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NON-EQUILIBRIUM PROCESSES IN THE EARLY UNIVERSE

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SUMMARY

In a recent paper Misner (1968) has suggested that neutrino viscosity is a highly efficient process for removing shear anisotropy during the early stages of the universe, so that the remarkable degree of isotropy observed in the microwave background would be expected to occur whatever the initial conditions. In this paper we suggest that the early universe may not have been in near thermal equilibrium as Misner assumed, so that under a wide range of initial conditions, arbitrarily large anisotropy could occur at any epoch. Therefore, the observed anisotropy does place some restrictions on early conditions in the universe.

1. INTRODUCTION

In the fifty years or so since its conception, relativistic cosmology has mainly concerned itself with trying to describe the universe in the large from the local observations made by astronomers. Recently however, Misner (1968) has suggested that in cosmological applications, Einstein's field equations should be considered as a predictive set which govern the evolution of the universe after the initial conditions have been specified. However one is not sure what the initial conditions are, and so Misner's programme consists of trying to show that whatever the starting conditions, certain physical processes occur which cause the universe to assume its present roughly homogeneous isotropic form. Our viewpoint in this paper is slightly different. We consider alternative states of the universe now and see whether there exist initial conditions which lead to them, taking into account some of the physical processes which might have occurred. If such initial conditions are possible, then we would also like to be sure that they do not form a set of measure zero within the class of possible initial conditions.

The object of the paper is to investigate the effectiveness of dissipative processes at removing shear anisotropy during the early stages of the universe. If we adopt the conventional view of a 'hot big bang' universe, then for each species of particle in the primeval plasma we would expect that the ratio $t_{\text{coll}}/t_{\text{exp}}$ of the mean collision time to the mean expansion time scale should tend to zero as the temperature tends to infinity. (Here and throughout this paper, t_{coll} is a harmonic mean of the collision times for the various interactions in which the species takes part.) If this occurred then we would say that the particles were in thermal equilibrium. If all the particles were in thermal equilibrium all of the time then the standard homogeneous models (e.g. Stewart & Ellis 1968; Ellis & MacCallum 1968; Thorne 1967) would give a good description of the early universe. (We are assuming for simplicity that the universe is spatially homogeneous on a large scale.)

However, certain particles which probably were in equilibrium during the early universe, e.g. neutrinos, gravitons (if they exist) now have extremely long collision times, $t_{coll}/t_{exp} \gg 1$, and so they would have passed through a phase when $t_{coll}/t_{exp} \sim 1$. During this phase the collision term in the relativistic Boltzmann equation (see e.g. Sachs 1968), is neither very large nor very small, and so one would expect dissipative processes to be important. Provided a fluid description is still adequate (which requires $t_{coll}/t_{exp} < 1$) one could characterize these processes by a bulk viscosity and a shear viscosity. The first of these is being considered elsewhere (Stewart & Anderson, to be published), but the second has recently received some attention from several authors. It should be made plain that this viscosity which tends to dissipate shear, occurs in a homogeneous fluid and is, therefore, a relativistic rather than a Newtonian phenomenon.

Its importance for cosmology was pointed out by Misner (1967a, b, 1968) who has considered the increasing ineffectiveness of e-neutrino–electron scattering in maintaining a collision dominated distribution of neutrinos as the temperature drops below about 2×10^{10} °K. Misner concluded that this process produced a remarkably effective viscous effect and that no large initial anisotropy could survive this epoch. However Stewart (1968), Doroshkevich, Zel'dovich & Novikov (1968, and private communication) have independently pointed out that while Misner's conclusion might be correct, his procedure is questionable since the viscosity only becomes large when $t_{coll}/t_{exp} > 1$, but a fluid description is then inadequate. A proper description requires the use of the relativistic Boltzmann equation with a collision term, and models incorporating a simple collision term (essentially equivalent) have been constructed by Stewart and by Doroshkevich (*op. cit.*). They suggest that viscosity is not so effective as was previously thought, but their conclusions are open to criticism*, because the collision term was chosen for mathematical convenience rather than physical plausibility.

Because a proper kinetic theory treatment is so difficult, we have adopted a more phenomenological approach to describe the critical period. In Section 2 we make the assumption that the shear stresses produced during any epoch never become so large that the total pressure in any direction becomes negative. We do not enquire too deeply into the exact physics of the process, but it is difficult to see how this assumption would become invalid for a short range scattering process. After setting up the basic equations we then prove the theorem that for such a short range process occurring in a homogeneous non-rotating universe, the space-like hypersurfaces of which have negative or zero curvature, the heating rate due to the shear stresses can never exceed one half of the adiabatic cooling rate. We use this extreme case to describe the particles when $t_{coll}/t_{exp} > 1$, instead of a kinetic theory approach. Of course the equations will look formally similar to those for a fluid, but the components of the energy–momentum tensor will have to be defined in terms of a distribution function (see e.g. Sachs 1968 for details). Our approach is more general than that of imposing an *ad hoc* collision term in the Boltzmann equation, because we should require of such a term that it reduced to the viscosity approximation when collisions were frequent, and did not produce negative pressures when collisions were rare, and our approach gives limits to the dissipation produced under such conditions.

* I thank Stephen Hawking and Richard Matzner for their criticisms of this simple model.

In the final section we reformulate the basic equations in such a way that they apply from soon after the singularity up until comparatively recent times, assuming that the only physically relevant processes are of the type described above. We consider a (hypothetical) model universe in which the shear anisotropy is extremely large at some finite time and temperature. We then integrate the equations backwards in time to as near to the singularity as we please, using either the viscosity approximation described above or the positive pressure criterion, ending with a possibly even larger anisotropy. If instead we had started from near the singularity with an anisotropy greater than or equal to our final calculated value, and then integrated the equations forward in time, the anisotropy at our final temperature would not have been less than our chosen value. Further, the set of such solutions does not form a set of measure zero, so that unless there is some reason why the anisotropy could not have been arbitrarily large at some initial time, the reason why the universe is so isotropic now remains a mystery.

The reasonableness of this method depends upon the physical details of the scattering process. A full discussion is given covering any temperature dependence for the scattering cross-section. There is no real conflict with the result of Misner for neutrino viscosity. His solution is shown to be consistent with the theory presented here, but it is not the only one.

2. THE KINEMATICS OF VISCOUS FLUIDS

In this section we consider a homogeneous non-rotating universe, the space-like hypersurfaces of which are flat unless stated otherwise. We introduce the fundamental form

$$ds^2 = g_{ab}\, dx^a\, dx^b = -dt^2 + e^{2\alpha}(e^{2\beta})_{ij}\, dx^i\, dx^j. \tag{1}$$

Here $a, b, c, \ldots$ range over 0 to 3, $i, j, k, \ldots$ range over 1 to 3, $x^0 = t$, '$\cdot$' $= d/dt$ and the units are chosen so that $c = 8\pi G = 1$. In equation (1) β_{ij} is a tracefree matrix representing the anisotropy, while e^α is the mean expansion scale length. We define the energy–momentum tensor as

$$T^{ab} = \int F(x, p)p^a p^b\, dP,$$

where $F(x, p)$ is the one particle distribution function in an eight dimensional phase space which satisfies Boltzmann's equation with a collision term, and dP is a coordinate invariant volume element on momentum space. Because there is no rotation, a coordinate system can be chosen so that g_{ab} and T_{ab} are diagonal, and T^{ab} can then be written in an invariant way as

$$T^{ab} = \operatorname{diag}(\mu, p, p, p) + \pi^{ab}. \tag{2}$$

Here μ is the energy density as measured by a comoving observer, p is the scalar pressure, and π^{ab} is a diagonal tracefree matrix representing the shearing stresses,

$$\pi_{ab} = \operatorname{diag}(0, \pi_1, \pi_2, \pi_3). \tag{3}$$

As an equation of state for p we shall adopt

$$p = \tfrac{1}{3}\mu,$$

corresponding to an ultrarelativistic gas. Note that we have made no assumption whatever about the gas being in thermal equilibrium. In a gas at very high temperatures, the rest mass of the particles is unimportant and so the trace of T^{ab} vanishes, which implies the above equation for p. The equation is exact for photons and neutrinos, and is a good approximation for electrons at temperatures above 10^{10} °K.

The basic equations we shall need are,

(i) a first integral of the Einstein field equations,

$$\mu = 3\dot{\alpha}^2 - \tfrac{1}{2}\sigma^2, \text{ where } \sigma^2 = \sigma_{ij}\sigma^{ij}, \ \sigma_{ij} = \dot{\beta}^{ij}. \tag{4}$$

(ii) five of the field equations,

$$\dot{\sigma}_{ij} + 3\dot{\alpha}\sigma_{ij} = \pi_{ij}, \tag{5}$$

and

(iii) the conservation equation $T^{0b}{}_{;b} = 0$,

$$\dot{\mu} + 4\mu\dot{\alpha} + \pi_{ij}\sigma^{ij} = 0. \tag{6}$$

The field equations do not form a closed system without the addition of an equation of state for the pressure (given above), and for the shearing stresses π_{ij}. The simplest approximation to the latter is the viscous one given by Misner (1967a, b, 1968),

$$\begin{aligned} \pi_{ij} &= -2\eta\sigma_{ij} \\ \eta &= \tfrac{1}{3}\mu t_{\text{coll}}. \end{aligned} \tag{7}$$

Here t_{coll} is the mean collision time as defined in the introduction. We have $t_{\text{coll}} = (\sqrt{2}\, n\Sigma)^{-1}$ where n is the number density proportional to T^3, and Σ is the mean scattering cross-section proportional to T^{λ} as a first approximation. λ is a slowly varying function of temperature. Here T is *not* the adiabatic temperature proportional to e^{α} because dissipative heating occurs. Instead we define $T \propto \mu^{1/4}$ so that T is a 'bolometric temperature'. The T^3 dependence for n follows from the equation of state, rather than from particle conservation arguments which involve the adiabatic temperature.

The form for the viscosity coefficient in equation (7) was suggested by Misner (1968) and is proportional to t_{coll} as one might expect from the classical case. We present here a heuristic argument to show why a viscous term of this form should occur in the energy equation (6). This point should be considered because the viscosity has no exact Newtonian analogue since it occurs in a homogeneous universe*. Imagine a mass of gas in a cylinder being uniformly and adiabatically compressed by a piston. The internal energy of the gas is being increased as a consequence of the work done at the boundary by the external forces, and the effect of collisions is to distribute the instantaneous total internal energy over the available modes of particle motion as described by the collision dominated Boltzmann equation. The first direct effect of the piston is to cause an increase in the translational energy in the direction of motion of the piston. Collisions then spread some of the excess energy into the other available modes. We would expect to obtain equipartition of energy between the three translational modes after a few collisions, and so for a collision dominated gas, isotropy of pressure is achieved almost instantaneously. If the length of the column of gas is decreasing with a steady negative logarithmic rate of extension $-\sigma$, the maintained difference

* This point seems to have been first noticed by G. Ellis (private communication).

between the pressure in the longitudinal and transverse directions over a time scale smaller than t_{coll} will be of the order of magnitude of the difference which movement of the piston would produce in the absence of collisions. In a time t_{coll} the length of the piston decreases by a fraction $-\sigma t_{coll}$ and by considering the work done per unit volume against the gas pressure, we see that the excess pressure π is of order

$$\pi \sim -p\sigma t_{coll}, \qquad \text{assuming } \sigma t_{coll} \ll 1.$$

Then the rate at which work is done by the gas per unit volume is $-\pi\sigma$. If we now superimpose this effect upon an isotropic expansion and set $p = \frac{1}{3}\mu$ we obtain equations (6), (7) because this argument obviously generalizes to three dimensions, and does not depend on the existence of the walls of the cylinder. The explanation is satisfactory while $t_{coll}/t_{exp} \lesssim 1$. When the collision time is longer than the expansion time more sophisticated arguments are desirable. However, we can extend the heuristic argument in the following way. (For the sake of brevity we consider only the 'cigar' mode of expansion.) Suppose our cylinder is made of rubber, and as the piston is depressed the cross-sectional area expands so as to maintain a constant volume. As the length of the cylinder decreases, collisions between particles travelling mainly in the direction of collapse, which are still relatively frequent, tend to distribute the excess energy among the other two modes, but since the transverse expansion tends to decrease the translational energy in these directions, there are relatively few collisions between particles moving in the transverse modes. As an extreme case we consider what happens when the collision time for particles moving in the transverse modes becomes extremely long so that the total stresses in these directions tend to zero. It then follows from the definition and tracefree properties of π_{ij} that we may take

$$\pi_{ij} = \text{diag}\,(2p, -p, -p), \tag{8}$$

where p is the scalar pressure and the x^1 axis is along the axis of the cylinder. This extreme case turns out to be the one in which the energy production is a maximum, as is shown by the following theorem.

Theorem 1

Consider a spatially homogeneous non-rotating universe, in which the spacelike hypersurfaces of homogeneity orthogonal to the fluid flow vector have negative or zero curvature. If a non-equilibrium process is occurring in such a way that the total stress in any direction is non-negative, and the isotropic stress is equal to one third of the total energy density, then the rate of work (heating) done by the tangential stresses is never greater than one half of the rate of adiabatic cooling (i.e. cooling due to the increase in the total volume).

Remark: In general in this paper we shall only consider space-times with flat spacelike hypersurfaces. However, this theorem is true even if the hypersurfaces have negative scalar curvature. It, therefore, holds in all spatially homogeneous non-rotating cosmologies in which the homogeneous surfaces are orthogonal to the fluid flow, except for a subset of Bianchi type IX (see e.g. Ellis & MacCallum 1969). The special case of positive curvature is being considered by Misner (private communication).

Proof. Instead of equation (4) we use the more general first integral,

$$\mu - \tfrac{1}{2}R^* = 3\dot{\alpha}^2 - \tfrac{1}{2}\sigma^2, \tag{9}$$

where the scalar curvature is $R^* \leqslant 0$. We also need equation (6),

$$\dot{\mu} + 4\dot{\alpha}\mu = -\pi_{ij}\sigma^{ij}. \tag{6}$$

In equation (6), $4\dot{\alpha}\mu$ is the rate of adiabatic cooling and $-\pi_{ij}\sigma^{ij}$ is the rate of heating due to the tangential stresses. It is a simple but tedious exercise (see appendix) to maximize $\pi_{ij}\sigma^{ij}$ subject to

(i) $\pi_i{}^i = \sigma_i{}^i = 0$;
(ii) π_{ij} is diagonal (this can always be achieved by the right choice of co-ordinates);
(iii) $\sigma^2 \leqslant 6\dot{\alpha}^2$, which follows from equation (9), and
(iv) $\pi_i{}^i \geqslant -\tfrac{1}{3}\mu$ (no summation convention), which is a necessary (but not sufficient) condition for positive total pressure in all directions.

The result is

$$-2\mu|\dot{\alpha}| \leqslant \pi_{ij}\sigma^{ij} \leqslant 2\mu|\dot{\alpha}|,$$

which proves the theorem. It is evident from the lemma in the appendix that the case of maximum heating or cooling occurs for the ' cigar ' mode described above, and that by suitable choice of axes the stress and strain tensors for the extreme case may be written,

$$\pi_{ij} = \mathrm{diag}\,(2p, -p, -p), \qquad \sigma_{ij} = \mathrm{diag}\,(\pm 2\dot{\alpha}, \mp\dot{\alpha}, \mp\dot{\alpha}),$$

and occur when $\mu/\dot{\alpha}^2$ or $\mu/\sigma^2 \ll 1$. It therefore follows that if this condition is satisfied we can describe the extreme case by a viscosity approximation with $t_{\text{coll}} = \pm\tfrac{1}{2}\dot{\alpha}^{-1}$. However in such a case, a fluid interpretation of the coefficients in the energy–momentum tensor is somewhat misleading, and the more general kinetic theory formalism must be used.

3. VISCOSITY IN COSMOLOGICAL MODELS

In this section we shall assume $R^* = 0$. It is convenient to redefine the equations in terms of new variables;

(i) $x = \mu/3\dot{\alpha}^2$. From equation (4) x is a measure of the ratio of anisotropy energy $\tfrac{1}{2}\sigma^2$ to the radiation density μ. Explicitly $\tfrac{1}{2}\sigma^2/\mu = (1-x)/x$, and so $x = 1$ corresponds to isotropy, $x \to 0$ corresponds to extreme anisotropy, and x must lie within these limits,
(ii) $y = \dot{\alpha}t_{\text{coll}}$. y is $t_{\text{coll}}/t_{\text{exp}}$ and for an expanding universe has to be positive and finite to be physically meaningful. Values of $y > \tfrac{1}{2}$ have to be treated with care as will be seen later, and
(iii) $z = \dot{\alpha}^{-2}$.

Then equations (4)–(7) can be written

$$dx/d\alpha = 2x(1-x)(1+2y), \tag{10}$$

$$dy/d\alpha = (\lambda+x)y - (1-x)(\lambda+3)y^2, \tag{11}$$

$$dz/d\alpha = 2(3-x)z, \tag{12}$$

where λ is given by $\Sigma \propto T^{\lambda}$, Σ the effective cross-section. The equation we originally used to define t_{coll} can now be used to obtain the temperature T. For

$$T^{4-2(3+\lambda)} \propto \tfrac{1}{3}\mu t_{\text{coll}}^2 = (\mu/3\dot{\alpha}^2)(\dot{\alpha} t_{\text{coll}})^2$$

or

$$xy^2 \propto T^{-2(1+\lambda)}. \tag{13}$$

Misner has calculated that for neutrino–electron scattering, $x \simeq 1$, $y = \frac{1}{2}$ corresponds to about 10^{10} °K and so

$$xy^2 \simeq T_{10}^{-2(1+\lambda)}, \qquad \text{where } T_{10} = T/10^{10}\ \text{°K}. \tag{14}$$

Equations (10)–(12) are not directly applicable to cosmological models when $y > \frac{1}{2}$ since the usual viscosity approximation has broken down. For $y \leqslant \frac{1}{2}$ it is possible to integrate the equations directly, but for values of y greater than $\frac{1}{2}$, a new equation of state for π replacing equation (7) is required. In this paper we are not seeking an exact equation of state, and the positive pressure criterion can be used instead to provide bounds on the possible values of π_{ij}. As was seen in the last section, we may continue to use the viscous formalism given by equation (7), provided we restrict the magnitude of y to be less than or equal to $\frac{1}{2}$. We therefore consider the two extreme cases where the effective values of y on the right-hand sides of equations (10)–(12) are given by $y^* = \min(\frac{1}{2}, y)$ and $y^* = \max(-\frac{1}{2}, y)$, respectively. The new equations are:

$$dx/d\alpha = 2x(1-x)(1+2y^*) \tag{15}$$

$$dy/d\alpha = (\lambda+x)y - (1-x)(\lambda+3)yy^* \tag{16}$$

$$dz/d\alpha = 2(3-x)z \tag{17}$$

$$xy^2 \simeq T_{10}^{-2(1+\lambda)}. \tag{14}$$

These equations are clearly invariant under a transformation $\alpha \to \alpha + \text{constant}$, corresponding to a change in the length scale factor. At any instant we can choose $\alpha = 0$. Then any two of x, y, z, T can be chosen arbitrarily, and the other two are uniquely determined from the defining relations. We now investigate the asymptotic solutions of this system of equations as we go backwards in time towards the singularity. First there is the solution,

$$x = 1, \qquad y \sim e^{(1+\lambda)\alpha}, \qquad z \sim e^{4\alpha}, \qquad \text{as } \alpha \to -\infty,$$

corresponding to an isotropic Robertson–Walker solution. Writing $\lambda_0 = \lim(\lambda)$ as $\alpha \to -\infty$, we can distinguish two cases. Firstly if $\lambda_0 > -1$, then $y \to 0$ as $\alpha \to -\infty$ and so the universe would have been in collision dominated equilibrium during the early stages. If $\lambda_0 < -1$, collisions would have been extremely rare during the early stages, while for $\lambda = -1$, y is constant and so either possibility may have occurred. However, in this isotropic case (and in no other) the terms involving y^* vanish, and so there is no dynamical difference between these two alternatives. Physically this is obvious, for in an isotropic universe there are no shear stresses, and, therefore, no non-equilibrium effects of the type considered here.

Leaving aside for the moment the special case $y^* = -\frac{1}{2}$, we have

$$x = o(e^{\rho\alpha}), \qquad z = o(e^{6\alpha}) \text{ as } \alpha \to -\infty, \qquad (\rho > 0), \tag{18}$$

The behaviour of y is more complicated and three cases can be distinguished.

If $\lambda_0 > 3$, then $y \to 0$ exponentially as $\alpha \to -\infty$, and so collision dominated equilibrium would have occurred. If $\lambda_0 < -1$, then $y \to \infty$ and so there would have been a gross deviation from collision dominated equilibrium during the early stages of the universe. Since the anisotropy would have been large then, large shear stresses would have developed, and a perfect fluid description is unsuitable for this epoch. If $-1 < \lambda_0 < 3$, then the two extreme cases considered in this section are incapable of distinguishing the actual behaviour of the model. Either possibility (equilibrium or disequilibrium) is a solution, and a more restrictive criterion than the positive pressure one must be adopted if more information is required. Unfortunately this ambiguous one is the one of physical interest according to Bahcall (1964) who finds $\lambda \equiv 2$. However Eden (1965) has suggested that weak interactions probably have an upper limit to their cross-section, and if this is correct it would appear that the assumption that weakly interacting particles were in thermal equilibrium during the early stages of the universe needs more justification. It is quite possible that gross disequilibrium occurred, and that the perfect fluid description is only relevant to recent times.

The special case $y^* = -\frac{1}{2}$ can only occur if $\lambda = -1$, and then x is constant. We now prove our second main result:

Theorem 2

Consider a homogeneous non-rotating cosmological model, the spacelike hypersurfaces of which are flat and orthogonal to the fluid 4-velocity. Suppose a non-equilibrium process is occurring which can either be described by a viscosity approximation, or satisfies the condition that the pressure in all directions is always positive. If the scattering cross-section is given by

$$\Sigma \propto T^{\lambda}, \qquad \lambda(T) \to \lambda_0 \text{ as } T \to \infty,$$

then,

(1) If $\lambda_0 < -1$, such a model could have been in gross thermal disequilibrium during the early stages of the universe.

(2) If $\lambda_0 > 3$, such a model was in thermal equilibrium during the early stages of the universe.

(3) If the universe was in disequilibrium, then the shear anisotropy could be arbitrarily large at any finite temperature T_0, if there existed a higher temperature T_1 such that the processes occurring before T_1 had not prevented the anisotropy from being arbitrarily large then.

(4) If the universe was in equilibrium during the early stages, it is possible for the anisotropy to be arbitrarily large at any finite temperature T_0.

Proof. Parts 1, 2 have already been proved. To prove (3) we note that if the conditions of the theorem are satisfied, then limits on the behaviour of the universe are given by the models described by equations (12), (14)–(16). We choose some arbitrarily small value of x, say $x = x_0$, at $T = T_0$, where we may also take $\alpha = 0$. It is crucial to take x_0 so small that the asymptotic form equation (17) holds. Let α_1 be an arbitrary negative value of α, at which $T = T_1$. In the range $\alpha_1 \leqslant \alpha \leqslant 0$ the right-hand sides of the differential equations (12), (14)–(16) satisfy a Lipschitz condition and in this range of α the equations have a unique solution with $x = x_0$ etc. at $\alpha = 0$. The theory of ordinary differential equations shows that the map (x, y, z) at $T_0 \to (x, y, z)$ at T_1 is both (1, 1) and continuous. Clearly the set of states at T_1 which lead to an anisotropy $< x_0$ at T_0 is not a set of measure zero, and so provided no physical processes have occurred at an earlier epoch to invalidate the hypothesis of this section, there is a finite probability of $x < x_0$ at $T = T_0$.

To prove (4) we note that if $y < \frac{1}{2}$ then the ambiguity in y^* disappears. In equilibrium $y \to 0$ as $\alpha \to -\infty$ and so there exists an α_1 such that $y < \frac{1}{2}$ for $\alpha < \alpha_1$. In the range $-\infty < \alpha < \alpha_1$ the right-hand sides of the differential equations are holomorphic in the complex (α, x, y, z) space. Therefore, there exists a unique solution having arbitrary values of x, T at α_1, which moreover is holomorphic there. Thus in this case the existence and uniqueness theorem obtained in Section 4 can be extended back to arbitrary negative values of α, which is the required result.

This result does not contradict that of Misner who has shown that for $\lambda = 1$, it is possible to have little residual anisotropy now. Misner's argument can be given as follows. One starts from thermal equilibrium at high temperatures with x and y small. Then as T decreases, y increases to a steady value of about $\lambda/(3+\lambda)$ which is maintained until x becomes $\simeq 1$ after which y increases. Thus Misner has shown that for $0 < \lambda < 3$ there exist solutions with small residual anisotropy. Our solutions in this range assume y large now. Because of the ambiguity in the y^* term, $dy/d\alpha$ is a decreasing function, and so y was always large in the past. The two solutions are not irreconcilable. They merely start from different initial conditions.

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APPENDIX

Proof of the Inequality $-2\mu|\dot{\alpha}| \leqslant \pi_{ij}\sigma^{ij} \leqslant 2|\dot{\alpha}|\mu$.

We must find the extremal values of $\pi_{ij}\sigma^{ij}$ subject to the conditions,

(i) $\sigma_i{}^i = \pi_i{}^i = 0$,
(ii) π_{ij} is diagonal,

(iii) $\pi_i{}^i > -\frac{1}{3}\mu$ (no summation convention), which is a necessary but not sufficient condition to ensure positive stresses in all directions, and

(iv) $\sigma^2 \leqslant 6\dot{\alpha}^2$.

Using the tracefree properties of π, σ, we can write $F = \pi_{ij}\sigma^{ij}$ in terms of $\pi_1 = \pi_1{}^1$, $\pi_2 = \pi_2{}^2$, $\sigma_1 = \sigma_1{}^1$, $\sigma^2 = \sigma_2{}^2$ as

$$F = 2\pi_1\sigma_1 + 2\pi_2\sigma_2 + (\pi_1\sigma_2 + \pi_2\sigma_1).$$

Because of (iii) (π_1, π_2) is restricted to lie inside the triangle in the π_1, π_2 plane whose vertices are $(-\frac{1}{3}\mu, -\frac{1}{3}\mu)$, $(-\frac{1}{3}\mu, \frac{2}{3}\mu)$, $(\frac{2}{3}\mu, -\frac{1}{3}\mu)$ and it is easy to see that the extremal values of F occur at any vertex. Taking $(-\frac{1}{3}\mu, -\frac{1}{3}\mu)$ we have

$$F = -\mu(\sigma_1 + \sigma_2).$$

But σ_1, σ_2 are constrained to lie within the ellipse,

$$\sigma_1{}^2 + \sigma_1\sigma_2 + \sigma_2{}^2 = 3\dot{\alpha}^2.$$

Elementary geometry then shows that $|\sigma_1 + \sigma_2| \leqslant 2|\dot{\alpha}|$ which proves the required result. It should be noted that this case of maximum dissipation occurs for an axisymmetric mode of expansion corresponding to a 'cigar'. For a 'pancake' mode we again have $\pi_1 = \pi_2$ but since both are positive, they are each equal to $\mu/6$ and the dissipation is only one half as great.