# Non-Geometric Cospectral Mates of Line Graphs with a Linear Representation 

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#### Abstract

For an incidence geometry $\mathcal{G}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ with a linear representation $\mathcal{T}_{n}^{*}(\mathcal{K})$, we apply WQH switching to construct a non-geometric graph $\Gamma^{\prime}$ cospectral with the line graph $\Gamma$ of $\mathcal{G}$.

As an application, we show that for $h \geq 2$ and $0<m<h$, there are strongly regular graphs with parameters $(v, k, \lambda, \mu)=\left(2^{2 h}\left(2^{m+h}+2^{m}-\right.\right.$ $\left.\left.2^{h}\right), 2^{h}\left(2^{h}+1\right)\left(2^{m}-1\right), 2^{h}\left(2^{m+1}-3\right), 2^{h}\left(2^{m}-1\right)\right)$ which are not point graphs of partial geometries of order $(s, t, \alpha)=\left(\left(2^{h}+1\right)\left(2^{m}-1\right), 2^{h}-1,2^{m}-1\right)$.


## 1 Introduction

Let $\mathcal{G}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space of order $(s, t)$, that is two points are incident with at most one line, each point is incident with $t+1$ lines, and each line is incident with $s+1$ points. If for any anti-flag $(P, L)$, there are precisely $\alpha$ lines through $P$ meeting $L$, then $\mathcal{G}$ is called a partial geometry with parameters $(s, t, \alpha)$. The line graph $\Gamma(\mathcal{L})$ of a partial linear space $\mathcal{G}$ has vertex set $\mathcal{L}$, two lines adjacent when they meet.

Linear representations are an important source for partial linear spaces: Let $n \geq 2$ and $q$ be a prime power. Let $\mathcal{P}$ be the points of $\mathrm{AG}(n+1, q)$ and $\mathcal{K}$ be a set of points in the hyperplane $H \cong \operatorname{PG}(n, q)$ at infinity. Let $\mathcal{L}$ be the lines of $\mathrm{AG}(n+1, q)$ which meet $H$ in a point of $\mathcal{K}$. Then $\mathcal{T}_{n}^{*}(\mathcal{K})$ denotes the incidence geometry $\mathcal{G}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ where incidence is inherited from $\operatorname{AG}(n+1, q)$. We call $\mathcal{T}_{n}^{*}(\mathcal{K})$ the linear representation of $\mathcal{G}$. The line graph $\Gamma(\mathcal{L})$ has $|\mathcal{K}| \cdot q^{n}$ vertices and degree $q \cdot(|\mathcal{K}|-1)$. We refer to [2, 4, 7] for various constructions of interesting geometries using linear representation.

If $q=2^{h}, 0<m<h$, and $n=2$, then a maximal arc $\mathcal{K}$ of Denniston type of size $\left(2^{h}+1\right)\left(2^{m}-1\right)+1$ (see [3]) yields a partial geometry with linear representation $\mathcal{T}_{2}^{*}(\mathcal{K})$ and parameters $(s, t, \alpha)=\left(2^{h}-1,\left(2^{h}+1\right)\left(2^{m}-1\right), 2^{m}-1\right)$. In particular, for $m=1$ we obtain generalized quadrangles of order $(s, t)=$ $(q-1, q+1)$.

Recall WQH-switching [10] (also see [6]):
Lemma 1.1 (WQH-Switching). Let $\Gamma$ be a graph with vertex set $X$ and let $\left\{C_{1}, C_{2}, D\right\}$ be a partition of $X$, where the subgraphs induced on $C_{1}, C_{2}$, and $C_{1} \cup C_{2}$ are regular, and $C_{1}$ and $C_{2}$ have the same size and degree. Suppose that $x \in D$ either has the same number of neighbors in $C_{1}$ and $C_{2}$, or satisfies $\Gamma(x) \cap$ $\left(C_{1} \cup C_{2}\right) \in\left\{C_{1}, C_{2}\right\}$. Construct a new graph $\Gamma^{\prime}$ by interchanging adjacency and
nonadjacency between $x \in D$ and $C_{1} \cup C_{2}$ when $\Gamma(x) \cap\left(C_{1} \cup C_{2}\right) \in\left\{C_{1}, C_{2}\right\}$. Then $\Gamma$ and $\Gamma^{\prime}$ are cospectral.

Only for this document, let us call a partial linear space $\mathcal{G}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order $(s, t)$ an incomplete $(s, t, \alpha)$-geometry when for any anti-flag $(P, L)$ of $\mathcal{G}$, $P$ is collinear with at most $\alpha$ points on $L$. We show the following:

Proposition 1.2. Let $\mathcal{G}=(\mathcal{P}, \mathcal{L}, I)$ be an incomplete $(q-1, t, \alpha)$-geometry with linear representation $\mathcal{T}_{n}^{*}(\mathcal{K})$ and line graph $\Gamma=\Gamma(\mathcal{L})($ so $t+1=|\mathcal{K}|)$. Suppose that $t>q(\alpha-1)$, and that $\mathcal{K}$ contains a line $K$ with $|K \cap \mathcal{K}| \geq 2$ and points $Q_{1}, Q_{2} \in \mathcal{K} \backslash K$ with $\left\langle Q_{1}, Q_{2}\right\rangle \cap K \notin \mathcal{K}$. Then there exists a graph $\Gamma^{\prime}$ cospectral with $\Gamma$ such that $\Gamma^{\prime}$ is not the line graph of an incomplete $\left(s^{\prime}, t, \alpha^{\prime}\right)$-geometry for any $s^{\prime}, \alpha^{\prime}$.

A graph (not edgeless, not complete) of order $v$ and degree $k$ is called strongly regular with parameters $(v, k, \lambda, \mu)$ if any two adjacent vertices have precisely $\lambda$ common neighbors, and any two nonadjacent vertices have precisely $\mu$ common neighbors. A partial geometry of order $(s, t, \alpha)$ yields a strongly regular graph with parameters

$$
(v, k, \lambda, \mu)=\left(\frac{(s+1)(s t+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right) .
$$

By applying Proposition 1.2 to the partial geometries from $\operatorname{arcs} \mathcal{K}$ of Denniston type with parameters $(s, t, \alpha)=\left(2^{h}-1,\left(2^{h}+1\right)\left(2^{m}-1\right), 2^{m}-1\right)$ mentioned above, we obtain the following:

Corollary 1.3. For $h \geq 2$, and $0<m<h$, there exists a strongly regular graph with parameters $(v, k, \lambda, \mu)=\left(2^{2 h}\left(2^{m+h}+2^{m}-2^{h}\right), 2^{h}\left(2^{h}+1\right)\left(2^{m}-\right.\right.$ 1), $\left.2^{h}\left(2^{m+1}-3\right), 2^{h}\left(2^{m}-1\right)\right)$ which is not the line graph of a partial geometry of order $(s, t, \alpha)=\left(2^{h}-1,\left(2^{h}+1\right)\left(2^{m}-1\right), 2^{m}-1\right)$.

Proof. From $q=2^{h}, t=\left(2^{h}+1\right)\left(2^{m}-1\right)$, and $\alpha=2^{m}-1$, the inequality $t>q(\alpha-1)$ follows. A line of $\mathrm{PG}(2, q)$ intersects a complete arc of size $\left(2^{h}+\right.$ 1) $\left(2^{m}-1\right)$ either in 0 or $2^{m}$ points, so $K, Q_{1}, Q_{2}$ exist.

If $m=1$, then we have the line graph of a generalized quadrangle of order $(q-1, q+1)$ with $q=2^{h}$. It is not too hard to see that a construction by Wallis 9 produces graphs cospectral with the line graph of $\mathcal{G}$ which are not line graphs themselves, so Corollary 1.3 is surely known for $m=1$. This note is motivated by [1] where the authors ask if there exists an infinite family of nongeometric strongly regular graphs cospectral with the line graph of a generalized quadrangle of order $(q-1, q+1)$. More generally, Wallis' construction with (in Wallis notation) an affine resolvable design of type $\operatorname{AR}\left(2^{h}, 1\right)$ and a block design with $(v, k)=\left(2^{h+m}+2^{m}-2^{h}, 2^{m}\right)$ works. Again, Denniston arcs imply the existence of these structures (for instance, see [8]).

When Proposition 1.2 is applicable, then one can most likely apply WQHswitching repeatedly and obtain large numbers of graphs. For instance, starting with the line graph of the unique generalized quadrangle of order $(3,5)$, so $(h, m)=(2,1)$ in Corollary 1.3, one obtains 133,005 strongly regular graphs by applying WQH-switching up to six times 5].

## 2 Proof of Proposition 1.2

Let $M_{1}, M_{2}$ be distinct planes of $\mathrm{AG}(n+1, q)$ through $K$. Pick $P \in \mathcal{K} \cap K$. For $i \in\{1,2\}$, let $C_{i}$ denote the lines in $M_{i} \cap \mathcal{L}$ which contain $P$.

Let us verify that we can apply Lemma 1.1. Clearly, $\left|C_{1}\right|=\left|C_{2}\right|$. Let $L$ be a line of $C_{i}$ for $\{i, j\}=\{1,2\}$. The line $L$ is adjacent to all lines in $C_{i}$ and no line in $C_{j}$. Hence, the induced subgraphs on $C_{1}, C_{2}$, and $C_{1} \cup C_{2}$ are all regular, and the induced subgraphs on $C_{1}$ and $C_{2}$ have the same orders and degrees.

Now let $L$ be a line of $\mathcal{L} \backslash\left(C_{1} \cup C_{2}\right)$. If $L \subseteq M_{i}$ for some $i \in\{1,2\}$, then $L$ meets all lines of $C_{i}$ and none of $C_{j}$ for $\{i, j\}=\{1,2\}$. In all other cases $L$ meets $M_{1}$ in a point $R_{1}$ and $M_{2}$ in a point $R_{2}$. Hence, $L$ meets one line of $C_{1}$ and $C_{2}$ each. Hence, we can apply Lemma 1.1 and obtain a graph $\Gamma^{\prime}$ cospectral with $\Gamma$.

The discussion above shows that the neighborhood of $L \in \mathcal{L} \backslash\left(C_{1} \cup C_{2}\right)$ only differs between $\Gamma$ and $\Gamma^{\prime}$ when $L \subseteq M_{i}$ and $P \notin L$. Such lines exist as we require $|L \cap \mathcal{K}| \geq 2$.

It remains to show that the resulting graph $\Gamma^{\prime}$ cannot be the line graph of an incomplete $\left(s^{\prime}, t, \alpha^{\prime}\right)$-geometry. Observe that cliques of $\Gamma$ either have size $t+1$ (when they consist of all lines through point) or size at most $q(\alpha-1)+1$ (when they are contained in a plane of $\operatorname{AG}(n+1, q))$. Suppose that $\Gamma^{\prime}$ is the line graph of an incomplete $\left(s^{\prime}, t, \alpha^{\prime}\right)$-geometry $\mathcal{G}^{\prime}$. Hence, if two lines are adjacent in $\Gamma^{\prime}$, then they lie together in a clique of size $t+1$.

Pick a point $R$ in $M_{1}$. For $i \in\{1,2\}$, let $L_{i}$ be the line through $Q_{i}$ and $R$. Note that $L_{1}$ and $L_{2}$ are not in $M_{1} \cup M_{2}$, so their neighborhoods are the same in $\Gamma$ and $\Gamma^{\prime}$. Hence, $L_{1}, L_{2}$ are adjacent in $\Gamma$ and $\Gamma^{\prime}$, so they lie in a clique of size $t+1$ in $\Gamma$ and $\Gamma^{\prime}$ each. For $\Gamma$, this clique is unique (as $t+1>q(\alpha-1)+1$ ) and consists of all lines through $R$.

The lines $L_{1}$ and $L_{2}$ have no common neighbor in $M_{2}$ : If $L_{1}$ or $L_{2}$ does not meet $M_{2}$, then this is clear. Hence, suppose that $L_{1} \cap M_{2}$ and $L_{2} \cap M_{2}$ are distinct points $S_{1}$ and $S_{2}$. Let $\tilde{L}$ be the line through $S_{1}$ and $S_{2}$. Then $\tilde{L} \cap K=\left\langle S_{1}, S_{2}\right\rangle \cap K=\left\langle Q_{1}, Q_{2}\right\rangle \cap K \notin \mathcal{K}$. Hence, $\tilde{L} \notin \mathcal{L}$.

Now we show that in $\Gamma^{\prime}$ a clique containing $L_{1}$ and $L_{2}$ has at most size $t$.
If $L_{1}, L_{2}$ lie in a clique $\mathcal{C}$ of $\Gamma^{\prime}$ which does not contain a line through $R$, then $|\mathcal{C}| \leq q(\alpha-1)+1<t+1$. Hence, $L_{1}, L_{2}$ lie in a clique of size $t+1$ which also contains a line $L \in \mathcal{L} \cap M_{1}$ with $R \in L$.

If $L \in C_{1}$, then let $L^{\prime}$ be a line of $\mathcal{L} \backslash C_{1}$ in $M_{1}$ with $R \in L^{\prime}$ (which exists as $|L \cap \mathcal{K}| \geq 2$ ). Then $L^{\prime}$ is nonadjacent to $L$ in $\Gamma^{\prime}$. The line $L$ only gains lines in $M_{2}$ as new neighbors in $\Gamma^{\prime}$, but $L_{1}$ and $L_{2}$ have no common neighbor in $M_{2}$ in $\Gamma^{\prime}$. Hence, $\left\{L, L_{1}, L_{2}\right\}$ lie in a clique of size at most $t$.

If $L \notin C_{1}$, then repeat the argument with switched roles for $L$ and $L^{\prime}$, that is $L^{\prime} \in \mathcal{L}$ with $R \in L^{\prime}$. Hence, $L_{1}$ and $L_{2}$ do not lie in a clique of size $t+1$, so $\Gamma^{\prime}$ is not the line graph of an incomplete $\left(s^{\prime}, t, \alpha^{\prime}\right)$-geometry.

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