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NON-EXISTENCE OF POSITIVE SOLUTIONS OF LANE-EMDEN SYSTEMS

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1. Introduction. In this paper we consider (component-wise) positive solutions of the weakly coupled system

$$\begin{aligned} \Delta u + v^p &= 0, \\ \Delta v + u^q &= 0, \end{aligned} \qquad x \in \mathbb{R}^n, \tag{I}$$

where $p, q > 0$ and $n \geq 3$ is the dimension of the space, and are concerned with the question of non-existence of such solutions. This system arises in chemical, biological and physical studies, and has been investigated by several authors, see for example [3, 8, 9] and references therein.

The system (I) is a natural extension of the celebrated Lane-Emden equation, and we thus refer to it as the Lane–Emden system. The Lane-Emden equation

$$\Delta u + u^p = 0, \qquad x \in \mathbb{R}^n,\ n > 2,\ p > 1 \tag{II}$$

has been extensively studied, going back to the pioneering work of Fowler (cf. [4], and the recent paper [12]). It is well-known that the Sobolev exponent

$$l = \frac{n+2}{n-2}$$

serves as the dividing number for existence and non-existence of solutions of (II), that is, equation (II) admits non-negative, non-trivial solutions if and only if $p \geq l$, see [4] and [5].

It is natural to ask if there exists a corresponding *dividing curve* in the $pq-$plane for the Lane-Emden system, that is, a curve with the property that (I) admits positive solutions if and only if (p, q) is on or above the curve.

Mitidieri [9] showed that (I) does not have any positive *radial* solutions if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \qquad p,\ q > 1,$$

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a result which was improved in [13] to be valid for $p, q > 0$. On the other hand, a concentrated - compactness argument (cf. [7]) implies that (I) admits a positive radial solution when (p, q) is on the hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}. \tag{1.1}$$

This suggests that the hyperbola (1.1) is the dividing curve for existence and nonexistence for (I). The purpose of the paper is to justify this assertion in a number of important cases, see Corollaries 1.1 and 1.2. We also note that condition (1.1) becomes exactly $p = l$ when $p = q$, as for the Lane–Emden equation.

We call a point (p, q) in the first quadrant of the $pq-$plane *critical, subcritical* or *supercritical* if it is respectively on, below or above the hyperbola (1.1). Also we say that a function w on $\mathbb{R}^n$ has algebraic growth at infinity if there exists a constant K such that

$$|w(x)| \leq \text{Const. } |x|^K, \qquad |x| > 1. \tag{1.2}$$

Throughout the paper we write vector-valued functions using bold face type, e.g., $\mathbf{u}(x) = (u(x), v(x))$, and we say that $\mathbf{u}$ is positive if both components are positive. We shall establish the following non-existence results concerning positive solutions of (I).

Theorem 1.1. *Let* $n = 3$ *and suppose that* (p, q) *is subcritical with respect to the hyperbola* (1.1). *Then* (I) *does not admit any non-negative and non-trivial solutions* $\mathbf{u}$ *with algebraic growth at infinity.*

Since by definition *ground states* of (I) tend to zero at infinity, *Theorem* 1.1 *excludes their existence for subcritical exponents when* $n = 3$, see Corollary 1.1.

Theorem 1.2. *The system* (I) *does not admit any non-negative and non-trivial solutions* $\mathbf{u}$, *provided either* $pq \leq 1$ *or*

$$pq > 1 \quad \textit{and} \quad \max\left\{\frac{2(p+1)}{pq-1}, \ \frac{2(q+1)}{pq-1}\right\} \geq n-2. \tag{1.3}$$

Remark. When $p,\ q > 1$, Theorem 1.2 was obtained in [9]. Condition (1.3) corresponds exactly to the relation $p \leq n/(n-2)$ when $p = q$.

The proofs depend heavily on asymptotic estimates at infinity. For the proof of Theorem 1.2, this amounts to obtaining estimates for spherical averages of solutions. Such an argument is no longer valid when the point (p, q) lies above the curve given by (1.3), as in the case of Theorem 1.1. A subtler approach is to apply a Pohozaev-type identity together with estimates for the solutions themselves.

There are various difficulties in obtaining such estimates, partly due to the imbalance between the two exponents p and q. This makes it generally impossible to apply techniques used in treating single equations. For instance, in contrast to the case of the Lane-Emden equation for which solutions have fast decay $|x|^{2-n}$ at infinity when $p = l$, the components u and v of solutions of (I) cannot both have fast decay everywhere on the critical hyperbola.

Problem (I) admits infinitely many positive radial solutions when (p, q) is critical or supercritical with respect to the hyperbola (1.1), see [14]. Combining the results of [9, 13] and Theorem 1.1 with this conclusion, we obtain the following corollaries showing that (1.1) is the dividing curve for (I) for radial solutions, and for all solutions with algebraic growth when $n = 3$.

Corollary 1.1. *Suppose* $n = 3$. *Then the hyperbola given by* (1.1) *is the dividing curve for positive solutions of* (I) *with algebraic growth at infinity, and, in particular, for ground states.*

Corollary 1.2. *The hyperbola given by* (1.1) *is the dividing curve for positive radial solutions of* (I).

General functions $f(v)$ and $g(u)$, replacing v^p and u^q, can also be treated in a similar manner, with corresponding existence and non-existence results under suitable conditions on f and g. We refer the reader to [13, 14] for details.

The outcomes could be quite different if there were special homogeneous relations between the exponents. For example, Bidaut-Veron and Raoux [1] considered the system

$$\begin{aligned} &\Delta u + |x|^\sigma u^q v^{p+1} = 0, \\ &\Delta v + |x|^\sigma u^{q+1} v^p = 0, \end{aligned} \qquad u, v > 0, \tag{III}$$

where $p,\ q > 0$. Estimates near isolated singularities were obtained when

$$p + q + 1 < \frac{n + 2 + 2\sigma}{n - 2} \tag{1.4}$$

by an argument similar to that used in [5]. The duality between u and v plays a major role here and methods used to treat single equations remain applicable. Moreover, for comparable n-component systems (with appropriate symmetry, and with (III) a special case), the moving plane method shows non-existence and symmetry of positive solutions when the nonlinearity is, respectively, subcritical or critical with respect to (1.4); see [11].

This approach also can be used for the Lane–Emden system in certain cases. Indeed when both exponents p and q are less than or equal to l, the moving plane method applies with the aid of a Kelvin transform. This shows in particular that the system (I) has no positive solutions if $p \leq l$ and $q \leq l$ with at most one equality holding, and that all solutions must be radially symmetric when $p = q = l$; see [3].

As a final remark, we observe that one can combine Theorem 1.1 with a blow-up argument to get a priori estimates for positive solutions of a large class of semilinear elliptic systems on bounded domains, especially systems without variational structure, see the forthcoming paper [16].

The organization of the paper is as follows. In Section 2, we present some preliminaries and our main estimates. In Section 4, appropriate asymptotic estimates are obtained for the case $n = 3$. Section 3 contains the proof of Theorem 1.2, and Section 5 the proof of Theorem 1.1.

2. Preliminaries and principal asymptotic estimates. Consider the weakly coupled system

$$\begin{cases} \Delta u + v^p = 0, \\ \Delta v + u^q = 0, \end{cases} \qquad x \in \mathbb{R}^n, \tag{I}$$

where $p, q > 0$ and $n \geq 3$ is an integer. We are interested in non-existence of non-negative and non-trivial C^2-solutions.

As mentioned in the introduction, throughout the paper we shall write vector-valued functions in the form $\mathbf{u}(x) = (u(x), v(x))$, and say that $\mathbf{u}$ is positive if both components u and v are positive. Generic constants will be denoted by c, M etc.; these depend on the arguments explicitly indicated, but may vary from line to line.

We begin with several preliminary results. The first is a standard comparison lemma.

Lemma 2.1. *Suppose that $w \geq 0$ is non-trivial and satisfies*

$$\Delta w \leq 0, \qquad x \in \mathbb{R}^n. \tag{2.1}$$

Then there exists a positive constant $c = c(w)$ such that

$$w(x) \geq c|x|^{2-n}, \qquad |x| \geq 1.$$

Proof. By the strong maximum principle, $w > 0$ (we take this granted without mention in the sequel). Take

$$c = \min_{|x|=1} w(x) > 0,$$

and put

$$w_1(x) = c|x|^{2-n}, \qquad |x| > 1.$$

Clearly $w - w_1$ is non-negative on the boundary of the set $\Omega = \mathbb{R}^n \backslash B_1$. Moreover

$$\liminf_{|x|\to\infty} (w - w_1) \geq 0$$

and

$$\Delta(w - w_1) \leq 0, \qquad x \in \Omega.$$

Therefore $w - w_1$ is non-negative in Ω by the maximum principle.

Lemma 2.2. *Suppose that $w \geq 0$ is non-trivial and satisfies* (2.1). *Then for $\eta \in C_0^\infty(\mathbb{R}^n)$ and $\gamma < 1$, there exists a constant $c = c(\gamma) > 0$ such that*

$$\int_{\mathbb{R}^n} \eta^2 |Dw|^2 w^{\gamma-2} \leq c \int_{\mathbb{R}^n} |D\eta|^2 w^\gamma. \tag{2.2}$$

Proof. Multiplying (2.1) by $\phi(x) = w^{\gamma-1}(x)\eta^2 \in C_0^\infty(\mathbb{R}^n)$ and integrating over $\mathbb{R}^n$, we obtain

$$(1-\gamma)\int_{\mathbb{R}^n} \eta^2|Dw|^2 w^{\gamma-2} - 2\int_{\mathbb{R}^n} \eta w^{\gamma-1} Dw\cdot D\eta \le 0.$$

It follows by the Schwarz inequality that

$$\begin{aligned}\int_{\mathbb{R}^n} \eta^2|Dw|^2 w^{\gamma-2} &\le \frac{2}{1-\gamma}\int_{\mathbb{R}^n} \eta|D\eta||Dw|w^{\gamma-1}\\ &\le \frac{2}{1-\gamma}\left(\int_{\mathbb{R}^n} \eta^2|Dw|^2 w^{\gamma-2}\right)^{1/2}\cdot\left(\int_{\mathbb{R}^n}|D\eta|^2 w^\gamma\right)^{1/2}\\ &\le \frac{1}{2}\int_{\mathbb{R}^n} \eta^2|Dw|^2 w^{\gamma-2} + \frac{2}{(1-\gamma)^2}\int_{\mathbb{R}^n}|D\eta|^2 w^\gamma,\end{aligned}$$

as required ($c = 4/(1-\gamma)^2$).

For $w \in C(\mathbb{R}^n)$, denote the spherical average of w by

$$\overline{w}(r) = \frac{1}{\omega_n}\int_{S^{n-1}} w(r,\theta), \qquad r > 0,$$

where (r,θ) are spherical-coordinates and $\omega_n = |S^{n-1}|$ is the area of the unit sphere in $\mathbb{R}^n$.

Lemma 2.3. *Suppose that $w \ge 0$ is non-trivial and satisfies* (2.1). *Then for $a \in (0,1]$, we have $(\overline{w^a})' \le 0$. In particular, $\overline{w^a}$ is a non-increasing function of r.*

Proof. For $a \in \mathbb{R}$, it follows by direct calculation and (2.1) that

$$\Delta(w^a) + a(1-a)w^{a-2}|\nabla w|^2 \le 0.$$

Taking the average over S^{n-1} yields, by a standard calculation,

$$\left(\overline{w^a}\right)'' + \frac{n-1}{r}\left(\overline{w^a}\right)' + a(1-a)\overline{w^{a-2}|\nabla w|^2} \le 0, \qquad r > 0,$$

with $'$ denoting differentiation with respect to r. In turn

$$(\overline{w^a})'' + \frac{n-1}{r}(\overline{w^a})' \le 0,$$

since $0 < a \le 1$. Thus $r^{n-1}(\overline{w^a})'$ is non-increasing. But this function vanishes as $r \to 0$, so that $r^{n-1}(\overline{w^a})' \le 0$, and in turn $(\overline{w^a})' \le 0$ and $\overline{w^a}$ is non-increasing.

We next state a useful version of the Poincaré-Sobolev embedding lemma.

Lemma 2.4. *Let k be a positive integer and $s > 1$. Suppose $w \in W^{k,s}(\Omega)$, where Ω is a smooth bounded connected domain in $\mathbb{R}^n$. Then there exists a positive number $C = C(n, s, k, \Omega)$ such that*

$$|w|_{L^\infty(\Omega)} \le C|D^k w|_{L^s(\Omega)} + |w_\Omega|,$$

provided $ks > n$, and

$$|w|_{L^{ns/(n-ks)}(\Omega)} \le C|D^k w|_{L^s(\Omega)} + |\Omega|^{(n-ks)/ns}|w_\Omega|,$$

provided $ks < n$, where

$$w_\Omega = \frac{1}{|\Omega|}\int_\Omega w.$$

When Ω is a compact smooth connected manifold in $\mathbb{R}^n$ the conclusions still hold, but with the conditions $ks > n$ and $ks < n$ respectively replaced by $ks > \dim(\Omega)$ and $ks < \dim(\Omega)$.

Proof. Let Ω be a smooth bounded connected domain. From [15], Corollary 4.2.3, $L(w) = T(w) = w_\Omega$ and $p = s, m = k + 1$, we obtain (replacing k by $k - 1$)

$$|w - w_\Omega|_{L^{s^*}(\Omega)} \le C|D^k w|_{L^s(\Omega)};$$

here one uses also that $T(1) = 1$ and that $\|T\|$ depends on Ω. The required estimates now follow from the triangle inequality.

The same conclusion obviously continues to hold (by a partition of unity argument) when Ω is a compact smooth connected manifold.

We remark that the constant C in Lemma 2.4 is invariant under geometric similarity (scaling, rotation and translation etc.) for the case when $ks < n$.

Lemma 2.5. (i) *Suppose that $w \ge 0$ is non-trivial and satisfies* (2.1). *Also assume that there exist $\gamma \in (0, 1)$ and $\mu > 0$ such that*

$$\overline{w^\gamma}(r) \le Mr^{-\mu}. \tag{2.3}$$

Then for any $R > 0$ and $\sigma > 1$ we have

$$\int_{B_{\sigma R}\setminus B_R} |Dw|^2 w^{\gamma-2} \le cMR^{n-2-\mu}$$

and

$$\int_{B_{\sigma R}\setminus B_R} w^{\gamma n/(n-2)} \le cM^{n/(n-2)}R^{n-\mu n/(n-2)},$$

where $c = c(n, \gamma, \sigma)$.

(ii) *If moreover $\gamma \in (0, (n-2)/n]$, then also*

$$\overline{w^{\gamma n/(n-2)}}(r) \le cM^{n/(n-2)}r^{-\mu n/(n-2)}.$$

Proof. Choose

$$\eta(x) \equiv \begin{cases} 0, & |x| > 2\sigma R \text{ or } |x| < R/2, \\ 1, & R < |x| < \sigma R, \end{cases} \qquad |\nabla \eta(x)| \le c/R \tag{2.4}$$

in Lemma 2.2. Then, recalling that $\gamma < 1$, we get

$$\begin{aligned} \int_{B_{\sigma R}\setminus B_R} |D(w^{\gamma/2})|^2 &= \frac{\gamma^2}{4}\int_{B_{\sigma R}\setminus B_R} |Dw|^2 w^{\gamma-2} \le \frac{c}{R^2}\int_{B_{2\sigma R}\setminus B_{R/2}} w^\gamma \\ &= \frac{c}{R^2}\int_{R/2}^{2\sigma R} r^{n-1}\overline{w^\gamma}(r)\,dr \le \frac{cM}{R^2}\int_{R/2}^{2\sigma R} r^{n-1-\mu}\,dr = cMR^{n-2-\mu} \end{aligned}$$

by (2.3) (note that the final step is valid for arbitrary μ). This gives the first result of (i), with the constant c clearly depending also on σ.

Next we apply Lemma 2.4 with

$$k = 1, \quad s = 2, \quad \Omega = B_{\sigma R}\setminus B_R$$

and with w replaced by $w^{\gamma/2}$. Since $ks < n$ in the present case, this gives, in view of the comment after the proof of Lemma 2.4,

$$J = |w^{\gamma/2}|^2_{L^{2n/(n-2)}(\Omega)} \le c|Dw^{\gamma/2}|^2_{L^2(\Omega)} + 2|\Omega|^{(n-2)/n}|w^{\gamma/2}|^2_{L^1(\Omega)}$$

where $c = c(n, \sigma)$. By the Hölder inequality

$$|w^{\gamma/2}|^2_{L^1(\Omega)} \le cR^n \int_{B_{\sigma R}\setminus B_R} w^\gamma \le cMR^{2n-\mu}$$

as in the last steps of the preceding calculation. Combining the previous lines then yields

$$J \le cMR^{n-2-\mu}.$$

The second result of (i) now follows from the calculation

$$\int_\Omega w^{\gamma n/(n-2)} = J^{n/(n-2)} \le (cMR^{n-2-\mu})^{n/(n-2)} = cM^{n/(n-2)}R^{n-\mu n/(n-2)}. \tag{2.5}$$

We now prove (ii). Since $\overline{w^{\gamma n/(n-2)}}(r)$ is non-increasing by Lemma 2.3 and the hypothesis $\gamma n/(n-2) \le 1$, it follows that

$$\int_\Omega w^{\gamma n/(n-2)} = \omega_n \int_R^{\sigma R} r^{n-1}\overline{w^{\gamma n/(n-2)}}(r)\,dr \ge \frac{(\sigma^n - 1)\omega_n}{n}\overline{w^{\gamma n/(n-2)}}(\sigma R)\cdot R^n.$$

Hence

$$\overline{w^{\gamma n/(n-2)}}(\sigma R) \le cR^{-n}\int_\Omega w^{\gamma n/(n-2)} \le cM^{n/(n-2)}(\sigma R)^{-\mu n/(n-2)},$$

where we have used (2.5) at the last step. This completes the proof of the lemma, when we put $r = \sigma R$.

The previous lemmas concerned solutions of (2.1). We now turn to the main problem of positive solutions of (I).

Lemma 2.6. *Let* $\mathbf{u}=(u,v)$ *be a positive solution of* (I). *Then*

$$U_1'' + \frac{n-1}{r}U_1' + V_1^p \le 0, \qquad \textit{for}\ \ 0 < p_1 \le p, \tag{2.6}$$

and

$$V_2'' + \frac{n-1}{r}V_2' + U_2^q \le 0, \qquad \textit{for}\ \ 0 < q_1 \le q, \tag{2.7}$$

where

$$U_1(r) = \left(\overline{u^{p_1/p}}\right)^{p/p_1}, \qquad V_1(r) = \left(\overline{v^{p_1}}\right)^{1/p_1};$$

and

$$U_2(r) = \left(\overline{u^{q_1}}\right)^{1/q_1}, \qquad V_2(r) = \left(\overline{v^{q_1/q}}\right)^{q/q_1}.$$

Proof. We only prove (2.6) for $p_1 < p$, the case (2.7) being the same and the case with equality being trivial. For $a \in \mathbb{R}$, it follows by direct calculation and $(\mathrm{I})_1$ that

$$\Delta(u^a) + a(1-a)u^{a-2}|\nabla u|^2 + au^{a-1}v^p = 0.$$

Taking the average over S^{n-1} yields

$$\left(\overline{u^a}\right)'' + \frac{n-1}{r}\left(\overline{u^a}\right)' + a(1-a)\overline{u^{a-2}|\nabla u|^2} + a\overline{v^p u^{a-1}} = 0, \qquad r > 0. \tag{2.8}$$

By the Hölder inequality, we have

$$\begin{aligned}\overline{v^{p_1}} &= \frac{1}{\omega_n}\int_{S^{n-1}} v^{p_1} \le \frac{1}{\omega_n}\Big(\int_{S^{n-1}} v^p u^{-1/p'}\Big)^{p_1/p} \cdot \Big(\int_{S^{n-1}} u^{p_1/p}\Big)^{1/p'} \\ &= \left(\overline{u^{p_1/p}}\right)^{1/p'} \cdot \left(\overline{v^p u^{-1/p'}}\right)^{p_1/p},\end{aligned}$$

where $p' = p/(p-p_1)$ is the conjugate of p/p_1, that is,

$$\overline{v^p u^{a-1}} \ge V_1^p \cdot \left(\overline{u^a}\right)^{1-p/p_1}$$

with $a = p_1/p \in (0,1)$. Therefore from (2.8)

$$\left(\overline{u^a}\right)'' + \frac{n-1}{r}\left(\overline{u^a}\right)' + a(1-a)\overline{u^{a-2}|\nabla u|^2} + aV_1^p \cdot \left(\overline{u^a}\right)^{1-p/p_1} \le 0, \qquad r > 0.$$

Multiply by $\left(\overline{u^a}\right)^{p/p_1-1}$ and rewrite in terms of $U_1(r) = \left(\overline{u^a}\right)^{1/a}$ to obtain after a short calculation

$$U_1'' + \frac{n-1}{r}U_1' + \frac{1-a}{a^2}\left[a^2\overline{u^a}\cdot\overline{u^{a-2}|\nabla u|^2} - \left(\overline{u^a}\right)'^2\right]U_1^{1-2a} + V_1^p \le 0. \tag{2.9}$$

By the Schwarz inequality, setting $u' = \partial u/\partial r$,

$$\begin{aligned}\left(\overline{u^a}\right)'^2 &= \Big[\Big(\frac{1}{\omega_n}\int_{S^{n-1}} u^a\Big)'\Big]^2 = \Big(\frac{a}{\omega_n}\int_{S^{n-1}} u'u^{a-1}\Big)^2 \\ &\le \frac{a^2}{\omega_n^2}\int_{S^{n-1}} u'^2u^{a-2}\cdot\int_{S^{n-1}} u^a \le a^2\overline{u^a}\cdot\overline{u^{a-2}|\nabla u|^2}.\end{aligned}$$

Thus (2.6) follows from (2.9), and the proof of the lemma is complete.

Lemma 2.7. *Suppose $z = z(r) > 0$ satisfies*

$$z'' + \frac{n-1}{r} z' + \phi(r) \le 0, \qquad r > 0,$$

with ϕ non-negative and non-increasing, and z' bounded for r near 0. Then

$$z(r) \ge c r^2 \phi(r),$$

where $c = c(n)$.

Proof. Clearly

$$-z'(r) \ge r^{1-n} \int_0^r s^{n-1} \phi(s) ds \ge r^{1-n} \int_0^{r/2} s^{n-1} \phi(s) ds \ge \frac{r}{2^n n} \phi(r/2).$$

Integrating from r to $2r$ then yields

$$z(r) \ge z(2r) + \frac{1}{2^n n} \int_r^{2r} s \phi(s/2) ds \ge \frac{r^2}{2^n n} \phi(r),$$

as required.

For $pq > 1$, put

$$\alpha = \frac{2(p+1)}{pq-1} > 0, \qquad \beta = \frac{2(q+1)}{pq-1} > 0. \tag{2.10}$$

Then we have the following proposition.

Proposition 2.1. *If $pq = 1$, then* (I) *admits no positive solutions. When $pq > 1$ there exists a positive constant $M = M(p,q,n)$ such that*

$$\overline{u}(r) \le M r^{-\alpha}, \qquad \overline{v}(r) \le M r^{-\beta} \quad \textit{for} \quad r > 0. \tag{2.11}$$

Proof. We first consider $p,\ q \ge 1$. Taking the average of (I) on S^{n-1} and using Jensen's inequality, we have

$$\overline{u}'' + \frac{n-1}{r} \overline{u}' + \overline{v}^p \le 0, \qquad \overline{v}'' + \frac{n-1}{r} \overline{v}' + \overline{u}^q \le 0, \qquad r > 0.$$

Since $\overline{u}$ and $\overline{v}$ are non-increasing by Lemma 2.3 with $a = 1$, we get from Lemma 2.7

$$\overline{u}(r) \ge c r^2 \overline{v}^p(r), \qquad \overline{v}(r) \ge c r^2 \overline{u}^q(r). \tag{2.12}$$

We immediately obtain a contradiction if $pq = 1$ ($p = q = 1$). Otherwise, solving the inequalities, we obtain (2.11).

Next, without loss of generality, assume $pq \geq 1$ and $q < 1$. Clearly from (I)

$$\overline{v}'' + \frac{n-1}{r}\overline{v}' + \overline{u^q} = 0.$$

Using (2.6) with $1 = p_1 < p$, and noticing that $\overline{u^q}$ $(q < 1)$ and $\overline{v}$ are non-increasing functions by Lemma 2.3, we again obtain from Lemma 2.7

$$U_1(r) = \left(\overline{u^{1/p}}\right)^p \geq cr^2\overline{v}^p, \qquad \overline{v}(r) \geq cr^2\overline{u^q}. \tag{2.13}$$

But from the Hölder inequality $\left(\overline{u^{1/p}}\right)^p \leq \left(\overline{u^q}\right)^{1/q}$, since $q \geq 1/p$. It follows that

$$\left(\overline{u^q}\right)^{1/q} \geq cr^2\overline{v}^p, \qquad \overline{v}(r) \geq r^2\overline{u^q},$$

again a contradiction if $pq = 1$. Thus we assume $pq > 1$ and solve the inequalities to get

$$\overline{u^q}(r) \leq Mr^{-\alpha q}, \quad \overline{v}(r) \leq Mr^{-\beta}. \tag{2.14}$$

We now require Lemma 2.5. Because $q \in (0,1)$ it is clear that there exists a positive number $\gamma_0 \leq q$ and an integer $N \geq 1$ such that

$$\left(\frac{n}{n-2}\right)^N \gamma_0 = 1.$$

Define

$$\gamma_i = \left(\frac{n}{n-2}\right)^i \gamma_0, \qquad i = 0, 1, 2, \dots ,$$

so

$$0 < \gamma_0 < \gamma_1 < \cdots < \gamma_{N-1} = \frac{n-2}{n} \quad \text{and} \quad \gamma_N = 1.$$

Now from (2.14) and the Hölder inequality

$$\overline{u^{\gamma_0}}(r) \leq \left(\overline{u^q}(r)\right)^{\gamma_0/q} \leq (Mr^{-\alpha q})^{\gamma_0/q} \leq Mr^{-\alpha\gamma_0}.$$

Hence from Lemma 2.5 (ii), we get by iteration (with $\gamma = \gamma_i$ and $\mu = \mu_i = \alpha\gamma_i$)

$$\overline{u^{\gamma_i}}(r) \leq Mr^{-\alpha\gamma_i}, \qquad i = 1, 2, \dots , N \tag{2.15}$$

since $\gamma_i \in (0, (n-2)/n]$ for $i = 1, 2, \dots , N-1$. Taking $i = N$ in (2.15) completes the proof of the proposition.

Corollary 2.1. *Suppose that* $pq > 1$ *and that* $\mathbf{u}$ *is a positive solution of* (I). *Then*

$$\int_{B_R} u^q \leq cR^{n-q\alpha}, \qquad \int_{B_R} v^p \leq cR^{n-p\beta} \tag{2.16}$$

where $c = c(p, q, n)$.

Proof. By (I) and (2.11),

$$\int_{B_R} u^q \leq cR^{n-2}\int_R^{2R} r^{1-n}\int_{B_r} u^q = -cR^{n-2}\int_R^{2R} r^{1-n}\int_{B_r}\Delta v$$
$$= -cR^{n-2}\int_R^{2R}\overline{v}' \leq cR^{n-2}\overline{v}(R) \leq cMR^{n-2-\beta} = cR^{n-q\alpha},$$

since $2 + \beta = q\alpha$, which yields $(2.16)_1$. Similarly one obtains $(2.16)_2$.

Corollary 2.2. *Suppose that $pq > 1$ and that* $\mathbf{u}$ *is a positive solution of* (I). *Then for* $\gamma \in (0,1)$, $R > 0$ *and* $\sigma > 1$, *there exists a constant* $c = c(\gamma, \sigma) > 0$ *such that*

$$\int_{B_{\sigma R} \setminus B_R} |Du|^2 u^{\gamma-2} \le cR^{n-2-\gamma\alpha}, \qquad \int_{B_{\sigma R} \setminus B_R} |Dv|^2 v^{\gamma-2} \le cR^{n-2-\gamma\beta}. \tag{2.17}$$

Moreover, for any $\delta < n/(n-2)$ *there exists a constant* $c = c(\delta, \sigma) > 0$ *such that*

$$\int_{B_{\sigma R} \setminus B_R} u^\delta \le cR^{n-\delta\alpha}, \qquad \int_{B_{\sigma R} \setminus B_R} v^\delta \le cR^{n-\delta\beta}. \tag{2.18}$$

Proof. Since $\gamma \in (0,1)$, we have $\overline{u^\gamma} \le \overline{u}^\gamma \le cr^{-\gamma\alpha}$ by Hölder's inequality and $(2.11)_1$. This is just (2.3) with $w = u$ and $\mu = \alpha\gamma$. Hence from Lemma 2.5 (i) we get $(2.17)_1$, and we derive $(2.17)_2$ similarly.

To obtain (2.18), choose $\gamma = \delta(n-2)/n$ and use the second part of Lemma 2.5 (i).

3. Non-existence I. In this section, we prove two non-existence results for positive solutions of (I). Consider the following region in the first quadrant of the $(p,q)-$plane,

$$\Sigma_2 = \{p,\ q > 0,\ pq > 1 : n - 2 \le \max\{\alpha, \beta\}\},$$

where α and β are given by (2.10). We first show non-existence in Σ_2.

Theorem 3.1. *The system* (I) *does not admit any non-negative and non-trivial solutions* $\mathbf{u}$ *if* $(p,q) \in \Sigma_2$.

Remark. When $p,\ q \ge 1$, Theorem 3.1 was proved in [9].

Proof. Suppose for contradiction that (I) admits a non-negative and non-trivial solution $\mathbf{u}$. Then $\mathbf{u}$ must be strictly positive by the maximum principle. Without loss of generality, we assume in what follows that $p \ge q$. Then necessarily $p > 1$, $\alpha \ge \beta$ and $\alpha \ge n-2$. Moreover,

$$n - p\beta = n - (2 + \alpha) = n - 2 - \alpha \le 0,$$

with equality holding if and only if $\alpha = n-2$. Thus by (2.16),

$$\int_{B_R} v^p \le cR^{n-p\beta} \to 0 \quad \text{as} \quad R \to \infty$$

when $\alpha > n-2$. It follows that $u \equiv v \equiv 0$ when $n - 2 < \alpha$, which is a contradiction.

Next suppose $\alpha = n-2$. Then by $(2.12)_2$ and Lemma 2.1 we have, when $q \ge 1$,

$$\overline{v} \ge cr^2 \left(\overline{u}\right)^q \ge cr^{2-(n-2)q}, \qquad r \ge 1,$$

and, by $(2.13)_2$ and Lemma 2.1, when $q < 1$,

$$\overline{v} \ge cr^2 \overline{u^q} \ge cr^{2-(n-2)q}, \qquad r \ge 1,$$

that is, in both cases, $\overline{v} \geq cr^{2-(n-2)q}$ when $r \geq 1$. Now, using (2.16) again,

$$c = cR^{n-2-\alpha} \geq \int_{B_R} v^p = \omega_n \int_0^R r^{n-1}\overline{v^p} \geq \omega_n \int_0^R r^{n-1}\overline{v}^p$$

by Jensen's inequality and the fact that $p > 1$. Thus, using the previous estimate for $\overline{v}$, we get

$$c \geq \int_0^R r^{n-1}\overline{v}^p \geq c\int_1^R r^{n-1+2p-(n-2)pq} = c\ln R \to \infty \quad \text{as} \quad R \to \infty$$

(since $n-1+2p-(n-2)pq = (pq-1)(\alpha-n+2)-1 = -1$), again a contradiction.

Next we show a non-existence result when $pq \leq 1$.

Theorem 3.2. *Suppose $pq \leq 1$. Then* (I) *has no non-trivial and non-negative solutions.*

Proof. The case $pq = 1$ was proved in Proposition 2.1, so we only need to consider the case $pq < 1$. Let s and θ be chosen so that

$$p < s < \frac{1}{q}, \qquad 0 < \theta < \min(1, 1/s),$$

and put $\gamma = \theta s$. Then clearly

$$\theta p < \gamma < 1, \qquad \gamma q < \theta < 1,$$

and

$$a = \frac{1-\theta}{1-\gamma q} \in (0,1), \qquad b = \frac{1-\gamma}{1-\theta p} \in (0,1).$$

By Hölder's inequality,

$$\overline{u^\theta} \leq \left(\overline{u^{\gamma q}}\right)^a \cdot \overline{u}^{1-a} \leq c\left(\overline{u^{\gamma q}}\right)^a. \tag{3.1}$$

since $\overline{u}$ is decreasing (Lemma 2.3) and thus bounded. Similarly

$$\overline{v^\gamma} \leq c\left(\overline{v^{\theta p}}\right)^b. \tag{3.2}$$

On the other hand, from (2.6) and (2.7) with $p_1 = \theta p$ and $q_1 = \gamma q$, we get

$$U_1'' + \frac{n-1}{r}U_1' + \left(\overline{v}^{\theta p}\right)^{1/\theta} \leq 0, \qquad V_1'' + \frac{n-1}{r}V_1' + \left(\overline{u}^{\gamma q}\right)^{1/\gamma} \leq 0.$$

Then from Lemma 2.7

$$U_1 = \left(\overline{u^\theta}\right)^{1/\theta} \geq cr^2\left(\overline{v^{\theta p}}\right)^{1/\theta}, \qquad V_1 = \left(\overline{v^\gamma}\right)^{1/\gamma} \geq cr^2\left(\overline{u^{\gamma q}}\right)^{1/\gamma}, \tag{3.3}$$

where again we have used the fact that $\overline{v^{\theta p}}$ and $\overline{u^{\gamma q}}$ are decreasing (Lemma 2.3). The four inequalities in (3.1)–(3.3) can be solved for $\overline{u^\theta}$, which yields after a short calculation

$$\overline{u^\theta} \le c r^{-2a\theta(s+b)/(1-ab)}.$$

In turn, from Lemma 2.1 and the fact that $\Delta(u^\theta) \le 0$, there follows

$$n-2 \ge \frac{2a(s+b)}{1-ab}. \tag{3.4}$$

But it is easy to see that (for fixed s)

$$\lim_{\theta\to 0} a = \lim_{\theta\to 0} b = 1.$$

Therefore

$$\lim_{\theta\to 0} \frac{2a(s+b)}{1-ab} = \infty,$$

which contradicts (3.4). Hence no solution can exist and the theorem is proved.

4. Further asymptotic estimates: the case $n = 3$. Let w be an arbitrary function on $\mathbb{R}^n$. We say that w has algebraic growth at infinity if there exists a constant K such that

$$|w| \le \text{Const.}|x|^K \qquad \text{for} \quad |x| > 1,$$

and that a vector-valued function $\mathbf{u} = (u, v)$ has algebraic growth if both components do. We shall deduce estimates for positive solutions $\mathbf{u}$ of (I) with algebraic growth at infinity.

In what follows we assume that all constants depend both on the explicit arguments indicated as well as on the structural numbers p, q, n and the solution itself.

We first prove a standard L^p estimate, valid for all $n \ge 3$.

Lemma 4.1 (L^p estimate)**.** *Let* $\mathbf{u}$ *be a positive solution of* (I) *with algebraic growth at infinity. Then, for* $R \ge 1$ *and* $\epsilon > 0$*, there exist two positive numbers* $m_1 = m_1(\epsilon) > 1$ *and* $c = c(m)$ *such that for* $m \in (1, m_1]$

$$\int_{B_{2R}\setminus B_R} |D^2 u|^m \le cR^{n+\epsilon-p\beta}, \qquad \int_{B_{2R}\setminus B_R} |D^2 v|^m \le cR^{n+\epsilon-q\alpha}. \tag{4.1}$$

Proof. It suffices to show $(4.1)_1$. Put

$$w(x) = u(x)\eta \in C_0^\infty(B_{4R}\setminus B_{R/2}),$$

where η is given by (2.4) with $\sigma = 2$. Then w satisfies the equation

$$\Delta w + \eta v^p - u\Delta\eta - 2Du\cdot D\eta = 0.$$

For $m > 1$, by standard L^p estimates (see [6], Corollary 9.10, page 235) we have

$$\begin{aligned}\int_{B_{2R}\setminus B_R} |D^2u|^m = \int_{B_{2R}\setminus B_R} |D^2w|^m \le \int_{B_{4R}\setminus B_{R/2}} |D^2w|^m \le c\int_{B_{4R}\setminus B_{R/2}} |\Delta w|^m \\ \le c\int_{B_{4R}\setminus B_{R/2}} v^{mp} + cR^{-2m}\int_{B_{4R}\setminus B_{R/2}} u^m + cR^{-m}\int_{B_{4R}\setminus B_{R/2}} |Du|^m\end{aligned} \tag{4.2}$$

where c depends on m. We shall estimate the right hand side of (4.2) term by term. From the algebraic growth condition of v at infinity and (2.16), we bound the first term by

$$\int_{B_{4R}\setminus B_{R/2}} v^{mp} \le cR^{Kp(m-1)}\int_{B_{4R}} v^p \le cR^{Kp(m-1)+n-p\beta} \le cR^{n+\epsilon-p\beta}, \tag{4.3}$$

provided that $R \ge 1$ and $Kp(m-1) \le \epsilon$. Next, for any $\gamma \in (0,1)$, we use Schwarz's inequality, (2.17) and (2.18) (with $\sigma = 8$ and R replaced by $R/2$) to obtain

$$\begin{aligned}\int_{B_{4R}\setminus B_{R/2}} |Du|^m \le \Big(\int_{B_{4R}\setminus B_{R/2}} |Du|^2u^{\gamma-2}\Big)^{m/2}\Big(\int_{B_{4R}\setminus B_{R/2}} u^{m(2-\gamma)/(2-m)}\Big)^{(2-m)/2} \\ \le cR^{m(n-2-\gamma\alpha)/2}\cdot R^{[n-m(2-\gamma)\alpha/(2-m)](2-m)/2} = cR^{n-m(1+\alpha)},\end{aligned} \tag{4.4}$$

provided that

$$m < 2, \qquad 0 < \frac{m(2-\gamma)}{2-m} < \frac{n}{n-2}. \tag{4.5}$$

Fix $\gamma \in (0,1)$ so that $\gamma > 2 - n/(n-2)$. Then there exists $m_1 \in (1,2)$ such that (4.5) holds for all $m \le m_1$. Note also

$$m_1 < \frac{2-m_1}{2-\gamma}\cdot\frac{n}{n-2} < \frac{n}{n-2}. \tag{4.6}$$

Thus for any $1 < m \le m_1$ the estimate $(2.18)_1$ is valid with $\delta = m$ and $\sigma = 8$, that is,

$$R^{-2m}\int_{B_{4R}\setminus B_{R/2}} u^m \le cR^{n-m(2+\alpha)} = cR^{n-mp\beta},$$

while by (4.1)

$$R^{-m}\int_{B_{4R}\setminus B_{R/2}} |Du|^m \le cR^{n-m(2+\alpha)} = cR^{n-mp\beta}.$$

If moreover m_1 is chosen even smaller if necessary, so that $m_1 \le 1 + \epsilon/Kp$, then (4.3) also holds. Putting these estimates into (4.2), we obtain $(4.1)_1$ as required.

After these preparations, we can derive the following estimates which are the key to non-existence for any subcritical point (p,q) when $n = 3$.

Theorem 4.1. *Suppose that $n = 3$, $pq > 1$ and* $\mathbf{u}$ *is a positive solution of* (I) *with algebraic growth at infinity. Then for any $\epsilon \in (0, 2)$, there exist a sequence $R_j \to \infty$ and a constant $c = c(\epsilon)$ such that*

$$\int_{S^2} |Du(R_j, \theta)|^2 \leq cR_j^{2\epsilon - 2(\alpha+1)}, \qquad u^*(R_j) \leq cR_j^{\epsilon - \alpha}, \tag{4.7}$$

and

$$\int_{S^2} |Dv(R_j, \theta)|^2 \leq cR_j^{2\epsilon - 2(\beta+1)}, \qquad v^*(R_j) \leq cR_j^{\epsilon - \beta} \tag{4.8}$$

for $j = 1,\ 2, \dots,$ where

$$u^*(r) = \max_{\theta \in S^{n-1}} u(r, \theta), \qquad v^*(r) = \max_{\theta \in S^{n-1}} v(r, \theta).$$

Proof. For any $n \geq 3$, fix $\gamma \in (0, 1)$ and put

$$f(r) = \int_{S^{n-1}} |Du(r, \theta)|^2 u^{\gamma - 2}(r, \theta), \qquad g(r) = \int_{S^{n-1}} |D^2 u(r, \theta)|^m,$$

and

$$\Gamma_{R,T} = \{\ r \in (R, 2R) : f(r) \geq T\}, \qquad \Lambda_{R,T} = \{\ r \in (R, 2R) : g(r) \geq T\},$$

where $m > 1$, $R > 0$ and $T > 0$. By (2.17) with $\sigma = 2$, and by (4.1), it follows that for every $\epsilon > 0$ there exist two positive constants $m_1 = m_1(\epsilon) > 1$ and $c = c(m)$ such that for $1 < m \leq m_1$

$$\int_R^{2R} f(r) r^{n-1} dr \leq cR^{n-2-\gamma\alpha}, \qquad \int_R^{2R} g(r) r^{n-1} dr \leq cR^{n+\epsilon/2-p\beta}.$$

Therefore one finds easily that

$$TR^{n-1} \Big|\Gamma_{R,T}\Big| \leq cR^{n-2-\gamma\alpha}, \qquad TR^{n-1} \Big|\Lambda_{R,T}\Big| \leq cR^{n+\epsilon/2-p\beta}.$$

In particular, if we take $T = T_1 = 4cR^{-2-\gamma\alpha}$ then

$$|\Gamma_{R,T_1}| \leq R/4,$$

and if $T = T_2 = 4cR^{\epsilon/2-p\beta}$, then

$$|\Lambda_{R,T_2}| \leq R/4.$$

It follows that there must be some point $R_0 \in [R, 2R]$ which is in neither Γ_{R,T_1} nor Λ_{R,T_2}. Hence at R_0 we have

$$f(R_0) \leq 4cR_0^{-2-\gamma\alpha}, \qquad g(R_0) \leq 4cR_0^{\epsilon/2-p\beta}.$$

Note that $R > 0$ is arbitrary, whence there exist a sequence $R_j \to \infty$ and a positive constant $c = c(m)$ such that

$$f(R_j) \le 4cR_j^{-2-\gamma\alpha}, \qquad g(R_j) \le 4cR_j^{\epsilon/2-p\beta}, \qquad j = 1, 2, \ldots . \tag{4.9}$$

We now wish to apply Lemma 2.4 on the manifold S^{n-1} to estimate the L^∞ norm $u^*(R_j)$. In view of (4.9) this requires that we take $k = 2$ and $s = m > 1$, and that $ks > \dim(S^{n-1})$, i.e.,

$$m > (n-1)/2. \tag{4.10}$$

However $m \le m_1 < n/(n-1)$ (as one easily sees from (4.6)), so that (4.10) requires essentially that $n = 3$. Taking this case from here on, and using Lemma 2.4 and Proposition 2.1, we deduce that

$$\begin{aligned} u^*(R_j) &\le c(|u(R_j)|_{W^{2,m}(S^2)} + |u_{S^2}(R_j)|) \\ &= c\Big(\int_{S^2} |D_\theta^2 u(R_j,\theta)|^m\Big)^{1/m} + c\overline{u}(R_j) \le cR_j^2 g^{1/m}(R_j) + c\overline{u}(R_j) \\ &\le cR_j^{2+\epsilon/2m-p\beta/m} + cR_j^{-\alpha} \le cR_j^{2+\epsilon/2m-p\beta/m}. \end{aligned} \tag{4.11}$$

Since $p\beta = 2 + \alpha$, it is easy to see that the required condition

$$2 + \frac{\epsilon}{2m} - \frac{p\beta}{m} = 2 + \frac{\epsilon}{2m} - \frac{2+\alpha}{m} \le \epsilon - \alpha$$

is equivalent to

$$D(m) = 2(2 + \alpha - \epsilon)m - 2(2 + \alpha) + \epsilon \le 0.$$

Recalling that $\epsilon < 2$, one checks that the (linear) equation $D(m) = 0$ has a unique root

$$m_0 = m_0(\alpha, \epsilon) = 1 + \epsilon/2(2 + \alpha - \epsilon) > 1$$

and that $D(m) \le 0$ when $m \le m_0$. Therefore

$$2 + \frac{\epsilon}{2m} - \frac{p\beta}{m} = \epsilon - \alpha + \frac{D(m)}{2m} \le \epsilon - \alpha \qquad \text{for} \quad 1 < m \le m_0.$$

Now $(4.7)_2$ follows from (4.11) by taking

$$m = \min\{m_0, m_1\} > 1.$$

(so $m = m(\epsilon)$ and in turn $c = c(\epsilon)$). On the other hand,

$$\int_{S^2} |Du(R_j,\theta)|^2 \le cf(R_j)[u^*(R_j)]^{2-\gamma} \le cR_j^{-2-\gamma\alpha} R_j^{\epsilon(2-\gamma)-(2-\gamma)\alpha} \le cR_j^{2\epsilon-2(\alpha+1)},$$

and $(4.7)_1$ is proved. The conclusions $(4.8)_1$ and $(4.8)_2$ are obtained in the same way, and the proof is complete.

5. Non-existence II: the case $n = 3$. In Section 3, we obtained non-existence for (p, q) in the region Σ_2 and also for $pq \le 1$. In this section, we consider the special case $n = 3$, and show non-existence of positive solutions with algebraic growth at infinity, as long as (p, q) is subcritical, namely, in the region

$$\Sigma_1 = \Big\{p,\ q > 0 : \frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}\Big\}.$$

Note that the relation $1/(p+1) + 1/(q+1) > (n-2)/n$ is equivalent to $\alpha + \beta > n - 2$ if $pq > 1$.

Theorem 5.1. *Suppose that $n = 3$ and $(p, q) \in \Sigma_1$. Then* (I) *does not admit any positive solutions* $\mathbf{u}$ *with algebraic growth at infinity.*

Before we proceed to the proof, we need a final lemma. It is obvious that the system (I) has the Lagrangian

$$\mathcal{F}(u, v) = \nabla u \cdot \nabla v - \frac{u^{q+1}}{q+1} - \frac{v^{p+1}}{p+1}.$$

Hence the following identity holds.

Proposition 5.1. *Let* $\mathbf{u}$ *be a positive solution of* (I) *and $B = B_R$ the ball centered at the origin with radius R. Then we have*

$$\begin{aligned}(\frac{n}{p+1} - a_1)\int_B v^{p+1} + (\frac{n}{q+1} - a_2)\int_B u^{q+1} = R^n \int_{S^{n-1}} (\frac{v^{p+1}}{p+1} + \frac{u^{q+1}}{q+1}) \\ + R^{n-1}\int_{S^{n-1}} (a_1 u'v + a_2 uv') + R^{n-2}\int_{S^{n-1}} (R^2 u'v' - \nabla_\theta u \nabla_\theta v), \qquad (5.1)\end{aligned}$$

where $a_1 + a_2 = n - 2$ and $' = \partial/\partial r$.

This follows directly from a result of Pucci and Serrin [10] by taking

$$\mathbf{h} = x, \qquad \mathbf{a} = (a_1, a_2).$$

The boundary terms in (5.1) appear due to the non-zero boundary conditions on ∂B_R.

Proof of Theorem 5.1. By Theorem 3.2 we may suppose that $pq > 1$. Now let $\mathbf{u}$ be a positive solution of (I). By Theorem 4.1, for any $\epsilon > 0$ there exist a sequence $R_j \to \infty$ and a positive constant $c = c(\epsilon)$ such that (4.7) and (4.8) hold. Denote by B_j $(j = 1, 2, \dots)$ the ball centered at the origin with radius R_j. We apply Proposition 5.1 on B_j for each j, and take a_1 and a_2 in (5.1) such that $a_1 + a_2 = n - 2 = 1$ and

$$(\frac{3}{p+1} - a_1) = (\frac{3}{q+1} - a_2) = \delta.$$

An easy calculation shows that

$$\delta = \frac{(pq-1)\sigma}{2(p+1)(q+1)}\,,$$

where $\sigma = \alpha + \beta - 1$ (recall that $pq > 1$). In turn $\sigma > 0$ since $(p,q) \in \Sigma_1$ and $n = 3$. Then (5.1) becomes

$$\begin{aligned}\delta \int_{B_j} \left(v^{p+1} + u^{q+1}\right) &= \frac{R_j^3}{p+1}\int_{S^2} v^{p+1} + \frac{R_j^3}{q+1}\int_{S^2} u^{q+1} + R_j^3 \int_{S^2} u'v' \\ &\quad - R_j \int_{S^2} \nabla_\theta u \nabla_\theta v + a_1 R_j^2 \int_{S^2} u'v + a_2 R_j^2 \int_{S^2} uv' \\ &= I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \qquad j = 1, 2, \dots .\end{aligned} \tag{5.2}$$

We shall show that the right hand side of (5.2) tends to zero as $j \to \infty$. Indeed, by Theorem 4.1

$$I_1 = R_j^3 \int_{S^2} v^{p+1} \le cR_j^{\epsilon(p+1)-\sigma}, \qquad I_2 = R_j^3 \int_{S^2} u^{q+1} \le cR_j^{\epsilon(q+1)-\sigma},$$

$$|I_3| \le R_j^3 \int_{S^2} |u'v'| \le cR_j^{2\epsilon-\sigma}, \qquad |I_4| \le R_j \int_{S^2} |\nabla_\theta u \nabla_\theta v| \le cR_j^{2\epsilon-\sigma},$$

and

$$|I_5| \le R_j^2 \int_{S^2} |u'v| \le cR_j^{2\epsilon-\sigma}, \qquad |I_6| \le R_j^2 \int_{S^2} |uv'| \le cR_j^{2\epsilon-\sigma}.$$

Taking

$$\epsilon < \min\left\{2, \frac{\sigma}{p+1}, \frac{\sigma}{q+1}\right\}$$

(so also $2\epsilon < \sigma$ since $pq > 1$), we infer that

$$|I| \le I_1 + I_2 + |I_3| + |I_4| + |I_5| + |I_6| \to 0 \qquad \text{as } j \to \infty.$$

This implies that $u \equiv v \equiv 0$, which contradicts our assumption and completes the proof.

We remark that Theorem 5.1 excludes the existence of ground states of (I) for subcritical exponents, since every ground state tends to zero at infinity. Combining this with the existence result in [14], we obtain the following corollary.

Corollary 5.1. *Suppose* $n = 3$. *Then* (I) *has a positive solution with algebraic growth at infinity if and only if* (p, q) *is critical or supercritical. In particular,* (I) *has a ground state if and only if* (p, q) *is critical or supercritical.*

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