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NON-FORMAL CO-SYMPLECTIC MANIFOLDS

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ABSTRACT. We study the formality of the mapping torus of an orientation-preserving diffeomorphism of a manifold. In particular, we give conditions under which a mapping torus has a non-zero Massey product. As an application we prove that there are non-formal compact co-symplectic manifolds of dimension m and with first Betti number b if and only if $m = 3$ and $b \geq 2$, or $m \geq 5$ and $b \geq 1$. Explicit examples for each one of these cases are given.

1. INTRODUCTION

In this paper we follow the nomenclature of [19], where *co-symplectic* manifolds are the odd-dimensional counterparts to symplectic manifolds. In terms of differential forms, a *co-symplectic structure* on a $(2n+1)$-dimensional manifold M is determined by a pair (F, η) of closed differential forms, where F is a 2-form and η is a 1-form such that $\eta \wedge F^n$ is a volume form, so that M is orientable. In this case, we say that (M, F, η) is a *co-symplectic manifold*. Earlier, such a manifold was called co-symplectic by Libermann [20], or almost-co-symplectic by Goldberg and Yano [17].

The simplest examples of co-symplectic manifolds are the manifolds called *co-Kähler* by Li in [19], or co-symplectic by Blair [3]. Such a manifold is locally a product of a Kähler manifold with a circle or a line. In fact, a *co-Kähler structure* on a $(2n+1)$-dimensional manifold M is a normal almost contact metric structure (ϕ, η, ξ, g) on M, that is, a tensor field ϕ of type $(1,1)$, a 1-form η, a vector field ξ (the Reeb vector field) with $\eta(\xi) = 1$, and a Riemannian metric g satisfying certain conditions (see section 3 for details) such that the 1-form η and the fundamental 2-form F given by $F(X, Y) = g(\phi X, Y)$, for any vector fields X and Y on M, are closed.

The topological description of co-symplectic and co-Kähler manifolds is due to Li [19]. There he proves that a compact manifold M has a co-symplectic structure if and only if M is the mapping torus of a symplectomorphism of a symplectic manifold, while M has a co-Kähler structure if and only if M is a Kähler mapping torus, that is, M is the mapping torus of a Hermitian isometry on a Kähler manifold. This result may be considered an extension to co-symplectic and co-Kähler manifolds of

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Tischler's Theorem [25] that asserts that a compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form.

The existence of a co-Kähler structure on a manifold M imposes strong restrictions on the underlying topology of M. Indeed, since co-Kähler manifolds are odd-dimensional analogues of Kähler manifolds, several known results from Kähler geometry carry over to co-Kähler manifolds. In particular, every compact co-Kähler manifold is formal. Another similarity is the monotone property for the Betti numbers of compact co-Kähler manifolds [7].

Intuitively, a simply connected manifold is formal if its rational homotopy type is determined by its rational cohomology algebra. Simply connected compact manifolds of dimension less than or equal to 6 are formal [13, 23]. We shall say that M is *formal* if its minimal model is formal or, equivalently, if the de Rham complex $(\Omega M, d)$ of M and the algebra of the de Rham cohomology $(H^*(M), d = 0)$ have the same minimal model (see section 2 for details).

It is well known that the existence of a non-zero Massey product is an obstruction to formality. In [13] the concept of formality is extended to a weaker notion called s-formality. There, the second and third authors prove that an orientable compact connected manifold, of dimension $2n$ or $2n-1$, is formal if and only if it is $(n-1)$-formal.

The importance of formality in symplectic geometry stems from the fact that it allows one to distinguish symplectic manifolds which admit Kähler structures from those which do not [8, 15, 24]. It seems thus interesting to analyze what happens for co-symplectic manifolds. In this paper we consider the following problem on the *geography* of co-symplectic manifolds:

> For which pairs $(m = 2n+1, b)$, with $n, b \geq 1$, are there compact co-symplectic manifolds of dimension m and with $b_1 = b$ which are non-formal?

We address this question in section 5. It will turn out that the answer is the same as for compact manifolds [14], i.e., that there are always non-formal examples except for $(m, b) = (3, 1)$.

On any compact co-symplectic manifold M, the first Betti number must satisfy $b_1(M) \geq 1$, since the $(2n+1)$-form $\eta \wedge F^n$ defines a non-zero cohomology class on M, and hence η defines a cohomology class $[\eta] \neq 0$. It is known that any orientable compact manifold of dimension ≤ 4 and with first Betti number $b_1 = 1$ is formal [14].

The main problem in order to answer the question above is to construct examples of non-formal compact co-symplectic manifolds of dimension $m = 3$ with $b_1 \geq 3$ as well as examples of dimension $m = 5$ with $b_1 = 1$. The other cases are covered in section 5, using essentially the 3-dimensional Heisenberg manifold to obtain non-formal co-symplectic manifolds of dimension $m \geq 3$ and with $b_1 = 2$ as well as non-formal co-symplectic manifolds of dimension $m \geq 5$ and with $b_1 \geq 2$, or from the non-formal compact simply connected symplectic manifold of dimension 8 given in [15] to exhibit non-formal co-symplectic manifolds of dimension $m \geq 9$ and with $b_1 = 1$.

To fill the gaps, we study in section 4 the formality of a (not necessarily symplectic) mapping torus N_φ obtained from $N \times [0,1]$ by identifying $N \times \{0\}$ with $\varphi(N) \times \{1\}$, where φ is a self-diffeomorphism of N. The description of a minimal model for a mapping torus can be very complicated even for low degrees. Nevertheless, in Theorem 15 we determine a minimal model of N_φ up to some degree $p \geq 2$ when φ satisfies some extra conditions, namely that the map induced on cohomology $\varphi^* : H^k(N) \to H^k(N)$ does not have the eigenvalue $\lambda = 1$, for any $k \leq (p-1)$, but $\varphi^* : H^p(N) \to H^p(N)$ has the eigenvalue $\lambda = 1$ with multiplicity $r \geq 1$. In particular (see Corollary 16), we show that if $r = 1$, N_φ is p-formal in the sense mentioned above.

Moreover, in Theorem 13 we prove that N_φ has a non-zero (triple) Massey product if there exists $p > 0$ such that the map

$$\varphi^* : H^p(N) \to H^p(N)$$

has the eigenvalue $\lambda = 1$ with multiplicity 2. In fact, we show that the Massey product $\langle [dt], [dt], [\tilde{\alpha}] \rangle$ is well-defined on N_φ and it does not vanish, where dt is the 1-form defined on N_φ by the volume form on S^1, and $[\tilde{\alpha}] \in H^p(N_\varphi)$ is the cohomology class induced on N_φ by a certain cohomology class $[\alpha] \in H^p(N)$ fixed by φ^*.

Regarding symplectic mapping torus manifolds, first we notice that if N is a compact symplectic $2n$-manifold, and $\varphi : N \to N$ is a symplectomorphism, then the map induced on cohomology $\varphi^* : H^2(N) \to H^2(N)$ always has the eigenvalue $\lambda = 1$. As a consequence of Theorem 15, we get that if N_φ is a symplectic mapping torus such that the map $\varphi^* : H^1(N) \to H^1(N)$ does not have the eigenvalue $\lambda = 1$, then N_φ is 2-formal if and only if the eigenvalue $\lambda = 1$ of $\varphi^* : H^2(N) \to H^2(N)$ has multiplicity $r = 1$. Thus, in these conditions, the co-symplectic manifold N_φ is formal when N has dimension four.

In section 5, using Theorem 13, we solve the case $m = 3$ with $b_1 \geq 3$ taking the mapping torus of a symplectomorphism of a surface of genus $k \geq 2$ (see Proposition 20). For $m = 5$ and $b_1 = 1$ we consider the mapping torus of a symplectomorphism of a 4-torus (see Proposition 22).

Let G be a connected, simply connected solvable Lie group, and let $\Gamma \subset G$ be a discrete, co-compact subgroup. Then $M = \Gamma\backslash G$ is a solvmanifold. The manifold constructed in Proposition 22 is not a solvmanifold according to our definition. However, it is the quotient of a solvable Lie group by a closed subgroup. In section 6 we present an explicit example of a non-formal compact co-symplectic 5-dimensional manifold S, with first Betti number $b_1(S) = 1$, which is a solvmanifold. We describe S as the mapping torus of a symplectomorphism of a 4-torus, so this example fits in the scope of Proposition 22.

2. Minimal models and formality

In this section we recall some fundamental facts of the theory of minimal models. For more details, see [9], [10] and [11].

We work over the field $\mathbb{R}$ of real numbers. Recall that a *commutative differential graded algebra* (A, d) (CDGA for short) is a graded algebra $A = \bigoplus_{k \geq 0} A^k$ which is graded commutative, i.e. $x \cdot y = (-1)^{|x||y|} y \cdot x$ for homogeneous elements x and y, together with a differential $d : A^k \to A^{k+1}$ such that $d^2 = 0$ and $d(x \cdot y) = dx \cdot y + (-1)^{|x|} x \cdot dy$ (here $|x|$ denotes the degree of the homogeneous element x).

Morphisms of CDGAs are required to preserve the degree and to commute with the differential. Notice that the cohomology of a CDGA is an algebra which can be turned into a CDGA by endowing it with the zero differential. A CDGA is said to be *connected* if $H^0(A,d) \cong \mathbb{R}$. The main example of a CDGA is the de Rham complex of a smooth manifold M, $(\Omega^*(M),d)$, where d is the exterior differential.

A CDGA (A,d) is said to be *minimal* (or *Sullivan*) if the following happens:

- $A = \bigwedge V$ is the free commutative algebra generated by a graded (real) vector space $V = \bigoplus_k V^k$;
- there exists a basis $\{x_i,\ i \in \mathcal{J}\}$ of V, for a well-ordered index set $\mathcal{J}$, such that $|x_i| \leq |x_j|$ if $i<j$ and the differential of a generator x_j is expressed in terms of the preceding x_i $(i<j)$; in particular, dx_j does not have a linear part.

We have the following fundamental result:

Proposition 1. *Every connected CDGA (A,d) has a minimal model, that is, there exist a minimal algebra $(\bigwedge V,d)$ together with a morphism of CDGAs $\varphi : (\bigwedge V,d) \to (A,d)$ which induces an isomorphism $\varphi^* : H^*(\bigwedge V,d) \to H^*(A,d)$. The minimal model is unique.*

The (real) *minimal model* of a differentiable manifold M is by definition the minimal model of its de Rham algebra $(\Omega^*(M),d)$.

Recall that a minimal algebra $(\bigwedge V,d)$ is *formal* if there exists a morphism of differential algebras $\psi\colon (\bigwedge V,d) \longrightarrow (H^*(\bigwedge V),0)$ that induces the identity on cohomology. Also a differentiable manifold M is *formal* if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, homogeneous spaces, flag manifolds, and compact Kähler manifolds.

In [9], the formality of a minimal algebra is characterized as follows.

Proposition 2. *A minimal algebra $(\bigwedge V,d)$ is formal if and only if the space V can be decomposed as a direct sum $V = C \oplus N$ with $d(C)=0$, d injective on N and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [13].

Definition 3. A minimal algebra $(\bigwedge V,d)$ is s-formal $(s>0)$ if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

(1) $d(C^i)=0$,
(2) the differential map $d\colon N^i \to \bigwedge V$ is injective,
(3) any closed element in the ideal $I_s = I(\bigoplus_{i\leq s} N^i)$, generated by the space $\bigoplus_{i\leq s} N^i$ in the free algebra $\bigwedge(\bigoplus_{i\leq s} V^i)$, is exact in $\bigwedge V$.

A smooth manifold M is s-formal if its minimal model is s-formal. Clearly, if M is formal, then M is s-formal, for any $s>0$. The main result of [13] shows that sometimes the weaker condition of s-formality implies formality.

Theorem 4. *Let M be a connected and orientable compact differentiable manifold of dimension $2n$, or $(2n-1)$. Then M is formal if and only if is $(n-1)$-formal.*

In order to detect non-formality, instead of computing the minimal model, which usually is a lengthy process, we can use Massey products, which are obstructions to formality. Let us recall their definition. The simplest type of Massey product is the triple Massey product. Let (A,d) be a CDGA and suppose $a,b,c \in H^*(A)$

are three cohomology classes such that $a \cdot b = b \cdot c = 0$. Take co-cycles x, y and z representing these cohomology classes and let s, t be elements of A such that

$$ds = (-1)^{|x|} x \cdot y, \quad dt = (-1)^{|y|} y \cdot z.$$

Then one checks that

$$w = (-1)^{|x|} x \cdot t + (-1)^{|x|+|y|-1} s \cdot z$$

is a co-cycle. The choice of different representatives gives an indeterminacy, represented by the space

$$\mathcal{I} = a \cdot H^{|y|+|z|-1}(A) + H^{|x|+|y|-1}(A) \cdot c.$$

We denote by $\langle a, b, c \rangle$ the image of the co-cycle w in $H^*(A)/\mathcal{I}$. As is proven in [9] (and which is essentially equivalent to Proposition 2), if a minimal CDGA is formal, then one can make uniform choices of co-cycles so that the classes representing (triple) Massey products are exact. In particular, if the real minimal model of a manifold contains a non-zero Massey product, then the manifold is not formal.

3. Co-symplectic manifolds

In this section we recall some definitions and results about co-symplectic manifolds, and we extend to co-symplectic Lie algebras the result of Fino-Vezzoni [16] for co-Kähler Lie algebras.

Definition 5. Let M be a $(2n+1)$-dimensional manifold. An *almost contact metric structure* on M consists of a quadruplet (ϕ, ξ, η, g), where ϕ is an endomorphism of the tangent bundle TM, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M satisfying the conditions

$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

for $X, Y \in \Gamma(TM)$.

Thus, ϕ maps the distribution $\ker(\eta)$ to itself and satisfies $\phi(\xi) = 0$. We call (M, ϕ, η, ξ, g) an *almost contact metric* manifold. The *fundamental* 2*-form* F on M is defined by

$$F(X, Y) = g(\phi X, Y),$$

for $X, Y \in \Gamma(TM)$.

Therefore, if (ϕ, ξ, η, g) is an almost contact metric structure on M with fundamental 2-form F, then $\eta \wedge F^n \neq 0$ everywhere. Conversely (see [4]), if M is a differentiable manifold of dimension $2n + 1$ with a 2-form F and a 1-form η such that $\eta \wedge F^n$ is a volume form on M, then there exists an almost contact metric structure (ϕ, ξ, η, g) on M having F as the fundamental form.

There are different classes of structures that can be considered on M in terms of F and η and their covariant derivatives. We recall here those that are needed in the present paper:

- M is *co-symplectic* iff $dF = d\eta = 0$;
- M is *normal* iff the Nijenhuis torsion N_ϕ satisfies $N_\phi = -2d\eta \otimes \xi$;
- M is *co-Kähler* iff it is normal and co-symplectic or, equivalently, ϕ is parallel,

where the Nijenhuis torsion N_ϕ is given by

$$N_\phi(X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for $X, Y \in \Gamma(TM)$.

In the literature, co-symplectic manifolds are often called *almost co-symplectic*, while co-Kähler manifolds are called *co-symplectic* (see [3, 5, 7, 16]).

Let us recall that a *symplectic* manifold (M, ω) is a pair consisting of a $2n$-dimensional differentiable manifold M with a closed 2-form ω which is non-degenerate (that is, ω^n never vanishes). The form ω is called symplectic. The following well-known result shows that co-symplectic manifolds are really the odd-dimensional analogue of symplectic manifolds; a proof can be found in Proposition 1 of [19].

Proposition 6. *A manifold M admits a co-symplectic structure if and only if the product $M \times S^1$ admits an S^1-invariant symplectic form.*

A theorem by Tischler [25] asserts that a compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form. This result was extended recently to co-symplectic manifolds by Li [19]. Let us first recall some definitions.

Let N be a differentiable manifold and let $\varphi : N \to N$ be a diffeomorphism. The *mapping torus N_φ* of φ is the manifold obtained from $N \times [0,1]$ by identifying the ends with φ, that is,

$$N_\varphi = \frac{N \times [0,1]}{(x,0) \sim (\varphi(x),1)}.$$

It is a differentiable manifold, because it is the quotient of $N \times \mathbb{R}$ by the infinite cyclic group generated by $(x,t) \to (\varphi(x), t+1)$. The natural map $\pi \colon N_\varphi \to S^1$ defined by $\pi(x,t) = e^{2\pi i t}$ is the projection of a locally trivial fiber bundle.

Definition 7. Let N_φ be a mapping torus of a diffeomorphism φ of N. We say that N_φ is a *symplectic* mapping torus if (N, ω) is a symplectic manifold and $\varphi : N \to N$ a symplectomorphism, that is, $\varphi^*\omega = \omega$.

Theorem 8 ([19, Theorem 1])**.** *A compact manifold M admits a co-symplectic structure if and only if it is a symplectic mapping torus $M = N_\varphi$.*

Notice that if M is a symplectic mapping torus $M = N_\varphi$, then the pair (F, η) defines a co-symplectic structure on M, where F is the closed 2-form on M defined by the symplectic form on N, and

$$\eta = \pi^*(\theta),$$

with θ the volume form on S^1. Moreover, notice that any 3-dimensional mapping torus is a symplectic mapping torus if the corresponding diffeomorphism preserves the orientation, since such a diffeomorphism is isotopic to an area-preserving one. However, in higher dimensions, there exist mapping tori without co-symplectic structures. That is, they are not symplectic mapping tori (see Remark 19 in section 5 and [19]).

Next, we consider a Lie algebra $\mathfrak{g}$ of dimension $2n+1$ with an *almost contact metric* structure, that is, with a quadruplet (ϕ, ξ, η, g), where ϕ is an endomorphism of $\mathfrak{g}$, ξ is a non-zero vector in $\mathfrak{g}$, $\eta \in \mathfrak{g}^*$ and g is a scalar product in $\mathfrak{g}$, satisfying (1). Then, $\mathfrak{g}$ is said to be *co-symplectic* iff $dF = d\eta = 0$; and $\mathfrak{g}$ is called *co-Kähler* iff it is normal and co-symplectic, where $d : \bigwedge^k \mathfrak{g}^* \to \bigwedge^{k+1} \mathfrak{g}^*$ is the Chevalley-Eilenberg differential.

The following result is proved in [16].

Proposition 9. *Co-Kähler Lie algebras in dimension $2n+1$ are in one-to-one correspondence with $2n$-dimensional Kähler Lie algebras endowed with a skew-adjoint derivation D which commutes with its complex structure.*

In order to extend this correspondence to co-symplectic Lie algebras we need to recall the following. Let (V,ω) be a symplectic vector space (hence ω is a skew-symmetric invertible matrix). An element $A \in \mathfrak{gl}(V)$ is an infinitesimal symplectic transformation if $A \in \mathfrak{sp}(V)$, that is, if

$$A^t\omega + \omega A = 0.$$

A scalar product g on (V,ω) is said to be compatible with ω if the endomorphism $J : V \to V$ defined by $\omega(u,v) = g(u, Jv)$ satisfies $J^2 = -\mathrm{Id}$. We prove the following:

Proposition 10. *Co-symplectic Lie algebras of dimension $2n+1$ are in one-to-one correspondence with $2n$-dimensional symplectic Lie algebras endowed with a compatible metric and a derivation D which is an infinitesimal symplectic transformation.*

Proof. Let (ϕ, ξ, η, g) be a co-symplectic structure on a Lie algebra $\mathfrak{g}$ of dimension $2n+1$. Set $\mathfrak{h} = \ker(\eta)$. For $u, v \in \mathfrak{h}$ we compute

$$\eta([u,v]) = -d\eta(u,v) = 0,$$

since η is closed (this is simply Cartan's formula applied to the case in which $\eta(u)$ and $\eta(v)$ are constant). Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Note that $\mathfrak{h}$ inherits an almost complex structure J and a metric g which are compatible. From ϕ and g we obtain the 2-form ω which is closed and non-degenerate by hypothesis. Thus $(\mathfrak{h}, \omega)$ is a symplectic Lie algebra.

Actually $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Indeed, the fact that $\eta(\xi) = 1$ implies that ξ does not belong to $[\mathfrak{g}, \mathfrak{g}]$, and then one has

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \quad \text{and} \quad [\xi, \mathfrak{h}] \subseteq \mathfrak{h}.$$

Thus one can write

$$\mathfrak{g} = \mathbb{R}\xi \oplus_{\mathrm{ad}_\xi} \mathfrak{h}.$$

Since ω is closed, we obtain

$$\begin{aligned} 0 = d\omega(\xi, u, v) &= -\omega([\xi, u], v) + \omega([u, v], \xi) - \omega([v, \xi], u) \\ &= -\omega(\mathrm{ad}_\xi(u), v) - \omega(u, \mathrm{ad}_\xi(v)). \end{aligned} \tag{2}$$

The correspondence $X \mapsto \mathrm{ad}_\xi(X)$ gives a derivation D of $\mathfrak{h}$ (this follows from the Jacobi identity in $\mathfrak{g}$), and the above equality shows that D is an infinitesimal symplectic transformation.

Next suppose we are given a symplectic Lie algebra $(\mathfrak{h}, \omega)$ endowed with a metric g and a derivation $D \in \mathfrak{sp}(\mathfrak{h})$. Set

$$\mathfrak{g} = \mathbb{R}\xi \oplus \mathfrak{h}$$

and define the following Lie algebra structure on $\mathfrak{g}$:

$$[u,v] := [u,v]_{\mathfrak{h}}, \quad [\xi, u] := D(u), \quad u, v \in \mathfrak{h}.$$

Since D is a derivation of $\mathfrak{h}$, the Jacobi identity holds in $\mathfrak{g}$. Let J denote the almost complex structure compatible with ω and g. Extend J to an endomorphism ϕ of $\mathfrak{g}$ setting $\phi(\xi) = 0$ and extend g so that ξ has length 1 and ξ is orthogonal to $\mathfrak{h}$. Also, let η be the dual 1-form with respect to the metric g. It is immediate to see that

$d\eta = 0$. On the other hand, equation (2) shows that $d\omega = 0$ as D is an infinitesimal symplectic transformation. Thus $\mathfrak{g}$ is a co-symplectic Lie algebra. $\square$

Remark 11. If one wants to obtain a co-symplectic *nilpotent* Lie algebra, then the initial data in Proposition 10 must be modified so that the symplectic Lie algebra and the derivation D are nilpotent. This gives a way to classify co-symplectic nilpotent Lie algebras in dimension $2n+1$ starting from nilpotent symplectic Lie algebras in dimension $2n$ and a nilpotent symplectic derivation.

4. Minimal models of mapping tori

In this section we study the formality of the mapping torus of an orientation-preserving diffeomorphism of a manifold. We start with some useful results.

Lemma 12. *Let N be a smooth manifold and let $\varphi : N \to N$ be a diffeomorphism. Let $M = N_\varphi$ denote the mapping torus of φ. Then the cohomology of M sits in an exact sequence*

$$0 \to C^{p-1} \to H^p(M) \to K^p \to 0,$$

where K^p is the kernel of $\varphi^* - \mathrm{Id}\colon H^p(N) \to H^p(N)$, and C^p is its cokernel.

Proof. This is a simple application of the Mayer-Vietoris sequence. Take U, V as two open intervals covering $S^1 = [0,1]/0 \sim 1$, where $U \cap V$ is the disjoint union of two intervals. Let $U' = \pi^{-1}(U)$, $V' = \pi^{-1}(V)$. Then $H^p(U') \cong H^p(N)$, $H^p(V') \cong H^p(N)$ and $H^p(U' \cap V') \cong H^p(N) \oplus H^p(N)$. The Mayer-Vietoris sequence associated to this covering becomes

$$\begin{aligned} &\ldots \to H^p(M) \to H^p(N) \oplus H^p(N) \xrightarrow{F} H^p(N) \oplus H^p(N) \to H^{p+1}(M) \\ &\quad \to H^{p+1}(N) \oplus H^{p+1}(N) \to \ldots, \end{aligned} \tag{3}$$

where the map F is $([\alpha],[\beta]) \mapsto ([\alpha]-[\beta], [\alpha] - \varphi^*[\beta])$.

Write

$$K = \ker\Big(\varphi^* - \mathrm{Id} : H^*(N) \to H^*(N)\Big) \text{ and } C = \operatorname{coker}\Big(\varphi^* - \mathrm{Id} : H^*(N) \to H^*(N)\Big).$$

These are graded vector spaces $K = \bigoplus K^p$, $C = \bigoplus C^p$. The exact sequence (3) then yields an exact sequence $0 \to C^{p-1} \to H^p(M) \to K^p \to 0$. $\square$

Let us look more closely at the exact sequence in Lemma 12. First take $[\beta] \in C^{p-1}$. Then $[\beta]$ can be thought of as an element in $H^{p-1}(N)$ modulo $\mathrm{Im}\,(\varphi^* - \mathrm{Id})$. The map $C^{p-1} \to H^p(M)$ in Lemma 12 is the connecting homomorphism δ^*. This is worked out as follows (see [6]): take a smooth function $\rho(t)$ on U which equals 1 in one of the intervals of $U \cap V$ and zero on the other. Then

$$\delta^*[\beta] = [d\rho \wedge \beta]. \tag{4}$$

Write $\tilde{\beta} = d\rho \wedge \beta$. If we put the point $t = 0$ in $U \cap V$, then clearly $\tilde{\beta}(x,0) = \tilde{\beta}(x,1) = 0$, so $\tilde{\beta}$ is a well-defined closed p-form on M. (Note that $[d\rho] = [\eta] \in H^1(S^1)$, where $\eta = \pi^*(\theta) = dt$, so $[\tilde{\beta}] \in H^p(M)$ is $[\eta \wedge \beta]$.)

On the other hand, if $[\alpha] \in K^p$, then $\varphi^*[\alpha] = [\alpha]$. So $\varphi^*\alpha = \alpha + d\theta$, for some $(p-1)$-form θ. Let us take a function $\rho : [0,1] \to [0,1]$ such that $\rho \equiv 0$ near $t = 0$ and $\rho \equiv 1$ near $t = 1$. Then, the closed p-form $\tilde{\alpha}$ on $N \times [0,1]$ given by

$$\tilde{\alpha}(x,t) = \alpha(x) + d(\rho(t)\theta(x)), \tag{5}$$

where $x \in N$ and $t \in [0,1]$, defines a closed p-form $\tilde{\alpha}$ on M. Indeed, $\varphi^*\tilde{\alpha}(x,0) = \varphi^*\alpha = \alpha + d\theta = \tilde{\alpha}(x,1)$. Moreover, the class $[\tilde{\alpha}] \in H^p(M)$ restricts to $[\alpha] \in H^p(N)$. This gives a splitting

$$H^p(M) \cong C^{p-1} \oplus K^p.$$

Theorem 13. *Let N be an oriented compact smooth manifold of dimension n, and let $\varphi : N \to N$ be an orientation-preserving diffeomorphism. Let $M = N_\varphi$ be the mapping torus of φ. Suppose that, for some $p > 0$, the homomorphism $\varphi^* : H^p(N) \to H^p(N)$ has eigenvalue $\lambda = 1$ with multiplicity*[1] *two. Then M is non-formal since there exists a non-zero (triple) Massey product. More precisely, if $[\alpha] \in K^p \subset H^p(N)$ is such that*

$$[\alpha] \in \mathrm{Im}\Big(\varphi^* - \mathrm{Id} : H^p(N) \to H^p(N)\Big),$$

then the Massey product $\langle [\eta], [\eta], [\tilde{\alpha}] \rangle$ does not vanish.

Proof. First, we notice that if the eigenvalue $\lambda = 1$ of $\varphi^* : H^p(N) \to H^p(N)$ has multiplicity two, then there exists $[\alpha] \in H^p(N)$ satisfying the conditions mentioned in Theorem 13. In fact, denote by

$$E = \ker(\varphi^* - \mathrm{Id})^2$$

the graded eigenspace corresponding to $\lambda = 1$. Then $K = \ker(\varphi^* - \mathrm{Id}) \subset E$ is a proper subspace. Take

$$[\beta] \in E^p \setminus K^p \subset H^p(N) \quad \text{and} \quad [\alpha] = \varphi^*[\beta] - [\beta]. \tag{6}$$

Thus $[\alpha] \in K^p \cap \mathrm{Im}\Big(\varphi^* - \mathrm{Id} : H^p(N) \to H^p(N)\Big)$. By (4) and Lemma 12, the Massey product $\langle [\eta], [\eta], [\tilde{\alpha}] \rangle$ is well-defined. In order to prove that it is non-zero we proceed as follows. Clearly,

$$C \cong E/I, \quad \text{where} \quad I = \mathrm{Im}(\varphi^* - \mathrm{Id}) \cap E.$$

As φ is an orientation-preserving diffeomorphism, the Poincaré duality pairing satisfies that $\langle \varphi^*(u), \varphi^*(v) \rangle = \langle u, v \rangle$, for $u \in H^p(N)$, $v \in H^{n-p}(N)$. Therefore the λ-eigenspace of φ^*, E_λ, pairs non-trivially only with $E_{1/\lambda}$. In particular, Poincaré duality gives a perfect pairing,

$$E^p \times E^{n-p} \to \mathbb{R}.$$

Now $K^p \times I^{n-p}$ is sent to zero: if $x \in \ker(\varphi^* - \mathrm{Id})$ and $y = \varphi^*(z) - z$, then $\langle x, y \rangle = \langle x, \varphi^*(z) - z \rangle = \langle x, \varphi^*(z) \rangle - \langle x, z \rangle = \langle \varphi^*(x), \varphi^*(z) \rangle - \langle x, z \rangle = 0$. Therefore there is a perfect pairing

$$E^p/K^p \times I^{n-p} \to \mathbb{R}.$$

Take $[\beta]$ and $[\alpha]$ as in (6). By the discussion above about Poincaré duality, there is some $[\xi] \in I^{n-p}$ such that

$$\langle [\beta], [\xi] \rangle \neq 0.$$

Note that in particular, $[\xi]$ pairs trivially with all elements in K^p.

Now consider the form $\tilde{\alpha}$ on M corresponding to α as in (5), $[\tilde{\alpha}] \in H^p(M)$. Let us take the p-form γ on N defined by

$$\gamma = \int_0^1 \tilde{\alpha}(x,s)\,ds.$$

[1] In this paper, by *multiplicity* of the eigenvalue λ of an endomorphism $A : V \to V$ we mean the multiplicity of λ as a root of the minimal polynomial of A.

Then $[\gamma] = [\alpha] = \varphi^*[\beta] - [\beta]$ on N. Hence we can write

$$\gamma = \varphi^*\beta - \beta + d\sigma,$$

for some $(p-1)$-form σ on N. Now let us set

$$\tilde{\gamma}(x,t) = \left(\int_0^t \tilde{\alpha}(x,s)ds\right) + \beta + d(\zeta(t)(\varphi^*)^{-1}\sigma),$$

where $\zeta(t)$, $t \in [0,1]$, equals 1 near $t=0$, and equals 0 near $t=1$. Then

$$\varphi^*(\tilde{\gamma}(x,0)) = \varphi^*(\beta + d((\varphi^*)^{-1}\sigma)) = \varphi^*\beta + d\sigma = \gamma + \beta = \tilde{\gamma}(x,1),$$

so $\tilde{\gamma}$ is a well-defined p-form on M. Moreover,

$$d(\tilde{\gamma}(x,t)) = dt \wedge \tilde{\alpha}(x,t)$$

on the mapping torus M. Therefore we have the Massey product

$$\langle [dt], [dt], [\tilde{\alpha}] \rangle = [dt \wedge \tilde{\gamma}]. \tag{7}$$

We need to see that this Massey product is non-zero. For this, we multiply against $[\tilde{\xi}]$, where $\tilde{\xi}$ is the $(n-p)$-form on M associated to ξ by the formula (5). Recall that $[\xi] \in I^{n-p} \subset K^{n-p} \subset H^{n-p}(M)$. We have

$$\langle [dt \wedge \tilde{\gamma}], [\tilde{\xi}] \rangle = \int_M dt \wedge \tilde{\gamma} \wedge \tilde{\xi} = \int_0^1 \left(\int_{N \times \{t\}} \tilde{\gamma} \wedge \tilde{\xi} \right) dt\,.$$

Restricting to the fibers, we have $[\tilde{\gamma}|_{N\times\{t\}}] = t[\alpha] + [\beta]$ and $[\tilde{\xi}|_{N\times\{t\}}] = [\xi]$. Moreover, $\langle [\alpha], [\xi] \rangle = 0$ and $\langle [\beta], [\xi] \rangle = \kappa \neq 0$. So $\int_{N\times\{t\}} \tilde{\gamma} \wedge \tilde{\xi} = \kappa \neq 0$. Therefore

$$\langle [dt \wedge \tilde{\gamma}], [\tilde{\xi}] \rangle = \kappa \neq 0\,.$$

Now the indeterminacy of the Massey product is in the space

$$\mathcal{I} = [\tilde{\alpha}] \wedge H^1(M) + [\eta] \wedge H^p(M).$$

To see that the Massey product (7) does not live in $\mathcal{I}$, it is enough to see that the elements in $\mathcal{I}$ pair trivially with $[\tilde{\xi}]$. On the one hand, $\tilde{\alpha} \wedge \tilde{\xi}$ is exact in every fiber (since $\langle [\alpha], [\xi] \rangle = 0$ on N). Therefore $[\tilde{\alpha}] \wedge [\tilde{\xi}] = 0$. On the other hand, $H^p(M) \cong C^{p-1} \oplus K^p$. The elements corresponding to C^{p-1} all have a dt-factor. Hence the elements in $[\eta] \wedge H^p(M)$ are of the form $[dt \wedge \tilde{\delta}]$, for some $[\delta] \in K^p \subset H^p(N)$. But then $\langle [dt \wedge \tilde{\delta}], [\tilde{\xi}] \rangle = \int_M dt \wedge \tilde{\delta} \wedge \tilde{\xi} = \langle [\delta], [\xi] \rangle = 0$. $\square$

Remark 14. The non-formality of the mapping torus M is proved in [12, Proposition 9] when $p = 1$ and the eigenvalue $\lambda = 1$ has multiplicity $r \geq 2$, by a different method.

We finish this section with the following result, which gives a partial computation of the minimal model of M.

From now on we write

$$\varphi_k^* : H^k(N) \to H^k(N),$$

for each $1 \leq k \leq n$, the induced morphism on cohomology by a diffeomorphism $\varphi : N \to N$.

Theorem 15. *With $M = N_\varphi$ as above, suppose that there is some $p \geq 2$ such that φ_k^* does not have the eigenvalue $\lambda = 1$ (i.e. $\varphi_k^* - \mathrm{Id}$ is invertible) for any $k \leq (p-1)$, and that φ_p^* does have the eigenvalue $\lambda = 1$ with some multiplicity $r \geq 1$. Denote*

$$K_j = \ker\Big((\varphi_p^* - \mathrm{Id})^j : H^p(N) \to H^p(N)\Big),$$

for $j = 0, \dots, r$. So $\{0\} = K_0 \subset K_1 \subset K_2 \subset \dots \subset K_r$. Write $G_j = K_j/K_{j-1}$, $j = 1, \dots, r$. The map $F = \varphi_p^* - \mathrm{Id}$ induces maps $F : G_j \to G_{j-1}$, $j = 1, \dots, r$ (here $G_0 = 0$).

Then the minimal model of M is, up to degree p, given by the following generators:

$$\begin{aligned} &W^1 = \langle a \rangle, \qquad da = 0, \\ &W^k = 0, \qquad k = 2, \dots, p-1, \\ &W^p = G_1 \oplus G_2 \oplus \dots \oplus G_r, \qquad dw = a \cdot F(w),\, w \in G_j. \end{aligned}$$

Proof. We need to construct a map of differential algebras

$$\rho : (\bigwedge(W^1 \oplus W^p), d) \to (\Omega^*(M), d)$$

which induces an isomorphism in cohomology up to degree p and an injection in degree $p+1$ (see [9]). By Lemma 12, we have that

$$\begin{aligned} H^1(M) &= \langle [dt] \rangle, \\ H^k(M) &= 0, \qquad 2 \le k \le p-1, \\ H^p(M) &= \ker(\varphi_p^* - \mathrm{Id}) = K_1, \\ H^{p+1}(M) &= \big([dt] \wedge \mathrm{coker}(\varphi_p^* - \mathrm{Id})\big) \oplus \ker(\varphi_{p+1}^* - \mathrm{Id}). \end{aligned}$$

We start by setting $\rho(a) = dt$, where t is the coordinate of $[0,1]$ in the description

$$M = (N \times [0,1])/(x,0) \sim (\varphi(x), 1)\,.$$

This automatically gives that ρ induces an isomorphism in cohomology up to degree $p-1$. Now let us go to degree p. Take a Jordan block of φ_p^* for the eigenvalue $\lambda = 1$. Let $1 \le j_0 \le r$ be its size. Then we may take $v \in K_{j_0} \setminus K_{j_0-1}$ in it. First, this implies that $v \notin I = \mathrm{Im}\,(\varphi_p^* - \mathrm{Id})$. Set

$$v_j = (\varphi_p^* - \mathrm{Id})^{j_0 - j} v \in K_j\,,$$

for $j = 1, \dots, j_0$. Now let b_j denote the class of v_j on $G_j = K_j/K_{j-1}$. Then $d(b_j) = a \cdot b_{j-1}$. We want to define ρ on $b_1, \dots, b_{j_0}$. For this, we need to construct forms $\tilde\alpha_1, \dots, \tilde\alpha_{j_0} \in \Omega^p(M)$ such that $[\tilde\alpha_1]$ represents $v_1 \in K_1 = H^p(M)$, and

$$d\tilde\alpha_j = dt \wedge \tilde\alpha_{j-1}\,.$$

Then we set $\rho(b_j) = \tilde\alpha_j$, and ρ is a map of differential algebras.

We work inductively. Let $v_j = [\alpha_j] \in H^p(N)$. Here $\varphi^*[\alpha_j] - [\alpha_j] = [\alpha_{j-1}]$. As $\varphi^*[\alpha_1] - [\alpha_1] = 0$, we have that $\varphi^*\alpha_1 = \alpha_1 + d\theta_1$. Set

$$\tilde\alpha_1(x,t) = \alpha_1(x) + d(\zeta(t)\theta_1(x)),$$

where $\zeta : [0,1] \to [0,1]$ is a smooth function such that $\zeta \equiv 0$ near $t = 0$ and $\zeta \equiv 1$ near $t = 1$. Clearly, $[\tilde\alpha_1] = [\alpha_1] = v_1$.

Assume by induction that $\tilde\alpha_1, \dots, \tilde\alpha_j$ have been constructed, and moreover satisfy that

$$[\tilde\alpha_k|_{N \times \{t\}}] = [\alpha_k] + \sum_{i=1}^{k-1} c_{ik}(t)[\alpha_i],$$

for some polynomials $c_{ik}(t)$, $k = 1, \dots, j$. Note that the result holds for $k = 1$. To construct $\tilde{\alpha}_{j+1}$, we work as follows. We define

$$\gamma_j(x) = \int_0^1 \left(\tilde{\alpha}_j - \sum_{i=1}^{j-1} c_i \tilde{\alpha}_i \right) dt.$$

This is a closed form on N. The constants c_i are adjusted so that $[\gamma_j] = [\alpha_j] = v_j = \varphi^*[\alpha_{j+1}] - [\alpha_{j+1}]$. So we can write

$$\gamma_j = \varphi^* \alpha_{j+1} - \alpha_{j+1} - d\theta_{j+1}$$

for some $(p-1)$-form θ_{j+1} on N. Write

$$\hat{\alpha}_{j+1} = \int_0^t \left(\tilde{\alpha}_j(x,s) - \sum_{i=1}^{j-1} c_i \tilde{\alpha}_i(x,s) \right) ds + \alpha_{j+1} + d(\zeta(t)\theta_{j+1}(x)) \, .$$

This is a p-form well-defined in M since $\varphi_p^*(\hat{\alpha}_{j+1}(x,0)) = \varphi_p^*(\alpha_{j+1}) = \gamma_j + \alpha_{j+1} + d\theta_{j+1} = \hat{\alpha}_{j+1}(x,1)$. Set

$$\tilde{\alpha}_{j+1} = \hat{\alpha}_{j+1} + \sum_{i<j} c_i \tilde{\alpha}_{i+1}.$$

Then

$$d\tilde{\alpha}_{j+1} = dt \wedge \tilde{\alpha}_j \, .$$

Finally,

$$[\tilde{\alpha}_{j+1}|_{N \times \{t\}}] = [\alpha_{j+1}] + \sum_{i=1}^{j} c_i(t)[\alpha_i] \, ,$$

for some $c_i(t)$, as required.

Repeating this procedure with all Jordan blocks, we finally get

$$\rho : (\bigwedge(W^1 \oplus W^p), d) \to (\Omega^*(M), d).$$

Clearly $H^p(\bigwedge(W^1 \oplus W^p)) = K_1$, so ρ^* is an isomorphism on degree p. For degree $p+1$, $H^{p+1}(\bigwedge(W^1 \oplus W^p))$ is generated by the elements $a \cdot b$, where $b \in G_{j_0}$ corresponds to some $v \in K_{j_0}$ generating a Jordan block (equivalently, $v \notin I$). These elements generate $\operatorname{coker}(\varphi_p^* - \mathrm{Id})$, i.e.

$$H^{p+1}(\bigwedge(W^1 \oplus W^p)) \cong \operatorname{coker}(\varphi_p^* - \mathrm{Id}).$$

An element $v = v_{j_0}$ is sent, by ρ, to a p-form $\tilde{\alpha}_{j_0}$ on M, which satisfies

$$[\tilde{\alpha}_{j_0}|_{N \times \{t\}}] = [\alpha_{j_0}] + \sum_{i=1}^{j_0 - 1} c_i [\alpha_i] \, ,$$

for some $c_i = c_i(t)$, following the previous notation. Therefore the class $[dt \wedge \tilde{\alpha}_{j_0}]$ corresponds to $[dt] \wedge [\alpha_{j_0}]$, in the notation of Lemma 12. So

$$\rho^* : H^{p+1}(\bigwedge(W^1 \oplus W^p)) \to H^{p+1}(M)$$

is the injection into the subspace $[dt] \wedge \operatorname{coker}(\varphi_p^* - \mathrm{Id})$. This completes the proof of the theorem. $\square$

Note that, in the notation of Proposition 2, we have that $C^1 = W^1$, $C^p = G_1$ and $N^p = G_2 \oplus \ldots \oplus G_r$. Also take $w \in G_r$. Then $a \cdot w \in I(N)$, $d(a \cdot w) = 0$, but $a \cdot w$ is not exact. Hence

Corollary 16. *Under the conditions of Theorem* 15*, if $r \geq 2$, then M is non-formal. Moreover, if $r = 1$, then M is p-formal (in the sense of Definition* 3*).*

Applying this to symplectic mapping tori, we have the following. Let N be a compact symplectic $2n$-manifold, and assume that $\varphi : N \to N$ is a symplectomorphism such that the map induced on cohomology $\varphi_1^* : H^1(N) \to H^1(N)$ does not have the eigenvalue $\lambda = 1$. As $\varphi_2^* : H^2(N) \to H^2(N)$ always has the eigenvalue $\lambda = 1$ (φ^* fixes the symplectic form), then we have that N_φ is 2-formal if and only if the eigenvalue $\lambda = 1$ of φ_2^* has multiplicity $r = 1$.

If $n = 2$, then N_φ is a 5-dimensional co-symplectic manifold with $b_1 = 1$. In dimension 5, Theorem 4 says that 2-formality is equivalent to formality. Therefore we have the following result:

Corollary 17. 5*-dimensional non-formal co-symplectic manifolds with $b_1 = 1$ are given as mapping tori of symplectomorphisms $\varphi : N \to N$ of compact symplectic* 4*-manifolds N, where φ_1^* does not have the eigenvalue $\lambda = 1$ and φ_2^* has the eigenvalue $\lambda = 1$ with multiplicity $r \geq 2$.*

Finally, let us mention that an analogue of Theorem 15 for $p = 1$ is harder to obtain. However, at least we can still say that if $\lambda = 1$ is an eigenvalue of φ_1^* with multiplicity $r \geq 2$, then $M = N_\varphi$ is non-formal (by Remark 14). In addition, one can also obtain a *non-formal* mapping torus such that $\lambda = 1$ is an eigenvalue of φ_1^* with multiplicity $r = 1$, e.g. by taking a non-formal symplectic nilmanifold N and multiplying it by S^1. Next, we give an example of a 5-dimensional *formal* mapping torus N_φ with no co-symplectic structure and such that $\lambda = 1$ is an eigenvalue of φ_1^* with multiplicity $r = 1$.

Let $G(k)$ be the simply connected completely solvable[2] 3-dimensional Lie group defined by the equations

$$de^1 = -ke^1 \wedge e^3, \quad de^2 = ke^2 \wedge e^3, \quad de^3 = 0,$$

where k is a real number such that $\exp(k) + \exp(-k)$ is an integer different from 2.

Let $\Gamma(k)$ be a discrete subgroup of $G(k)$ such that the quotient space $P(k) = \Gamma(k) \backslash G(k)$ is compact (such a subgroup $\Gamma(k)$ always exists; see [24] for example). Then $P(k)$ is a completely solvable solvmanifold.

We can use Hattori's theorem [18] which asserts that the de Rham cohomology ring $H^*(P(k))$ is isomorphic to the cohomology ring $H^*(\mathfrak{g}^*)$ of the Lie algebra $\mathfrak{g}$ of $G(k)$. For simplicity we denote the left invariant forms $\{e^i\}$, $i = 1, 2, 3$, on $G(k)$ and their projections on $P(k)$ by the same symbols. Thus, we obtain

- $H^0(P(k)) = \langle 1 \rangle$,
- $H^1(P(k)) = \langle [e^3] \rangle$,
- $H^2(P(k)) = \langle [e^{12}] \rangle$,
- $H^3(P(k)) = \langle [e^{123}] \rangle$.

Therefore, there exists a real number a such that the cohomology class $a[e^{12}]$ is integral. Hence there exists a principal circle bundle $\pi : N(k) \to P(k)$ with Euler

[2] A solvable Lie group G is *completely solvable* if for every $X \in \mathfrak{g}$, the eigenvalues of the map ad_X are real.

class $a[e^{12}]$ and a connection 1-form e^4 whose curvature form is ae^{12} (we use the same notation for differential forms on the base space $P(k)$ and their pull-backs via π to the total space $N(k)$).

One can check that the de Rham cohomology groups $H^*(N(k))$ are:

- $H^0(N(k)) = \langle 1\rangle$,
- $H^1(N(k)) = \langle [e^3]\rangle$,
- $H^2(N(k)) = 0$,
- $H^3(N(k)) = \langle [e^{124}]\rangle$,
- $H^4(N(k)) = \langle [e^{1234}]\rangle$.

Moreover, the manifold $N(k)$ is formal. In fact, let $(\Omega^*(N(k)), d)$ be the de Rham complex of differential forms on $N(k)$. The minimal model of $N(k)$ is a differential graded algebra $(\mathcal{M}, d)$, with

$$\mathcal{M} = \bigwedge(a, b),$$

where the generator a has degree 1, the generator b has degree 3, and d is given by $da = db = 0$. The morphism $\rho\colon \mathcal{M} \to \Omega^*(N(k))$, inducing an isomorphism on cohomology, is defined by

$$\rho(a) = e^3,$$
$$\rho(b) = e^{124}.$$

According to Definition 3, we have $C^1 = \langle a\rangle$ and $N^1 = 0$. Thus $N(k)$ *is* 1*-formal* and hence it is formal by Theorem 4.

Now, let M be the 5-dimensional compact manifold defined as $M = N(k) \times S^1$. Denote by e^5 the canonical 1-form on S^1. Then M is formal. Clearly M is a mapping torus. But M does not admit co-symplectic structures since $H^2(M) = \langle [e^{35}]\rangle$, and so any closed 2-form F satisfies $F^2 = 0$.

5. Geography of non-formal compact co-symplectic manifolds

In this section we consider the following problem:

> For which pairs $(m = 2n+1, b)$, with $n, b \geq 1$, are there compact co-symplectic manifolds of dimension m and with $b_1 = b$ which are non-formal?

It will turn out that the answer is the same as for compact smooth manifolds [14], i.e., that there are non-formal examples if and only if $m = 3$ and $b \geq 2$, or $m \geq 5$ and $b \geq 1$. We start with some straightforward examples:

- For $b = 1$ and $m \geq 9$, we may take a compact non-formal symplectic manifold N of dimension $m-1 \geq 8$ and simply connected. Such a manifold exists for dimensions ≥ 10 by [1], and for dimension equal to 8 by [15]. Then consider $M = N \times S^1$.
- For $m = 3$, $b = 2$, we may take the 3-dimensional nilmanifold M_0 defined by the structure equations $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$. This is non-formal since it is not a torus. The pair $\eta = e^1$, $F = e^2 \wedge e^3$ defines a co-symplectic structure on M_0 since $d\eta = dF = 0$ and $\eta \wedge F \neq 0$.
- For $m \geq 5$ and $b \geq 2$ even, take the co-symplectic compact manifold $M = M_0 \times \Sigma_k \times (S^2)^\ell$, where Σ_k is the surface of genus $k \geq 0$, $\ell \geq 0$, and $(S^2)^\ell$ is the product of ℓ copies of S^2. Then $\dim M = m = 5 + 2\ell$ and $b_1(M) = 2 + 2k$.

- For $m = 5$ and $b = 3$, we can take $M_1 = N \times S^1$, where N is a compact 4-dimensional symplectic manifold with $b_1 = 2$. For example, take N as the compact nilmanifold defined by the equations $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$, $de^4 = e^1 \wedge e^3$, which is non-formal and symplectic with $\omega = e^1 \wedge e^4 + e^2 \wedge e^3$.
- For $m \geq 7$ and $b \geq 3$ odd, take $M = M_1 \times \Sigma_k \times (S^2)^\ell$, $k, \ell \geq 0$.

Other examples with $b_1 = 2$ and $m = 5$ can be obtained from the list of 5-dimensional compact nilmanifolds. According to the classification in [2, 21] of nilpotent Lie algebras of dimension < 7, there are 9 nilpotent Lie algebras $\mathfrak{g}$ of dimension 5, and only 3 of them satisfy $\dim H^1(\mathfrak{g}^*) = 2$, namely

$$(0, 0, 12, 13, 14 + 23), \quad (0, 0, 12, 13, 14), \quad (0, 0, 12, 13, 23).$$

In the description of the Lie algebras $\mathfrak{g}$, we are using the structure equations with respect to a basis $e^1, \ldots, e^5$ of the dual space $\mathfrak{g}^*$. For instance, $(0, 0, 12, 13, 14+23)$ means that there is a basis $\{e^j\}_{j=1}^5$ satisfying $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$, $de^4 = e^1 \wedge e^3$ and $de^5 = e^1 \wedge e^4 + e^2 \wedge e^3$; equivalently, the Lie bracket is given in terms of its dual basis $\{e_j\}_{j=1}^5$ by $[e_1, e_2] = -e_3$, $[e_1, e_3] = -e_4$, $[e_1, e_4] = [e_2, e_3] = -e_5$. Also, from now on we write $e^{ij} = e^i \wedge e^j$.

Proposition 18. *Among the* 3 *nilpotent Lie algebras* $\mathfrak{g}$ *of dimension* 5 *with* $\dim H^1(\mathfrak{g}^*) = 2$*, those that have a co-symplectic structure are*

$$(0, 0, 12, 13, 14 + 23), \quad (0, 0, 12, 13, 14).$$

Proof. Clearly the forms η and F given by

$$\eta = e^1, \quad F = e^{25} - e^{34}$$

satisfy $d\eta = dF = 0$ and $\eta \wedge F^2 \neq 0$, and so they define a co-sympectic structure on each of those Lie algebras.

To prove that the Lie algebra $(0, 0, 12, 13, 23)$ does not admit a co-sympectic structure, one can check it directly or use the fact that the direct sum of $(0, 0, 12, 13, 23)$ with the 1-dimensional Lie algebra has no symplectic form [2]. $\square$

Remark 19. Let N denote the 5-dimensional compact nilmanifold associated to the Lie algebra $\mathfrak{n}$ with structure $(0, 0, 12, 13, 23)$. Then N has a closed 1-form; indeed, $de^1 = de^2 = 0$. By Tischler's Theorem [25], N is a mapping torus. However, it is not a *symplectic* mapping torus, since it is not co-symplectic. We describe this mapping torus explicitly. Since N is a nilmanifold, we can describe the structure at the level of Lie algebras. The map $\mathfrak{n} \to \mathbb{R}$, $(e_1, \ldots, e_5) \to e_1$ gives an exact sequence

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{n} \longrightarrow \mathbb{R} \longrightarrow 0 \tag{8}$$

of Lie algebras, and one sees immediately that $\mathfrak{k}$ is a 4-dimensional symplectic nilpotent Lie algebra, spanned by $e_2, \ldots, e_5$, with structure $(0, 0, 0, 23)$ (with respect to the dual basis of $\mathfrak{k}^*$). The fiber of the corresponding bundle over S^1 is the Kodaira-Thurston manifold KT. Taking into account the proof of Proposition 10, the Lie algebra extension (8) is associated to the derivation $D = \mathrm{ad}(e_1)$ of $\mathfrak{k}$. In other words, $\mathfrak{n} = \mathbb{R} \oplus_D \mathfrak{k}$. A computation shows that this derivation is not symplectic with respect to *any* symplectic form on $\mathfrak{k}$, and Proposition 10 implies that $\mathfrak{n}$ is not co-symplectic. The map $\varphi := \exp(D)$ is a diffeomorphism of KT which does not preserve any symplectic structure of KT, and $N = KT_\varphi$.

The previous examples leave some gaps, notably the cases $m = 3$, $b \geq 3$, and $m = 5$, $b = 1$. By [14], we know that there are compact non-formal manifolds with these Betti numbers and dimensions. Let us see that there are also non-formal co-symplectic manifolds in these cases.

Proposition 20. *There are non-formal compact co-symplectic manifolds with $m \geq 3$, $b_1 \geq 2$.*

Proof. We consider the symplectic surface Σ_k of genus $k \geq 1$. Consider a symplectomorphism $\varphi : \Sigma_k \to \Sigma_k$ such that $\varphi^* : H^1(\Sigma_k) \to H^1(\Sigma_k)$ has the form

$$\varphi^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with respect to a symplectic basis $\xi_1, \xi_2, \dots, \xi_{2k-1}, \xi_{2k}$ of $H^1(\Sigma_k)$. Consider the mapping torus M of φ. The symplectic form of Σ_k induces a closed 2-form F on M. The pull-back η of the volume form of S^1 under $M \to S^1$ is closed and satisfies that $\eta \wedge F > 0$. Therefore M is co-symplectic.

Now $\varphi^*\xi_1 = \xi_1 + \xi_2$ and $\varphi^*\xi_i = \xi_i$, for $2 \leq i \leq 2k$. By Lemma 12, the cohomology of M is

$$\begin{aligned} H^1(M) &= \langle a, \xi_2, \dots, \xi_{2k-1}, \xi_{2k} \rangle, \\ H^2(M) &= \langle F, a\,\xi_1, a\,\xi_3, \dots, a\,\xi_{2k-1}, a\,\xi_{2k} \rangle, \end{aligned}$$

where $a = [\eta]$. So $b_1 = 2k \geq 2$. By Theorem 13, the Massey product $\langle a, a, \xi_2 \rangle$ does not vanish, and so M is non-formal.

Similarly, take Σ_k where $k \geq 2$. We consider a symplectomorphism $\psi : \Sigma_k \to \Sigma_k$ such that $\psi^* : H^1(\Sigma_k) \to H^1(\Sigma_k)$ has the form

$$\psi^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the mapping torus M of ψ has $b_1 = 2k - 1 \geq 3$ and odd, and M is co-symplectic and non-formal.

For higher dimensions, take $M \times (S^2)^\ell$, $\ell \geq 0$. □

Remark 21. Notice that the case $k = 1$ in the first part of the previous proposition yields another description of the Heisenberg manifold.

Proposition 22. *There are non-formal compact co-symplectic manifolds with $m \geq 5$, $b_1 = 1$.*

Proof. It is enough to construct an example for $m = 5$. Take the torus T^4 and the mapping torus T^4_φ of the symplectomorphism $\varphi : T^4 \to T^4$ such that

$$\varphi^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{9}$$

on $H^1(T^4)$. Taking η as the pull-back of the 1-form θ on S^1 and $F = e^1 \wedge e^2 + e^3 \wedge e^4$, we have that T^4_φ is co-symplectic. The map φ^* on $H^2(T^4)$ satisfies:

$$\begin{aligned}
\varphi^*([e^1 \wedge e^2]) &= [e^1 \wedge e^2],\\
\varphi^*([e^1 \wedge e^3]) &= [e^1 \wedge e^3] - [e^1 \wedge e^4],\\
\varphi^*([e^1 \wedge e^4]) &= [e^1 \wedge e^4],\\
\varphi^*([e^2 \wedge e^3]) &= [e^2 \wedge e^3] - [e^2 \wedge e^4],\\
\varphi^*([e^2 \wedge e^4]) &= [e^2 \wedge e^4],\\
\varphi^*([e^3 \wedge e^4]) &= [e^3 \wedge e^4].
\end{aligned}$$

Then $b_1(T^4_\varphi) = 1$ as $H^1(T^4_\varphi) = \langle a \rangle$, with $a = [\eta]$. Also $H^2(T^4_\varphi) = \langle [e^{12}], [e^{14}], [e^{24}], [e^{34}] \rangle$. In particular, notice that $\mathrm{Im}\,(\varphi^* - \mathrm{Id}) = \langle [e^{14}], [e^{24}] \rangle$. Then $[e^{14}] \in \ker(\varphi^* - \mathrm{Id})$ and $[e^{14}] \in \mathrm{Im}\,(\varphi^* - \mathrm{Id})$. So Theorem 13 gives us the non-formality of T^4_φ.

For higher dimensions, take $M = N \times (S^2)^\ell$, where $\ell \geq 0$. Then $\dim M = 5 + 2\ell$ and $b_1(M) = 1$. $\square$

Remark 23. Let us show that the 5-manifold T^4_φ is not a solvmanifold, that is, it cannot be written as a quotient of a simply connected solvable Lie group by a discrete co-compact subgroup.[3] The fiber bundle

$$T^4 \longrightarrow T^4_\varphi \longrightarrow S^1$$

gives a short exact sequence at the level of fundamental groups,

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{10}$$

where $H = \pi_1(T^4_\varphi)$. Since $\mathbb{Z}$ is free and $\mathbb{Z}^4$ is abelian, one has $H = \mathbb{Z} \ltimes \mathbb{Z}^4$. Now suppose that T^4_φ is a solvmanifold of the form $\Gamma \backslash G$. Clearly, it is $\Gamma \cong H$. According to [22], we have a fibration

$$N \longrightarrow T^4_\varphi \longrightarrow T^k$$

where N is a nilmanifold and T^k is a k-torus. Since $b_1(T^4_\varphi) = 1$, we have $k = 1$ and N is a 4-dimensional nilmanifold. This gives another short exact sequence of groups

$$0 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\Delta = \pi_1(N)$. But we know that there is a unique surjection $H_1(\Gamma) = \mathbb{Z} \oplus T \longrightarrow \mathbb{Z}$ (where T is a torsion group) and that, composed with the natural surjection $\Gamma \longrightarrow \Gamma/[\Gamma,\Gamma] = H_1(\Gamma)$, this gives a unique homomorphism $\Gamma \longrightarrow \mathbb{Z}$. Hence, the extension $\Delta \longrightarrow \Gamma \longrightarrow \mathbb{Z}$ is the same as (10). Therefore $\Delta = \mathbb{Z}^4$. The Mostow fibration of $\Gamma \backslash G = T^4_\varphi$ coincides with the mapping torus bundle. At the level of Lie groups, it must be $G = \mathbb{R} \ltimes \mathbb{R}^4$ with semidirect product

$$(t, x) \cdot (t', x') = (t + t', x + f(t)x')$$

with f a 1-parameter subgroup in $\mathrm{GL}(4, \mathbb{R})$, i.e., $f(t) = \exp(tg)$ for some matrix g. Moreover, $f(1) = \exp(g) = \varphi^*$. But φ^* cannot be the exponential of a matrix. Indeed, if g has real eigenvalues, then φ^* has positive eigenvalues. If g has purely imaginary eigenvalues and diagonalizes, so does φ^*. Also, if g has complex conjugate

[3] If we define a solvmanifold as a quotient $\Gamma \backslash G$, where G is a simply connected solvable Lie group and $\Gamma \subset G$ is a closed (not necessarily discrete) subgroup, then any mapping torus N_φ, where N is a nilmanifold, is of this type (see [22]).

eigenvalues but does not diagonalize, then φ^* has two Jordan blocks. None of these cases occur.

Remark 24. The example constructed in the proof of Proposition 22 can be used to give another example of a 5-dimensional non-formal co-symplectic manifold with $b_1 = 1$ which is not a solvmanifold.

Take $N = T^4$ and $\varphi : N \to N$ satisfying (9). We may arrange it so that φ fixes the neighborhood of a point $p \in N$. Take the (symplectic) blow-up of N at p, $\widetilde{N} = N \# \overline{\mathbb{C}P}^2$, and the induced symplectomorphism $\tilde{\varphi} : \widetilde{N} \to \widetilde{N}$. Let $M = \widetilde{N}_{\tilde{\varphi}}$ be the corresponding mapping torus. Clearly, M is co-symplectic, it has $b_1(M) = 1$ and the eigenvalue $\lambda = 1$ of $\varphi^* : H^2(\widetilde{N}) \to H^2(\widetilde{N})$ has multiplicity 2, hence M is non-formal. But M cannot be a solvmanifold since $\pi_2(M) = \pi_2(\widetilde{N}) = \mathbb{Z}$ is non-trivial.

6. A non-formal solvmanifold of dimension 5 with $b_1 = 1$

In this section we show an example of a non-formal compact co-symplectic[4] 5-dimensional solvmanifold S with first Betti number $b_1(S) = 1$. Actually, S is the mapping torus of a certain diffeomorphism φ of a 4-torus preserving the orientation, so this example fits in the scope of Proposition 22.

Let $\mathfrak{g}$ be the abelian Lie algebra of dimension 4. Suppose $\mathfrak{g} = \langle e_1, e_2, e_3, e_4 \rangle$, and take the symplectic form $\omega = e^{14} + e^{23}$ on $\mathfrak{g}$, where $\langle e^1, e^2, e^3, e^4 \rangle$ is the dual basis for the dual space $\mathfrak{g}^*$ such that the first cohomology group $H^1(\mathfrak{g}^*) = \langle [e^1], [e^2], [e^3], [e^4] \rangle$. Consider the endomorphism of $\mathfrak{g}$ represented, with respect to the chosen basis, by the matrix

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

It is immediate to see that D is an infinitesimal symplectic transformation. Since $\mathfrak{g}$ is abelian, it is also a derivation. Applying Proposition 10 we obtain a co-symplectic Lie algebra

$$\mathfrak{h} = \mathbb{R}\xi \oplus \mathfrak{g}$$

with brackets defined by

$$[\xi, e_1] = -e_1 - e_3, \quad [\xi, e_2] = e_2 - e_4, \quad [\xi, e_3] = -e_3 \quad \text{and} \quad [\xi, e_4] = e_4.$$

One can check that $\mathfrak{h} = \langle e_1, e_2, e_3, e_4, e_5 = \xi \rangle$ is a completely solvable non-nilpotent Lie algebra. We denote by $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ the dual basis for $\mathfrak{h}^*$. The Chevalley-Eilenberg complex of $\mathfrak{h}^*$ is

$$(\bigwedge(\alpha_1, \ldots, \alpha_5), d)$$

[4]Recall that the definition of a co-symplectic manifold in this paper differs from that used in other papers, such as [16].

with differential d defined by

$$\begin{aligned} d\alpha_1 &= -\alpha_1 \wedge \alpha_5, \\ d\alpha_2 &= \alpha_2 \wedge \alpha_5, \\ d\alpha_3 &= -\alpha_1 \wedge \alpha_5 - \alpha_3 \wedge \alpha_5, \\ d\alpha_4 &= -\alpha_2 \wedge \alpha_5 + \alpha_4 \wedge \alpha_5, \\ d\alpha_5 &= 0. \end{aligned}$$

Let H be the simply connected and completely solvable Lie group of dimension 5 consisting of matrices of the form

$$a = \begin{pmatrix} e^{-x_5} & 0 & 0 & 0 & 0 & x_1 \\ 0 & e^{x_5} & 0 & 0 & 0 & x_2 \\ -x_5e^{-x_5} & 0 & e^{-x_5} & 0 & 0 & x_3 \\ 0 & -x_5e^{x_5} & 0 & e^{x_5} & 0 & x_4 \\ 0 & 0 & 0 & 0 & 1 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_i \in \mathbb{R}$, for $1 \leq i \leq 5$. Then a global system of coordinates $\{x_i, 1 \leq i \leq 5\}$ for H is defined by $x_i(a) = x_i$, and a standard calculation shows that a basis for the left invariant 1-forms on H consists of

$$\alpha_1 = e^{x_5}dx_1, \quad \alpha_2 = e^{-x_5}dx_2, \quad \alpha_3 = x_5e^{x_5}dx_1 + e^{x_5}dx_3,$$
$$\alpha_4 = x_5e^{-x_5}dx_2 + e^{-x_5}dx_4, \quad \alpha_5 = dx_5.$$

This means that $\mathfrak{h}$ is the Lie algebra of H. We notice that the Lie group H may be described as a semidirect product $H = \mathbb{R} \ltimes_\rho \mathbb{R}^4$, where $\mathbb{R}$ acts on $\mathbb{R}^4$ via the linear transformation $\rho(t)$ of $\mathbb{R}^4$ given by the matrix

$$\rho(t) = \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ -te^{-t} & 0 & e^{-t} & 0 \\ 0 & -te^t & 0 & e^t \end{pmatrix}.$$

Thus the operation on the group H is given by

$$\mathbf{a}\cdot\mathbf{x} = (a_1+x_1e^{-a_5}, a_2+x_2e^{a_5}, a_3+x_3e^{-a_5}-a_5x_1e^{-a_5}, a_4+x_4e^{a_5}-a_5x_2e^{a_5}, a_5+x_5),$$

where $\mathbf{a} = (a_1, \ldots, a_5)$ and similarly for $\mathbf{x}$. Therefore $H = \mathbb{R} \ltimes_\rho \mathbb{R}^4$, where $\mathbb{R}$ is a connected abelian subgroup, and $\mathbb{R}^4$ is the nilpotent commutator subgroup.

Now we show that there exists a discrete subgroup Γ of H such that the quotient space $\Gamma\backslash H$ is compact. To construct Γ it suffices to find some real number t_0 such that the matrix defining $\rho(t_0)$ is conjugate to an element A of the special linear group $\mathrm{SL}(4,\mathbb{Z})$ with distinct real eigenvalues λ and λ^{-1}. Indeed, we could then find a lattice Γ_0 in $\mathbb{R}^4$ which is invariant under $\rho(t_0)$, and take $\Gamma = (t_0\mathbb{Z}) \ltimes_\rho \Gamma_0$. To this end, we choose the matrix $A \in \mathrm{SL}(4,\mathbb{Z})$ given by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{11}$$

with double eigenvalues $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. Taking $t_0 = \log(\frac{3+\sqrt{5}}{2})$, we have that the matrices $\rho(t_0)$ and A are conjugate. Indeed, put

$$P = \begin{pmatrix} 1 & \frac{-2(2+\sqrt{5})}{3+\sqrt{5}} & 0 & 0 \\ 1 & \frac{1+\sqrt{5}}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & \log(\frac{2}{3+\sqrt{5}}) & \frac{2(2+\sqrt{5})\log(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \\ 0 & 0 & \log(\frac{2}{3+\sqrt{5}}) & -\frac{(1+\sqrt{5})\log(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \end{pmatrix}; \tag{12}$$

then a direct calculation shows that $PA = \rho(t_0)P$. So the lattice Γ_0 in $\mathbb{R}^4$ defined by

$$\Gamma_0 = P(m_1, m_2, m_3, m_4)^t,$$

where $m_1, m_2, m_3, m_4 \in \mathbb{Z}$ and $(m_1, m_2, m_3, m_4)^t$ is the transpose of the vector (m_1, m_2, m_3, m_4), is invariant under the subgroup $t_0\mathbb{Z}$. Thus $\Gamma = (t_0\mathbb{Z}) \ltimes_\rho \Gamma_0$ is a co-compact subgroup of H.

We denote by $S = \Gamma\backslash H$ the compact quotient manifold. Then S is a 5-dimensional (non-nilpotent) completely solvable solvmanifold.

Alternatively, S may be viewed as the total space of a T^4-bundle over the circle S^1. In fact, let $T^4 = \Gamma_0\backslash\mathbb{R}^4$ be the 4-dimensional torus and $\varphi\colon \mathbb{Z} \to \mathrm{Diff}(T^4)$ the representation defined as follows: $\varphi(m)$ is the transformation of T^4 covered by the linear transformation of $\mathbb{R}^4$ given by the matrix

$$\rho(mt_0) = \begin{pmatrix} e^{-mt_0} & 0 & 0 & 0 \\ 0 & e^{mt_0} & 0 & 0 \\ -mt_0e^{-mt_0} & 0 & e^{-mt_0} & 0 \\ 0 & -mt_0e^{mt_0} & 0 & e^{mt_0} \end{pmatrix}.$$

So $\mathbb{Z}$ acts on $T^4 \times \mathbb{R}$ by

$$((x_1, x_2, x_3, x_4), x_5) \mapsto (\rho(mt_0) \cdot (x_1, x_2, x_3, x_4)^t, x_5 + m),$$

and S is the quotient $(T^4 \times \mathbb{R})/\mathbb{Z}$. The projection π is given by

$$\pi[(x_1, x_2, x_3, x_4), x_5] = [x_5].$$

Remark 25. We notice that S is a mapping torus associated to a certain symplectomorphism $\Phi : T^4 \to T^4$. Indeed, since D is an infinitesimal symplectic transformation, its exponential $\exp(tD)$ is a 1-parameter group of symplectomorphisms of $\mathbb{R}^4$. Notice that $\exp(tD) = \rho(t)$. We saw that there exists a number $t_0 \in \mathbb{R}$ such that $\rho(t_0)$ preserves a lattice $\Gamma_0 \cong \mathbb{Z}^4 \subset \mathbb{R}^4$. Therefore the symplectomorphism $\rho(t_0)$ descends to a symplectomorphism Φ of the 4-torus $\Gamma_0\backslash\mathbb{R}^4$, whose mapping torus is precisely $\Gamma\backslash H$.

Next, we compute the real cohomology of S. Since S is completely solvable, Hattori's theorem [18] says that the de Rham cohomology ring $H^*(S)$ is isomorphic to the cohomology ring $H^*(\mathfrak{h}^*)$ of the Lie algebra $\mathfrak{h}$ of H. For simplicity we denote the left invariant forms $\{\alpha_i\}$, $i = 1, \ldots, 5$, on H and their projections on S by the same symbols. Thus, we obtain

- $H^0(S) = \langle 1 \rangle$,
- $H^1(S) = \langle [\alpha_5] \rangle$,
- $H^2(S) = \langle [\alpha_1 \wedge \alpha_2], [\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3] \rangle$,

- $H^3(S) = \langle [\alpha_3 \wedge \alpha_4 \wedge \alpha_5], [(\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3) \wedge \alpha_5] \rangle$,
- $H^4(S) = \langle [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4] \rangle$,
- $H^5(S) = \langle [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5] \rangle$.

The product $H^1(S) \otimes H^2(S) \to H^3(S)$ is given by

$$[\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3] \wedge [\alpha_5] = [(\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3) \wedge \alpha_5] \quad \text{and} \quad [\alpha_1 \wedge \alpha_2] \wedge [\alpha_5] = 0.$$

Theorem 26. *S is a compact co-symplectic 5-manifold which is non-formal and with first Betti number $b_1(S) = 1$.*

Proof. Take the 1-form $\eta = \alpha_5$, and let F be the 2-form on S given by

$$F = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3.$$

Then (F, η) defines a co-symplectic structure on S since $dF = d\eta = 0$ and $\eta \wedge F^2 \neq 0$.

We prove the non-formality of S from its minimal model [24]. The minimal model of S is a differential graded algebra $(\mathcal{M}, d)$, with

$$\mathcal{M} = \bigwedge(a) \otimes \bigwedge(b_1, b_2, b_3, b_4) \otimes \bigwedge V^{\geq 3},$$

where the generator a has degree 1, the generators b_i have degree 2, and d is given by $da = db_1 = db_2 = 0$, $db_3 = a \cdot b_2$, $db_4 = a \cdot b_3$. The morphism $\rho\colon \mathcal{M} \to \Omega^*(S)$, inducing an isomorphism on cohomology, is defined by

$$\begin{aligned} \rho(a) &= \alpha_5, \\ \rho(b_1) &= \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3, \\ \rho(b_2) &= \alpha_1 \wedge \alpha_2, \\ \rho(b_3) &= \frac{1}{2}(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3), \\ \rho(b_4) &= \frac{1}{2}\alpha_3 \wedge \alpha_4. \end{aligned}$$

Following the notation in Definition 3, we have $C^1 = \langle a \rangle$ and $N^1 = 0$; thus *S is 1-formal.* We see that *S is not 2-formal.* In fact, the element $b_4 \cdot a \in N^2 \cdot V^1$ is closed but not exact, which implies that $(\mathcal{M}, d)$ is not 2-formal. Therefore, $(\mathcal{M}, d)$ is not formal. □

Remark 27. It can be seen that S is non-formal by computing a *quadruple* Massey product [24] $\langle [\alpha_1 \wedge \alpha_2], [\alpha_5], [\alpha_5], [\alpha_5] \rangle$. As $\alpha_1 \wedge \alpha_2 \wedge \alpha_5 = \frac{1}{2} d(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3)$ and $(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3) \wedge \alpha_5 = d(\alpha_3 \wedge \alpha_4)$, we have

$$\langle [\alpha_1 \wedge \alpha_2], [\alpha_5], [\alpha_5], [\alpha_5] \rangle = \frac{1}{2}[\alpha_3 \wedge \alpha_4 \wedge \alpha_5].$$

This is easily seen to be non-zero modulo the indeterminacies.

Remark 28. Theorem 26 can also be proved with the techniques of section 5. By Remark 25, S is the mapping torus of a diffeomorphism $\rho(t_0)$ of $T^4 = \Gamma_0 \backslash \mathbb{R}^4$. Conjugating by the matrix P in (12), we have that S is the mapping torus of A in (11) acting on the standard 4-torus $T^4 = \mathbb{Z}^4 \backslash \mathbb{R}^4$. The action of A on 1-forms leaves no invariant forms, so $b_1(S) = 1$. The action of A on 2-forms is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 \\ -1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix},$$

with respect to the basis $\{e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34}\}$. This matrix has eigenvalues $\lambda = \frac{1}{2}(7 \pm 3\sqrt{5})$ (simple) and $\lambda = 1$, with multiplicity 3 (one block of size 1 and another of size 3). Theorem 15 implies the non-formality of S.

Remark 29. We notice that the previous example S may be generalized to dimension $2n+1$ with $n \geq 2$. For this, it is enough to consider the $(2n+1)$-dimensional completely solvable Lie group H^{2n+1} defined by the structure equations

- $d\alpha_j = (-1)^j \alpha_j \wedge \alpha_{2n+1}$, $j = 1, \ldots, 2n-2$;
- $d\alpha_{2n-1} = -\alpha_1 \wedge \alpha_{2n+1} - \alpha_{2n-1} \wedge \alpha_{2n+1}$;
- $d\alpha_{2n} = -\alpha_2 \wedge \alpha_{2n+1} + \alpha_{2n} \wedge \alpha_{2n+1}$;
- $d\alpha_{2n+1} = 0$.

The co-symplectic structure (η, F) is defined by $\eta = \alpha_{2n+1}$, and $F = \alpha_1 \wedge \alpha_{2n} + \alpha_2 \wedge \alpha_{2n-1} + \alpha_3 \wedge \alpha_4 + \cdots + \alpha_{2n-3} \wedge \alpha_{2n-2}$.

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