Non-Gaussian Cloning of Quantum Coherent States is Optimal

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We consider the optimal cloning of quantum coherent states with single-clone and joint fidelity as figures of merit. While the latter is maximized by a Gaussian cloner, the former is not: the *optimal* single-clone fidelity for a symmetric 1-to-2 cloner is 0.6826, compared to $2/3$ in a Gaussian setting. This cloner can be realized with an optical parametric amplifier and certain non-Gaussian bimodal states. Finally, we show that the single-clone fidelity of the optimal 1-to-∞ cloner is $1/2$. It is achieved by a Gaussian scheme and cannot be surpassed even with supplemental bound entangled states.

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The no-cloning theorem states that there is no quantum apparatus capable of perfectly duplicating an arbitrary input state [1]. This is a direct consequence of the linearity of quantum mechanics and a fundamental difference between classical and quantum information. This theorem enables one of the most promising applications of quantum information theory, namely, secure quantum key distribution. Moreover, the impossibility of perfect cloning machines is intimately connected to other impossible tasks in quantum mechanics [2].

Soon after the observation of the no-cloning theorem as a fundamental feature of quantum mechanics the question arose how well an approximative cloning machine could work. For the case of universal cloning of finite-dimensional pure states this question was addressed and answered in [3–9]. There, the figure of merit was the fidelity, i.e., the overlap between hypothetically perfect clones and the actual output of the imperfect cloner. In particular, it was shown that judging single clones leads to the same optimal cloner as when comparing the joint output with a tensor product of perfect clones [6,7].

Recently, more and more attention has been devoted to continuous variable systems, especially to states with Gaussian Wigner function—so-called Gaussian states. Besides their outstanding importance in quantum optics, quantum communication [10], and various other fields (atomic ensembles, ion traps, etc.), they provide a closed test-bed within which many of the otherwise hardly tractable problems in quantum information become feasible. Restricting to the Gaussian world, i.e., to Gaussian operations on Gaussian states, led, for instance, to solutions to open problems in the theory of entanglement measures [11], quantum channels [12], and secret key distillation [13]. Similarly, the problem of cloning, in particular, coherent states by Gaussian operations has been addressed in [14,15]. The obtained cloner was shown to be optimal within the class of Gaussian operations by exploiting the connection with the state estimation [16]. However, it remained unclear whether Gaussian operations really lead to the optimum, even under the assumptions typically made in the literature such as phase space translation covariance or output symmetry.

The present Letter is concerned with the problem of optimally cloning coherent states without imposing any restrictions on the cloning operation. After recalling some preliminaries, we prove that without loss of generality one can restrict to covariant cloners, for which a powerful characterization is provided. Based on this, we show that, in contrast to the finite-dimensional case, the optimal cloner depends on whether we judge single clones or joint clones. Surprisingly, in the latter case the known Gaussian cloners turn out to be optimal, whereas with respect to the single-clone fidelity, non-Gaussian operations can perform better. The problem of finding the optimal cloner reduces to finding the dominant eigenstate of an appropriate operator. For the optimal 1-to-2 cloner this eigenstate is directly linked to an optical implementation: it is the bimodal state of light that has to be injected on the idler mode of an optical parametric amplifier and the input port of a beam splitter. We envision that a few-photon approximation of this cloner, only suboptimal but yet non-Gaussian, might be feasible, making it possible to experimentally demonstrate this fidelity enhancement. In addition, we show that a 1-to-∞ cloner based on a measure-and-prepare scheme cannot exceed a fidelity of $1/2$, not even with supplemental bound entangled states. Extended discussions of the mathematical details [17] and the quantum optical aspects [18] will be reported elsewhere.

Phase space and coherent states.—Consider a system of n harmonic oscillators with respective canonical operators, or optical field quadratures, $(Q_1, P_1, \ldots, Q_n, P_n) =: R$ and the corresponding phase space $\Xi = \mathbb{R}^{2n}$, which is equipped with an antilinear symplectic form $\sigma(\xi, \eta)$. Translations in this phase space are governed by the *Weyl* or *displacement operators* $W_\xi = e^{i\sigma(\xi,R)}$, $\xi \in \Xi$, which in turn obey the Weyl relations

$$W_\xi W_\eta = e^{-(i/2)\sigma(\eta,\xi)} W_{\xi+\eta}, \quad \text{where } \sigma = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

implements the symplectic form via $\sigma(\xi, \eta) = \xi^T \cdot \sigma \cdot \eta$.

Tensor products in Hilbert space correspond to direct sums in phase space, and in particular $\bigotimes_i W_{\xi_i} = W_{\oplus_i \xi_i}$, where each of the $\xi_i \in \mathbb{R}^2$ belongs to a single mode.

The expectation values of all Weyl operators completely determine a state, and the resulting function, which is the Fourier transform of the Wigner function, is called the *characteristic function* [19]. For a *coherent state,* it is a Gaussian of the form $\chi(\xi) = \mathrm{tr}[\rho W_\xi] = \exp(-\xi^T \cdot \gamma \cdot \xi/4 - i d^T \cdot \xi)$, with covariance matrix $\gamma = \mathbb{1}$ and displacement vector d. Coherent states are translations of the harmonic oscillator ground state $W_\xi|0\rangle = |\xi\rangle$ with $d = \sigma \cdot \xi$. In quantum optical settings, position and momentum coordinates correspond to the real and imaginary parts of the complex field amplitude.

Figures of merit.—The *fidelity* quantifies how close two states ρ_1 and ρ_2 are [20]. Here, we consider only the case of pure input states, so we can simply set $f(\rho_1, \rho_2) = \mathrm{tr}[\rho_1 \rho_2]$. A 1-to-$n$ cloning transformation T (a "cloner" for short) by definition takes systems in the pure input state ρ into n systems whose state is close to n copies of ρ. We can express this by requiring the fidelity

$$f_{\mathrm{joint}}(T, \rho) = \mathrm{tr}[T(\rho)\, \rho^{\otimes n}] \tag{1}$$

to be as large as possible. This is a very demanding criterion, as it also depends on the correlations between the clones. Instead, we might just evaluate the quality of an individual clone, say the ith,

$$f_i(T, \rho) = \mathrm{tr}[T(\rho)\, (\mathbb{1} \otimes \cdots \mathbb{1} \otimes \rho^{(i)} \otimes \mathbb{1} \cdots \otimes \mathbb{1})], \tag{2}$$

where the upper index denotes the position in the tensor product. Since any such fidelity can be put to one by copying the input onto the ith clone, we have to maximize a weighted sum $\sum_i \lambda_i f_i(T, \rho)$ with positive weights λ_i.

Further options arise from the choice of the set of states ρ that we want to clone optimally. Here, we consider the family of coherent states $\rho = |\xi\rangle\langle\xi|$, with $|\xi\rangle = W_\xi|0\rangle$. We define $f_{\mathrm{joint}}(T)$ and $f_i(T)$ as the respective *worst-case fidelities*, i.e., the minima of (1) and (2) over all coherent states ρ. Note that this is different from the usual case of universal cloners in finite-dimensional Hilbert spaces, where one considers the minimum with respect to all pure states. This is connected to the infinite number of dimensions of the continuous variable Hilbert space: Even minimizing (1) or (2) over all pure squeezed Gaussian states (a larger though still very small subset of all states) would already yield a zero fidelity for all T.

Our goal is thus to find the optimal worst-case joint fidelity

$$f_{\mathrm{joint}} = \sup_T f_{\mathrm{joint}}(T) = \sup_T \inf_{\rho \in \mathrm{coh}} f_{\mathrm{joint}}(T, \rho)$$

as well as the convex set of achievable n tuples of single-clone fidelities $(f_1(T), f_2(T), \ldots, f_n(T))$ as T varies over all cloners. This can be simplified as both fidelities are invariant under displacements in phase space, so we can choose the optimal cloner to be covariant. Consequently, they are optimal with respect to worst-case and average fidelities.

Covariance.—Let T be a 1-to-n cloning map. If displacing the input in phase space is equivalent to displacing the outputs by the same amount, then T is called (displacement) covariant:

$$T(\rho) = W_\xi^{\otimes n \dagger} T(W_\xi \rho W_\xi^\dagger) W_\xi^{\otimes n} \equiv T_\xi(\rho)$$

for all ξ and ρ, where we have defined the shifted cloner T_ξ for later reference. The cloners investigated in [14,15] were restricted to be covariant. However, this need not be assumed, but rather comes out as a property of the optimal cloners. As in the case of cloning of finite-dimensional systems [7], the core of the argument is averaging over the symmetry group: we have, for $f = f_{\mathrm{joint}}$ or $f = \sum_i \lambda_i f_i$, respectively,

$$\begin{aligned} f(T) &= \inf_\xi f(T, |\xi\rangle\langle\xi|) \leq \mathbf{M}_\xi f(T_\xi, |0\rangle\langle 0|) \\ &= f(\mathbf{M}_\xi T_\xi, |0\rangle\langle 0|) = f(\mathbf{M}_\xi T_\xi). \end{aligned} \tag{3}$$

Here $\mathbf{M}_\xi$ stands for "mean with respect to ξ" and is implemented by an invariant mean [21]. So, the averaged and thus covariant cloner is at least as good as T for all T, and we can restrict the search to the covariant case. Note that the output of such cloners could be singular for this phase space average. A detailed argumentation shows, however, that this is not optimal for the fidelities considered [17].

Optimizing covariant cloners.—In the Heisenberg picture, (the adjoint of) a covariant cloner maps Weyl operators onto multiples of Weyl operators, $T_*(W_{\xi_1,\ldots,\xi_n}) = t(\xi_1, \ldots, \xi_n)\, W_{\sum_i \xi_i}$, where ξ_i is the pair of phase space variables of the ith clone [17]. In terms of characteristic functions of input and output states, t acts as a characteristic function of the cloner:

$$\chi_{\mathrm{out}}(\xi_1, \ldots, \xi_n) = t(\xi_1, \ldots, \xi_n)\, \chi_{\mathrm{in}}\Big(\sum_i \xi_i\Big).$$

The condition of complete positivity requires that t is the characteristic function of a state ρ_T, plus a fixed linear transformation [22]. We call a cloner *Gaussian* if t has a Gaussian form and it thus maps Gaussian input states onto Gaussian output states. Since fidelities are linear in T, and hence linear in ρ_T, they can be expressed as expectation values of linear operators:

$$f(T, \rho) = \mathrm{tr}[\rho_T F]. \tag{4}$$

The appropriate operators F_{joint} and F_i do not depend on T, which allows us to reduce the supremum of the left-hand side of (4) to finding the state ρ_T (hence the map T) corresponding to the largest eigenvalue of F. This is the core of our method. Physically, the state ρ_T is directly related to the bimodal state that needs to be injected on the idler mode of an optical parametric amplifier together with the input port of a beam splitter in order to realize the cloner T (see below).

Optimal fidelities.—Since by Eq. (3) the maximum fidelities are reached by covariant cloners, we can restrict the further discussion to a vacuum input state $\rho = |0\rangle\langle 0|$. For Gaussian input states, the operators F in Eq. (4) are themselves Gaussian, so that the respective fidelities f are optimized by Gaussian pure states ρ_T, hence by Gaussian cloners T. Consequently, the joint fidelity $f_{\text{joint}}(T) = f_{\text{joint}}(T, |0\rangle\langle 0|) = \text{tr}[\rho_T F_{\text{joint}}]$ is maximized by a Gaussian cloner. The maximum fidelity is given by the largest eigenvalue of the appropriately defined operator F_{joint}, that is,

$$\sup_T f_{\text{joint}}(T) = \max \text{spec}(F_{\text{joint}}) = \frac{1}{n}.$$

Thus, the unique optimal cloner in this case is the known Gaussian cloner of [14–16].

For the single-clone fidelity, we have to maximize the weighted sum $\sum_{i=1}^n \lambda_i f_i = \text{tr}[\rho_T \sum_{i=1}^n \lambda_i F_i]$. Since a linear combination of Gaussian operators does not, in general, have Gaussian eigenfunctions, the optimal cloners with respect to single-clone fidelities are, in fact, *not* Gaussian. For simplicity, we restrict in the following to the 1-to-2 cloning problem. In this case the maximum of the weighted sum of single-copy fidelities $\lambda_1 f_1 + \lambda_2 f_2 = \text{tr}[\rho_T F]$ is the largest eigenvalue of the operator

$$F = \lambda_1 e^{-(Q_1^2+P_2^2)/2} + \lambda_2 e^{-(Q_2^2+P_1^2)/2}. \tag{5}$$

A simple numerical method to find this eigenvalue is to iterate $\phi_{n+1} = F\phi_n/\|F\phi_n\|$. Varying the weights λ_i yields the fidelity pairs (f_1, f_2) along the solid curve in Fig. 1. In comparison, the best Gaussian cloners are given by rotation invariant Gaussian wave functions with appropriate squeezing, and the resulting fidelity pairs are plotted in Fig. 1 as a dotted curve. At the intersection with the diagonal of symmetric fidelities lie the respective optimal cloners. For the optimal non-Gaussian cloner, we obtain $f_1 = f_2 \approx 0.6826$, which is strictly higher than the fidelity of the optimal Gaussian cloner, $f_1 = f_2 = 2/3$ (cf. [15]).

Studying cloners that are described by highly squeezed non-Gaussian states ρ_T reveals that on the curve of optimal fidelity pairs the points with $f_1 = 1$ and $f_2 = 1$ are approached with infinite slope [17]. It is thus clear that the iteration for the largest eigenvalue does not become singular. This regime is of potential interest in quantum key distribution, since nearly perfect clones for the legitimate recipient combined with clones of nontrivial fidelity for the eavesdropper would be the hallmark of a successful cloning attack. On the other hand, the potential room for this regime is tiny as it is already proven that Gaussian attacks are optimal for a large class of quantum key distribution protocols where the channel is probed via second-order moments of the quadratures [23].

Optical implementation.—The Gaussian symmetric cloner can be realized by linear amplification of the input state, followed by distributing the output state into the two clones with a balanced beam splitter [24]. This corresponds to the setup shown in Fig. 2 where the idler mode of the amplifier (b_1) and the second input mode of the beam splitter (b_2) are both initially in the vacuum state. Let us now analyze the cloning transformation that results from injecting an arbitrary two-mode state at modes b_1 and b_2. If the intensity gain of the optical parametric amplifier is 2, the modes where the two clones emerge are related to the input modes via the canonical transformation

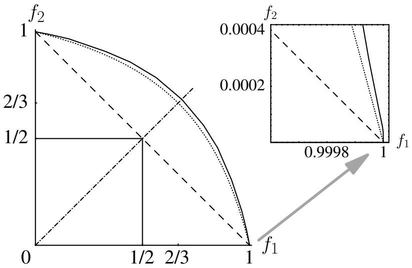

FIG. 1. Achievable pairs (f_1, f_2) of single-clone fidelities in 1-to-2 cloning of coherent states. The dots represent the optimal Gaussian cloner, while the solid curve indicates optimal non-Gaussian operations. Fidelities in the lower left quadrant are accessible to measure-and-prepare schemes. The inset shows the infinite slope at $f_1 = 1$ for non-Gaussian cloners as opposed to the Gaussian case.

$$a_1 = a_{\text{in}} + (b_1^\dagger + b_2)/\sqrt{2},$$
$$a_2 = a_{\text{in}} + (b_1^\dagger - b_2)/\sqrt{2}.$$

From this expression, it is straightforward to check that the underlying cloner is displacement covariant. Moreover, if the input is in the vacuum state $\rho = |0\rangle\langle 0|$, the single-clone fidelities amount to expectation values of the observables

$$F_1 = e^{-(Q_1+Q_2)^2/4-(P_1-P_2)^2/4},$$
$$F_2 = e^{-(Q_1-Q_2)^2/4-(P_1+P_2)^2/4},$$

where (Q_1, P_1) and (Q_2, P_2) are the canonically conjugate field quadratures of modes b_1 and b_2, respectively. This exactly coincides with expression (5) up to a symplectic rotation, namely, a beam splitter transformation.

Consequently, the problem of finding the optimal cloner reduces to finding the eigenstate with the highest eigenvalue of $\lambda_1 F_1 + \lambda_2 F_2$, that is, to find the optimal bimodal state $|\psi\rangle$ to be injected in modes b_1 and b_2. Note that if $|\psi\rangle$ is an EPR state, i.e., a suitable infinitely squeezed state [25], then this corresponds to the two extreme points of the solid curve in Fig. 1. The symmetric case $\lambda_1 = \lambda_2$ is obtained by choosing

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |2n\rangle|2n\rangle,$$

where $|n\rangle$ are Fock states and the probability amplitudes c_n correspond to the dominant eigenstate of $F_1 + F_2$. Truncations of this state to finite photon numbers correspond to suboptimal cloners: keeping only the vacuum

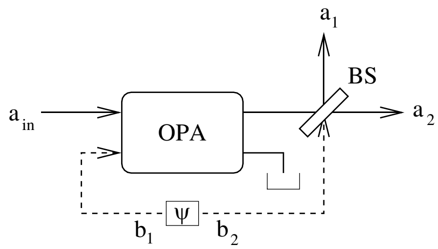

FIG. 2. Optical scheme of a displacement-covariant cloner. The input mode $a_{\rm in}$ is injected on the signal mode of an optical parametric amplifier (OPA) of gain 2, the idler mode being denoted as b_1. After amplification, the signal mode is divided at a balanced beam splitter (BS), resulting in two clones in modes a_1 and a_2. The second input mode of the beam splitter is noted b_2. If both b_1 and b_2 are initially in the vacuum state, the corresponding cloner is the Gaussian cloner of [14–16]. In contrast, if we inject a specific two-mode state $|\psi\rangle$ into b_1 and b_2, we can generate the whole set of displacement-covariant cloners, in particular, the non-Gaussian optimal one.

term $n = 0$, we get the optimal Gaussian cloner with fidelities $2/3$, while allowing for $n \leq 2$ yields the higher fidelities $f_1 = f_2 \approx 0.6801 > 2/3$. The experimental realization of this cloner does not seem unrealistic, given the recently proposed schemes for conditionally preparing arbitrary bimodal states of light based on linear optics [26]. In the limit $n \rightarrow \infty$, we arrive at the optimal cloner with $f_1 = f_2 \approx 0.6826$. Independent studies [27] of the cloning fidelities of coherent states in finite-dimensional Hilbert spaces and their numerical extrapolation have indicated that the optimal fidelity ranges between 2/3 and 0.699, which fits our result.

Optimal classical cloning.—Let us finally consider a classical 1-to-∞ cloning map T which is realized by measuring and repreparing the system. From the line of arguments above, T can be assumed to be covariant. Since composing this cloner with time reversal τ leads to a completely positive map, we get $\tau \circ T_*(W_{p,q}) = \chi_T(\sqrt{2}p, \sqrt{2}q)W_{-p,q}$ with $\chi_T(p, q)$ the characteristic function of a state. Computing the fidelity for coherent input states immediately yields

$$f_{\rm classical}(T, |0\rangle\langle 0|) = \tfrac{1}{2}\,{\rm tr}[\rho_T|0\rangle\langle 0|] \leq \tfrac{1}{2}.$$

The bound is reached by a heterodyne measurement and repreparation of coherent states, i.e., by a Gaussian scheme. This limit cannot be surpassed even with the assistance of PPT bound entanglement [28], since the respective maps are included in the above argumentation. In the case of an unassisted measure-and-prepare scheme an independent proof was recently given in [29].

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