

## CHAPTER 36

### NON-GAUSSIAN PROBABILITY DISTRIBUTION OF COASTAL WAVES

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#### Abstract

This paper presents the development of a probability density function applicable to waves in finite water depth (which can be considered to be a nonlinear, non-Gaussian random process) in closed form. The derivation of the density function is based on the Kac-Siegert solution developed for a nonlinear mechanical system, but the parameters involved in the solution are evaluated from the wave record only. Further, the probability density function is asymptotically expressed in closed form. Comparisons between the presently developed probability density function and histograms constructed from wave records show good agreement.

#### Introduction

The wind-generated wave profile observed in a sea of finite water depth is significantly different from that observed in deep water in that there is a definite excess of high crests and shallow troughs in contrast to those of waves in deep water. This is attributed to nonlinear wave-wave interaction (energy transfer) between component waves, and such waves are considered to be a typical non-Gaussian random process.

Probability distributions applicable for presenting non-Gaussian random waves have been derived through three different approaches; (a) application of the orthogonal polynomials to the probability density function [Longuet-Higgins 1963], (b) application of Stokes wave theory [Tayfun 1980, Huang, et al., 1983], and (c) application of the Kac-Siegert solution [Langley 1987].

Derivation of the non-Gaussian probability density function by applying the concept of orthogonal polynomials is well-known as the

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Gram-Charlier series probability density function. The probability density function, however, is given in series form. Therefore, the density function usually has a negative value at some part of the distribution caused by the finite number of waves used in computing the density function.

The probability density function derived based on the application of Stokes waves imposes a preliminary form on the wave profile such that the individual waves are expressed as a Stokes expansion to the 2nd or 3rd order components. It is highly desirable to justify the validity of the assumption involved for random waves, particularly waves in shallow water. On the other hand, the probability density function derived by application of the Kac-Siebert solution appears to be pertinent for waves in finite water depth, since the solution represents a nonlinear, non-Gaussian random process. The probability density function, however, cannot be presented in closed form. Hence, it is not possible to develop a probability density function applicable to wave height therefrom.

This paper presents a probability density function applicable to waves in finite water depth which can be considered to be a nonlinear, non-Gaussian random process. The density function is derived based on the Kac-Siebert solution through spectral analysis. During the course of analysis, the wave spectrum is decomposed into linear and nonlinear components in order to clarify the degree of nonlinearity involved in shallow water waves. Furthermore, the probability density function is derived in closed form so that it can be used for derivation of the distribution functions applicable to peaks and troughs of waves in finite water depth.

### Basic Concept

The basic concept of the analysis applied in the present study is that the stochastic characteristics of waves in finite water depth may be considered to be the same as those of the output of a nonlinear mechanical system. In order to elaborate on this concept, let us examine the variation of wave profiles observed from deep to shallow water.

Figure 1 shows portions of wave records obtained by the Coastal Engineering Research Center, US Army, during the Atlantic Ocean Remote Sensing Land-Ocean Experiment (ARSLOE) project in 1980 at Duck, North Carolina. Included in each figure is the distance from the shoreline at the location where data were taken. As can be seen, the wave profile recorded by Gage 710 (obtained at the deepest of the three locations) is almost the same as that observed in deep water waves. As the water depth decreases, the wave profile shows a definite excess of high crests and shallow troughs which is a typical feature of a non-Gaussian random process. In other words, the wave profile transforms from a Gaussian random process to a non-Gaussian random process as they move from deep to shallow wave areas.

Let us compare the transformation of wave profiles from deep to shallow water with the output of a nonlinear mechanical system having a Gaussian random input with various degrees of nonlinearity. The

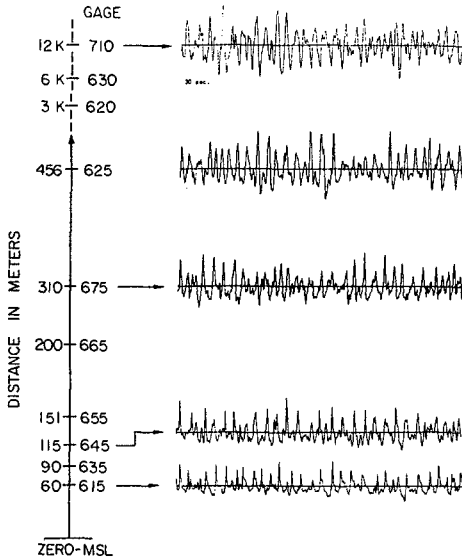


Figure 1:  
Portion of wave records obtained during ARSLOE Project at locations Gage 710, 625, 675 and 615.

output of a system with very weak nonlinearity may be considered as a Gaussian random process. However, the statistical properties of the system's output show increasing non-Gaussian characteristics with increase in the intensity of system's nonlinearity. Thus, we may consider the augmentation of the non-Gaussian characteristics of waves with decreasing water depth to be analogous to that observed in the output of a nonlinear system as its nonlinearity increases.

With this basic concept in mind, we may apply the unique solution of the output of a nonlinear system developed by Kac-Siegert [1947] to nonlinear waves. That is, the output (response) of a nonlinear system which can be presented by Volterra's stochastic series expansion can be presented in terms of the standardized normal random variable as follows:

$$y(t) = \sum_{j=1}^N (\beta_j Z_j + \lambda_j Z_j^2) \quad (1)$$

where  $y(t)$  = output of a nonlinear system  
 $Z_j$  = standardized normal variate.

The parameters  $\beta_j$  and  $\lambda_j$  are evaluated by finding the eigenfunction and eigenvalues of the integral equation given by

$$\int K(\omega_1, \omega_2) \psi_j(\omega_2) d\omega_2 = \lambda_j \psi_j(\omega_1) \quad (2)$$

$$\begin{aligned} \text{where } K(\omega_1, \omega_2) &= H(\omega_1, \omega_2) \sqrt{S(\omega_1) S(\omega_2)} \\ \psi_j(\omega) &= \text{orthogonal eigenfunction} \\ H(\omega_1, \omega_2) &= \text{2nd order frequency response function} \\ S(\omega) &= \text{output spectral density function.} \end{aligned}$$

As can be seen, knowledge of the 2nd order frequency response function,  $H(\omega_1, \omega_2)$  is necessary in order to solve the integral equation given in Eq.(2), but there is no way to evaluate it for random waves. Recently, however, the authors developed a probability density function applicable to the response of a nonlinear mechanical system in closed form based on Kac-Siebert's solution without knowledge of the second order frequency response function [Ochi and Ahn 1994]. Based on the analogy between the nonlinear mechanical system and transformation of wave profiles as they move from deep to shallow water, this paper presents the application of the authors' method in the above reference to the analysis of waves in finite water depth.

An approach to evaluate the nonlinear properties of random waves by applying the Kac-Siebert solution was considered earlier by Langley [1987]. Our approach, however, differs from Langley's approach in that (i) the wave spectrum is decomposed into linear and nonlinear components in order to examine how notably the nonlinearity increases with decreasing water depth, (ii) spectral analysis methodology instead of wave potential theory is applied in evaluating the parameters  $\beta_j$  and  $\lambda_j$ , and (iii) the probability density function applicable to wave profile  $y(t)$  is presented in closed form.

#### Presentation of Nonlinear Waves

Let us write the surface profile of nonlinear waves as follows:

$$\begin{aligned} y(t) = \text{Re} \sum_{k=1}^N c_k e^{i(\omega_k t + \varepsilon_k)} + \text{Re} \sum_{k=1}^N \sum_{\ell=1}^N c_k c_\ell \left[ q_{k\ell} e^{i\{(\omega_k + \omega_\ell)t + \varepsilon_+ \}} \right. \\ \left. + r_{k\ell} e^{i\{(\omega_k - \omega_\ell)t + \varepsilon_- \}} \right] \end{aligned} \quad (3)$$

where  $\omega$  = frequency,  $\varepsilon$  = phase lag.

$q_{k\ell}$  is a coefficient associated with  $(\omega_k + \omega_\ell)$  which is the sum of the interaction between two frequency components  $\omega_k$  and  $\omega_\ell$ , while  $r_{k\ell}$  is associated with the interaction differential between  $\omega_k$  and  $\omega_\ell$ . Since there is a phase shift between the frequencies,  $\omega_k$  and  $\omega_\ell$ , in general, these interaction coefficients are complex numbers.

Equation (3) has often been considered for simulation studies as well as for analysis of nonlinear ocean waves in which the interaction coefficients are evaluated from second order nonlinear wave

theory [Hasselmann 1962, Hudspeth and Chen 1979, Sharma and Dean 1979, Anastasiou et al. 1982, etc.].

Let us write the first term of Eq.(3) as follows:

$$y_1(t) = \sum_{k=1}^N c_k e^{i(\omega_k t + \epsilon_k)} = \sum_{k=1}^N c_k \{ \cos(\omega_k t + \epsilon_k) + i \sin(\omega_k t + \epsilon_k) \} \quad (4)$$

We assume that the linear component  $y_1(t)$  is a narrow-band Gaussian random process. By ignoring the factor  $\rho g$ ,  $(1/2)c_k^2$  represents the spectral density,  $S_L(\omega_k)\Delta\omega$ , where the subscript  $L$  stands for the spectral density of the linear wave components. That is, the spectral density at  $\omega = \omega_k$  is written by

$$s_k = \{S_L(\omega_k)\Delta\omega\}^{1/2} = \left\{ \int_{\omega_k - (\Delta\omega/2)}^{\omega_k + (\Delta\omega/2)} S_L(\omega) d\omega \right\}^{1/2}. \quad (5)$$

Then by defining

$$c_k \cos(\omega_k t + \epsilon_k) = s_k \cdot u_k, \quad \text{and} \quad c_k \sin(\omega_k t + \epsilon_k) = s_k \cdot v_k, \quad (6)$$

the linear component can be written as

$$y_1(t) = \text{Re} \sum_{k=1}^N s_k (u_k + i v_k). \quad (7)$$

The second term of Eq.(3) can be similarly written as

$$y_2(t) = \text{Re} \sum_{k=1}^N \sum_{\ell=1}^N s_k s_\ell \left\{ q_{k\ell} (u_k + i v_k) (u_\ell + i v_\ell) + r_{k\ell} (u_k + i v_k) (u_\ell - i v_\ell) \right\}, \quad \text{where} \quad s_\ell = \{S_L(\omega_\ell)\Delta\omega\}^{1/2} \quad (8)$$

Then, by taking the real-part of Eqs.(7) and (8), the nonlinear wave profile can be presented as follows:

$$y(t) = y_1(t) + y_2(t) = \sum_{k=1}^N s_k u_k + \sum_{k=1}^N \sum_{\ell=1}^N \left\{ s_k s_\ell (q_{k\ell} + r_{k\ell}) u_k u_\ell + s_k s_\ell (q_{k\ell} - r_{k\ell}) v_k v_\ell \right\} \quad (9)$$

Separation of Linear and Nonlinear Components of Wave Spectrum

Several methods for decomposing a wave spectrum into linear and nonlinear components have been developed to date. These include Tick [1959, 1961], Hamada [1965], Hudspeth and Chen [1979], Masuda, et al. [1979] and Anastasiou et al. [1982], among others. Almost all of these methods for evaluating the nonlinear part are based on the second order interaction kernel of a weakly nonlinear solution. Kim and Power [1979], on the other hand, have developed a method using the bicoherent spectrum to separate the nonlinear wave-wave interaction of coherent waves in plasma fluctuation data. Their method may be applied for separation of the spectral energy density of a random process with strong nonlinear characteristics.

Since the Kim-Power method was developed to evaluate the wave-wave interaction associated with two arbitrarily chosen constant frequencies, the method is extended in the present study so that any two frequencies associated with wave-wave interaction are not constant; instead they are variables. This implies that the nonlinear component of the spectral density at a frequency  $\omega_m$  is equal to the accumulation of nonlinear interaction associated with various pairs of frequency components  $\omega_k$  and  $\omega_l$  under the condition that  $\omega_k + \omega_l = \omega_m$ . Furthermore, we consider the interaction not only at the frequency  $(\omega_k + \omega_l)$  but also at the frequency  $(\omega_k - \omega_l)$ , where  $\omega_k > \omega_l$ . The latter is equivalent to the sum interaction between  $\omega_l$  and  $(\omega_k - \omega_l)$ . In evaluating the interaction at the frequency  $(\omega_k - \omega_l)$ , it is assumed that the spectral energy density at frequencies smaller than the minimum frequency of the main energy density in the spectrum ( $\omega_s$  in Figure 2) is due to nonlinear interaction associated with the difference between various combinations of the two frequency components at  $\omega_k$  and  $\omega_l$ . The spectral density for frequencies greater than  $\omega_s$  is due to the nonlinear interaction associated with the sum of various combinations of two frequency components.

Let us evaluate the interaction due to the sum of two frequency components. We may write the Fourier transform of  $y(t)$  at the frequency  $\omega_m$  as follows:

$$Y(\omega_m) = Y_L(\omega_m) + \sum_{\omega_k + \omega_l = \omega_m} A_L(\omega_k, \omega_l) Y_L(\omega_k) Y_L(\omega_l), \quad (10)$$

where  $Y_L(\omega_j)$  = Fourier transform of the linear component  $y_1(t)$  at the frequency  $j$ ,

$A_L(\omega_k, \omega_l)$  = coupling coefficient.

The summation in Eq.(10) is for various combinations of components at  $\omega_k$  and  $\omega_l$  where  $\omega_k + \omega_l = \omega_m$ , and that the second term is a convolution in discrete form. The coupling coefficient  $A_L(\omega_k, \omega_l)$  can be obtained by multiplying each side of Eq.(10) by the product of the conjugates of  $Y_L(\omega_k)$  and  $Y_L(\omega_l)$ ,  $Y_L^*(\omega_k) Y_L^*(\omega_l)$ , and by taking the expected value. That is,

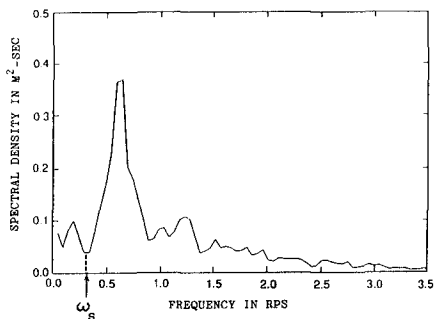


Figure 2: Definition of frequency  $\omega_s$ .

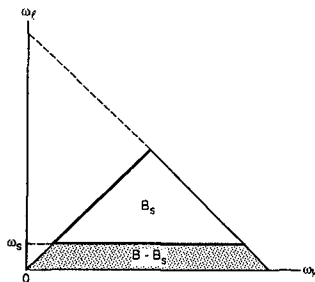


Figure 3: Domains for computing bispectra  $B(\omega_k, \omega_l)$  and  $B(\omega_l, \omega_k - \omega_l)$ .

$$A_L(\omega_k, \omega_l) = \frac{B^*(\omega_k, \omega_l)}{E \left[ |Y_L(\omega_k) Y_L(\omega_l)|^2 \right]}, \tag{11}$$

where  $B^*(\omega_k, \omega_l)$  = conjugate of the bispectrum  $B(\omega_k, \omega_l)$ . It should be noted that the domain of the bispectrum in this case is limited to that in  $B_s$  as indicated in Figure 3.

The spectral density function at the frequency  $\omega_m$  can be obtained from Eqs.(10) and (11) as

$$\begin{aligned} S(\omega_m) &= E \left[ |Y_L(\omega_m)|^2 \right] + \sum_{\omega_k + \omega_l = \omega_m} |A_L(\omega_k, \omega_l)|^2 \cdot E \left[ |Y_L(\omega_k) Y_L(\omega_l)|^2 \right] \\ &= E \left[ |Y_L(\omega_m)|^2 \right] + \sum_{\omega_k + \omega_l = \omega_m} \{b(\omega_k, \omega_l)\}^2 \cdot S(\omega_m), \end{aligned} \tag{12}$$

where 
$$b(\omega_k, \omega_l) = \left\{ \frac{|B(\omega_k, \omega_l)|^2}{E \left[ |Y_L(\omega_k) Y_L(\omega_l)|^2 \right] E \left[ |Y(\omega_m)|^2 \right]} \right\}^{1/2}. \tag{13}$$

The second term of Eq.(12) represents the accumulation of energy densities associated with interactions which occur at the frequency  $\omega_k + \omega_l$ . It is noted that  $E \left[ |Y_L(\omega_k) Y_L(\omega_l)|^2 \right]$  in Eq.(12) is unknown in advance; hence we may evaluate  $E \left[ |Y(\omega_k) Y(\omega_l)|^2 \right]$  for a given spectrum, and use it as an initial value in finding  $b(\omega_k, \omega_l)$  by iteration.

For the nonlinear components which occur at the frequency  $(\omega_k - \omega_l)$ ,  $B^*(\omega_k, \omega_l)$  as well as  $B(\omega_k, \omega_l)$  in Eq.(12) are replaced by

$B^*(\omega_\ell, \omega_k - \omega_\ell)$  and  $B(\omega_\ell, \omega_k - \omega_\ell)$ , respectively. Note that the domain of the bispectrum  $B(\omega_\ell, \omega_k - \omega_\ell)$  is limited to that in  $(B - B_s)$  as illustrated in Figure 3.

As an example of application of the method for separating the spectrum into linear and nonlinear components, Figures 4 through 6 show the results of computations carried out on three wave records, Gages 710, 625 and 615; a portion of each record is shown in Figure 1. It can be clearly seen in these figures that the ratio of nonlinear energy to total energy increases as water depth decreases. That is, for the Gage 710 wave record (depth 21.4 m), there exists no appreciable nonlinear energy component in its spectrum. On the other hand, for the Gage 625 record (depth 9.94 m), an appreciable amount of nonlinear energy exists at low and high frequencies, and the same trend can be observed on the Gage 615 wave record (depth 2.28 m) with substantial increase of nonlinear energy. It is noted that no nonlinear component exists in the neighborhood of the frequency where the spectrum peaks irrespective of sea severity.

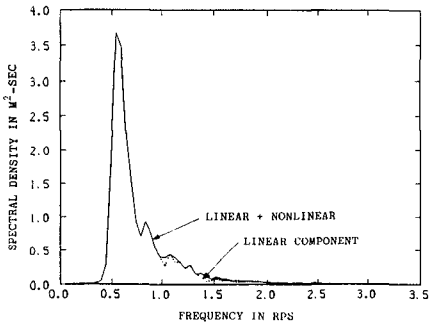


Figure 4: Separation of linear & nonlinear components of spectrum for wave record Gage 710.

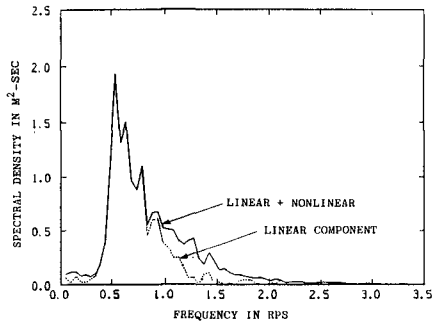


Figure 5: Separation of linear & nonlinear components of spectrum for wave record Gage 625.

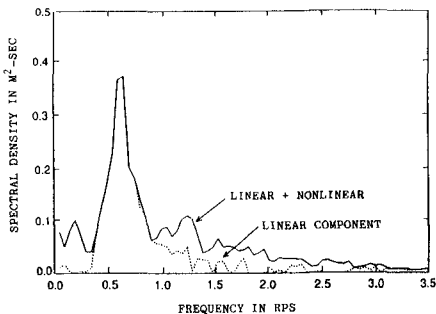


Figure 6: Separation of linear and nonlinear components of spectrum for wave record Gage 615.



Evaluation of Interaction Coefficients  $q_{k\ell}$  and  $r_{k\ell}$

The interaction coefficients  $q_{k\ell}$  and  $r_{k\ell}$  in Eq.(9) can be evaluated through bispectral analysis of the wave  $y(t)$ .

The bispectrum  $B(\omega_k, \omega_\ell)$  is most commonly evaluated in the fundamental region, which is the octant, the domain defined by  $0 \leq \omega_\ell \leq \omega_k$  and  $0 \leq \omega_k \leq \infty$  as shown in Figure 3. The volume under the bispectrum is equal to the 3rd moment of  $y(t)$  if the mean value of  $y(t)$  is zero. That is,

$$E \left[ \{y(t)\}^3 \right] = 6 \sum_{k=1}^N \sum_{\ell=1}^N \text{Re} \{B(\omega_k, \omega_\ell)\}. \quad (14)$$

On the other hand, the 3rd moment of  $y(t) = y_1(t) + y_2(t)$  can be obtained as

$$E \left[ \{y(t)\}^3 \right] = 3 E \left[ \{y_1(t)\}^2 y_2(t) \right] + E \left[ \{y_2(t)\}^3 \right]. \quad (15)$$

Note that  $y_1(t)$  is a normal distribution with zero mean, therefore  $E[\{y_1(t)\}^3]$  and  $E[y_1(t)\{y_2(t)\}^2]$  are zero.

Since the linear component  $y_1(t)$  is usually much greater than the nonlinear component  $y_2(t)$ , the second term of Eq.(15) may be neglected in comparison with the first term. Hence, we have from Eq.(9)

$$E \left[ \{y(t)\}^3 \right] = 3 E \left[ \{y_1(t)\}^2 y_2(t) \right] = 6 \sum_{k=1}^N \sum_{\ell=1}^N s_k^2 s_\ell^2 (q_{k\ell} + r_{k\ell}). \quad (16)$$

Then, Eqs.(14) and (16) yield

$$\sum_{k=1}^N \sum_{\ell=1}^N (q_{k\ell} + r_{k\ell}) = \sum_{k=1}^N \sum_{\ell=1}^N \frac{\text{Re} \{B(\omega_k, \omega_\ell)\}}{s_k^2 s_\ell^2}. \quad (17)$$

As stated in earlier, the domain of the bispectrum applicable for the interactions associated with the sum of two frequency components is  $B_s$  as shown in Figure 3. Therefore, from Eq.(17), the interaction coefficient  $q_{k\ell}$  can be obtained as

$$q_{k\ell} = \frac{1}{s_k^2 s_\ell^2} B_s(\omega_k, \omega_\ell). \quad (18)$$

Similarly, the interaction coefficient  $r_{k\ell}$  can be evaluated as

$$r_{k\ell} = \frac{1}{s_k s_\ell} \{B(\omega_\ell, \omega_k - \omega_\ell) - B_s(\omega_k, \omega_\ell)\}. \tag{19}$$

Application of the Kac-Siegert Solution

The functional relationship between parameters  $\beta_j$  and  $\lambda_j$  in the Kac-Siegert solution and the spectral densities of the linear wave components,  $s_k$  and  $s_\ell$ , as well as the interaction coefficients  $q_{k\ell}$  and  $r_{k\ell}$  are presented by Langley [1987] in a concise matrix form. Although our methods of deriving the linear wave component as well as the interaction coefficients are quite different from Langley's methods, the functional relationship between  $(\beta_j, \lambda_j)$  and  $(s_k, s_\ell, q_{k\ell}, r_{k\ell})$  can still be applied to the present problem. That is, Langley presents Eq.(9) as follows:

$$y(t) = \underline{s}' \underline{u} + \underline{u}' (\underline{Q} + \underline{R}) \underline{u} + \underline{v}' (\underline{Q} - \underline{R}) \underline{u}. \tag{20}$$

where  $\underline{Q}$  and  $\underline{R}$  are real symmetric matrices whose  $k\ell$ -th components are  $s_k s_\ell q_{k\ell}$  and  $s_k s_\ell r_{k\ell}$ , and  $\underline{s}$ ,  $\underline{u}$  and  $\underline{v}$  are vectors whose  $k$ -th components are  $s_k$ ,  $u_k$  and  $v_k$ , respectively. We may write the two matrices in Eq.(20) as

$$\begin{aligned} \underline{Q} + \underline{R} &= \underline{w}'_1 \underline{\Lambda}_1 \underline{w}_1 \\ \underline{Q} - \underline{R} &= \underline{w}'_2 \underline{\Lambda}_2 \underline{w}_2 \end{aligned} \tag{21}$$

where the column vectors of  $\underline{w}_j$  ( $j = 1, 2$ ) are normalized eigenvectors. They are orthonormal vectors satisfying the condition  $\underline{w}'_j \underline{w}_j = \underline{I}$ , where  $\underline{I}$  is the identity matrix. The elements of the diagonal matrices  $\underline{\Lambda}_j$  ( $j = 1, 2$ ) are eigenvalues of matrix  $\underline{Q} + \underline{R}$  and  $\underline{Q} - \underline{R}$ , respectively. Since  $\underline{Q} + \underline{R}$  and  $\underline{Q} - \underline{R}$  are  $N$  by  $N$  symmetric matrices, there is a total of  $2N$  eigenvalues.

Let us write

$$z_j = \begin{cases} (\underline{w}'_1 \underline{u})_j, & j = 1, 2, \dots, N \\ (\underline{w}'_2 \underline{u})_{j-N}, & j = (N+1), \dots, 2N \end{cases} \tag{22}$$

where  $z_j$  are independent standardized Gaussian random variables. Then, Langley has shown that Eq.(20) can be reduced to the Kac-Siegert formulation given in Eq.(1) with the following relationships:

$$\beta_j = \begin{cases} (\underline{s}' \underline{w}_1)_j, & j = 1, 2, \dots, N \\ 0 & j = (N+1) \dots, 2N \end{cases}$$

$$\lambda_j = \begin{cases} (\Delta_1)_j, & j = 1, 2, \dots, N \\ (\Delta_2)_j, & j = (N+1), \dots, 2N \end{cases} \quad (23)$$

Since the parameters of the Kac-Siebert solution are obtained through spectral analysis, the probability density function applicable to nonlinear, non-Gaussian waves can be derived numerically with the aid of the characteristic function as suggested by Kac-Siebert.

As an example, the method is applied to the Gage 615 wave record which indicates strong nonlinear characteristics. A comparison between the probability density function and the histogram constructed from the data is shown in Figure 7. Excellent agreement between them can be seen in the figure. This result implies that the Kac-Siebert solution evaluated from the time history of waves yields a probability density function representing the statistical properties of waves in finite water depth which have strong nonlinear characteristics. It is unfortunate, however, that the probability density function cannot be obtained in closed form.

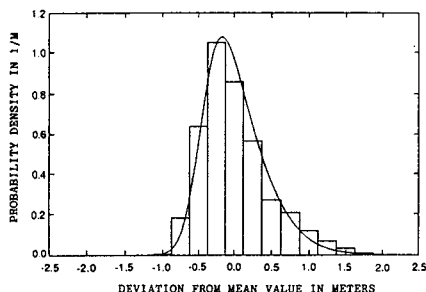


Figure 7:  
Comparison of probability density function obtained from the Kac-Siebert solution and histogram constructed from data for wave record Gage 615.

#### Asymptotic Probability Distribution for Non-Gaussian Waves

One method for deriving the probability density function in closed form is to present the Kac-Siebert solution (Eq.1) as a function of a single random variable instead of the summation of the standardized normal distribution and its squared quantity. For this, let us present Eq.(1) as

$$Y = U + aU^2 \quad (24)$$

where "a" is a constant (unknown) and U is a normal variate with mean  $\mu_*$  and variance  $\sigma_*^2$ , both of which are also unknown. The value of these unknowns will be determined from the following three equations which are derived by equating the cumulant generating function of Eq.(24) with that of Kac-Siebert's solution given in Eq.(1) (see Ochi and Ahn 1994).

$$\left\{ \begin{array}{l} a\sigma_*^2 + a\mu_*^2 + \mu_* = 0 \\ \left( \sum_{j=1}^{2N} \beta_j^2 \right) + 2 \left( \sum_{j=1}^{2N} \lambda_j^2 \right) = \sigma_*^2 - 2a^2\sigma_*^4 \\ 3 \left( \sum_{j=1}^{2N} \beta_j^2 \lambda_j \right) + 4 \left( \sum_{j=1}^{2N} \lambda_j^3 \right) = 3a\sigma_*^4 - 8a^3\sigma_*^6. \end{array} \right. \quad (25)$$

It should be noted that the left side of Eq.(25) can be presented in terms of cumulants,  $k_1$ ,  $k_2$  and  $k_3$ . If we have a wave record obtained for a sufficiently long time (on the order of 20 minutes) and if we let the mean value be the zero line, we have  $k_1 = 0$ , and thereby  $k_2$  and  $k_3$  are equal to the sample moments  $E[y^2]$  and  $E[y^3]$ , respectively. Thus, we can determine the unknown parameters  $a$ ,  $\mu_*$  and  $\sigma_*^2$  by simply evaluating the sample moments from the wave record  $y(t)$  and by applying the following relationship:

$$\begin{aligned} a\sigma_*^2 + a\mu_*^2 + \mu_* &= 0 \\ \sigma_*^2 - 2a^2\sigma_*^4 &= E[x^2] \\ 2a\sigma_*^4 (3 - 8a^2\sigma_*^2) &= E[x^3]. \end{aligned} \quad (26)$$

Since the random variable  $U$  in Eq.(24) is now a normal variate with known mean  $\mu_*$  and variance  $\sigma_*^2$ , the probability density function of  $Y$  can be derived by applying the technique of change of random variables from  $U$  to  $Y$ . Unfortunately, however, the density function thusly derived vanishes at a point  $y = -(1/4a)$  due to a singularity involved in the density function.

In order to circumvent this drawback, let us present the functional relationship between  $Y$  and  $U$  given in Eq.(24) inversely such that the random variable  $U$  is expressed as a function of  $Y$  as follows:

$$U = \frac{1}{\gamma a} \left( 1 - e^{-\gamma a Y} \right), \quad (27)$$

where  $\gamma$  is a constant; 1.28 for  $y \geq 0$  and 3.00 for  $y \leq 0$ . Justification for selecting these constant values of  $\gamma$  is given in Ochi and Ahn 1994. It may suffice here to say that these constants are valid even for a random process with very strong nonlinear characteristics.

It is noted that the values of  $\gamma$  are different for positive and negative  $y$ -values. This results in a slight difference in the slope of the probability density function at  $y = 0$ , although the density function is continuous at this point.

By using the functional relationship given in Eq.(27), we may derive the probability density function of Y from the random variable U which obeys the normal distribution with mean  $\mu_*$  and variance  $\sigma_*^2$ . The change of random variables technique yields the probability density function of Y as follows:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_*} e^{-\frac{1}{2(\gamma a \sigma_*)^2} (1 - \gamma a \mu_* - e^{-\gamma a y})^2 - \gamma a y} \quad , \quad (28)$$

where  $\gamma = \begin{cases} 1.28 & y \geq 0 \\ 3.00 & y < 0. \end{cases}$

By applying the method for determining the limit of an indefinite function, it can be easily proved that Eq.(28) reduces to a normal probability density function with mean  $\mu_*$  and variance  $\sigma_*^2$  if  $a = 0$ ; namely, for a linear system.

Figure 8 shows a comparison of the newly developed asymptotic probability density function with the histogram of the Gage 615 wave record. The parameters  $a$ ,  $\mu_*$  and  $\sigma_*^2$  are determined from Eq.(26). Included also in the figure is the normal probability density function with zero mean and the variance evaluated from the record. As can be seen in the figure, the histogram deviates from the normal density function to a great extent; but, the agreement between the histogram and the newly developed probability density function is excellent.

Figures 9 and 10 show comparisons between the probability density function given in Eq.(28) and histograms for the Gage 625 (water depth 9.94 m) and the Gauge 710 (water depth 21.4 m), respectively. In the latter case, the probability density function is normally distributed and agrees very well with the histogram.

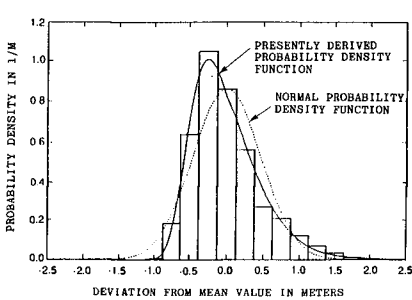


Figure 8: Comparison of presently developed probability density function and histogram constructed from data for wave Gage 615.

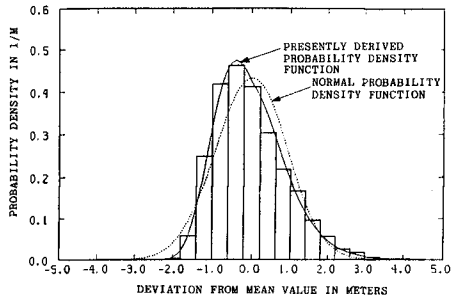


Figure 9: Comparison of presently developed probability density function and histogram constructed from data for wave Gage 625.

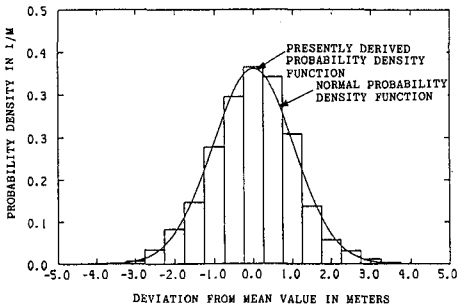


Figure 10:  
Comparison of presently developed probability density function and histogram constructed from data for wave record Gage 710.

From the results of these comparisons, it can be concluded that the probability density function covers the non-Gaussian distribution observed for random waves in shallow water as well as the Gaussian distribution observed for random waves in deep water, and that the probability density function agrees very well with the histograms constructed from wave data.

### Conclusions

This paper presents the results of a study on a probability density function applicable to waves in finite water depth developed based on the concept of Kac-Siegert's solution for the output of a nonlinear mechanical system. Although the Kac-Siegert solution requires knowledge of the second order frequency response function of the system, we developed a method to evaluate the parameters involved in the solution only from measured waves through spectral analysis. Further, the probability density function associated with the Kac-Siegert solution is asymptotically expressed in closed form (Eq.28). The presently developed probability density function covers the non-Gaussian distribution observed for random waves in shallow water as well as the Gaussian distribution observed for random waves in deep water. Comparison between the probability density function and histograms constructed from wave records show good agreement.

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