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Non-existence of some natural operators on connections

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Abstract. Let n, r, k be natural numbers such that $n \geq k+1$. Non-existence of natural operators $C_0^r \rightsquigarrow Q(\operatorname{reg} T_k^r \to K_k^r)$ and $C_0^r \rightsquigarrow Q(\operatorname{reg} T_k^{r*} \to K_k^{r*})$ over n-manifolds is proved. Some generalizations are obtained.

0. Introduction. Let n, r and k be natural numbers such that $n \geq k+1$. In [1], C. Ehresmann constructed functorially the fiber bundle $K_k^r M = \operatorname{reg} T_k^r M / L_k^r$ of contact (k,r)-elements over an n-dimensional manifold M and obtained the bundle functor $K_k^r : \mathcal{M}f_n \to \mathcal{FM}$ from the category $\mathcal{M}f_n$ of n-dimensional manifolds and their embeddings into the category $\mathcal{FM}$ of fibered manifolds and their fibered maps. In [5], I. Kolář, P. W. Michor and J. Slovák studied the problem of how a vector field X on M induces a vector field $A(X)$ on $K_k^r M$ and proved that for sufficiently large n every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow TK_k^r$ is a constant multiple of the complete lifting $\mathcal{K}_k^r$.

In [6], I. Kolář and the author investigated the naturality problem for bundle mappings $B : K_k^r M \to K_k^r M$ and deduced the so called rigidity theorem for K_k^r saying that the only natural transformation $B : K_k^r \to K_k^r$ over n-manifolds is the identity. They also studied the naturality problem for affinors (i.e. tensor fields of type $(1,1)$) $C : TK_k^r M \to TK_k^r M$ on $K_k^r M$ and derived that for sufficiently large n every natural affinor $C : TK_k^r \to TK_k^r$ on K_k^r over n-manifolds is a constant multiple of the identity. Moreover they analysed how a 1-form ω on M can induce a 1-form $D(\omega)$ on $K_k^r M$ and showed that for sufficiently large n every natural operator $D : T^*_{|\mathcal{M}f_n} \rightsquigarrow T^* K_k^r$ is a constant multiple of the vertical lifting. Some generalizations of the above results can be found in [8].

Similarly to (k,r)-elements, C. Ehresmann introduced the fiber bundle $K_k^{r*} M = \operatorname{reg} T_k^{r*} M / L_k^r$ of contact (k,r)-coelements over an n-dimensional manifold M. So, we have the bundle functor $K_k^{r*} : \mathcal{M}f_n \to \mathcal{FM}$. In [9], we

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studied for $K^{r*}_k M$ the same naturality problems as for $K^r_k M$ and proved the same results.

In the present paper we continue the investigation of $K^r_k M$ and $K^{r*}_k M$.

In the first part of the paper, we study the problem of whether a torsion free linear r-order connection $\lambda : TM \to J^r TM$ on an n-manifold M induces a connection $A(\lambda) : T(\mathrm{reg}\, T^r_k M) \to l^r_k = \mathrm{Lie}(L^r_k)$ on the principal fiber bundle $\mathrm{reg}\, T^r_k M \to K^r_k M$ with structure group L^r_k. This problem is reflected in the concept of natural operators $A : C^r_0 \rightsquigarrow Q(\mathrm{reg}\, T^r_k \to K^r_k)$ in the sense of [5]. We prove that if $n \geq k+1$, then there are no natural operators $C^r_0 \rightsquigarrow Q(\mathrm{reg}\, T^r_k \to K^r_k)$ over n-manifolds. We find an assumption on a bundle functor $F : \mathcal{M}f_n \to \mathcal{FM}$ under which there are no natural operators $F \rightsquigarrow Q(\mathrm{reg}\, T^r_k \to K^r_k)$. We give an example of a bundle functor $F : \mathcal{M}f_n \to \mathcal{FM}$ $(n \geq k+1)$ and a natural operator $A : F \rightsquigarrow Q(\mathrm{reg}\, T^r_k \to K^r_k)$.

In the second part we study the problem of whether a torsion free linear r-order connection $\lambda : TM \to J^r TM$ on an n-manifold M induces a connection $A(\lambda) : T(\mathrm{reg}\, T^{r*}_k M) \to l^r_k$ on the principal fiber bundle $\mathrm{reg}\, T^{r*}_k M \to K^{r*}_k M$ with structure group L^r_k. We prove that if $n \geq k+1$ then there are no natural operators $C^r_0 \rightsquigarrow Q(\mathrm{reg}\, T^{r*}_k \to K^{r*}_k)$ over n-manifolds. We find an assumption on a bundle functor $F : \mathcal{M}f_n \to \mathcal{FM}$ under which there are no natural operators $F \rightsquigarrow Q(\mathrm{reg}\, T^{r*}_k \to K^{r*}_k)$. We observe that there is a bundle functor $F : \mathcal{M}f_n \to \mathcal{FM}$ with $n \geq k+1$ and a natural operator $A : F \rightsquigarrow Q(\mathrm{reg}\, T^{r*}_k \to K^{r*}_k)$.

Natural operations with connections have been studied by many authors; see e.g. [2], [4], [5], [7], etc.

From now on $x^1, \ldots, x^n$ and $t^1, \ldots, t^k$ are the usual coordinates on $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively.

All manifolds are assumed to be finite-dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

I. Torsion free linear connections of order r and the bundle of contact (k,r)-elements

1. The bundle functor K^r_k of contact (k,r)-elements. Let n and k be natural numbers. For every n-manifold M we have the bundle $T^r_k M = J^r_0(\mathbb{R}^k, M)$ over M of so-called (k,r)-*velocities* on M. Every embedding $\varphi : M \to N$ of n-manifolds induces a bundle map $T^r_k\varphi : T^r_k M \to T^r_k N$ by $T^r_k\varphi(j^r_0\gamma) = j^r_0(\varphi \circ \gamma)$ for $\gamma : \mathbb{R}^k \to M$. The correspondence $T^r_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order r from the category $\mathcal{M}f_n$ of n-dimensional manifolds and their embeddings into the category $\mathcal{FM}$ of fibered manifolds and their fibered maps.

Every $\xi = j^r_0\psi \in L^r_k = \mathrm{inv}\, J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0$, the Lie group of invertible r-jets $\mathbb{R}^k \to \mathbb{R}^k$ with source and target $0 \in \mathbb{R}^k$, induces a natural automorphism

$\overline{\xi} : T^r_k \to T^r_k$, $\overline{\xi} : T^r_k M \to T^r_k M$ by $\overline{\xi}(j^r_0 \gamma) = j^r_0(\gamma \circ \psi^{-1})$ for $\gamma : \mathbb{R}^k \to M$. This defines a group homomorphism $L^r_k \to \mathrm{Aut}(T^r_k)$.

It is well known that if $n \geq k+1$ then the above homomorphism is an isomorphism, i.e. $\mathrm{Aut}(T^r_k) \cong L^r_k$.

Assume $n \geq k+1$. For every n-manifold M, $\widetilde{T}^r_k M = \mathrm{reg}\, J^r_0(\mathbb{R}^k, M) = \{j^r_0 \gamma \mid \gamma : \mathbb{R}^k \to M,\ \mathrm{rank}(d_0\gamma) = k\}$ is an open subbundle of $T^r_k M$. Elements of $\widetilde{T}^r_k M$ are called *regular* (k,r)*-velocities* on M. For every embedding $\varphi : M \to N$ of n-manifolds, $T^r_k \varphi(\widetilde{T}^r_k M) \subset \widetilde{T}^r_k N$, and we let $\widetilde{T}^r_k \varphi : \widetilde{T}^r_k M \to \widetilde{T}^r_k N$ be the restriction of $T^r_k \varphi$. The correspondence $\widetilde{T}^r_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order r.

For every n-manifold M, $\widetilde{T}^r_k M$ is invariant with respect to the action of $L^r_k = \mathrm{Aut}(T^r_k)$ on $T^r_k M$. So, we have (by restriction) the left action of L^r_k on $\widetilde{T}^r_k M$, and the quotient bundle $K^r_k M = \widetilde{T}^r_k M / L^r_k$ over M of so-called *contact* (k,r)*-elements*. Let $\pi : K^r_k M \to M$ be the bundle projection. This bundle was introduced by C. Ehresmann [1]. The quotient projection $\kappa : \widetilde{T}^r_k M \to K^r_k M$ is a principal fiber bundle with structure group L^r_k. The right principal bundle action of L^r_k on $\widetilde{T}^r_k M$ is given by $v.\xi = \overline{\xi}^{-1}(v)$ for $\xi \in L^r_k$, $v \in \widetilde{T}^r_k M$. For every embedding $\varphi : M \to N$ of n-manifolds, $\widetilde{T}^r_k \varphi$ commutes with the left action of L^r_k on $\widetilde{T}^r_k M$ and we have the quotient map $K^r_k \varphi : K^r_k M \to K^r_k N$. Then $\widetilde{T}^r_k \varphi$ is a principal bundle morphism covering $K^r_k \varphi$. The correspondence $K^r_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order r.

2. Principal fiber bundle $\kappa^0 : P^0 \to Q^0$ and non-existence of $\mathrm{GL}(\mathbb{R}^n)$-invariant connections on P^0. Let $n \geq k+1$. Let $\kappa^0 : P^0 \to Q^0$ be the restriction of $\kappa : \widetilde{T}^r_k \mathbb{R}^n \to K^r_k \mathbb{R}^n$ to the fibers over $0 \in \mathbb{R}^n$. Then κ^0 is a principal fiber bundle (a principal subbundle of κ) with structure group L^r_k. The right action of L^r_k on P^0 is the restriction of the right action of L^r_k on $\widetilde{T}^r_k \mathbb{R}^n$.

There is a left action $\alpha^0 : \mathrm{GL}(\mathbb{R}^n) \times P^0 \to P^0$ given by $\alpha^0(\eta, p) = \widetilde{T}^r_k \eta(p)$ for $\eta \in \mathrm{GL}(\mathbb{R}^n)$, $p \in P^0$. This action covers the left action $\beta^0 : \mathrm{GL}(\mathbb{R}^n) \times Q^0 \to Q^0$ defined by $\beta^0(\eta, q) = K^r_k \eta(q)$ for $\eta \in \mathrm{GL}(\mathbb{R}^n)$, $q \in Q^0$. For every $\eta \in \mathrm{GL}(\mathbb{R}^n)$ the mapping $\alpha^0_\eta = \alpha^0(\eta, \cdot) : P^0 \to P^0$ is a principal fiber bundle isomorphism covering $\beta^0_\eta = \beta^0(\eta, \cdot) : Q^0 \to Q^0$.

A connection $\omega : TP^0 \to l^r_k = \mathrm{Lie}(L^r_k)$ on P^0 is called $\mathrm{GL}(\mathbb{R}^n)$*-invariant* if $(\alpha^0_\eta)^* \omega = \omega$ for every $\eta \in \mathrm{GL}(\mathbb{R}^n)$.

Proposition 1. *There are no* $\mathrm{GL}(\mathbb{R}^n)$*-invariant connections* $\omega : TP^0 \to l^r_k$ *on* P^0.

Proof. Suppose $\omega : TP^0 \to l^r_n$ is a $\mathrm{GL}(\mathbb{R}^n)$-invariant connection. According to the general theory of invariant connections (see Kobayashi and

Nomizu [3]), there is a linear map $\Lambda : \mathrm{gl}(\mathbb{R}^n) \to l^r_k$ satisfying the following two conditions:

(1) $\Lambda(X) = \lambda(X)$ for $X \in \mathrm{Lie}(J)$;
(2) $\Lambda(\mathrm{ad}(j)(X)) = \mathrm{ad}(\lambda(j))\Lambda(X)$ for $j \in J$ and $X \in \mathrm{gl}(\mathbb{R}^n)$.

Here $J \subset \mathrm{GL}(\mathbb{R}^n)$ is the stabilizer of $\kappa^0(\sigma_0) \in Q^0$, $\sigma_0 = j^r_0(t^1, \ldots, t^k, 0, \ldots, 0) \in P^0$, and $\lambda : \mathrm{Lie}(J) \to l^r_k$ is the Lie algebra homomorphism corresponding to the group homomorphism $\lambda : J \to L^r_k$, that is, $j.\sigma_0 = \sigma_0.\lambda(j)$ for $j \in J$. We recall that $\Lambda(X) = \omega_{\sigma_0}(\widetilde{X})$ for $X \in \mathrm{gl}(\mathbb{R}^n)$, where $\widetilde{X}$ is the vector field on P^0 induced by X.

We have the Lie algebra isomorphism $l^r_n = J^r_0(T\mathbb{R}^n)_0 := \{j^r_0 X \mid X \in \mathcal{X}(\mathbb{R}^n),\ X_0 = 0\}$, where the bracket in $J^r_0(T\mathbb{R}^n)_0$ is given by $[j^r_0X, j^r_0Y] = j^r_0([Y, X])$ (see [5]).

Consider the elements

$$X^1 = j^r_0\left(x^1 \frac{\partial}{\partial x^{k+1}}\right), \quad X^3 = j^r_0\left(x^{k+1} \frac{\partial}{\partial x^{k+1}}\right),$$
$$X^2 = j^r_0\left(x^1 \frac{\partial}{\partial x^1}\right), \quad X^4 = j^r_0\left(x^{k+1} \frac{\partial}{\partial x^1}\right)$$

of $\mathrm{gl}(\mathbb{R}^n) \subset l^r_n$. Their one-parameter subgroups in $\mathrm{GL}(\mathbb{R}^n) \subset L^r_n$ are

$$\begin{aligned}
a^1_t &= j^r_0(x^1, \ldots, x^k, x^{k+1} + tx^1, x^{k+2}, \ldots, x^n),\\
a^2_t &= j^r_0(e^t x^1, x^2, \ldots, x^n),\\
a^3_t &= j^r_0(x^1, \ldots, x^k, e^t x^{k+1}, x^{k+2}, \ldots, x^n),\\
a^4_t &= j^r_0(x^1 + tx^{k+1}, x^2, \ldots, x^n).
\end{aligned}$$

Of course $a^2_t, a^3_t, a^4_t \in J$, $\lambda(a^2_t) \neq \mathrm{id}$, $\lambda(a^3_t) = \mathrm{id}$, $\lambda(a^4_t) = \mathrm{id}$ and $a^1_t \in \mathrm{GL}(\mathbb{R}^n) \subset L^r_n$. Hence $X^2, X^3, X^4 \in \mathrm{Lie}(J)$, $X^1 \in \mathrm{gl}(\mathbb{R}^n)$, $\Lambda(X^3) = \Lambda(X^4) = 0$ and $\Lambda(X^2) \neq 0$.

On the other hand if $\varphi = (x^1 + x^{k+1}, x^2, \ldots, x^n)$ then $j = j^r_0\varphi \in J$ and $\lambda(j) = \mathrm{id}$. So,

$$\begin{aligned}
\Lambda(X^1) &= \mathrm{ad}(\lambda(j))\Lambda(X^1) = \Lambda(\mathrm{ad}(j)X^1) = \Lambda\left(j^r_0\left(\varphi_*\left(x^1 \frac{\partial}{\partial x^{k+1}}\right)\right)\right)\\
&= \Lambda(X^1 + X^2 - X^3 - X^4) = \Lambda(X^1) + \Lambda(X^2),
\end{aligned}$$

i.e. $\Lambda(X^2) = 0$.

This contradiction ends the proof of Proposition 1. ■

3. Linear connections of order r. A *linear r-order connection* on an n-manifold M is a vector bundle morphism $\lambda : TM \to J^rTM$ such that $\pi^r_0 \circ \lambda = \mathrm{id}_{TM}$, where $\pi^r_0 : J^rTM \to TM$ is the target projection (see [10]).

REMARK 1. If M is an n-manifold we have the principal fiber bundle $P^rM = \operatorname{inv} J^r_0(\mathbb{R}^n, M)$ over M with standard group L^r_n. The right action of L^r_n on P^rM is given by the composition of jets. If $\varphi : M \to N$ is an embedding of n-manifolds, then we define $P^r(\varphi) : P(M) \to P(N)$ by composition of jets. There is a canonical bijection between connections $\omega : TP^rM \to l^r_n$ on P^rM and linear r-order connections $\lambda : TM \to J^rTM$ on M by $H^\omega = \mathcal{P}^r \circ (\lambda \times_M \mathrm{id}_{P^rM})$, where $H^\omega : TM \times_M P^rM \to TP^rM$ is the horizontal lifting morphism of ω, and $\mathcal{P}^r : J^rTM \times_M P^rM \to TP^rM$ is the flow morphism of P^r (see [10]).

Given an n-manifold M we define $C^r(M) = (\mathrm{id}_{T^*M} \otimes \pi^r_0)^{-1}(\mathrm{id}_{TM}) \subset T^*M \otimes J^rTM$ to be the subbundle in $T^*M \otimes J^rTM$, where $\pi^r_0 : J^rTM \to TM$ is the target projection. It is called the *bundle of linear r-order connections* on M. The sections of $C^r(M)$ are exactly the linear r-order connections on M. For every embedding $\varphi : M \to N$ the mapping $T^*\varphi \otimes J^rT\varphi : T^*M \otimes J^rTM \to T^*N \otimes J^rTN$ sends $C^r(M)$ into $C^r(N)$ and we have (by restriction) a fiber bundle map $C^r(\varphi) : C^r(M) \to C^r(N)$. The correspondence $C^r : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order $r + 1$.

An r-order linear connection $\lambda : TM \to J^rTM$ is called *torsion free* if $\{\lambda(u), \lambda(v)\} = 0$ for any $u, v \in T_xM$, $x \in M$. Here $\{\cdot,\cdot\} : J^rTM \times_M J^rTM \to J^{r-1}TM$ is the algebraic bracket given by

$$\{j^r_xX, j^r_xY\} = j^{r-1}_x([X, Y])$$

(see [10]).

By the Frobenius theorem, an r-order connection λ on an n-manifold M is torsion free iff for each $x \in M$ there is a chart Φ near x such that $\Phi(x) = 0$ and $C^r(\Phi)(\lambda_x) = \lambda^0$, where $\lambda^0 : T_0\mathbb{R}^n \to J^r_0T\mathbb{R}^n$ is defined by $\lambda^0(u) = j^r_0\widetilde{u}$ for $u \in T_0\mathbb{R}^n$ and $\widetilde{u}$ is the constant vector field on $\mathbb{R}^n$ such that $\widetilde{u}_0 = u$. The coordinates Φ corresponding to Φ are called *normal coordinates* of λ at x (see [10]). Clearly, if $\bar{\Phi}$ are another normal coordinates of λ at x then $\bar{\Phi} = A \circ \Phi$ for some $A \in \mathrm{GL}(\mathbb{R}^n)$.

Given an n-manifold M we define $C^r_0(M)$ to be the orbit in $T^*M \otimes J^rTM$ of λ^0 with respect to embeddings $\psi : \mathbb{R}^n \to M$, where $\lambda^0 \in (T^*_0\mathbb{R}^n) \otimes (J^r_0T\mathbb{R}^n)$ is as above. It is a subbundle of $T^*M \otimes J^rTM$ and it is called the *bundle of torsion free linear r-order connections* on M. The sections of $C^r_0(M)$ are exactly the torsion free linear r-order connections on M. For every embedding $\varphi : M \to N$ the mapping $T^*\varphi \otimes J^rT\varphi : T^*M \otimes J^rTM \to T^*N \otimes J^rTN$ sends $C^r_0(M)$ into $C^r_0(N)$ and we have (by restriction) a fiber bundle map $C^r_0(\varphi) : C^r_0(M) \to C^r_0(N)$. The correspondence $C^r_0 : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order $r + 1$.

4. Non-existence of natural operators $C^r_0 \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$. Let $P \to \underline{P}$ be a principal fiber bundle with structure group L^r_k and m-dimensional base. According to the general theory, connections $TP \to l^r_k$ on P can be identified with sections of some fiber bundle $QP = Q(P \to \underline{P})$ over $\underline{P}$, called the *bundle of connections* on P. It is known that every local principal bundle isomorphism $\Psi : (P \to \underline{P}) \to (H \to \underline{H})$ covering $\underline{\Psi} : \underline{P} \to \underline{H}$ induces functorially a fiber bundle map $Q\Psi : QP \to QH$ over $\underline{\Psi}$. The correspondence $Q : \mathcal{P}_m(L^r_k) \to \mathcal{FM}$ is a gauge bundle functor from the category $\mathcal{P}_m(L^r_k)$ of principal fiber bundles with structure group L^r_k and m-dimensional bases and their local principal bundle isomorphisms (see [5]).

According to [5] a natural operator $A : C^r_0 \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$ over n-manifolds is a family of regular operators

$$A : C^\infty_M(C^r_0(M)) \to C^\infty_{K^r_k M}(Q(\widetilde{T}^r_k M \to K^r_k M))$$

from the set of torsion free r-order linear connections on M (sections of $C^r_0(M)$) into the set of connections on $\kappa : \widetilde{T}^r_k M \to K^r_k M$ (sections of $Q(\widetilde{T}^r_k M \to K^r_k M)$) for every n-manifold M such that the naturality condition with respect to $\mathcal{M}f_n$-morphisms is satisfied. This means that for every $\lambda \in C^\infty_M(C^r_0(M))$ and $\overline{\lambda} \in C^\infty_N(C^r_0(N))$ and every $\mathcal{M}f_n$-morphism $\varphi : M \to N$, if $\overline{\lambda}$ and λ are φ-related, then $A(\overline{\lambda})$ and $A(\lambda)$ are $Q\widetilde{T}^r_k\varphi$-related.

The first main result of this paper is the following theorem.

THEOREM 1. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no natural operators $C^r_0 \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$ over n-manifolds.*

Proof. Suppose that $A : C^r_0 \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$ is a natural operator over n-manifolds. We define $H^\omega : TQ^0 \times_{Q^0} P^0 \to TP^0$ by

$$H^\omega(u, v) = H^{A(\widetilde{\lambda}^0)}(u, v)$$

for $u \in T_\sigma Q^0$, $\sigma \in Q^0$, $v \in P^0$, $\kappa^0(v) = \sigma$, where $H^{A(\widetilde{\lambda}^0)} : TK^r_k\mathbb{R}^n \times_{K^r_k\mathbb{R}^n} \widetilde{T}^r_k\mathbb{R}^n \to T\widetilde{T}^r_k\mathbb{R}^n$ is the horizontal lifting morphism of $A(\widetilde{\lambda}^0)$ and where $\widetilde{\lambda}^0$ is the translation invariant section of $C^r_0(\mathbb{R}^n)$ such that $\widetilde{\lambda}^0_{|0} = \lambda^0$. Here λ_0 is the element from $C^r_0(\mathbb{R}^n)$ as in Section 3.

Clearly, $H^\omega(u, v) \in T_v\widetilde{T}^r_k\mathbb{R}^n$ and it projects on u under $T\kappa$. Hence $H^\omega(u, v)$ is $(\widetilde{T}^r_k\mathbb{R}^n \to \mathbb{R}^n)$-vertical because u is π-vertical. Therefore, $H^\omega(u, v) \in TP^0$. Since $H^{A(\widetilde{\lambda}^0)}$ is a horizontal lifting, so is H^ω.

Let $\omega : TP^0 \to l^r_k$ be the corresponding connection on P^0. Since $\widetilde{\lambda}^0$ is $\mathrm{GL}(\mathbb{R}^n)$-invariant, so is $\Lambda(\widetilde{\lambda}^0)$ because of the naturality od Λ. Hence ω is $\mathrm{GL}(\mathbb{R}^n)$-invariant.

This is a contradiction by Proposition 1. ∎

We have the following obvious corollaries of Theorem 1.

COROLLARY 1. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no natural operators $C^r \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$ over n-manifolds.*

COROLLARY 2. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no canonical connections $\omega : T\widetilde{T}^r_k M \to l^r_k$ on $T\widetilde{T}^r_k M$ over n-manifolds.*

5. A generalization. Using the same method as in the proof of Theorem 1 we obtain the following general fact.

THEOREM 2. *Let n, k and r be natural numbers such that $n \geq k+1$. Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a bundle functor such that there is a* $\mathrm{GL}(\mathbb{R}^n)$*-invariant element $\mu^0 \in F_0\mathbb{R}^n$. There are no natural operators $A : F \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$.*

Proof. In the proof of Theorem 1 we replace $\widetilde{\lambda}^0$ by the translation invariant section $\widetilde{\mu}^0$ of $F\mathbb{R}^n$ such that $\widetilde{\mu}^0_{|0} = \mu^0$. ∎

REMARK 2. There are many natural bundles FM satisfying the assumption of Theorem 2. For example, $C^s_0(M)$ and $C^s(M)$ for all s, fiber products $C^{s_1}_0(M) \times_M \ldots \times_M C^{s_K}_0(M)$, $T^A M$ for all Weil algebras A, all natural vector bundles ($\bigotimes^p T^*M \otimes \bigotimes^q TM$, $T^r_k M$, $T^{r*}_k M, \ldots$), etc.

COROLLARY 3. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no natural operators Λ sending a generalized connection Γ on $\widetilde{T}^r_k M \to M$ (or $K^r_k M \to M$) and a linear connection ∇ on M to a connection $\Lambda(\Gamma, \nabla) : T\widetilde{T}^r_k M \to l^r_k$.*

Proof. Every r-order linear connection $\lambda : TM \to J^r TM$ on M gives rise to a generalized connection Γ on $\widetilde{T}^r_k M \to M$ (or $K^r_k M \to M$) because $\widetilde{T}^r_k$ (or K^r_k) is of order r. Next we apply Theorem 2 and Remark 2. ∎

The following example shows that Theorem 1 is not true for an arbitrary natural bundle F instead of C^r_0.

EXAMPLE 1. Let $F = P^{r+1} : \mathcal{M}f_n \to \mathcal{FM}$ be the bundle functor from Remark 1. Consider a connection $\omega : T\widetilde{T}^r_k\mathbb{R}^n \to l^r_k$ on $\kappa : \widetilde{T}^r_k\mathbb{R}^n \to K^r_k\mathbb{R}^n$. Given a section ϱ of FM define $A(\varrho) : T\widetilde{T}^r_k M \to l^r_k$ by $A(\varrho)(v) = \omega(T\widetilde{T}^r_k\varphi^{-1}(v))$ for $j^{r+1}_0\varphi = \varrho(x)$, $v \in (T\widetilde{T}^r_k M)_x$, $x \in M$. Then $A(\varrho)$ is a connection on $\kappa : \widetilde{T}^r_k M \to K^r_k M$. In this way we obtain a natural operator $A : F \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$.

II. Torsion free linear connections of order r and the bundle functor of contact (k, r)-coelements

6. The bundle functor K^{r*}_k of contact (k, r)-coelements. Let n and k be natural numbers. For every n-manifold M we have the bundle $T^{r*}_k M = J^r(M, \mathbb{R}^k)_0$ over M of so-called (k, r)-*covelocities* on M. Every embedding $\varphi : M \to N$ of n-manifolds induces a bundle map $T^{r*}_k\varphi : T^{r*}_k M \to T^{r*}_k N$

by $T^{r*}_k\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma\circ\varphi^{-1})$ for $\gamma : M \to \mathbb{R}^k$, $x \in M$, $\gamma(x) = 0$. The correspondence $T^{r*}_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order r.

Every $\xi = j^r_0\psi \in L^r_k$ induces a natural automorphism $\overline{\xi} : T^{r*}_k \to T^{r*}_k$, $\overline{\xi} : T^{r*}_k M \to T^{r*}_k M$, defined by $\overline{\xi}(j^r_x\gamma) = j^r_x(\psi\circ\gamma)$ for $\gamma : M \to \mathbb{R}^k$, $x \in M$, $\gamma(x) = 0$. This defines a group homomorphism $L^r_k \to \mathrm{Aut}(T^{r*}_k)$

If $n \geq k+1$ then the above homomorphism is an isomorphism, i.e. $\mathrm{Aut}(T^{r*}_k) \cong L^r_k$.

Assume $n \geq k+1$. For every n-manifold M, $\widetilde{T}^{r*}_k M = \mathrm{reg}\, J^r(M,\mathbb{R}^k)_0 = \{j^r_x\gamma \mid \gamma : M \to \mathbb{R}^k,\ x \in M,\ \gamma(x) = 0,\ \mathrm{rank}(d_x\gamma) = k\}$ is an open subbundle of $T^{r*}_k M$. Elements of $\widetilde{T}^{r*}_k M$ are called *regular* (k,r)*-covelocities* on M. For every embedding $\varphi : M \to N$ of n-manifolds, $T^{r*}_k\varphi(\widetilde{T}^{r*}_k M) \subset \widetilde{T}^{r*}_k N$, and we let $\widetilde{T}^{r*}_k\varphi : \widetilde{T}^{r*}_k M \to \widetilde{T}^{r*}_k N$ be the restriction of $T^{r*}_k\varphi$. The correspondence $\widetilde{T}^{r*}_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order r.

For every n-manifold M, $\widetilde{T}^{r*}_k M$ is invariant with respect to the action of $L^r_k = \mathrm{Aut}(T^{r*}_k)$ on $T^{r*}_k M$. So, we have (by restriction) the left action of L^r_k on $\widetilde{T}^{r*}_k M$, and the quotient bundle $K^{r*}_k M = \widetilde{T}^{r*}_k M/L^r_k$ over M of so called *contact* (k,r)*-coelements*. This bundle was introduced in [9]. The quotient projection $\kappa^* : \widetilde{T}^{r*}_k M \to K^{r*}_k M$ is a principal fiber bundle with structure group L^r_k. The right principal bundle action of L^r_k on $\widetilde{T}^{r*}_k M$ is given by $v.\xi = \overline{\xi}^{-1}(v)$ for $\xi \in L^r_k$, $v \in \widetilde{T}^{r*}_k M$. For every embedding $\varphi : M \to N$ of n-manifolds, $\widetilde{T}^{r*}_k\varphi$ commutes with the left action of L^r_k on $\widetilde{T}^{r*}_k M$ and we have the quotient map $K^{r*}_k\varphi : K^{r*}_k M \to K^{r*}_k N$. Then $\widetilde{T}^{r*}_k\varphi$ is a principal bundle morphism covering $K^{r*}_k\varphi$. The correspondence $K^{r*}_k : \mathcal{M}f_n \to \mathcal{FM}$ is a bundle functor of order r (see [9]).

7. Principal fiber bundle $\kappa^{*0} : P^{*0} \to Q^{*0}$ and non-existence of $\mathrm{GL}(\mathbb{R}^n)$-invariant connections on P^{*0}. Let $n \geq k+1$. Let $\kappa^{*0} : P^{*0} \to Q^{*0}$ be the restriction of $\kappa^* : \widetilde{T}^{r*}_k\mathbb{R}^n \to K^{r*}_k\mathbb{R}^n$ to the fibers over $0 \in \mathbb{R}^n$. Then κ^{*0} is a principal fiber bundle (a principal subbundle of κ^*) with structure group L^r_k. Its right action of L^r_k on P^{*0} is the restriction of the right action of L^r_k on $\widetilde{T}^{r*}_k\mathbb{R}^n$.

There is a left action $\alpha^{*0} : \mathrm{GL}(\mathbb{R}^n) \times P^{*0} \to P^{*0}$ defined by $\alpha^{*0}(\eta,p) = \widetilde{T}^{r*}_k\eta(p)$ for $\eta \in \mathrm{GL}(\mathbb{R}^n)$, $p \in P^{*0}$. This action covers the left action $\beta^{*0} : \mathrm{GL}(\mathbb{R}^n)\times Q^{*0} \to Q^{*0}$ given by $\beta^{*0}(\eta,q) = K^{r*}_k\eta(q)$ for $\eta \in \mathrm{GL}(\mathbb{R}^n)$, $q \in Q^{*0}$. For every $\eta \in \mathrm{GL}(\mathbb{R}^n)$ the mapping $\alpha^{*0}_\eta = \alpha^{*0}(\eta,\cdot) : P^{*0} \to P^{*0}$ is a principal fiber bundle isomorphism covering $\beta^{*0}_\eta = \beta^{*0}(\eta,\cdot) : Q^{*0} \to Q^{*0}$.

A connection $\omega : TP^{*0} \to l^r_k = \mathcal{L}(L^r_k)$ on P^{*0} is called $\mathrm{GL}(\mathbb{R}^n)$*-invariant* if $(\alpha^{*0}_\eta)^*\omega = \omega$ for every $\eta \in \mathrm{GL}(\mathbb{R}^n)$.

Proposition 2. *There are no* $\mathrm{GL}(\mathbb{R}^n)$*-invariant connections* $\omega : TP^{*0} \to l^r_k$ *on* P^{*0}.

Proof. Similar to the proof of Proposition 1. ∎

8. Non-existence of natural operators $C^r_0 \rightsquigarrow Q(\widetilde{T}^{r*}_k \to K^r_k)$. The definition of natural operators $C^r_0 \rightsquigarrow Q(\widetilde{T}^{r*}_k \to K^{r*}_k)$ is similar to that of natural operators $C^r_0 \rightsquigarrow Q(\widetilde{T}^r_k \to K^r_k)$.

The second main result of this paper is the following theorem.

THEOREM 3. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no natural operators $C^r_0 \rightsquigarrow Q(\widetilde{T}^{r*}_k \to K^{r*}_k)$ over n-manifolds.*

Proof. The proof is similar to that of Theorem 1. One uses Proposition 2 instead of Proposition 1. ∎

We have the following obvious corollaries of Theorem 3.

COROLLARY 4. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no natural operators $C^r \rightsquigarrow Q(\widetilde{T}^{r*}_k \to K^{r*}_k)$ over n-manifolds.*

COROLLARY 5. *Let n, k and r be natural numbers such that $n \geq k+1$. There are no canonical connections $\omega : T\widetilde{T}^{r*}_k M \to l^r_k$ on $\widetilde{T}^{r*}_k M$ over n-manifolds.*

9. A generalization. Using the same method as in the proof of Theorem 2 we obtain the following general fact.

THEOREM 4. *Let n, r, k be natural numbers such that $n \geq k+1$. Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a bundle functor such that there is a $\mathrm{GL}(\mathbb{R}^n)$-invariant element $\mu^0 \in F_0\mathbb{R}^n$. Then there are no natural operators $A : F \rightsquigarrow Q(\widetilde{T}^{r*}_k \to K^{r*}_k)$.*

COROLLARY 6. *Let n, r, k be natural numbers such that $n \geq k+1$. There are no natural operators Λ sending a generalized connection Γ on $\widetilde{T}^{r*}_k M \to M$ (or $K^{r*}_k M \to M$) and a linear connection ∇ on M to a connection $\Lambda(\Gamma, \nabla) : T\widetilde{T}^{r*}_k M \to l^r_k$.*

REMARK 3. Modifying Example 1 we can show that Theorem 3 is not true for an arbitrary natural bundle F instead of C^r_0.

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