Ergod. Th. & Dynam. Sys. (2001), **21**, 1447–1457
Printed in the United Kingdom

Non-expanding maps and Busemann functions

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(*Received* 11 *November* 1999 *and accepted in revised form* 20 *October* 2000)

Abstract. We give stronger versions and alternative simple proofs of some results of Beardon, [**Be1**] and [**Be2**]. These results concern contractions of locally compact metric spaces and generalize the theorems of Wolff and Denjoy about the iteration of a holomorphic map of the unit disk. In the case of unbounded orbits, there are two types of statements which can sometimes be proven; first, about invariant horoballs, and second, about the convergence of the iterates to a point on the boundary. A few further remarks of similar type are made concerning certain random products of semicontractions and also concerning semicontractions of Gromov hyperbolic spaces.

1. *Introduction*

Let (Y, d) be a metric space. A *contraction* is a map $\phi : Y \to Y$, such that

$$d(\phi(x), \phi(y)) < d(x, y),$$

for any distinct $x, y \in Y$. A *semicontraction* (or *non-expanding/non-expansive map*) is a map $\phi : Y \to Y$ such that

$$d(\phi(x), \phi(y)) \leq d(x, y)$$

for any $x, y \in Y$. In particular, any isometry is a semicontraction.

In this paper we are interested in (the iteration of) semicontractions of locally compact, complete metric spaces.

Recall that the Schwarz–Pick lemma asserts that any holomorphic map $f : D \to D$, where D is the open unit disk in the complex plane, is a semicontraction with respect to the hyperbolic metric on D. Therefore the following two classic theorems are the first results of the type we are concerned with in this paper.

THEOREM 1.1. [**Wo1, De**] *Let f be a holomorphic map of the unit disk D into itself. Then either f has a fixed point in D or there is a point $\xi \in \partial D$ such that the iterates f^n converge uniformly on compact sets in D to the constant map taking the value ξ.*

THEOREM 1.2. [**Wo2**] *Let f be a holomorphic map of the unit disk D into itself. Then either f has a fixed point (in which case every ball centered at this point is an f-invariant set) or there is a point ξ for which every horoball $\mathcal{H}$ based at ξ is an f-invariant set, that is $f(\mathcal{H}) \subset \mathcal{H}$.*

Generalizations of these results in various directions have been obtained in several papers, the majority of which consider holomorphic maps of more general complex spaces. We refer to the introduction and references of the recent papers [**KKR, M**], and to the book [**Ko**]. Beardon's two papers [**Be1**] and [**Be2**] advanced the understanding of the general metric space mechanisms behind the above two results and gave simple proofs of significant extensions of these results. These two papers had a crucial impact on the present paper.

We will here give an elementary argument that makes the proofs of some of the statements in [**Be1**] and [**Be2**] somewhat simpler. At the same time, more general results can be obtained in this way. It turns out that, fairly generally, a semicontraction with unbounded orbits interacts nicely with some Busemann function of the space.

Examples of semicontractions of locally compact metric spaces include holomorphic maps of a (Kobayashi) hyperbolic complex space, affine maps of a convex domain with the Hilbert metric, invertible matrices acting on the corresponding symmetric space and elements of the fundamental group of a Riemannian manifold acting on the universal covering manifold.

The paper is organized as follows: the first section lists some of the properties that metric spaces can have and that will be used in this paper. (Examples of spaces having or failing to have these properties are mentioned in the last section.) §3 describes the main argument of this paper and its corollaries. §4 combines the main subadditive ergodic lemma in [**KaMa**] with some additional axiom imposed on the metric space to give a statement about certain random products of semicontractions. In §5 we make one remark about semicontractions of (not necessarily locally compact) Gromov hyperbolic spaces and describe one corollary concerning fixed points of isometries of Gromov hyperbolic Hadamard spaces. In the last section we wish to point out some examples and draw attention to some related works. The list of references is far from being complete.

2. *Some axioms on metric spaces*

A metric space is *proper* if every closed ball is compact. A proper metric space can be compactified, $\overline{Y}^B = Y \cup Y(\infty)$ (finer than the one-point compactification) using Busemann functions. Namely, fixing a point y, the space Y is injected into $C(Y)$ by the map

$$z \mapsto d(\cdot, z) - d(z, y).$$

The compactification is now obtained by taking the closure of this image in $C(Y)$, where the topology is given by uniform convergence on compact subsets. So a sequence of points y_n going to infinity converges to a point ξ on the boundary $Y(\infty)$ if and only if

$$d(\cdot, y_n) - d(y_n, y)$$

converges uniformly on compact sets. This limit in $C(Y)$ is called a *Busemann function*, denoted by b_ξ, and a sublevel set $\{z : b_\xi(z) \leq C\}$ is called a *horoball centered at* ξ. For the details of this compactification, see for example [**Ba**].

A *geodesic space* is a metric space (Y, d) such that for, every two points x, y in Y, there is an isometry $\alpha : [0, d(x, y)] \to Y$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$. This path α, which is not required to be unique, is called a *geodesic* connecting x and y.

Any locally compact, complete (meant in the usual sense), geodesic space (Y, d) is proper. For such a space Y there is also another boundary, *the ray boundary* or *the visual sphere*, consisting of equivalence classes of geodesic rays starting from y. Two rays are equivalent if they stay a bounded distance from each other. The topology is uniform convergence on compact sets. Any geodesic ray in a proper metric space defines a Busemann function.

The following two axioms for metric spaces (Y, d) appeared in [**Be2**]. The first axiom says that Y can be compactified by adding a boundary at infinity.

AXIOM 1. *The metric space Y is an open dense subset of a compact Hausdorff topological space $\overline{Y}$, whose relative topology coincides with the metric topology. For any $y \in Y$, if x_n is a sequence in Y converging to a point in $\partial Y := \overline{Y} - Y$, then $d(y, x_n) \to \infty$.*

The second axiom says that the asymptotic geometry resembles that of hyperbolic spaces.

AXIOM 2. *If x_n and y_n converge to two distinct points γ and ξ, respectively, in ∂Y, then for any y in Y*

$$d(x_n, y_n) - \max\{d(y, x_n), d(y, y_n)\} \to +\infty.$$

An alternative to Axiom 2 is the following.

AXIOM 3. *Let $x_n \to \gamma \in \partial Y$ and let y_n be a sequence in Y. If, for some y,*

$$d(x_n, y_n) - d(y, y_n) \to -\infty,$$

then $y_n \to \gamma$.

Another property we need to note is the following.

AXIOM 4. *Let $x_n \to \gamma \in \partial Y$ and let y_n be a sequence in Y. If*

$$d(x_n, y_n) < C$$

for some constant C, then $y_n \to \gamma$.

Note that Axiom 2 implies Axiom 3. Under the assumption that all points of ∂Y lie on infinite distance (Axiom 1), Axiom 4 is implied by either of Axiom 2 or Axiom 3. Note also that in the case $\overline{Y} = \overline{Y}^B$, Axiom 2 implies the property that any sequence of points y_n going to infinity inside a horoball based at γ must converge to γ, in other words, any horoball meets the boundary at infinity at only one point.

Recall the definition of the so-called Gromov product. Fix a point y and let

$$(z, w)_y = \tfrac{1}{2}(d(y, z) + d(y, w) - d(z, w)).$$

A metric space is called *hyperbolic in the sense of Gromov* or *δ-hyperbolic* if there is a constant $\delta \geq 0$ such that

$$(z, w)_y \geq \min\{(z, x)_y, (x, w)_y\} - \delta$$

holds for all $z, w, x \in Y$. For a Gromov hyperbolic space, one defines the boundary ∂Y so that y_n converges to a point on the boundary if and only if

$$(y_n, y_m)_y \to \infty$$

as $n, m \to \infty$.

It follows immediately from the definitions that Gromov hyperbolic spaces satisfy Axioms 2, 3 and 4. It is known that the ray boundary of a proper, geodesic Gromov hyperbolic space is naturally isomorphic to the hyperbolic boundary ∂Y and it is Hausdorff, see [**CDP**].

3. *Semicontractions of locally compact spaces*

Several of the arguments in this paper use one of the following two simple observations. The first one is trivial and the second one follows from the first one applied to $a_n - (A - \epsilon)n$.

OBSERVATION 3.1. *Let a_n be a sequence of real numbers which is unbounded from above. Then there are infinitely many n such that*

$$a_m < a_n$$

for all $m < n$.

OBSERVATION 3.2. *Let a_n be a sequence of real numbers and assume $A := \limsup a_n/n$ is finite. For any $\epsilon > 0$ there are infinitely many n such that*

$$a_n - a_{n-k} \geq (A - \epsilon)k$$

for all k, $1 \leq k \leq n$.

These type of statements are also useful in other dynamical contexts (cf. Pliss' lemma and Alves' hyperbolic times, see [**A**]).

It is an elementary fact that

$$A := \lim_{n \to \infty} \frac{1}{n} d(y, \phi^n(y))$$

exists. The following two results are generalizations of Wolff's theorem.

THEOREM 3.3. *Let (Y, d) be a proper metric space and let ϕ be a semicontraction. Then any orbit of ϕ lies inside a (horo)ball. In the unbounded orbit case, there is in fact a point ξ in $Y(\infty)$ such that for all $k \geq 0$,*

$$b_\xi(\phi^k(y)) \leq -Ak.$$

In particular,

$$\lim_{n \to \infty} -\frac{1}{n} b_\xi(\phi^n(y)) = A.$$

Proof. Let $a_n = d(y, \phi^n(y))$. In the case of an unbounded orbit, Observation 3.2 implies that given a sequence ϵ_i decreasing to zero we can find indices n_i such that

$$a_{n_i} - a_{n_i - k} \geq (A - \epsilon_i)k$$

for all $k < n_i$, and by compactness we can assume that

$$\phi^{n_i}(y) \to \xi \in Y(\infty).$$

Now let b_ξ be the Busemann function based at ξ with y as the reference point. We have by the triangle inequality that

$$-d(y, \phi^k(y)) \leq b_\xi(\phi^k(y)) = \lim_{i\to\infty} d(\phi^k(y), \phi^{n_i}(y)) - d(\phi^{n_i}(y), y)$$
$$\leq \liminf_{i\to\infty} a_{n_i - k} - a_{n_i} \leq -Ak.$$

This means in particular that the orbit $\{\phi^k(y)\}$ lies inside the horoball $\{z : b_\xi(z) \leq 0\}$ and that the limit in question exists. □

The following statement was proved in [**Be1**] for the special case of Cartan–Hadamard manifolds.

THEOREM 3.4. *Let (Y, d) be a proper metric space satisfying Axiom 4. If ϕ is a semicontraction, then either the orbit is bounded or there is a point ξ in $Y(\infty)$ such that every horoball $\mathcal{H}$ based at ξ, is a ϕ-invariant set, that is $\phi(\mathcal{H}) \subset \mathcal{H}$.*

Proof. By Observation 3.1 and compactness we can find n_i such that

$$d(y, \phi^m(y)) < d(y, \phi^{n_i}(y)),$$

for all $m < n_i$, and

$$\phi^{n_i}(y) \to \xi \in Y(\infty).$$

Now for any z in Y we have

$$\begin{aligned} b_\xi(\phi(z)) &= \lim_{i\to\infty} d(\phi(z), \phi^{n_i}(y)) - d(\phi^{n_i}(y), y) \\ &\leq \liminf_{i\to\infty} d(z, \phi^{n_i - 1}(y)) - d(\phi^{n_i}(y), y) \\ &\leq \liminf_{i\to\infty} d(z, \phi^{n_i - 1}(y)) - d(\phi^{n_i - 1}(y), y) \\ &= b_\xi(z), \end{aligned}$$

since $\phi^{n_i - 1}(y) \to \xi$ by Axiom 4. This proves that all horoballs based at ξ are invariant under ϕ. □

Remark. Given a compactification as in Axiom 1, one can define some kind of horoballs using balls centered at points converging to a point on the boundary, see also the discussion in [**KKR**]. These horoballs may not be as natural, since they depend not only on the boundary point, but, at least *a priori*, also on the sequence of points defining them.

Remark. If ϕ is a contraction and the orbit $\phi^n(y)$ has a bounded subsequence, then there exists a unique fixed point of ϕ, any orbit of ϕ converges to this fixed point and the balls centered at the fixed point are invariant. This is a simple fact; we reproduce the argument in [**Be1**]. First note that $d(\phi^n(y), \phi^{n+1}(y)) \to \delta$ as $n \to \infty$, for some $\delta \geq 0$. Now if there is a convergent subsequence $\phi^{k_i}(y) \to z \in Y$, then by continuity we have that $d(z, \phi(z)) = \delta$ and also $d(\phi(z), \phi^2(z)) = \delta$. Since ϕ is a (strict) contraction $\phi(z) = z$ and $\delta = 0$.

Remark. Let (Y, d) be a uniformly convex, complete metric space. (For a definition of uniform convexity, see for example [**KaMa**].) Then any semicontraction with bounded orbit has a fixed point. This is basically a well-known fact; it can be proved using circumcenters and Zorn's lemma.

From the second remark above and by Observation 3.1 we can now prove the following generalization of the Wolff–Denjoy theorem. It is straightforward to verify that the convergence is uniform on compact sets, see [**Be2**].

THEOREM 3.5. (Cf. [**Be2**]) *Let (Y, d) be a metric space satisfying Axioms 1 and 2. Then the iterates ϕ^n of a contraction ϕ converge uniformly on compact sets in Y to some point ξ in $\overline{Y}$.*

This shows that it is not necessary to make the assumption, as is done in [**Be2**], that ϕ is the pointwise limit of a sequence of contractions ϕ_j, each of which has a fixed point in Y. However, in many concrete situations it is possible to construct such maps ϕ_j. In [**Be2**], Beardon wrote that '[i]t would be desirable to avoid [this assumption] but this seems difficult'.

In view of the second remark above, the following statement, which in the special case of visibility manifolds was proved in [**Be1**], is a corollary of Theorem 3.3.

COROLLARY 3.6. *Let (Y, d) be a proper metric space and assume that every horoball meets $Y(\infty)$ at only one point. Then the iterates $\phi^n(y)$ of a contraction ϕ converge to some point in $\overline{Y}^B$.*

The Schwarz–Pick lemma asserts that the holomorphic map in question is in fact either an isometry or a contraction. The argument in the second remark above is only valid for (strict) contractions. In order to get the analogous result for semicontractions, we use a theorem proved by Calka in [**Ca**]. This result asserts that if there is a bounded subsequence of $\{\phi^n(y)\}$, where ϕ is a semicontraction of a proper metric space, then in fact the whole orbit is bounded.

In view of the theorem of Calka just described, the following statement is now a corollary of Theorem 3.3.

COROLLARY 3.7. *Let (Y, d) be a proper metric space and assume that every horoball meets $Y(\infty)$ at only one point or let (Y, d) be a metric space satisfying Axioms 1 and 2. Assume ϕ is a semicontraction. Then either the orbit of ϕ is bounded or $\phi^n(y)$ converges to a point on the boundary of Y.*

Note that without local compactness this might fail. Edelstein considered the following map of the space l^2 of square summable sequences of complex numbers:

$$\{x_n\} \mapsto \{e^{2\pi i/n!}x_n + 1 - e^{2\pi i/n!}\}.$$

This map is a fixed point free isometry and the orbit of zero is unbounded but accumulates at zero, see [**Ed**].

4. *Cocycles of semicontractions*

Let S be a semigroup of semicontractions of a metric space (Y, d) and fix a point y in Y. Let (X, μ) be a measure space with $\mu(X) = 1$ and let L be an ergodic measure-preserving transformation.

Given a measurable map $w : X \to S$, let

$$u(n, x) = w(x)w(Lx) \cdots w(L^{n-1}x),$$

and $a(n, x) = d(y, u(n, x)y)$. Assume that

$$\int_X d(y, w(x)y)\, d\mu(x) < \infty$$

and let

$$A = \lim_{n\to\infty} \frac{1}{n} \int_X a(n, x)\, d\mu(x).$$

A proof of the following subadditive ergodic lemma was given by Margulis and the present author in [**KaMa**]. This lemma is the key ergodic theoretic ingredient of that paper.

LEMMA 4.1. *For each $\epsilon > 0$, let E_ϵ be the set of x in X for which there exist an integer $K = K(x)$ and infinitely many n such that*

$$a(n, x) - a(n - k, L^k x) \geq (A - \epsilon)k$$

for all k, $K \leq k \leq n$. Then $\mu(\bigcap_{\epsilon>0} E_\epsilon) = 1$.

The following result is now a consequence of the lemma.

THEOREM 4.2. *Assume that (Y, d) satisfies Axioms 1 and 3. If $A > 0$, then for μ-almost every x*

$$u(n, x)y \to \gamma(x) \in \partial Y$$

as $n \to \infty$.

Proof. Pick ϵ such that $0 < A - \epsilon < A$. Fix $x \in E_\epsilon$ and note that obviously $a(n, x) \to \infty$. From Axiom 1 and Lemma 4.1 we can pick n_i such that

$$a(n_i, x) - a(n_i - k, L^k x) \geq (A - \epsilon)k$$

for all k, $K \leq k \leq n_i$, and

$$u(n_i, x)y \to \xi \in \partial Y.$$

For any other convergent subsequence $u(k_j,x)y \to \gamma \in \partial Y$, pick $n_j \in \{n_i\}$ such that $n_j \geq k_j$. We then have, for $k_j \geq K$, that

$$\begin{aligned} &d(u(k_j,x)y, u(n_j,x)y) - d(y,u(n_j,x)y) \\ &\quad \leq d(y,u(n_j-k_j,L^{k_j}x)y) - d(y,u(n_j,x)y) \\ &\quad = a(n_j-k_j,L^{k_j}x) - a(n_j,x) \leq -(A-\epsilon)k_j \end{aligned}$$

and since $k_j \to \infty$ as $j \to \infty$, we get from Axiom 3 that $\gamma = \xi$. □

Note that in the case where there is no randomness ($X = \{x\}$), this is a statement about the iteration of a single semicontraction $\phi = w(x)$. We also want to point out that in some cases it is possible to guarantee that $A > 0$, but in general it is difficult to determine whether $A = 0$ or $A > 0$.

In many cases it is also possible to conclude from Lemma 4.1 that for almost every x there is $\gamma(x) \in Y(\infty)$ such that

$$\lim_{n\to\infty} -\frac{1}{n} b_{\gamma(x)}(u(n,x)y) = A = \lim_{n\to\infty} \frac{1}{n} d(y,u(n,x)y).$$

5. *Gromov hyperbolicity and orbits of semicontractions*

The purpose of this section is to show that compactness is sometimes not needed if assuming Gromov hyperbolicity.

PROPOSITION 5.1. *Let (Y,d) be a Gromov hyperbolic metric space and suppose that ϕ is a semicontraction such that $d(y,\phi^n(y)) \to \infty$. Then the orbit $\phi^n(y)$ converges to a point γ on the hyperbolic boundary of Y.*

Proof. Let $a_n = d(y,\phi^n(y))$. From Observation 3.1, we get a subsequence $\phi^{n_i}(y)$ such that for $k \leq n_i$,

$$\begin{aligned} (\phi^{n_i}(y),\phi^k(y)) &= \tfrac{1}{2}(a_{n_i} + a_k - d(\phi^{n_i}(y),\phi^k(y))) \\ &\geq \tfrac{1}{2}(a_k + a_{n_i} - a_{n_i-k}) \geq \tfrac{1}{2}a_k. \end{aligned}$$

Since by assumption $a_k \to \infty$ and in view of the definition of the hyperbolic boundary, we get (from the above inequality with $k = n_j$) that $\phi^{n_i}(y)$ converges to a point γ on the boundary and then also $\phi^k(y) \to \gamma$ as k tends to infinity. □

The analog of Theorem 4.2 (or rather the theorem in [**KaMa**]) can of course also be obtained. This is well known, a proof was given by Delzant, see [**K**]. The following result was proved in quite a different way in the recent paper [**Bu**].

THEOREM 5.2. [**Bu**] *Let (Y,d) be a Gromov hyperbolic Hadamard space. Any isometry of Y fixes a point in the closure $\overline{Y} = Y \cup \partial Y$.*

Proof. If the orbit is bounded, then the circumcenter of the orbit is a fixed point, see [**Ba**]. Otherwise, the orbit contains an unbounded subsequence $\phi^{n_i}(y)$. Arguing in the same way as in the previous proposition, we can assume that $\phi^{n_i}(y)$ converges to a point γ on the boundary. By continuity and since Axiom 4 holds, any accumulation point on the boundary is a fixed point. □

In the locally compact case this result is simple, well known and holds for any Hadamard space (not necessarily hyperbolic). In contrast, Edelstein's isometry described above (with a minor modification) is a parabolic isometry of the infinite-dimensional Hadamard space l^2 without fixed points in the closure.

Note that for a Gromov hyperbolic Hadamard space it is possible with these arguments to get invariant horoballs in the unbounded orbit case even in the non-locally compact situation.

6. *Examples*

The Poincaré metric can be generalized in several directions; for example, the Carathéodory and Kobayashi pseudometrics on complex spaces, the Hilbert metric on convex domains in vector spaces and the symmetric spaces of non-positive curvature.

By the Hopf–Rinow theorem, any complete, connected, finite-dimensional Riemannian manifold with its usual distance function is a locally compact, complete, geodesic space. Any finitely generated group can be turned into a proper metric space using a word metric.

The Hilbert metric on a strictly convex bounded domain in $\mathbb{R}^k$ (with the usual boundary) satisfies Axioms 1 and 2 (see [**Be2**]). Any visibility manifold, see [**EO'N**] for the definition, satisfies Axiom 2. The Teichmüller space with the Teichmüller metric and the Thurston compactification satisfies Axiom 3 for almost every boundary point, see Lemma 1.4.2 in [**KM**]. It is known that there are Teichmüller geodesic rays starting from one point that stay within bounded distance from each other and that the visual sphere is non-Hausdorff, see [**McCP**]. This situation is similar to the Hilbert metric on a non-strictly convex domain; there are rays which do not diverge, so Axiom 4 fails for the natural (extrinsic) boundary.

Simply connected spaces of non-positive curvature, sometimes called CAT(0)-spaces or (Cartan–)Hadamard spaces, constitute an important class of nice geodesic spaces, see [**Ba**]. The ray boundary of these metric spaces is isomorphic to $Y(\infty)$ and Axiom 4 always holds. Any Hadamard space has a large supply of semicontractions, in fact any point η in $\overline{Y}$ defines a one-parameter family of semicontractions by pushing every point in Y a length t towards η along geodesics. Compare this to Theorem 3.4.

The main theorem of [**KaMa**] asserts, in particular, that the orbit of certain semicontractions of Hadamard spaces (for example any isometry ϕ for which $\inf_{z\in Y} d(z,\phi(z)) > 0$) converges to a point on the boundary. There are also some papers, starting with [**Pa**], that treat similar questions for semicontractions ('non-expansive maps') of Banach spaces.

A system which to each complex space assigns a pseudometric is called a *Schwarz–Pick system* if it assigns the Poincaré metric to the unit disk and any holomorphic map between two spaces semidecreases distances. The Carathéodory and the Kobayashi pseudometrics are two examples. It is known that on convex bounded domains all such Schwarz–Pick pseudometrics coincide. A complex space, for which the Kobayashi pseudeometric actually is a metric, is called a hyperbolic complex space. The book [**Ko**] is a comprehensive account of this theory.

It is reasonable to wonder what geometric properties (such as various convexity properties, the axioms in this paper, Gromov hyperbolicity, etc.) these intrinsic distances enjoy. In general, most properties can fail, which is analogous to the situation with the Hilbert metrics. One known fact is that a complete (Kobayashi) hyperbolic space is a geodesic space. (How do the various boundaries compare?) Another result, proven in [**BBo**], asserts that the Bergman, Carathéodory and Kobayashi metrics on a bounded, strictly pseudoconvex domain with C^2-smooth boundary are hyperbolic in the sense of Gromov. In contrast, Royden's theorem asserts that the Kobayashi metric on the Teichmüller spaces actually coincides with the Teichmüller metric, hence these spaces are hyperbolic in the sense of complex analysis, but Masur and Wolf showed that they are not hyperbolic in the sense of Gromov (for the genus ≥ 2 cases). Some other questions about this metric, for example concerning convexity properties, are still unresolved.

Acknowledgements. I am most grateful to my advisor Professor Gregory Margulis for all his encouragement, support and guidance. I also owe many thanks to Professor Torbjörn Kolsrud. I was an Alfred P. Sloan Doctoral Dissertation Fellow at Yale University when writing this paper. I also received support from EU TMR project Stochastic Analysis and Applications, ERB-FMRX-CT96-0075. Finally, I thank the referee for useful comments.

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