BULLETIN
of the
MALAYSIAN MATHEMATICAL SOCIETY

Bull. Malaysian Math. Soc. (Second Series) **22** (1999) 179-183

Non-Existence of Proper Semi-Invariant Submanifolds of a Lorentzian Para-Sasakian Manifold

U.C. DE AND ABSOS ALI SHAIKH
Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India
e-mail: ucde@klyuniv.ernet.in

Abstract. The main objective of the paper is to prove that a Lorentzian para-Sasakian manifold does not admit any proper semi-invariant submanifold.

1. Introduction

Recently, in [1], semi-invariant submanifolds of a Sasakian manifold have been defined and studied. It has been shown that a Sasakian manifold always admits a proper semi-invariant submanifold [1]. In [2] Matsumoto introduced the idea of Lorentzian para-contact structure and studied its several properties. Subsequently, in [3] semi-invariant submanifolds of a Lorentzian para-Sasakian (LP-Sasakian) manifold have been studied. In the present paper we prove that there does not exist any proper semi-invariant submanifold of an LP-Sasakian manifold and also such a manifold do not admit proper mixed foliated semi-invariant submanifold. Some interesting results concerning integrability of the distributions which arise naturally on the semi-invariant submanifold have also been established.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞-vector valued linear function φ, a C^∞ vector field ξ, 1-form η and Lorentzian metric g of type (0, 2) such that for each point $p \in \overline{M}$, the tensor $g_p : T_p\overline{M} \times T_p\overline{M} \to R$ is a non-degenerate inner product of signature (–, +, +, …, +), where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and R is the real number space, which satisfies

$$\varphi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \xi) = \eta(X) \tag{2.3}$$

for all vector field X, Y tangent to $\overline{M}$. Such a structure (φ, η, ξ, g) is termed as Lorentzian para-contact structure [2].

Also in a Lorentzian para-contact structure the following relations hold:

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \text{rank}(\varphi) = n - 1.$$

A Lorentzian para-contact manifold $\overline{M}$ is called Lorentzian para-Sasakian (LP-Sasakian) manifold if [2]

$$\left(\overline{\nabla}_X \varphi\right)(Y) = g(X, Y)\,\xi + \eta(Y)X + 2\eta(X)\eta(Y)\,\xi, \tag{2.4}$$

and from (2.3), we find

$$\overline{\nabla}_X \xi = \varphi X \tag{2.5}$$

for all X, Y tangent to $\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection with respect to g.

Again if we put

$$\Phi(X, Y) = g(X, \varphi Y), \tag{2.6}$$

then $\Phi(X, Y)$ is a symmetric (0,2) tensor field ([2]), that is,

$$\Phi(X, Y) = \Phi(Y, X). \tag{2.7}$$

A submanifold M of an LP-Sasakian manifold $\overline{M}$ is said to be semi-invariant submanifold if the following conditions are satisfied

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D, $D^\perp$ are orthogonal differentiable distributions on M and $\{\xi\}$ is the 1-dimensional distribution spanned by ξ,

(ii) The distribution D is invariant by φ, that is, $\varphi D_x = D_x$ for each $x \in M$,

(iii) The distribution $D^\perp$ is anti-invariant under φ, that is, $\varphi D^\perp \subset T_x M^\perp$ for each $x \in M$.

If both the distribution D and $D^{\perp}$ are non-zero then the semi-invariant submanifold is called a proper semi-invariant submanifold. For any vector bundle H on M [resp., $\overline{M}$], we denote by $\Gamma(H)$ the module of all differentiable section of H on a neighbourhood co-ordinate on M [resp., $\overline{M}$].

The equations of Gauss and Weingarten of the immersion of M in $\overline{M}$ are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X{}^{\perp} N \tag{2.9}$$

for any $X,\ Y \in \Gamma(TM)$ and $N \in TM^{\perp}$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the linear connection induced by $\overline{\nabla}$ on the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. From (2.8) and (2.9), it follows that

$$g\big(h(X, Y), N\big) = g(A_N X, Y) \tag{2.10}$$

for any $X, Y \in \Gamma(TM), N \in \Gamma(TM^{\perp})$.

We denote by the same symbol g both metrices on $\overline{M}$ and M.

3. Non-existence of proper semi-invariant submanifolds

We first prove a lemma.

Lemma 3.1. *On an LP-Sasakian manifold $\overline{M}$, the distribution T determined by η is involutive.*

Proof. Let $X,\ Y \in \Gamma(T)$. Here $\eta(X) = 0,\ \eta(Y) = 0$ and consequently in view of (2.3), (2.5), (2.6) and (2.7), it follows that $\eta([X, Y]) = 0,$ which completes the proof of the lemma.

The above lemma provides the proof of the following:

Theorem 3.1. *The distribution $D \oplus D^{\perp}$ of a semi-invariant submanifold of an LP-Sasakian manifold is always integrable.*

Now we prove the main theorem of the paper.

Theorem 3.2. *For a semi-invariant submanifold M of an LP-Sasakian manifold* $\overline{M}$, *dim* $D^{\perp} = 0$. *Consequently, an LP-Sasakian manifold* $\overline{M}$ *does not admit any proper semi-invariant submanifold.*

To prove the theorem we state the following:

Lemma 3.2. [3]. *Let M be a semi-invariant submanifold of an LP-Sasakian manifold* $\overline{M}$. *Then we have* $A_{\varphi X} Y + A_{\varphi Y} X = 0$, *for all* $X, Y \in \Gamma(D^{\perp})$.

Proof of the Main Theorem. Let $X, Y \in \Gamma(D^{\perp})$. Hence $\varphi X, \varphi Y \in \Gamma(TM^{\perp})$. In view of (2.3), (2.9) and (2.5), we get

$$\eta(A_{\varphi Y} X) = -g(\overline{\nabla}_X (\varphi Y), \xi) = g(\varphi Y, \overline{\nabla}_X \xi) = g(\varphi Y, \varphi X) = g(X, Y); \quad X, Y \in \Gamma(D^{\perp}).$$

Interchanging X and Y in this equation and then adding both the equations, in view of Lemma 3.2, we have

$$2g(X, Y) = \eta(A_{\varphi X} Y + A_{\varphi Y} X) = 0; \quad \text{for all } X, Y \in \Gamma(D^{\perp}).$$

Hence dim $D^{\perp} = 0$. This completes the proof of the theorem.

Theorem 3.1 and Theorem 3.2 lead to

Theorem 3.3. *If M is a semi-invariant submanifold of an LP-Sasakian manifold, then*

(i) *the distribution D is integrable;*

(ii) $\nabla_X (\varphi Y) - \nabla_Y (\varphi X) = \varphi([X, Y])$;

(iii) $h(X, \varphi Y) = h(\varphi X, Y)$; $X, Y \in \Gamma(D)$.

Proof. Since in view of Theorem 3.2, dim $D^{\perp} = 0$; taking into account of Theorem 3.1, the distribution D is integrable. From (2.4) we get

$$(\overline{\nabla}_X \varphi)(Y) - (\overline{\nabla}_Y \varphi)(X) = 0; \; X, Y \in \Gamma(D). \tag{3.1}$$

Using (2.8) in (3.1), we have

$$0 = (\overline{\nabla}_X \varphi)(Y) - (\overline{\nabla}_Y \varphi)(X) = \nabla_X(\varphi Y) - \nabla_Y(\varphi X) - \varphi([X, Y])$$
$$+ h(X, \varphi Y) - h(\varphi X, Y); \quad X, Y \in \Gamma(D),$$

which on equating tangential and normal parts, yeilds (ii) and (iii) respectively.

4. Non-existence of proper mixed foliated semi-invariant submanifolds

In [4], a semi-invariant submanifold is said to be foliated if $D \oplus \{\xi\}$ is integrable and $h(Z + \xi, X) = 0$ for all $Z \in D$ and $X \in D^{\perp}$.

Here we prove

Theorem 4.1. *LP-Sasakian manifolds do not admit proper mixed foliated semi-invariant submanifolds.*

Proof. From (2.5) and (2.8) we have

$$\varphi X = \nabla_X \xi + h(X, \xi), \; X \in TM.$$

If $X \in D^{\perp}$ then $\nabla_X \xi = 0$ and $h(X, \xi) = \varphi X$. Moreover, if M is mixed foliated then the above equation yields $\varphi X = 0$, $X \in D^{\perp}$. Thus $D^{\perp} = \{0\}$ and M cannot be a proper mixed foliated semi-invariant submanifold.

References

1. A. Bejancu, N. Papaghiuc, Semi-invariant submanifold of a Sasakian manifold, *An Stiint. Univ. "Al. I.Cuza" Iasi* **27** (1981), 163-170.
2. K. Matsumoto, Lorentzian para-contact manifolds, *Bull. of Yamagata Univ., Nat. Sci.* **12** (1989), 151-156.
3. B. Prasad, Semi-invariant submanifolds of a Lorentzian Para-Sasakian manifold, *Bull. Malaysian Math. Soc. (Second Series)* **21** (1998), 21-26.
4. S.M. Khursheed Haider, V.A. Khan and S.I. Husain, Reduction in co-dimension of mixed foliated semi-invariant submanifold of a Sasakian space form $\overline{M}$ (−3), *Riv. Mat. Univ. Parma* **1** (1992), 147-153.