspondence[30] between analytic cut-elimination, coherence and finite Nmatrices:

THEOREM 188. *Let G be a canonical calculus for a language L and a finite set of signs $\mathcal{V}$. The following statements concerning G are equivalent:*

1. *G is coherent.*

2. *G has a strongly characteristic finite Nmatrix.*

3. *G admits strong analytic cut-elimination.*

[30]Cut-elimination for a general family of sequent calculi with generalized quantifiers (of which the canonical calculi are specific instances) is investigated in [Ciabattoni and Terui, 2006b] (extending [Ciabattoni and Terui, 2006a], see Remark 45). Their reductivity condition can again be shown to be equivalent to coherence.

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