Non-Geometric Cospectral Mates of Line Graphs with a Linear Representation

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Abstract

For an incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ with a linear representation $\mathcal{T}_n^*(\mathcal{K})$, we apply WQH switching to construct a non-geometric graph Γ' cospectral with the line graph Γ of \mathcal{G} .

As an application, we show that for $h \ge 2$ and 0 < m < h, there are strongly regular graphs with parameters $(v, k, \lambda, \mu) = (2^{2h}(2^{m+h} + 2^m - 2^h), 2^h(2^h+1)(2^m-1), 2^h(2^{m+1}-3), 2^h(2^m-1))$ which are not point graphs of partial geometries of order $(s, t, \alpha) = ((2^h + 1)(2^m - 1), 2^h - 1, 2^m - 1)$.

1 Introduction

Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a partial linear space of order (s, t), that is two points are incident with at most one line, each point is incident with t + 1 lines, and each line is incident with s + 1 points. If for any anti-flag $(\mathcal{P}, \mathcal{L})$, there are precisely α lines through \mathcal{P} meeting \mathcal{L} , then \mathcal{G} is called a *partial geometry* with parameters (s, t, α) . The line graph $\Gamma(\mathcal{L})$ of a partial linear space \mathcal{G} has vertex set \mathcal{L} , two lines adjacent when they meet.

Linear representations are an important source for partial linear spaces: Let $n \geq 2$ and q be a prime power. Let \mathcal{P} be the points of $\operatorname{AG}(n+1,q)$ and \mathcal{K} be a set of points in the hyperplane $H \cong \operatorname{PG}(n,q)$ at infinity. Let \mathcal{L} be the lines of $\operatorname{AG}(n+1,q)$ which meet H in a point of \mathcal{K} . Then $\mathcal{T}_n^*(\mathcal{K})$ denotes the incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ where incidence is inherited from $\operatorname{AG}(n+1,q)$. We call $\mathcal{T}_n^*(\mathcal{K})$ the *linear representation* of \mathcal{G} . The line graph $\Gamma(\mathcal{L})$ has $|\mathcal{K}| \cdot q^n$ vertices and degree $q \cdot (|\mathcal{K}| - 1)$. We refer to [2, 4, 7] for various constructions of interesting geometries using linear representation.

If $q = 2^{h}$, 0 < m < h, and n = 2, then a maximal arc \mathcal{K} of Denniston type of size $(2^{h} + 1)(2^{m} - 1) + 1$ (see [3]) yields a partial geometry with linear representation $\mathcal{T}_{2}^{*}(\mathcal{K})$ and parameters $(s, t, \alpha) = (2^{h} - 1, (2^{h} + 1)(2^{m} - 1), 2^{m} - 1)$. In particular, for m = 1 we obtain generalized quadrangles of order (s, t) = (q - 1, q + 1).

Recall WQH-switching [10] (also see [6]):

Lemma 1.1 (WQH-Switching). Let Γ be a graph with vertex set X and let $\{C_1, C_2, D\}$ be a partition of X, where the subgraphs induced on C_1, C_2 , and $C_1 \cup C_2$ are regular, and C_1 and C_2 have the same size and degree. Suppose that $x \in D$ either has the same number of neighbors in C_1 and C_2 , or satisfies $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. Construct a new graph Γ' by interchanging adjacency and

nonadjacency between $x \in D$ and $C_1 \cup C_2$ when $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. Then Γ and Γ' are cospectral.

Only for this document, let us call a partial linear space $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ of order (s, t) an *incomplete* (s, t, α) -geometry when for any anti-flag (P, L) of \mathcal{G} , P is collinear with at most α points on L. We show the following:

Proposition 1.2. Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, I)$ be an incomplete $(q - 1, t, \alpha)$ -geometry with linear representation $\mathcal{T}_n^*(\mathcal{K})$ and line graph $\Gamma = \Gamma(\mathcal{L})$ (so $t + 1 = |\mathcal{K}|$). Suppose that $t > q(\alpha - 1)$, and that \mathcal{K} contains a line K with $|K \cap \mathcal{K}| \ge 2$ and points $Q_1, Q_2 \in \mathcal{K} \setminus K$ with $\langle Q_1, Q_2 \rangle \cap K \notin \mathcal{K}$. Then there exists a graph Γ' cospectral with Γ such that Γ' is not the line graph of an incomplete (s', t, α') -geometry for any s', α' .

A graph (not edgeless, not complete) of order v and degree k is called strongly regular with parameters (v, k, λ, μ) if any two adjacent vertices have precisely λ common neighbors, and any two nonadjacent vertices have precisely μ common neighbors. A partial geometry of order (s, t, α) yields a strongly regular graph with parameters

$$(v, k, \lambda, \mu) = (\frac{(s+1)(st+\alpha)}{\alpha}, s(t+1), s-1 + t(\alpha - 1), \alpha(t+1)).$$

By applying Proposition 1.2 to the partial geometries from arcs \mathcal{K} of Denniston type with parameters $(s, t, \alpha) = (2^h - 1, (2^h + 1)(2^m - 1), 2^m - 1)$ mentioned above, we obtain the following:

Corollary 1.3. For $h \ge 2$, and 0 < m < h, there exists a strongly regular graph with parameters $(v, k, \lambda, \mu) = (2^{2h}(2^{m+h} + 2^m - 2^h), 2^h(2^h + 1)(2^m - 1), 2^h(2^{m+1} - 3), 2^h(2^m - 1))$ which is not the line graph of a partial geometry of order $(s, t, \alpha) = (2^h - 1, (2^h + 1)(2^m - 1), 2^m - 1)$.

Proof. From $q = 2^h$, $t = (2^h + 1)(2^m - 1)$, and $\alpha = 2^m - 1$, the inequality $t > q(\alpha - 1)$ follows. A line of PG(2, q) intersects a complete arc of size $(2^h + 1)(2^m - 1)$ either in 0 or 2^m points, so K, Q_1 , Q_2 exist.

If m = 1, then we have the line graph of a generalized quadrangle of order (q - 1, q + 1) with $q = 2^h$. It is not too hard to see that a construction by Wallis [9] produces graphs cospectral with the line graph of \mathcal{G} which are not line graphs themselves, so Corollary 1.3 is surely known for m = 1. This note is motivated by [1] where the authors ask if there exists an infinite family of non-geometric strongly regular graphs cospectral with the line graph of a generalized quadrangle of order (q - 1, q + 1). More generally, Wallis' construction with (in Wallis notation) an affine resolvable design of type AR $(2^h, 1)$ and a block design with $(v, k) = (2^{h+m} + 2^m - 2^h, 2^m)$ works. Again, Denniston arcs imply the existence of these structures (for instance, see [8]).

When Proposition 1.2 is applicable, then one can most likely apply WQHswitching repeatedly and obtain large numbers of graphs. For instance, starting with the line graph of the unique generalized quadrangle of order (3, 5), so (h, m) = (2, 1) in Corollary 1.3, one obtains 133,005 strongly regular graphs by applying WQH-switching up to six times [5].

2 Proof of Proposition 1.2

Let M_1, M_2 be distinct planes of AG(n+1, q) through K. Pick $P \in \mathcal{K} \cap K$. For $i \in \{1, 2\}$, let C_i denote the lines in $M_i \cap \mathcal{L}$ which contain P.

Let us verify that we can apply Lemma 1.1: Clearly, $|C_1| = |C_2|$. Let L be a line of C_i for $\{i, j\} = \{1, 2\}$. The line L is adjacent to all lines in C_i and no line in C_j . Hence, the induced subgraphs on C_1 , C_2 , and $C_1 \cup C_2$ are all regular, and the induced subgraphs on C_1 and C_2 have the same orders and degrees.

Now let L be a line of $\mathcal{L} \setminus (C_1 \cup C_2)$. If $L \subseteq M_i$ for some $i \in \{1, 2\}$, then L meets all lines of C_i and none of C_j for $\{i, j\} = \{1, 2\}$. In all other cases L meets M_1 in a point R_1 and M_2 in a point R_2 . Hence, L meets one line of C_1 and C_2 each. Hence, we can apply Lemma 1.1 and obtain a graph Γ' cospectral with Γ .

The discussion above shows that the neighborhood of $L \in \mathcal{L} \setminus (C_1 \cup C_2)$ only differs between Γ and Γ' when $L \subseteq M_i$ and $P \notin L$. Such lines exist as we require $|L \cap \mathcal{K}| \geq 2$.

It remains to show that the resulting graph Γ' cannot be the line graph of an incomplete (s', t, α') -geometry. Observe that cliques of Γ either have size t + 1 (when they consist of all lines through point) or size at most $q(\alpha - 1) + 1$ (when they are contained in a plane of AG(n+1, q)). Suppose that Γ' is the line graph of an incomplete (s', t, α') -geometry \mathcal{G}' . Hence, if two lines are adjacent in Γ' , then they lie together in a clique of size t + 1.

Pick a point R in M_1 . For $i \in \{1, 2\}$, let L_i be the line through Q_i and R. Note that L_1 and L_2 are not in $M_1 \cup M_2$, so their neighborhoods are the same in Γ and Γ' . Hence, L_1, L_2 are adjacent in Γ and Γ' , so they lie in a clique of size t + 1 in Γ and Γ' each. For Γ , this clique is unique (as $t + 1 > q(\alpha - 1) + 1$) and consists of all lines through R.

The lines L_1 and L_2 have no common neighbor in M_2 : If L_1 or L_2 does not meet M_2 , then this is clear. Hence, suppose that $L_1 \cap M_2$ and $L_2 \cap M_2$ are distinct points S_1 and S_2 . Let \tilde{L} be the line through S_1 and S_2 . Then $\tilde{L} \cap K = \langle S_1, S_2 \rangle \cap K = \langle Q_1, Q_2 \rangle \cap K \notin \mathcal{K}$. Hence, $\tilde{L} \notin \mathcal{L}$.

Now we show that in Γ' a clique containing L_1 and L_2 has at most size t.

If L_1, L_2 lie in a clique C of Γ' which does not contain a line through R, then $|C| \leq q(\alpha - 1) + 1 < t + 1$. Hence, L_1, L_2 lie in a clique of size t + 1 which also contains a line $L \in \mathcal{L} \cap M_1$ with $R \in L$.

If $L \in C_1$, then let L' be a line of $\mathcal{L} \setminus C_1$ in M_1 with $R \in L'$ (which exists as $|L \cap \mathcal{K}| \geq 2$). Then L' is nonadjacent to L in Γ' . The line L only gains lines in M_2 as new neighbors in Γ' , but L_1 and L_2 have no common neighbor in M_2 in Γ' . Hence, $\{L, L_1, L_2\}$ lie in a clique of size at most t.

If $L \notin C_1$, then repeat the argument with switched roles for L and L', that is $L' \in \mathcal{L}$ with $R \in L'$. Hence, L_1 and L_2 do not lie in a clique of size t + 1, so Γ' is not the line graph of an incomplete (s', t, α') -geometry.

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