



Heriot-Watt University  
Research Gateway

## Non-geometric fluxes, quasi-Hopf twist deformations, and nonassociative quantum mechanics

### Citation for published version:

Mylonas, D, Schupp, P & Szabo, RJ 2014, 'Non-geometric fluxes, quasi-Hopf twist deformations, and nonassociative quantum mechanics', *Journal of Mathematical Physics*, vol. 55, no. 12, 1.4902378. <https://doi.org/10.1063/1.4902378>

### Digital Object Identifier (DOI):

[10.1063/1.4902378](https://doi.org/10.1063/1.4902378)

### Link:

[Link to publication record in Heriot-Watt Research Portal](#)

### Document Version:

Publisher's PDF, also known as Version of record

### Published In:

Journal of Mathematical Physics

### Publisher Rights Statement:

Copyright © 2014 AIP Publishing LLC.

### General rights

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### Take down policy

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [open.access@hw.ac.uk](mailto:open.access@hw.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.

## Non-geometric fluxes, quasi-Hopf twist deformations, and nonassociative quantum mechanics

Dionysios Mylonas, Peter Schupp, and Richard J. Szabo

Citation: *Journal of Mathematical Physics* **55**, 122301 (2014); doi: 10.1063/1.4902378

View online: <http://dx.doi.org/10.1063/1.4902378>

View Table of Contents: <http://scitation.aip.org/content/aip/journal/jmp/55/12?ver=pdfcov>

Published by the [AIP Publishing](#)

---

### Articles you may be interested in

[Exotic orientifolds in non-geometric flux cosmology](#)

*AIP Conf. Proc.* **1548**, 205 (2013); 10.1063/1.4817045

[Noncommutative oscillators from a Hopf algebra twist deformation. A first principles derivation](#)

*J. Math. Phys.* **52**, 032102 (2011); 10.1063/1.3562510

[A Note on 3-Quasi-Sasakian Geometry](#)

*AIP Conf. Proc.* **1023**, 132 (2008); 10.1063/1.2958163

[Two-Dimensional Quantum Gravity as String Bit Hamiltonian Models](#)

*AIP Conf. Proc.* **646**, 161 (2002); 10.1063/1.1524567

[Geometric quantization of mechanical systems with time-dependent parameters](#)

*J. Math. Phys.* **43**, 2882 (2002); 10.1063/1.1477262

---

Did your publisher get  
**18 MILLION DOWNLOADS** in 2014?  
AIP Publishing did.



THERE'S POWER IN NUMBERS. Reach the world with AIP Publishing.



# Non-geometric fluxes, quasi-Hopf twist deformations, and nonassociative quantum mechanics

Dionysios Mylonas,<sup>1,a)</sup> Peter Schupp,<sup>2,b)</sup> and Richard J. Szabo<sup>1,c)</sup>

<sup>1</sup>*Department of Mathematics, Heriot-Watt University Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, United Kingdom; and Maxwell Institute for Mathematical Sciences, Edinburgh, United Kingdom; and The Tait Institute, Edinburgh, United Kingdom*

<sup>2</sup>*Jacobs University Bremen, 28759 Bremen, Germany*

(Received 15 January 2014; accepted 8 November 2014; published online 3 December 2014)

We analyse the symmetries underlying nonassociative deformations of geometry in non-geometric  $R$ -flux compactifications which arise via T-duality from closed strings with constant geometric fluxes. Starting from the non-abelian Lie algebra of translations and Bopp shifts in phase space, together with a suitable cochain twist, we construct the quasi-Hopf algebra of symmetries that deforms the algebra of functions and the exterior differential calculus in the phase space description of nonassociative  $R$ -space. In this setting, nonassociativity is characterised by the associator 3-cocycle which controls non-coassociativity of the quasi-Hopf algebra. We use abelian 2-cocycle twists to construct maps between the dynamical nonassociative star product and a family of associative star products parametrized by constant momentum surfaces in phase space. We define a suitable integration on these nonassociative spaces and find that the usual cyclicity of associative noncommutative deformations is replaced by weaker notions of 2-cyclicity and 3-cyclicity. Using this star product quantization on phase space together with 3-cyclicity, we formulate a consistent version of nonassociative quantum mechanics, in which we calculate the expectation values of area and volume operators, and find coarse-graining of the string background due to the  $R$ -flux. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4902378>]

## I. INTRODUCTION AND SUMMARY

Flux compactifications of superstring theory (see, e.g., Refs. 31, 28, and 18 for reviews) enjoy T-duality symmetry which in some instances map them to non-geometric spaces,<sup>50</sup> and in this way non-geometric flux backgrounds can also arise as consistent closed string vacua. In this paper, we consider the emergence of non-geometry under the action of T-duality on torus fibrations with fluxes. In particular, the geometry probed by closed strings propagating in a flat three-torus  $\mathbb{T}^3$  endowed with a constant Neveu-Schwarz  $H$ -flux can be regarded as the identity fibration  $\mathbb{T}^3 \rightarrow \mathbb{T}^3$  with zero-dimensional fibres. It is mapped under T-duality along a cycle to a circle bundle over  $\mathbb{T}^2$  of degree equal to the cohomology class of the  $H$ -flux in  $H^3(\mathbb{T}^3; \mathbb{Z}) = \mathbb{Z}$ ; this is the twisted three-torus with zero flux but non-vanishing metric torsion. A further T-duality along a cycle of the base yields a  $\mathbb{T}^2$ -fibration over  $S^1$ , which is a non-geometric space with  $Q$ -flux (the flux dual to the  $H$ -flux); this space is an example of a T-fold,<sup>36</sup> which is locally Riemannian but globally the transition functions include T-duality transformations in addition to the standard diffeomorphisms. As a result of non-trivial  $SL(2, \mathbb{Z})$  monodromies of the  $\mathbb{T}^2$  fibres induced by wrapping closed strings around the

<sup>a)</sup>Email: [mylonas.dionysios@gmail.com](mailto:mylonas.dionysios@gmail.com)

<sup>b)</sup>Email: [p.schupp@jacobs-university.de](mailto:p.schupp@jacobs-university.de)

<sup>c)</sup>Email: [R.J.Szabo@hw.ac.uk](mailto:R.J.Szabo@hw.ac.uk)

$S^1$  base, the fibre directions acquire a noncommutative deformation induced by both the  $Q$ -flux and the winding number along the base direction.<sup>42,25,4,33</sup> (Non-geometric  $Q$ -flux backgrounds dual to three-spheres were also recently constructed in Ref. 48). After a final T-duality of the base  $S^1$ , one obtains a  $\mathbb{T}^3$ -fibration over a point. Then, the  $Q$ -flux is mapped to its T-dual  $R$ -flux while the winding number is mapped to the string momentum, thus yielding the quantum phase space algebra<sup>42</sup>

$$[x^i, x^j] = i \hbar R^{ijk} p_k, \quad [x^i, p_j] = i \hbar \delta^i_j, \quad [p_i, p_j] = 0, \quad (1.1)$$

where  $x^i$ ,  $p_i$  are the zero modes of the string positions and conjugate momenta, respectively (Throughout this paper, we use units in which Planck's constant  $\hbar$  is dimensionless. We also use implicit summation over repeated upper and lower indices throughout.); in a general  $d$ -dimensional non-geometric parabolic  $R$ -flux background, which is the main focus of this paper, the transition functions cannot even be defined locally and a short calculation reveals that it also acquires a *nonassociative* deformation since the algebra (1.1) has a non-trivial Jacobiator given by

$$[[x^i, x^j], x^k] := [[x^i, x^j], x^k] + \text{cyclic permutations} = 3 \hbar^2 R^{ijk}. \quad (1.2)$$

In the following, we also consider suitable decompactification limits of the parabolic  $R$ -flux model to spaces with trivial topology.

Non-geometric backgrounds can also be studied in the context of worldsheet conformal field theory. In this setting, geometric flux backgrounds correspond to conformal field theories on freely acting orbifolds with left-right symmetric twisted sectors with respect to the action of the  $SL(2, \mathbb{Z})$  monodromy on  $\mathbb{T}^2$ , while non-geometric backgrounds are regarded as left-right asymmetric orbifold theories whose asymmetric twisting is related to the presence of non-geometric fluxes.<sup>25,26</sup> Such conformal field theories exhibit the nonassociative structure of the underlying target space as a discontinuity of the three-point functions.<sup>15</sup> In fact, one can read off a deformed product of three functions up to linear order in the background flux by calculating off-shell correlation functions of tachyon vertex operators.<sup>19</sup>

From a target space perspective, a suitable framework for describing non-geometry in string theory is provided by both double field theory<sup>38</sup> (see, e.g., Refs. 1, 14, and 35 for reviews) and generalized geometry (see, e.g., Ref. 32). Double field theory implements  $O(d, d)$  transformations as a symmetry of the string theory effective action via a doubling of the dimension of space through the introduction of dual coordinates on equal footing with the original space coordinates; hence, it encompasses all T-duality frames in an invariant way, including those exhibiting closed string noncommutativity and nonassociativity. On the other hand, generalized geometry amounts to the extension of differential geometry to algebroids which accommodates string symmetries, through an  $O(d, d)$ -invariant bilinear form on sections of the algebroid. These approaches can be used in a complimentary way to construct string actions in non-geometric frames in which diffeomorphism and gauge symmetries are expressed via generalised geometry.<sup>5</sup> In fact, it is possible to formulate a bi-invariant action, i.e., invariant under both diffeomorphisms and  $\beta$ -transformations of the algebroid, for closed strings in non-geometric flux backgrounds;<sup>16</sup> in particular, each  $O(d, d)$  transformation is associated with a Lie algebroid.<sup>20</sup> In addition to providing the appropriate geometric formalism for describing non-geometric backgrounds, these methods go a step further in providing a suitable context for a desired deformation quantization of these spaces which is not present in the usual description of closed string theory; the structure of nonassociative deformations of geometry in double field theory is analysed in Ref. 17. Some of these non-geometries can also be equivalently described entirely within the geometric framework of the ten-dimensional supergravity theory proposed by Ref. 3, without recourse to any worldsheet formalism or the target space formalisms based on double field theory and generalized geometry.

An alternative geometric interpretation of the non-geometric  $R$ -flux background along these lines was given in Ref. 46 in terms of certain membrane sigma-models, proposed as topological sectors of closed string dynamics in flux compactifications. Courant algebroids provide the appropriate target spaces for these sigma-models in order to accommodate the 3-tensor fluxes. In particular, for constant, fluxes the topological sigma-model on the standard Courant algebroid  $TM \oplus T^*M$  (in a suitable frame for the  $\beta$ -transformation symmetry) over a manifold  $M$  of dimension  $d$  reduces on the (closed string) boundary of the membrane to sigma-models whose target spaces are

Lie algebroids over twisted Poisson manifolds. We showed in Ref. 46 that the geometric  $H$ -flux background corresponds in this way to a twisted Poisson sigma-model with target space  $M$ , while the non-geometric  $R$ -flux background corresponds to a twisted Poisson sigma-model whose target space is the cotangent bundle  $\mathcal{M} = T^*M$ , thus reproducing the twisted Poisson structure on phase space which is quantized by (1.1).

Based on an open/closed string duality in  $R$ -space arising from the asymmetric twisted sectors of the orbifold conformal field theory,<sup>46</sup> this interpretation enables an explicit construction of a nonassociative star product on the algebra of functions  $C^\infty(\mathcal{M})$  via two techniques: Kontsevich's deformation quantization of a twisted Poisson manifold<sup>41</sup> and strict deformation quantization of the dual of a Lie 2-algebra via convolution in an integrating Lie 2-group. The first method is related to the quantization of Nambu-Poisson structures on the original compactification manifold  $M$ , while the second method leads to a categorification of Weyl's quantization map and clarifies the relation with the twisted convolution products on nonassociative torus bundles.<sup>21</sup> Equivalence of the two methods was demonstrated by a categorified version of Kathotia's theorem which asserts the equivalence between Baker-Campbell-Hausdorff quantization and Kontsevich's deformation quantization in the case of nilpotent Lie algebras.<sup>40</sup> In particular, the non-trivial associator calculated in Ref. 46 is determined by a classifying 3-cocycle in the Chevalley-Eilenberg cohomology with values in the trivial representation of the  $d$ -dimensional Heisenberg group which integrates the phase space algebra in the  $Q$ -flux duality frame. This 3-cocycle provides the twisting of the pertinent horizontal product of the Lie 2-group, which was calculated explicitly for the case of toroidal backgrounds using the Baker-Campbell-Hausdorff formula. At the algebraic level, this 3-cocycle is exactly the one that appeared recently in Ref. 11, where  $R$ -space nonassociativity was similarly characterised in terms of 3-cocycles of the abelian group of translations in phase space with and without central extension using the deformation theory of Lie algebras (see also Ref. 22).

In this paper, we describe a third way of quantizing non-geometric  $R$ -flux backgrounds using twist deformation techniques. The terminology "twist" refers to a deformation of a Hopf algebra  $H$  which is constructed from the universal enveloping algebra of a Lie algebra of symmetries acting on the phase space description of  $R$ -space. Such deformations are typically provided by a 2-cocycle  $F \in H \otimes H$  called a *twist* (see, e.g., Ref. 43); gauge and gravity theories on a noncommutative space as deformations of their classical counterparts using twisting techniques can be found, e.g., in Refs. 9 and 8 (see also Ref. 7). The advantage of twist deformation quantization is that it accommodates nonassociativity in a natural and concrete way which overcomes the difficulties encountered in quantizing nonassociative algebras using (higher) Lie algebraic methods, such as Baker-Campbell-Hausdorff quantization: One simply requires that the usual coassociativity of the Hopf algebra  $H$  holds only up to a 3-cocycle  $\phi \in H \otimes H \otimes H$  called the *associator*; this yields a quasi-Hopf algebra.<sup>29</sup> If  $\phi$  is trivial, i.e., it is the coboundary of a 2-cochain  $F \in H \otimes H$ , then the twisting is provided by  $F$ . Once a twist is known, it is just a matter of applying the cochain twist machinery to deform all geometric structures which are covariant under the symmetries of a manifold; such cochain twists were employed in Ref. 12 to describe nonassociative differential calculus and in Ref. 44 to formulate gauge theory on nonassociative algebras (see also Ref. 13).

The use of trivial 3-cocycles as sources of nonassociativity first appeared in the physics literature in the description of magnetic translations of charged particles in the background of a magnetic monopole, where it was shown that demanding associativity yields the Dirac quantization condition.<sup>39</sup> In this case, one finds an associative representation of the global translations, even though the Jacobi identity for the infinitesimal generators continues to fail. This point of view is taken in the description of non-geometric toroidal flux backgrounds within the framework of Matrix theory compactifications in Ref. 24. Thus, although one finds a non-trivial Jacobiator (1.2) on  $R$ -space, demanding associativity of global quantities may teach us something about the structure of non-geometric fluxes, such as flux quantization; indeed, the on-shell worldsheet tachyon scattering amplitudes computed in Ref. 19 exhibit no violations of associativity once momentum conservation is taken into account in accord with the standard crossing symmetry of correlation functions in two-dimensional conformal field theory. This point of view of nonassociative  $R$ -space is addressed in the context of double field theory in Ref. 17, while the parallels between nonassociative parabolic

$R$ -flux string models and the dynamics of charged particles in uniform magnetic charge distributions are elucidated in Ref. 11.

Twist deformations can also be formulated in a categorical framework, based on the observation that every braided monoidal category whose objects are vector spaces is equivalent to the representation category of some (quasi-)Hopf algebra  $H$  (see, e.g., Ref. 43, Chap. 9 for details). The associator in this case is a functorial isomorphism which satisfies the pentagonal coherence relations in the braided monoidal category of left  $H$ -modules. This categorical formulation connects the strict deformation quantization techniques developed in Ref. 46 for non-geometric  $R$ -flux backgrounds with the twist deformation quantization techniques which are developed in the present paper; in particular, it *explicitly* yields the generalisation of Kathotia's theorem to Lie 2-algebras. The formalism of twist deformation quantization is reviewed in Sec. II.

In this paper, we demonstrate that twist deformation quantization allows for the introduction of a 3-form  $R$ -flux in phase space in a natural way. In Sec. III, we do this in three steps which illustrate that the nonassociativity of  $R$ -space is not completely arbitrary, in that it results from the fact that the star product is a function of momentum, i.e., it is dynamical. We shall find a whole family of associative star products for constant momentum slices, all interrelated among themselves and to the nonassociative star product by twists; these relations were described by Seiberg-Witten maps in Ref. 46. First, we construct the Hopf algebra  $K$  related to the abelian Lie algebra of translations in  $2d$ -dimensional phase space  $\mathcal{M}$  and deform it using an abelian 2-cocycle twist  $F \in K \otimes K$ ; the action of the twisted Hopf algebra  $K_F$  on the algebra of functions  $C^\infty(\mathcal{M})$  yields the canonical Moyal-Weyl star product on phase space. We then endow  $\mathcal{M}$  with a trivector  $R$  which is T-dual to the 3-form of a uniform background  $H$ -flux. To bring  $R$  into the twist quantization scheme, we introduce a unique family of twist elements associated to the translation algebra which are parametrised by constant momentum, and hence deform the pointwise product of functions on phase space to a family of associative noncommutative Moyal-Weyl type star products; these sorts of deformed products were derived in Ref. 46 for  $Q$ -flux backgrounds (T-folds). Finally, we promote constant momenta to dynamical momenta appropriate to  $R$ -space and twist the pertinent Hopf algebra  $H$  using a cochain twist, which is tantamount to an abelian cocycle twist of the canonical Moyal-Weyl product; the underlying Lie algebra of symmetries of  $R$ -space is non-abelian, nilpotent, and includes non-local Bopp shifts on  $\mathcal{M}$  that mix positions with momenta. The resulting twisted Hopf algebra is a quasi-Hopf algebra whose action on  $C^\infty(\mathcal{M})$  quantizes the phase space structure of constant  $R$ -flux backgrounds and yields the nonassociative star product on  $R$ -space that was first proposed in Ref. 46.

Equipped with our cochain twist, in Sec. IV, we deform the differential calculus on  $\mathcal{M}$ , and thus formulate nonassociative deformations of the exterior differential algebra and of the  $C^\infty(\mathcal{M})$ -bimodule structure on  $R$ -flux backgrounds. In order to set up a framework in which to study field theories on  $R$ -space, we define integration on the deformed algebra of forms on  $\mathcal{M}$  to be the standard integration; the integral of multiple exterior star products of differential forms is not (graded) cyclic but rather satisfies weaker notions of 2-cyclicity and 3-cyclicity that we describe, which turn out to be crucial for a consistent formulation of quantum theory on  $R$ -space. 3-cyclicity is also crucial for ensuring that nonassociative deformations of field theory can be made consistent with the requirements of crossing symmetry of conformal field theory scattering amplitudes, which on-shell show no violations of associativity.

Nonassociative quantum theory on non-geometric  $R$ -flux backgrounds cannot be treated by conventional means in terms of linear operators on separable Hilbert spaces; instead, one should resort to our phase space star product quantization (this point is also emphasised by Ref. 11). We elucidate this point in Sec. V by considering quantum mechanics in the nonassociative phase space formalism; in this approach, we emphasise the roles of observables without concern about their representations. We demonstrate that, against all odds, a consistent formulation of nonassociative quantum mechanics is indeed possible; an important input is the 3-cyclicity of the nonassociative star product. This analysis demonstrates that quantum theory on the nonassociative string background can be treated in a systematic way, and our treatment is the first step towards realising more elaborate models, such as field theory or gravity, on non-geometric flux compactifications. Through this formalism, we find that a triple of operators that do not associate does not have

common eigenstates, which is a clear sign of position space quantization in the presence of  $R$ -flux. An uncertainty relation proportional to the transverse momentum for the measurement of a pair of position coordinates is induced. We also find non-zero expectation values for (the uncertainty of) suitably defined area and volume operators in configuration space, leading in particular to a minimal volume element; this formalism thus provides a concrete and rigorous derivation for the uncertainty relations anticipated by Refs. 42 and 15. We shall further find that operator time evolution in the Heisenberg picture is not a derivation of the star product algebra of operators.

Finally, some generalizations of our twist deformation methods to non-constant  $R$ -flux backgrounds as well as more generic  $R$ -flux string vacua are briefly considered in Sec. VI. We consider the case of position-dependent  $R$ -fluxes in (1.1) and the conditions under which the techniques of twist deformation quantization developed in this paper carry through at least locally, leaving the difficult problem of globalization, which is analogous to that of the star products in Kontsevich's approach,<sup>23</sup> to future work; in particular, by restricting to functions of the position coordinates in  $M$ , this technique provides a framework for quantizing generic Nambu-Poisson 3-brackets determined by the trivector field  $R$ . We also consider the extension of the phase space algebra (1.1) to allow for twist deformations provided by a class of quasi-Poisson structures that are generic non-linear functions of the momenta, which is the case of  $R$ -flux backgrounds that arise from monodromies lying in certain non-parabolic conjugacy classes of the  $SL(2, \mathbb{Z})$  automorphism group of  $\mathbb{T}^2$ . In this case, we apply Kontsevich's deformation quantization of phase space to compute the nonassociative star product and associator up to third order in a derivative expansion in the  $R$ -flux, which is used to identify the pertinent Hopf algebra of symmetries and a (non-unique) cochain twist. The generality of this setting allows for deformation quantization of closed string  $R$ -flux backgrounds determined by the elliptic model of Refs. 42 and 25 and illustrates the power of our twist deformation techniques: The passage to these non-parabolic  $R$ -flux models amounts to extending the Lie algebra of translations and Bopp shifts in phase space to a certain infinite-dimensional Lie algebra of diffeomorphisms.

## II. TWIST DEFORMATION QUANTIZATION

Twist deformation techniques provide a very precise and systematic way of quantizing any algebraic structure acted upon by a (quasi-)Hopf algebra. Such is the case for the algebra of functions on a space acted upon by a Lie group of symmetries which will be our main application in this paper. In this section, we briefly review standard deformation quantization by cocycle twists as well as the more general case of cochain twists which is our main case of interest.

### A. Hopf algebras and cocycle twist quantization

We begin by defining some of the basic algebraic structures that we will encounter in the following. We then describe deformation quantization via a Drinfel'd cocycle twist.

A *bialgebra*  $H$  over  $\mathbb{C}$  is an associative unital algebra with a counital coalgebra structure that satisfies the properties

$$(\text{id}_H \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_H) \circ \Delta, \quad (2.1)$$

$$(\text{id}_H \otimes \varepsilon) \circ \Delta = \text{id}_H = (\varepsilon \otimes \text{id}_H) \circ \Delta, \quad (2.2)$$

where  $\varepsilon : H \rightarrow \mathbb{C}$  is the counit and  $\Delta : H \rightarrow H \otimes H$  is the coproduct. The relation (2.1) means that the coalgebra is *coassociative*. Throughout we use the usual Sweedler notation  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$  with  $h, h_{(1)}, h_{(2)} \in H$  and suppress the summation.

A *quasi-triangular bialgebra* is a pair  $(H, \mathcal{R})$  where  $H$  is a bialgebra, and  $\mathcal{R} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \in H \otimes H$  is an invertible element which obeys

$$(\Delta \otimes \text{id}_H)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (\tau \circ \Delta)(h) = \mathcal{R} \Delta(h) \mathcal{R}^{-1} \quad (2.3)$$

for all  $h \in H$ , where  $\mathcal{R}_{12} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \otimes 1_H$ ,  $\mathcal{R}_{13} = \mathcal{R}_{(1)} \otimes 1_H \otimes \mathcal{R}_{(2)}$ , and  $\mathcal{R}_{23} = 1_H \otimes \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}$  with  $1_H$  the unit of  $H$ , and we abbreviate the product map  $\mu : H \otimes H \rightarrow H$  by  $\mu(h \otimes h') = h h'$  for all

$h, h' \in H$ . Here, we have defined the *transposition map*  $\tau : H \otimes H \rightarrow H \otimes H$  as

$$\tau(h \otimes h') := h' \otimes h \tag{2.4}$$

for all  $h, h' \in H$ .

Through the transposition map, the *co-opposite coproduct*  $\Delta^{\text{op}} : H \rightarrow H \otimes H$  is defined by

$$\Delta^{\text{op}}(h) := (\tau \circ \Delta)(h) = h_{(2)} \otimes h_{(1)}. \tag{2.5}$$

Then,  $H$  is a cocommutative coalgebra if  $\Delta^{\text{op}}(h) = \Delta(h)$  for all  $h \in H$ . If  $(H, \mathcal{R})$  is a quasi-triangular bialgebra, then cocommutativity simply means that  $\Delta(h)\mathcal{R} = \mathcal{R}\Delta(h)$  for all  $h \in H$  as can be easily seen from (2.3); in general the element  $\mathcal{R}$  intertwines the action of the coproduct  $\Delta$  with the co-opposite coproduct  $\Delta^{\text{op}}$ .

A *Hopf algebra* over  $\mathbb{C}$  is a bialgebra  $H$  equipped with an algebra anti-automorphism  $S : H \rightarrow H$  called the *antipode* satisfying

$$\mu \circ (\text{id}_H \otimes S) \circ \Delta = \eta_{1_H} \circ \varepsilon = \mu \circ (S \otimes \text{id}_H) \circ \Delta, \tag{2.6}$$

where  $\eta_h : \mathbb{C} \rightarrow H$  is the unit homomorphism with  $\eta_h(1) = h$  for  $h \in H$ . A quasi-triangular Hopf algebra  $(H, \mathcal{R})$  consists of a quasi-triangular structure  $\mathcal{R}$  on the underlying bialgebra of  $H$ .

In this paper, we will be primarily interested in the large class of Hopf algebras  $H$  which arise as universal enveloping algebras  $U(\mathfrak{g})$  of Lie algebras  $\mathfrak{g}$ . The algebra  $U(\mathfrak{g})$  is constructed by taking the quotient of the tensor algebra  $T(\mathfrak{g}) = \bigoplus_{k \geq 0} \mathfrak{g}^{\otimes k} = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$  by the two-sided ideal  $\mathcal{I}$  generated by elements of the form  $x \otimes y - y \otimes x - [x, y]$ , where  $x, y \in \mathfrak{g}$ . Next, we equip  $U(\mathfrak{g})$  with the symmetric coalgebra structure

$$\begin{aligned} \Delta : U(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), & \Delta(x) &= x \otimes 1 + 1 \otimes x, \\ \varepsilon : U(\mathfrak{g}) &\longrightarrow \mathbb{C}, & \varepsilon(x) &= 0, \\ S : U(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}), & S(x) &= -x \end{aligned} \tag{2.7}$$

defined on primitive elements  $x \in \mathfrak{g}$  and extended to all of  $U(\mathfrak{g})$  as algebra (anti-)homomorphisms. The desired Hopf algebra  $H$  is then  $U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}$  with these structure maps. Finally, we further equip  $H$  with the trivial quasi-triangular structure

$$\mathcal{R}_0 = 1 \otimes 1, \quad 1_H := 1 \tag{2.8}$$

which turns it into a cocommutative quasi-triangular Hopf algebra.

A Hopf algebra  $H$  can act on a complex vector space  $V$  to give a representation of  $H$  on  $V$ . In particular, a *left action* of  $H$  on  $V$  is a pair  $(\lambda, V)$ , where  $\lambda : H \otimes V \rightarrow V$  is a linear map,  $\lambda(h \otimes v) =: \lambda_h(v)$ , such that  $\lambda_{hg}(v) = \lambda_h(\lambda_g(v))$  and  $\lambda(1_H \otimes v) = v$ , where  $g, h \in H$  and  $v \in V$ . It is customary to denote the action of  $H$  on  $V$  by  $\triangleright$  and write the above relations as

$$h \triangleright v \in V, \quad (hg) \triangleright v = h \triangleright (g \triangleright v), \quad 1_H \triangleright v = v. \tag{2.9}$$

Such a vector space is called a *left H-module*. If  $V$  carries additional structure, for example, if it is an algebra  $(A, \mu_A)$  where  $\mu_A : A \otimes A \rightarrow A$  is the product on  $A$  or a coalgebra  $(C, \Delta_C)$  where  $\Delta_C : C \rightarrow C \otimes C$  is the coproduct on  $C$ , then we demand that the action of  $H$  is covariant in the sense that it preserves the additional structure of  $V$ . Thus, we say that a unital algebra  $(A, \mu_A)$  over  $\mathbb{C}$  is a *left H-module algebra* if  $A$  is a left  $H$ -module and

$$h \triangleright (ab) = h \triangleright \mu_A(a \otimes b) = \mu_A(\Delta(h) \triangleright (a \otimes b)) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \tag{2.10}$$

where  $h \in H$  and  $a, b \in A$ . Here, we abbreviate  $\mu_A(a \otimes b) = ab$ ; we use this notation throughout when no confusion arises. Likewise, a counital coalgebra  $(C, \Delta_C)$  is a *left H-module coalgebra* if  $C$  is a left  $H$ -module and

$$(h \triangleright c)_{(1)} \otimes (h \triangleright c)_{(2)} = \Delta_C(h \triangleright c) = \Delta(h) \triangleright \Delta_C(c) = (h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)}), \tag{2.11}$$

where  $h \in H$  and  $c \in C$ . Similar definitions hold for right  $H$ -modules, and right  $H$ -module algebras and coalgebras.

From the above definitions, it is understood that a left or right  $H$ -module is a representation by an algebra  $A$  or a coalgebra  $C$  of the Hopf algebra  $H$ ; if  $H$  is modified then so is the representation.



Such modifications were introduced by Drinfel'd. Let  $H[[\hbar]]$  denote the  $\hbar$ -adic completion of  $H$  consisting of all formal  $H$ -valued power series in a deformation parameter  $\hbar$ . A *Drinfel'd twist* is an invertible element  $F = F_{(1)} \otimes F_{(2)} \in H[[\hbar]] \otimes H[[\hbar]]$  that satisfies the two conditions

$$(F \otimes 1_H)(\Delta \otimes \text{id}_H)(F) = (1_H \otimes F)(\text{id}_H \otimes \Delta)(F), \tag{2.12}$$

$$(\varepsilon \otimes \text{id}_H)(F) = 1_H = (\text{id}_H \otimes \varepsilon)(F). \tag{2.13}$$

We will further demand that  $F = 1_H \otimes 1_H + \mathcal{O}(\hbar)$  is a deformation of the trivial twist, which is always formally invertible for sufficiently small  $\hbar$ . By these two conditions,  $F$  is a counital 2-cocycle which can be used to define a new Hopf algebra  $H_F$  with the same underlying algebra as  $H[[\hbar]]$  but with a twisted coalgebraic structure given by the twisted coproduct

$$\Delta_F(h) = F \Delta(h) F^{-1} \tag{2.14}$$

and the twisted antipode

$$S_F(h) = U_F S(h) U_F^{-1}, \quad \text{where } U_F = \mu \circ (\text{id}_H \otimes S)(F) \tag{2.15}$$

for  $h \in H$ . This new bialgebra  $H_F$  is called a *twisted Hopf algebra*; coassociativity and counitality of (2.14) follow, respectively, from the 2-cocycle condition (2.12) and the counital condition (2.13). If  $(H, \mathcal{R})$  is a quasi-triangular Hopf algebra, then the quasi-triangular structure is also twisted by the formula

$$\mathcal{R}_F := \tau(F) \mathcal{R} F^{-1} = F_{21} \mathcal{R} F^{-1}, \tag{2.16}$$

where  $F_{21} = F_{(2)} \otimes F_{(1)}$  in Sweedler notation. The Hopf algebra  $H_F$  need not be cocommutative even if  $H$  is cocommutative; this may be checked by calculating the twisted co-opposite coproduct

$$\Delta_F^{\text{op}}(h) = \mathcal{R}_F \Delta_F(h) \mathcal{R}_F^{-1}, \tag{2.17}$$

where  $h \in H$ .

For the twisted Hopf algebra  $H_F$  to act covariantly on a left  $H$ -module algebra  $(A, \mu_A)$ , we need to twist (deform) the binary product  $\mu_A : A \otimes A \rightarrow A$  to a new product defined by

$$a \star b = \mu_A(F^{-1} \triangleright (a \otimes b)) = (F_{(1)}^{-1} \triangleright a)(F_{(2)}^{-1} \triangleright b) \tag{2.18}$$

for  $a, b \in A$ . The deformed product is called a *star product* and  $(A[[\hbar]], \star)$  is a deformation quantization of  $(A, \mu_A)$ ; indeed, one has  $a \star b = ab + \mathcal{O}(\hbar)$ . The twist cocycle condition (2.12) ensures associativity of the star product (2.18), while the counital condition (2.13) implies that if  $(A, \mu_A)$  is unital with unit  $1_A$ , then  $(A[[\hbar]], \star)$  is also unital with the same unit.

### B. Quasi-Hopf algebras and cochain twist quantization

In this paper, we are concerned with nonassociative twist deformations, therefore we will be using an appropriate generalisation of a Hopf algebra, called a quasi-Hopf algebra.<sup>29</sup> To explain what a quasi-Hopf algebra is let us begin by defining the notion of a *quasi-bialgebra*. This is simply a bialgebra  $H$  where coassociativity is required to hold only up to a 3-cocycle  $\phi$ , i.e., the condition (2.1) is substituted by

$$(\text{id}_H \otimes \Delta) \circ \Delta(h) = \phi [(\Delta \otimes \text{id}_H) \circ \Delta(h)] \phi^{-1}, \tag{2.19}$$

where  $h \in H$  and  $\phi = \phi_{(1)} \otimes \phi_{(2)} \otimes \phi_{(3)} \in H \otimes H \otimes H$  is an invertible 3-cocycle (see, e.g., Ref. 43) in the sense that

$$(1_H \otimes \phi) [(\text{id}_H \otimes \Delta \otimes \text{id}_H)(\phi)] (\phi \otimes 1_H) = [(\text{id}_H \otimes \text{id}_H \otimes \Delta)(\phi)] [(\Delta \otimes \text{id}_H \otimes \text{id}_H)(\phi)]. \tag{2.20}$$

We say that  $\phi$  is counital if it additionally satisfies the condition

$$(\varepsilon \otimes \text{id}_H \otimes \text{id}_H)(\phi) = (\text{id}_H \otimes \varepsilon \otimes \text{id}_H)(\phi) = (\text{id}_H \otimes \text{id}_H \otimes \varepsilon)(\phi) = 1_H \otimes 1_H. \tag{2.21}$$

These two conditions on  $\phi$  ensure that all distinct orderings of higher coproducts by insertions of  $\phi$  yield the same result and are consistent with the counital condition (2.2).

The definition of a quasi-triangular quasi-bialgebra is that of a quasi-triangular bialgebra with the first two axioms of (2.3) modified by  $\phi$  to

$$(\Delta \otimes \text{id}_H)(\mathcal{R}) = \phi_{321} \mathcal{R}_{13} \phi_{132}^{-1} \mathcal{R}_{23} \phi, \quad (\text{id}_H \otimes \Delta)(\mathcal{R}) = \phi_{231}^{-1} \mathcal{R}_{13} \phi_{213} \mathcal{R}_{12} \phi^{-1} \tag{2.22}$$

in the notation of Sec. II A with  $\phi_{abc} := \phi_{(a)} \otimes \phi_{(b)} \otimes \phi_{(c)}$ , while the third axiom of (2.3) remains unchanged.

A quasi-Hopf algebra  $\mathcal{H} = (H, \phi)$  is a quasi-bialgebra  $H$  equipped with an antipode that consists of two elements  $\alpha, \beta \in H$  and an algebra anti-automorphism  $S : H \rightarrow H$  obeying

$$S(h_{(1)}) \alpha h_{(2)} = \varepsilon(h) \alpha, \quad h_{(1)} \beta S(h_{(2)}) = \varepsilon(h) \beta, \tag{2.23}$$

$$\phi_{(1)} \beta S(\phi_{(2)}) \alpha \phi_{(3)} = 1_H, \quad S(\phi_{(1)}^{-1}) \alpha \phi_{(2)}^{-1} \beta S(\phi_{(3)}^{-1}) = 1_H \tag{2.24}$$

for all  $h \in H$ . The antipode is determined uniquely only up to the transformations

$$S'(h) = u S(h) u^{-1}, \quad \alpha' = u \alpha, \quad \beta' = \beta u^{-1} \tag{2.25}$$

for any invertible element  $u \in H$  and any  $h \in H$ . When  $\phi = 1_H \otimes 1_H \otimes 1_H$  is the trivial 3-cocycle, the conditions (2.24) imply that  $\alpha \beta = \beta \alpha = 1_H$  and the symmetry (2.25) allows us to suppose without loss of generality that  $\alpha = \beta = 1_H$ . Then (2.19) reduces to the coassociativity condition (2.1) and (2.23) to the usual definition of an antipode given by (2.6), thus the quasi-Hopf algebra  $\mathcal{H}$  becomes a coassociative Hopf algebra.

A useful way to construct a quasi-Hopf algebra is to start with a Hopf algebra  $H$  and an invertible twist element  $F \in H[[\hbar]] \otimes H[[\hbar]]$  that does not satisfy the cocycle condition (2.12). In particular, if  $(H, \phi, \mathcal{R})$  is a quasi-triangular quasi-Hopf algebra and  $F$  is an arbitrary invertible element in  $H[[\hbar]] \otimes H[[\hbar]]$  obeying (2.13), then  $(H_F, \phi_F, \mathcal{R}_F)$  defined as follows is also a quasi-triangular quasi-Hopf algebra. It has the same algebra and counit as  $H$ , with twisted coproduct and quasi-triangular structure defined by the same formulas (2.14) and (2.16), with twisted antipode

$$S_F = S, \quad \alpha_F = S(F_{(1)}^{-1}) \alpha F_{(2)}^{-1}, \quad \beta_F = F_{(1)} \beta S(F_{(2)}), \tag{2.26}$$

and with twisted 3-cocycle given by the coboundary

$$\phi_F = \delta^* F := F_{23} [(\text{id}_H \otimes \Delta)(F)] \phi [(\Delta \otimes \text{id}_H)(F^{-1})] F_{12}^{-1}, \tag{2.27}$$

where  $F_{23} = 1_H \otimes F$ ,  $F_{12}^{-1} = F^{-1} \otimes 1_H$  and  $\phi_F \in H[[\hbar]] \otimes H[[\hbar]] \otimes H[[\hbar]]$  is called the *associator* (see, e.g., Ref. 44). A Hopf algebra  $H$  viewed as a trivial quasi-Hopf algebra has  $\alpha = \beta = 1_H$  and the symmetry (2.25). Twisting  $H$  with the counital 2-cochain twist  $F$  then provides a quasi-Hopf algebra  $\mathcal{H}_F = (H_F, \phi_F)$  with  $S_F = S$ ,  $\alpha_F = \mu \circ (S \otimes \text{id}_H)(F^{-1})$  and  $\beta_F = \mu \circ (\text{id}_H \otimes S)(F) = \alpha_F^{-1}$  which by (2.25) is equivalent to (2.15). The twisted coproduct  $\Delta_F$  fails to satisfy (2.1) and, in particular, (2.19) is a consequence of this definition.

A left  $H$ -module algebra  $(A, \mu_A)$  is then twisted to a *nonassociative* algebra  $(A[[\hbar]], \star)$  by the same formula (2.18) with the associator appearing when we rebracket products of three elements as

$$(a \star b) \star c = (\phi_{F(1)} \triangleright a) \star [(\phi_{F(2)} \triangleright b) \star (\phi_{F(3)} \triangleright c)] \tag{2.28}$$

for  $a, b, c \in A$ . The cocycle condition (2.20) on  $\phi_F$  ensures that the distinct ways of rebracketing higher order products by inserting  $\phi_F$  all yield the same result.

### C. Twist quantization functor

A natural way to deal with both noncommutative and nonassociative structures arising as above is through the formalism of braided monoidal categories. The algebras encountered above are “braided-commutative” and “quasi-associative”, in the sense that they are noncommutative and nonassociative but in a controlled way by means of a braiding and a multiplicative associator, respectively. This means that the algebras *are* commutative and associative when regarded as objects of a suitable braided monoidal category which is different from the usual category of complex vector spaces. The twist deformation quantization described above can then be regarded as a functor that yields algebras in such a braided monoidal category and at the same time quantises all other

covariant structures with respect to a symmetry. We briefly review this framework here as we will make reference to it later on and because it connects with some of the constructions of Ref. 46 as explained in Sec. I.

A monoidal category  $\mathcal{C}$  consists of a collection of objects  $V, W, Z, \dots$  with a tensor product between any two objects and a natural associativity isomorphism  $\Phi_{V,W,Z} : (V \otimes W) \otimes Z \rightarrow V \otimes (W \otimes Z)$  for any three objects obeying the pentagon identity, which states that the two ways of rebracketing morphisms  $((U \otimes V) \otimes W) \otimes Z \rightarrow U \otimes (V \otimes (W \otimes Z))$  are the same. Then, MacLane's coherence theorem states that all different ways of inserting associators  $\Phi$  as needed to make sense of higher order rebracketed expressions yield the same result. A braiding on  $\mathcal{C}$  is a natural commutativity isomorphism  $\Psi_{V,W} : V \otimes W \rightarrow W \otimes V$  for any pair of objects which is compatible with the associativity structure in a natural way. If  $\mathcal{C}$  is the category of complex vector spaces, then the associator  $\Phi$  is the identity morphism and the braiding  $\Psi$  is the transposition morphism.

If  $\mathcal{H} = (H, \phi)$  is a quasi-Hopf algebra, we take  $\mathcal{C}$  to be the category  ${}_H\mathcal{M}$  of left  $H$ -modules. This is a monoidal category with tensor product defined via the coproduct  $\Delta$  and with associator given by

$$\Phi_{V,W,Z}((v \otimes w) \otimes z) = (\phi_{(1)} \triangleright v) \otimes [(\phi_{(2)} \triangleright w) \otimes (\phi_{(3)} \triangleright z)] \quad (2.29)$$

for all  $v \in V$ ,  $w \in W$ , and  $z \in Z$ . If in addition  $H$  is quasi-triangular, then there is a braiding defined by

$$\Psi_{V,W}(v \otimes w) = (\mathcal{R}_{(1)} \triangleright w) \otimes (\mathcal{R}_{(2)} \triangleright v) \quad (2.30)$$

for all  $v \in V$  and  $w \in W$ .

Given a cochain twist  $F \in H[[\hbar]] \otimes H[[\hbar]]$ , the constructions of this section determine a functorial isomorphism of braided monoidal categories

$$\mathcal{F}_F : {}_H\mathcal{M} \longrightarrow {}_{H_F}\mathcal{M} \quad (2.31)$$

which acts as the identity on objects and morphisms, but intertwines the tensor, braiding, and associativity structures. In particular, the covariance condition (2.10) means that the product map  $\mu_A$  is a morphism in the category  ${}_H\mathcal{M}$ ; hence,  $\mathcal{F}_F$  functorially deforms  $H$ -module algebras into  $H_F$ -module algebras and in this sense it may be regarded as a "twist quantization functor".

In our main case of interest in this paper, we will take  $H = U(\mathfrak{g})$  to be the universal enveloping algebra of a Lie algebra  $\mathfrak{g}$  of symmetries acting on a manifold  $\mathcal{M}$ ; then the Hopf algebra  $H$  acts on the algebra of smooth functions  $A = C^\infty(\mathcal{M})$ , and by functoriality of the twist deformation, it is also quantized to a generically noncommutative and nonassociative algebra  $A_F$ , which is in fact commutative and associative in the category  ${}_{H_F}\mathcal{M}$ . Similarly, the exterior algebra of differential forms  $\Omega^*(\mathcal{M})$  is quantized to  $\Omega_F^*(\mathcal{M})$  as a differential calculus on  $A_F$ ; in this way, any geometry can be systematically quantized with respect to a symmetry and a choice of 2-cochain  $F$ .

### III. COCHAIN TWIST QUANTIZATION OF PARABOLIC $R$ -FLUX BACKGROUNDS

In this section, we employ the formalism of Sec. II to study nonassociative deformations of certain non-geometric closed string backgrounds. In order to make contact with the key ideas of Ref. 46, we will initially study standard deformation quantization on the cotangent bundle of a closed string vacuum and subsequently add a constant background  $R$ -flux. This approach has the advantage of illuminating pertinent non-local and non-geometric symmetry transformations analogous to the ones which arise on T-folds induced by parabolic monodromies.

#### A. Quantum phase space

Let us begin by considering a manifold  $M$  of dimension  $d$  with trivial cotangent bundle  $\mathcal{M} := T^*M \cong M \times (\mathbb{R}^d)^*$  and coordinates  $x^I = (x^i, p_i)$ , where  $I = 1, \dots, 2d$ ,  $(x^i) \in M$ ,  $(p_i) \in (\mathbb{R}^d)^*$  and  $i = 1, \dots, d$ . Throughout, we use upper case indices for the full phase space while lower case indices will be reserved for position or momentum space individually. Consider the abelian Lie algebra  $\mathfrak{h} = \mathbb{R}^d \oplus (\mathbb{R}^d)^*$  of dimension  $2d$  generated by  $P_i$  and  $\tilde{P}^i$ . It is realised on  $\mathcal{M}$  by its action on the algebra

of smooth complex functions  $C^\infty(\mathcal{M})$  which we take to be given by the vector fields

$$P_i \triangleright f := \partial_i f \quad \text{and} \quad \tilde{P}^i \triangleright f := \tilde{\partial}^i f, \tag{3.1}$$

where  $f \in C^\infty(\mathcal{M})$ ,  $\partial_i = \frac{\partial}{\partial x^i}$  and  $\tilde{\partial}^i = \frac{\partial}{\partial p_i}$ . For constant vectors  $a = (a^i) \in \mathbb{R}^d$  and  $\tilde{a} = (\tilde{a}_i) \in (\mathbb{R}^d)^*$ , we can define  $P_a = a^i P_i$  and  $\tilde{P}_{\tilde{a}} = \tilde{a}_i \tilde{P}^i$  which as vector fields on  $\mathcal{M}$  translate  $x^i$  and  $p_i$  by  $a^i$  and  $\tilde{a}_i$ , respectively, hence  $\mathfrak{h}$  is the classical phase space translation algebra on  $M$ .

We can now construct the related Hopf algebra  $K$  that acts on an algebra  $(A, \mu_A)$  in the usual way, i.e., we consider the universal enveloping algebra  $K = U(\mathfrak{h})$  and equip it with the coalgebra structure from Sec. II A. In particular, the action of  $K$  on  $A = C^\infty(\mathcal{M})$  is given by (3.1) extended covariantly to all elements in  $K$  using linearity and the Leibniz rule for the vector fields  $\partial_i$  and  $\tilde{\partial}^i$ .

Phase space quantization is carried out simply by twisting  $K$  in the manner described in Sec. II A. A suitable abelian twist  $F \in K[[\hbar]] \otimes K[[\hbar]]$  is given by

$$F = \exp\left[-\frac{i\hbar}{2} (P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i)\right], \tag{3.2}$$

where  $\hbar$  is the deformation parameter. In this simple case  $\Delta_F = \Delta$ , where the coproduct  $\Delta$  is defined in (2.7), and thus the twisted Hopf algebra  $K_F$  is cocommutative. The twisted quasi-triangular structure is easily calculated from (2.16) and is given by

$$Q = F^{-2} = \exp\left[i\hbar (P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i)\right]. \tag{3.3}$$

We may now deform any left (or right)  $K$ -module algebra  $(A, \mu_A)$  using (2.18) and the relevant action. Let us do this for the algebra of functions on  $\mathcal{M}$ , i.e., we set  $A = C^\infty(\mathcal{M})$  and  $\mu_A(f \otimes g) = f g$  the pointwise multiplication of functions and derive its deformation quantization  $(C^\infty(\mathcal{M})[[\hbar]], \star)$ ; the star product given by (2.18) is

$$f \star g = \mu_A\left(\exp\left[\frac{i\hbar}{2} (\partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i)\right](f \otimes g)\right), \tag{3.4}$$

where the action (3.1) has been used. This *noncommutative* star product is the canonical associative Moyal-Weyl star product familiar from quantum mechanics. With its use, the usual quantum phase space commutation relations are calculated as

$$[x^i, p_j] \star = i\hbar \delta^i_j, \quad [x^i, x^j] \star = 0 = [p_i, p_j] \star, \tag{3.5}$$

where  $[f, g] \star := f \star g - g \star f$  for all  $f, g \in C^\infty(\mathcal{M})$ .

### B. Noncommutative quantum phase space

Let us endow  $M$  with a constant trivector  $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$  which is T-dual to the background  $H$ -flux of a non-trivial closed string  $B$ -field. To bring  $R$  into the twist quantization scheme, we introduce a family of antisymmetric linear combinations of the generators of  $\mathfrak{h}$  as

$$\tilde{M}_{ij} := M_{ij}^{\tilde{p}} = \tilde{p}_i P_j - \tilde{p}_j P_i \tag{3.6}$$

which we will regard as parametrized by constant momentum surfaces  $\tilde{p} = (\tilde{p}_i) \in (\mathbb{R}^d)^*$ . The generators  $\tilde{M}_{ij}$  are unique in the sense that they are the only rank two tensors constructed by primitive elements in  $\mathfrak{h}$  to lowest order that can be non-trivially contracted with the constant antisymmetric trivector components  $R^{ijk}$ . The restriction to constant momentum surfaces here will ensure (co)associativity, but will be relaxed in the Sec. III C.

The unique twist element that can be constructed in this way from generators of  $\mathfrak{h}$  is the abelian twist  $\tilde{F}_R \in K[[\hbar]] \otimes K[[\hbar]]$  given by

$$\tilde{F}_R = \exp\left(-\frac{i\hbar}{8} R^{ijk} (\tilde{M}_{ij} \otimes P_k - P_i \otimes \tilde{M}_{jk})\right), \tag{3.7}$$

where  $\hbar$  is the deformation parameter. The Hopf algebra  $K$  is twisted to a new Hopf algebra  $K_{\tilde{F}_R}$  which is cocommutative with quasi-triangular structure given by

$$\tilde{\mathcal{R}} = \tilde{F}_R^{-2}, \tag{3.8}$$

and the algebra of functions  $(C^\infty(\mathcal{M}), \mu_A)$  is quantized to  $(C^\infty(\mathcal{M})[[\hbar]], \bar{\star} := \star_{\bar{p}})$ . The star product  $\bar{\star}$  is calculated by (2.18) as

$$f \bar{\star} g = \mu_A \left( \exp \left[ \frac{i\hbar}{2} R^{ijk} \bar{p}_k \partial_i \otimes \partial_j \right] (f \otimes g) \right) \tag{3.9}$$

for all  $f, g \in C^\infty(\mathcal{M})$ , and it is a family of Moyal-Weyl products. This can be seen by calculating the  $\bar{\star}$ -commutators  $[f, g]_{\bar{\star}} := f \bar{\star} g - g \bar{\star} f$  on phase space coordinate functions where we find

$$[x^i, x^j]_{\bar{\star}} = i \hbar \theta^{ij}(\bar{p}), \quad [x^i, p_j]_{\bar{\star}} = 0 = [p_i, p_j]_{\bar{\star}}, \tag{3.10}$$

where

$$\theta^{ij}(\bar{p}) = R^{ijk} \bar{p}_k. \tag{3.11}$$

This reveals that upon twisting, the cotangent bundle  $\mathcal{M} = T^*M$  is deformed to  $M_{\theta(\bar{p})} \times (\mathbb{R}^d)^*$ ; hence configuration space is quantized to a family of noncommutative spaces  $M_{\theta(\bar{p})}$  with constant noncommutativity parameters proportional to the constant background  $R$ -flux and parametrized by the surfaces of constant momentum  $\bar{p} = (\bar{p}_i) \in (\mathbb{R}^d)^*$ . This is similar to what happens in the associative  $Q$ -flux T-duality frame with parabolic monodromy, where the configuration space noncommutativity is proportional to the winding numbers  $w \in \mathbb{Z}^d$  of closed strings, i.e.,  $[x^i, x^j] = i \hbar Q^{ij}_k w^k$ .

The above construction is further extended to give a noncommutative quantum phase space if we use the abelian twist

$$\bar{\mathcal{F}} := \bar{F}_R F = F \bar{F}_R = \exp \left[ -\frac{i\hbar}{2} \left( \frac{1}{4} R^{ijk} (\bar{M}_{ij} \otimes P_k - P_i \otimes \bar{M}_{jk}) + P_i \otimes \bar{P}^i - \bar{P}^i \otimes P_i \right) \right]. \tag{3.12}$$

The quasi-triangular Hopf algebra  $(K, \mathcal{R}_0)$  is twisted to the cocommutative quasi-triangular Hopf algebra  $(K_{\bar{\mathcal{F}}}, \bar{Q})$  with quasi-triangular structure

$$\bar{Q} = \bar{\mathcal{F}}^{-2} = Q \bar{\mathcal{R}} = \bar{\mathcal{R}} Q, \tag{3.13}$$

and quantization of  $(C^\infty(\mathcal{M}), \mu_A)$  is given by  $(C^\infty(\mathcal{M})[[\hbar]], \bar{\star} := \star_{\bar{p}})$ , where

$$f \bar{\star} g = \mu_A \left( \exp \left[ \frac{i\hbar}{2} (R^{ijk} \bar{p}_k \partial_i \otimes \partial_j + \partial_i \otimes \bar{\delta}^i - \bar{\delta}^i \otimes \partial_i) \right] (f \otimes g) \right) \tag{3.14}$$

for all  $f, g \in C^\infty(\mathcal{M})$ . In this case, the full cotangent bundle  $\mathcal{M}$  becomes a noncommutative quantum phase space with commutation relations

$$[x^i, x^j]_{\bar{\star}} = i \hbar \theta^{ij}(\bar{p}), \quad [x^i, p_j]_{\bar{\star}} = i \hbar \delta^i_j, \quad [p_i, p_j]_{\bar{\star}} = 0, \tag{3.15}$$

where  $[f, g]_{\bar{\star}} := f \bar{\star} g - g \bar{\star} f$  for all  $f, g \in C^\infty(\mathcal{M})$ . In particular, the zero momentum surface  $\bar{p} = 0$  recovers the canonical Moyal-Weyl product  $\star = \star_0$  on phase space from Sec. III A.

The star product (3.14) first appeared in Ref. 46 as the restriction of a nonassociative star product to slices of constant momentum in phase space. However, in the context of cocycle twist quantization, its origin can be traced back to the unique choice of contraction of the  $R$ -flux  $R^{ijk}$  which is compatible with the ‘‘minimal’’ (translation) symmetries of  $\mathcal{M}$  and associativity. We will see below how this choice naturally extends to dynamical momentum and leads to the nonassociative star product of Ref. 46, thus further clarifying the operations of restricting to constant momentum and of reinstating dynamical momentum dependence that were used for the derivation of the quantized associator in Ref. 46.

### C. Let’s twist again

As discussed in Ref. 46, the trivector  $R$  has no natural geometric interpretation on configuration space except via T-duality. On the other hand, it is a 3-form on phase space  $\mathcal{M}$  which is in fact the curvature of a non-flat  $U(1)$  gerbe in momentum space. The presence of this 3-form enhances the symmetries of  $\mathcal{M} = T^*M$ , and thus the abelian Lie algebra  $\mathfrak{h}$  should be enlarged in order to accommodate the new symmetries. For this, we extend  $\mathfrak{h}$  to the non-abelian nilpotent Lie algebra  $\mathfrak{g}$  of dimension  $\frac{1}{2} d(d+3)$  generated by  $P_i, \bar{P}^i$  and  $M_{ij} = -M_{ji}$  with commutation relations given by

$$[\bar{P}^i, M_{jk}] = \delta^i_j P_k - \delta^i_k P_j, \tag{3.16}$$

while all other commutators are equal to zero. The actions of  $P_i$  and  $\tilde{P}^i$  on  $C^\infty(\mathcal{M})$  are still given by (3.1) from which we find the action of  $M_{ij}$  on  $C^\infty(\mathcal{M})$  to be

$$M_{ij} \triangleright f := p_i \partial_j f - p_j \partial_i f \tag{3.17}$$

for all  $f \in C^\infty(\mathcal{M})$ . Introducing elements  $M_\sigma = \frac{1}{2} \sigma^{ij} M_{ij} \in \mathfrak{g}$ , where  $\sigma^{ij} = -\sigma^{ji} \in \mathbb{R}$ , we see that  $M_\sigma$  generates the non-local coordinate transformations

$$x^i \mapsto x^i + \sigma^{ij} p_j, \quad p_i \mapsto p_i \tag{3.18}$$

that mix positions and momenta, which in quantum mechanics are called *Bopp shifts*. This symmetry is reminiscent of those encountered in T-folds ( $Q$ -space), where diffeomorphism symmetries include T-duality transformations that mix positions with winding numbers which are T-dual to the conjugate momenta. This points to the use of doubled geometry (it is possible to extend our star products below to T-duality covariant star products defined on double phase space, as in Ref. 11; a field theory written in this formalism is manifestly  $O(d, d)$ -invariant; however, in order to avoid overly cumbersome equations with essentially the same generic features, for simplicity we write all formulas below only in the  $R$ -flux duality frame), while here we are on a phase space of  $2d$  coordinates. In this sense, the symmetries (3.18) can be regarded as the analog of T-duality transformations in our algebraic framework.

One may now proceed as we did previously to find the related quasi-triangular Hopf algebra  $(H, \mathcal{R}_0) = (U(\mathfrak{g}), \Delta, \varepsilon, S, \mathcal{R}_0)$ , i.e., we equip the universal enveloping algebra  $U(\mathfrak{g})$  with the structure maps (2.7) and the quasi-triangular structure (2.8). Then,  $H$  can be twisted by the abelian Drinfel'd twist

$$F_R = \exp\left(-\frac{i\hbar}{8} R^{ijk} (M_{ij} \otimes P_k - P_i \otimes M_{jk})\right). \tag{3.19}$$

The result is a cocommutative twisted Hopf algebra  $H_{F_R}$  with quasi-triangular structure  $\mathcal{R} = F_R^{-2}$ . We can use  $H_{F_R}$  to twist the algebra of functions  $(C^\infty(\mathcal{M}), \mu_A)$ , and the resulting star product has the form

$$f \star_p g := f \star g = \mu_A\left(\exp\left[\frac{i\hbar}{2} R^{ijk} p_k \partial_i \otimes \partial_j\right](f \otimes g)\right) \tag{3.20}$$

for all  $f, g \in C^\infty(\mathcal{M})$ , and it is a noncommutative, associative Moyal-Weyl type star product similar to the one found in Sec. III B. The algebra  $(C^\infty(\mathcal{M}), \mu_A)$  is hence quantized to  $(C^\infty(\mathcal{M})[[\hbar]], \star_p := \star)$  and  $\mathcal{M}$  acquires a spatial noncommutativity since

$$[x^i, x^j]_\star = i\hbar \theta^{ij}(p), \quad [x^i, p_j]_\star = 0 = [p_i, p_j]_\star, \tag{3.21}$$

where  $[f, g]_\star := f \star g - g \star f$  for all  $f, g \in C^\infty(\mathcal{M})$  and  $\theta^{ij}(p)$  is defined by (3.11).

We can incorporate quantum phase space in this description using the method of Sec. III B. The pertinent non-abelian twist  $\mathcal{F} \in H[[\hbar]] \otimes H[[\hbar]]$  is

$$\mathcal{F} := F_R F = F F_R = \exp\left[-\frac{i\hbar}{2} \left(\frac{1}{4} R^{ijk} (M_{ij} \otimes P_k - P_i \otimes M_{jk}) + P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i\right)\right], \tag{3.22}$$

where we have used the antisymmetry of  $R^{ijk}$ . Equivalently, instead of (3.22), we can use the action of (3.12) on  $C^\infty(\mathcal{M})$  to write

$$\mathcal{F} = [\tilde{\mathcal{F}}]_{\bar{p} \rightarrow p}, \tag{3.23}$$

where the operation  $[-]_{\bar{p} \rightarrow p}$  denotes the change from constant to dynamical momentum. (We can also restrict to constant momentum by taking a double scaling limit  $\hbar \rightarrow 0$ ,  $R^{ijk} \rightarrow \infty$  with  $\bar{R}^{ijk} := \hbar R^{ijk}$  held constant. This limit is equivalent to the restriction  $\bar{F}_R = [F_R]_{p \rightarrow \bar{p}}$  when acting on  $C^\infty(\mathcal{M})$ .)

The twist  $\mathcal{F}$  is an invertible counital 2-cochain, hence  $H_{\mathcal{F}}$  defines a quasi-Hopf algebra  $\mathcal{H} = (H_{\mathcal{F}}, \phi)$  where the associator  $\phi = \phi_{\mathcal{F}}$  calculated from (2.27) is

$$\phi = \exp\left(\frac{\hbar^2}{2} R^{ijk} P_i \otimes P_j \otimes P_k\right). \tag{3.24}$$

Its coproduct  $\Delta_{\mathcal{F}} : H_{\mathcal{F}} \rightarrow H_{\mathcal{F}} \otimes H_{\mathcal{F}}$  is given by (2.14); calculating this on the generating primitive elements, we get

$$\begin{aligned} \Delta_{\mathcal{F}}(P_i) &= \Delta(P_i), \\ \Delta_{\mathcal{F}}(\tilde{P}^i) &= \Delta(\tilde{P}^i) + \frac{i\hbar}{2} R^{ijk} P_j \otimes P_k, \\ \Delta_{\mathcal{F}}(M_{ij}) &= \Delta(M_{ij}) - i\hbar (P_i \otimes P_j - P_j \otimes P_i). \end{aligned} \tag{3.25}$$

In particular,  $\mathcal{H}$  is a non-cocommutative quasi-Hopf algebra with quasi-triangular structure  $\mathcal{R} = \mathcal{F}^{-2}$ , as a straightforward calculation of the co-opposite coproduct  $\Delta_{\mathcal{F}}^{\text{op}}$  on primitive elements using (2.17) reveals.

A left (or right)  $H$ -module algebra  $(A, \mu_A)$  can now be deformed using (2.18) and the relevant action. Let us do this for the algebra of functions on  $\mathcal{M}$ , i.e., we set  $A = C^\infty(\mathcal{M})$  and  $\mu_A(f \otimes g) = fg$  and derive its deformation quantization  $(C^\infty(\mathcal{M})[[\hbar]], \star)$  with the star product given by (2.18). We find

$$f \star g = \mu_A \left( \exp \left[ \frac{i\hbar}{2} (R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i) \right] (f \otimes g) \right), \tag{3.26}$$

where the actions (3.1) and (3.17) have been used. This is a *nonassociative star product*, and hence  $(C^\infty(\mathcal{M})[[\hbar]], \star)$  is a nonassociative algebra, i.e., the product of three functions is associative only up to the associator (3.24). This is expressed by (2.28) which in this case can be written in the more explicit form

$$\begin{aligned} (f \star g) \star h &= f \star (g \star h) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\hbar^2}{2} \right)^n R^{i_1 j_1 k_1} \dots R^{i_n j_n k_n} (\partial_{i_1} \dots \partial_{i_n} f) \star \\ &\quad \star ((\partial_{j_1} \dots \partial_{j_n} g) \star (\partial_{k_1} \dots \partial_{k_n} h)). \end{aligned} \tag{3.27}$$

From (3.25), we find the (modified) Leibniz rules

$$\begin{aligned} \partial_i(f \star g) &= (\partial_i f) \star g + f \star (\partial_i g), \\ \tilde{\partial}^i(f \star g) &= (\tilde{\partial}^i f) \star g + f \star (\tilde{\partial}^i g) + \frac{i\hbar}{2} R^{ijk} (\partial_j f) \star (\partial_k g), \end{aligned} \tag{3.28}$$

and in particular  $\partial_i$  is a derivation of the star product. These relations simply reflect the fact that the nonassociative  $R$ -flux background exhibits space translation symmetry, but not momentum translation symmetry, due to the dynamical nonassociativity. The loss of translation invariance in momentum space is related to the violation of Bopp shift symmetry through

$$\Delta_{\mathcal{F}}(\tilde{P}^i) - \Delta(\tilde{P}^i) = -\frac{1}{4} R^{ijk} (\Delta_{\mathcal{F}}(M_{jk}) - \Delta(M_{jk})) \tag{3.29}$$

indicating that the Bopp shift generators  $M_{ij}$  are observables which detect the effects of nonassociativity.

The nonassociative star product (3.26) coincides with the one previously proposed in Ref. 46 for functions on  $\mathcal{M}$ . To see that (3.27) coincides with the associator derived in Ref. 46, it is enough to compute it on the position space plane waves  $U_{\tilde{a}} = e^{i\tilde{a}_i x^i}$  where  $\tilde{a} = (\tilde{a}_i) \in (\mathbb{R}^d)^*$  are constant vectors. One easily finds

$$(U_{\tilde{a}} \star U_{\tilde{b}}) \star U_{\tilde{c}} = \varphi_R(\tilde{a}, \tilde{b}, \tilde{c}) U_{\tilde{a}} \star (U_{\tilde{b}} \star U_{\tilde{c}}), \tag{3.30}$$

where

$$\varphi_R(\tilde{a}, \tilde{b}, \tilde{c}) := \exp\left(-\frac{i\hbar^2}{2} R^{ijk} \tilde{a}_i \tilde{b}_j \tilde{c}_k\right) \tag{3.31}$$

is the 3-cocycle of the group of translations that was obtained in Refs. 46 and 11.

Using (3.26), we can calculate the  $\star$ -commutation relations on the coordinate functions of  $\mathcal{M}$ . We find

$$[x^i, x^j]_{\star} = i\hbar \theta^{ij}(p), \quad [x^i, p_j]_{\star} = i\hbar \delta^i_j, \quad [p_i, p_j]_{\star} = 0, \tag{3.32}$$

where  $[f, g]_{\star} := f \star g - g \star f$  for all  $f, g \in C^\infty(\mathcal{M})$  and  $i, j, k = 1, \dots, d$ . Nonassociativity of (3.26) can also be seen through the failure of the Jacobi identity for this  $\star$ -commutator, analogously to (1.2). As explained in Ref. 46, the antisymmetric bracket  $[-, -]_{\star}$  gives rise to a Lie 2-algebra

structure on the function algebra  $C^\infty(\mathcal{M})$ ; while Lie algebraic structures give rise to associative deformations, Lie 2-algebras lead to nonassociative deformations.

**D. Integral formulas**

For later use, let us note that the star product (3.26) can be written in upper case index notation as

$$f \star g = \mu_A \left( \exp\left(\frac{i\hbar}{2} \Theta^{IJ} \partial_I \otimes \partial_J\right) (f \otimes g) \right), \tag{3.33}$$

where  $\partial_I = \frac{\partial}{\partial x^I}$  and

$$\Theta = (\Theta^{IJ}) = \begin{pmatrix} R^{ijk} p_k & \delta^i_j \\ -\delta_i^j & 0 \end{pmatrix} \tag{3.34}$$

with  $I, J \in \{1, \dots, 2d\}$ . The formula (3.33) for the nonassociative star product was first derived in Ref. 46 using Kontsevich’s deformation quantization; it is formally identical to the Moyal-Weyl star product, but it is nonassociative and its derivation is non-trivial. In practise, for explicit calculations, it is useful to employ an integral representation of the star product; a Fourier integral formula for this product was derived in Ref. 46, Sec. 4.3 and in Ref. 11, Sec. 4.2. A related but more useful formula can be easily derived by expressing  $g(x)$  in terms of its Fourier transform  $\hat{g}(k)$ , where  $k \in \mathcal{M}^*$ ; here  $\mathcal{M} := T^*M \cong M \times (\mathbb{R}^d)^* \cong \mathbb{R}^d \times (\mathbb{R}^d)^*$  and  $\mathcal{M}^* \cong (\mathbb{R}^d)^* \times \mathbb{R}^d$ . Then, the derivative  $\partial_J$  turns into multiplication by  $i k_J$  and we can interpret the exponential  $\exp(-\frac{\hbar}{2} \Theta^{IJ} k_J \partial_I)$  as a shift operator to rewrite (3.33) as

$$(f \star g)(x) = \int_{\mathcal{M}^*} d^{2d}k f(x - \frac{\hbar}{2} \Theta k) \hat{g}(k) e^{i k_I x^I}, \tag{3.35}$$

where  $(\Theta k)^I := \Theta^{IJ} k_J$ . Using the inverse Fourier transformation, this becomes

$$(f \star g)(x) = \frac{1}{(2\pi)^{2d}} \int_{\mathcal{M}^*} d^{2d}k \int_{\mathcal{M}} d^{2d}x' f(x - \frac{\hbar}{2} \Theta k) g(x') e^{i k_I (x-x')^I}. \tag{3.36}$$

Since the matrix  $\Theta$  is unimodular, we can use the inverse matrix ( $B$ -field)

$$\Theta^{-1} = (\Theta_{IJ}^{-1}) = \begin{pmatrix} 0 & -\delta^i_j \\ \delta_i^j & R^{ijk} p_k \end{pmatrix} \tag{3.37}$$

to change variables to  $z = -\frac{\hbar}{2} \Theta k$  and  $w = x' - x$ , and in this way, we finally obtain a nice expression in terms of a twisted convolution product

$$(f \star g)(x) = \left(\frac{1}{\pi\hbar}\right)^{2d} \int_{\mathcal{M}} d^{2d}z \int_{\mathcal{M}} d^{2d}w f(x+z) g(x+w) e^{-\frac{2i}{\hbar} z^I \Theta_{IJ}^{-1} w^J} \tag{3.38}$$

that is often more convenient for computations than (3.33); it is also well-defined as a nonperturbative formula on the larger class of Schwartz functions on phase space  $\mathcal{M}$ . One should not forget that the matrix  $\Theta^{-1}$  in (3.38) depends on  $p_i$  and hence on  $x^I$ , but otherwise the expression is again formally identical to the standard twisted convolution product formula for the Moyal-Weyl product (see, e.g., Ref. 51 for a review).

There are also integral formulas available for our twists. Changing the sign of  $\Theta$  and dropping the multiplication operator  $\mu_A$  in (3.33), we can similarly derive an integral formula for the action of the twist on a pair of functions  $f$  and  $g$  (evaluated at  $x$  and  $y$ , respectively) given by

$$\begin{aligned} (\mathcal{F} \triangleright (f \otimes g))(x, y) &= \left(\frac{1}{\pi\hbar}\right)^{2d} \int_{\mathcal{M}} d^{2d}z \int_{\mathcal{M}} d^{2d}w (e^{\frac{i}{\hbar} z^I \Theta_{IJ}^{-1} w^J} f) \otimes (e^{\frac{i}{\hbar} z^I \Theta_{IJ}^{-1} w^J} g)(x+z, y+w). \end{aligned} \tag{3.39}$$



Introducing the shift operator  $(\mathcal{T}_a f)(x) := f(x + a)$  for any  $2d$ -vector  $a$ , the twist element acting on  $C^\infty(\mathcal{M})$  thus becomes

$$\mathcal{F} = \left( \frac{1}{\pi \hbar} \right)^{2d} \int_{\mathcal{M}} d^{2d}z \int_{\mathcal{M}} d^{2d}w \left( e^{\frac{i}{\hbar} z^I \Theta_{IJ}^{-1} w^J} \mathcal{T}_z \right) \otimes \left( e^{\frac{i}{\hbar} z^I \Theta_{IJ}^{-1} w^J} \mathcal{T}_w \right). \tag{3.40}$$

#### IV. NONASSOCIATIVE EXTERIOR DIFFERENTIAL CALCULUS ON $R$ -SPACE

The approach of Sec. III has the great virtue of enabling a systematic development of exterior differential calculus on these deformations. In this section, we develop the basic ingredients necessary for both an investigation of nonassociative quantum mechanics within a phase space quantization formalism, which we undertake in Sec. V, as well as a putative formulation of field theories on the nonassociative parabolic  $R$ -flux backgrounds.

##### A. Covariant differential calculus

In this section, we will use the cochain twist (3.22) to deform the exterior algebra of differential forms  $(\Omega^\bullet, \mu_\wedge, d)$ , where  $\Omega^\bullet := \bigoplus_{n \geq 0} \Omega^n$  with  $\Omega^n = \Omega^n(\mathcal{M})$  the vector space of complex smooth  $n$ -forms on  $\mathcal{M}$ ,  $\mu_\wedge : \Omega^n \otimes \Omega^m \rightarrow \Omega^{n+m}$  the usual exterior product  $\mu_\wedge(\omega \otimes \omega') := \omega \wedge \omega'$ , and  $d : \Omega^n \rightarrow \Omega^{n+1}$  the exterior derivative with  $d^2 = 0$ .

We demand that the action of the Hopf algebra  $H = U(\mathfrak{g})$  from Sec. III C on  $(\Omega^\bullet, \mu_\wedge, d)$  is covariant in the sense that (c.f. (2.10))

$$h \triangleright (\omega \wedge \omega') = (h_{(1)} \triangleright \omega) \wedge (h_{(2)} \triangleright \omega') \tag{4.1}$$

and that  $d$  is equivariant under the action of  $H$  in the sense that

$$h \triangleright (d\omega) = d(h \triangleright \omega) \tag{4.2}$$

for all  $\omega, \omega' \in \Omega^\bullet$  and  $h \in H$ ; in the framework of Sec. II C, these two conditions, respectively, mean that the structure maps  $\mu_\wedge$  and  $d$  are both morphisms in the category  ${}_H\mathcal{M}$  of left  $H$ -modules. The action of  $H$  on  $\Omega^\bullet$  can be determined by finding the action on  $\Omega^1$  and extending it to  $\Omega^\bullet$  as an algebra homomorphism using the Leibniz rule

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega' \tag{4.3}$$

for all  $\omega, \omega' \in \Omega^\bullet$ . Employing (4.2), the action of  $H$  on  $\Omega^0$  given by (3.1) and (3.17), and the fact that  $d$  commutes with the Lie derivative along any vector field, we conclude that the action of  $H$  on  $\Omega^\bullet$  is given by the Lie derivative  $\mathcal{L}_h$  along elements  $h \in H$ . As previously the action is defined on primitive elements of  $H$  as an algebra homomorphism, i.e.,  $\mathcal{L}_{\xi \xi'} := \mathcal{L}_\xi \circ \mathcal{L}_{\xi'}$  for  $\xi, \xi' \in \mathfrak{g} \subset U(\mathfrak{g})$ , and it extends to a left action via linearity of the Lie derivative and the Leibniz rule to get  $\mathcal{L}_{hh'} = \mathcal{L}_h \circ \mathcal{L}_{h'}$  for all  $h, h' \in H$ . Calculating this action on the generating 1-forms gives

$$M_{ij} \triangleright dx^k := \mathcal{L}_{M_{ij}}(dx^k) = \delta_j^k dp_i - \delta_i^k dp_j, \tag{4.4}$$

with all other generators  $dx^I$  invariant under the action of  $H$ .

Following the methods of Sec. II (with  $A = \Omega^\bullet$  and  $\mu_A = \mu_\wedge$ ), we ensure that  $\Omega^\bullet$  is covariant under the action of  $\mathcal{H} = (H_{\mathcal{F}}, \phi)$  by introducing a deformed exterior product  $\wedge_\star$  on  $\Omega^n[[\hbar]] \otimes \Omega^m[[\hbar]] \rightarrow \Omega^{n+m}[[\hbar]]$  given by the formula

$$\omega \wedge_\star \omega' = \mu_\wedge(\mathcal{F}^{-1} \triangleright (\omega \otimes \omega')) = (\mathcal{F}_{(1)}^{-1} \triangleright \omega) \wedge (\mathcal{F}_{(2)}^{-1} \triangleright \omega') \tag{4.5}$$

for all  $\omega, \omega' \in \Omega^\bullet$ . The exterior derivative is still a derivation for the deformed exterior product, and thus we call the twisted exterior algebra  $(\Omega^\bullet[[\hbar]], \wedge_\star, d)$  the *nonassociative exterior differential calculus*. Using (4.5) on the generating 1-forms, we find the relations

$$dx^I \wedge_\star dx^J = -dx^J \wedge_\star dx^I = dx^I \wedge dx^J, \tag{4.6}$$

where  $I, J \in \{1, \dots, 2d\}$ . We can again write (2.28) in a more enlightening form for the case at hand as

$$(\omega \wedge_\star \omega') \wedge_\star \omega'' = \omega \wedge_\star (\omega' \wedge_\star \omega'') + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\hbar^2}{2}\right)^n R^{i_1 j_1 k_1} \dots R^{i_n j_n k_n} \mathcal{L}_{i_1} \dots \mathcal{L}_{i_n}(\omega) \wedge_\star \wedge_\star (\mathcal{L}_{j_1} \dots \mathcal{L}_{j_n}(\omega') \wedge_\star \mathcal{L}_{k_1} \dots \mathcal{L}_{k_n}(\omega'')) \tag{4.7}$$

for all  $\omega, \omega', \omega'' \in \Omega^\bullet$ , where we have abbreviated  $\mathcal{L}_i := \mathcal{L}_{\partial_i}$ . When  $\omega, \omega', \omega'' \in \Omega^1$ , this formula takes an even simpler form given by

$$(\omega \wedge_\star \omega') \wedge_\star \omega'' = \omega \wedge_\star (\omega' \wedge_\star \omega'') + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\hbar^2}{2}\right)^n R^{i_1 j_1 k_1} \dots R^{i_n j_n k_n} (\partial_{i_1} \dots \partial_{i_n} \omega_L) dx^L \wedge_\star \wedge_\star ((\partial_{j_1} \dots \partial_{j_n} \omega'_M) dx^M \wedge_\star (\partial_{k_1} \dots \partial_{k_n} \omega''_N) dx^N), \tag{4.8}$$

where  $i_l, j_l, k_l \in \{1, \dots, d\}$  and  $L, M, N \in \{1, \dots, 2d\}$ . It follows that:

$$(dx^I \wedge_\star dx^J) \wedge_\star dx^K = dx^I \wedge_\star (dx^J \wedge_\star dx^K) =: dx^I \wedge_\star dx^J \wedge_\star dx^K, \tag{4.9}$$

where  $I, J, K \in \{1, \dots, 2d\}$ .

The exterior product provides an  $A$ -bimodule structure on  $\Omega^\bullet$ , where  $A = (C^\infty(\mathcal{M}), \mu_A)$ , with right and left action given by the pointwise multiplication of an  $n$ -form by a function. Let us denote this action by  $\blacktriangleright$  and  $\blacktriangleleft$  respectively; then covariance of the bimodule under the action of the Hopf algebra  $H$  means

$$h \triangleright (f \blacktriangleright \omega) = (h_{(1)} \triangleright f) \blacktriangleright (h_{(2)} \triangleright \omega), \quad h \triangleright (\omega \blacktriangleleft f) = (h_{(1)} \triangleright \omega) \blacktriangleleft (h_{(2)} \triangleright f), \tag{4.10}$$

for all  $h \in H, f \in C^\infty(\mathcal{M})$  and  $\omega \in \Omega^\bullet$ . To ensure that the bimodule is covariant under the action of the quasi-Hopf algebra  $\mathcal{H}$  we must replace its action by the deformed right and left actions given respectively by the formulas

$$f \blacktriangleright_\star \omega = \lambda_\blacktriangleright(\mathcal{F}^{-1} \triangleright (f \otimes \omega)), \quad \omega \blacktriangleleft_\star f = \lambda_\blacktriangleleft(\mathcal{F}^{-1} \triangleright (\omega \otimes f)), \tag{4.11}$$

where  $\lambda_\blacktriangleright(f \otimes \omega) = f \blacktriangleright \omega = f \omega$  for all  $\omega \in \Omega^\bullet$  and  $f \in C^\infty(\mathcal{M})$ , and similarly for  $\lambda_\blacktriangleleft: \Omega^\bullet \otimes A \rightarrow \Omega^\bullet$ , which yields the deformed  $A_\star$ -bimodule, where  $A_\star = (C^\infty(\mathcal{M})[[\hbar]], \star)$ . Since  $f \blacktriangleright \omega = \omega \blacktriangleleft f$ , here and throughout we will abuse notation for the sake of simplicity by denoting  $\blacktriangleright_\star$  and  $\blacktriangleleft_\star$  by  $\star$  where no confusion arises. A short calculation then reveals the non-trivial bimodule relations between coordinates and 1-forms given by

$$x^i \star dx^j = dx^j \star x^i + \frac{i\hbar}{2} R^{ijk} dp_k, \tag{4.12}$$

while all other left and right  $A_\star$ -actions coincide.

**B. Integration**

To compute quantum mechanical averages, and also to set up a Lagrangian formalism for field theory, we need a suitable definition of integration  $\int$  on  $(S(\mathcal{M})[[\hbar]], \star)$ , where  $S(\mathcal{M}) \subset C^\infty(\mathcal{M})$  is the subalgebra of Schwartz functions on  $\mathcal{M} = T^*M$ .

Let us first notice that the star product (3.26) satisfies

$$f \star g = fg + \text{total derivative}. \tag{4.13}$$

This can be easily verified if we write the star product in the form (3.33), and keep in mind that a total derivative in phase space includes both position and momentum derivatives. The order  $\hbar^n$  term can then be written as

$$\Theta^{I_1 J_1} \dots \Theta^{I_n J_n} (\partial_{I_1} \dots \partial_{I_n} f) (\partial_{J_1} \dots \partial_{J_n} g) = \partial_{I_1} \dots \partial_{I_n} (\Theta^{I_1 J_1} \dots \Theta^{I_n J_n} f \partial_{J_1} \dots \partial_{J_n} g) \tag{4.14}$$

since no momentum derivatives act on the upper left block of  $\Theta$ , which means that (4.13) is satisfied to all orders in  $\hbar$ . Then, the usual integration on  $\mathcal{M}$  satisfies the 2-cyclicity condition

$$\int_{\mathcal{M}} d^{2d}x f \star g = \int_{\mathcal{M}} d^{2d}x g \star f = \int_{\mathcal{M}} d^{2d}x f g \tag{4.15}$$

for all  $f, g \in \mathcal{S}(\mathcal{M})$ , i.e., the standard integral on  $(\mathcal{S}(\mathcal{M})[[\hbar]], \star)$  is 2-cyclic.

In addition to the 2-cyclicity condition, the standard integral on  $(\mathcal{S}(\mathcal{M})[[\hbar]], \star)$  satisfies a cyclicity condition on the associator derived from the property

$$f \star (g \star h) = (f \star g) \star h + \text{total derivative}, \tag{4.16}$$

which easily follows from (3.27) and (3.28). Hence, the standard integral also satisfies the property

$$\int_{\mathcal{M}} d^{2d}x f \star (g \star h) = \int_{\mathcal{M}} d^{2d}x (f \star g) \star h =: \int_{\mathcal{M}} d^{2d}x f \star g \star h \tag{4.17}$$

for all  $f, g, h \in \mathcal{S}(\mathcal{M})$ , which we call the 3-cyclicity condition; this property was also derived in Ref. 46 from a slightly different perspective.

Our proofs of 2-cyclicity (4.15) and 3-cyclicity (4.17) are only carried out for constant  $R$ -flux in the standard coordinate frame. As changing coordinates alters this situation, let us comment briefly on what restrictions are generically required. As we discuss at greater length in Sec. VI A, the antisymmetric tensor  $R^{ijk}$  defines a 3-bracket which is Nambu-Poisson, and this property is invariant under change of coordinates. A Nambu-Poisson tensor implies locally a foliation of the underlying manifold, while our proofs require this property globally. It follows that 2-cyclicity and 3-cyclicity hold whenever  $R$  is a Nambu-Poisson trivector that defines a global foliation of  $M$ .

The 2-cyclicity condition does *not* generally guarantee the usual cyclicity property involving integration of  $n$ -fold star products of functions. This is because the star product is nonassociative and thus a bracketing for the star product of  $n$  functions has to be specified. Once this is done, one cannot freely move functions cyclically under integration using (4.15) as one would normally do in the associative case; instead the 3-cyclicity condition (4.17) can be used to rebracket the integrated expression and to investigate its equivalence with expressions involving different bracketings. In general, the total number of ways to bracket a star product of  $n$  functions is given by the Catalan number  $C_{n-1}$ , where

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} \tag{4.18}$$

for  $n \geq 0$ . Starting from the integral

$$\int_{\mathcal{M}} d^{2d}x f_1 \star (f_2 \star (f_3 \star (\dots \star f_n \dots))), \tag{4.19}$$

one can prove that it is equal to a number of different bracketings but always inequivalent to any other bracketing of the form  $f_1 \star (\text{bracketed expression})$ , i.e., under 3-cyclicity, the distinct ways of bracketing an integrated  $n$ -fold star product of functions are organised into  $C_{n-2}$  classes, one for each different bracketing where  $f_1$  is free at the front. For example, for  $n = 4$ , there are five different bracketings out of which two are of the form (4.19), and thus we have two different classes of equivalent bracketings, namely,

$$\int_{\mathcal{M}} d^{2d}x f_1 \star (f_2 \star (f_3 \star f_4)) = \int_{\mathcal{M}} d^{2d}x (f_1 \star f_2) \star (f_3 \star f_4) = \int_{\mathcal{M}} d^{2d}x ((f_1 \star f_2) \star f_3) \star f_4 \tag{4.20}$$

and

$$\int_{\mathcal{M}} d^{2d}x f_1 \star ((f_2 \star f_3) \star f_4) = \int_{\mathcal{M}} d^{2d}x (f_1 \star (f_2 \star f_3)) \star f_4. \tag{4.21}$$

A graded 2-cyclicity condition could also be derived for the deformed exterior product, provided that we can write an equation similar to (4.13) for it. This seems complicated since generally

(4.5) cannot be written explicitly in closed form, but fortunately there is a way around this problem: We can use the result of Ref. 6 where it was shown that if the identity

$$U_{\mathcal{F}} = \mathcal{F}_{(1)} S(\mathcal{F}_{(2)}) = 1_H \tag{4.22}$$

holds, where  $U_{\mathcal{F}} = \mu \circ (\text{id}_H \otimes S)(\mathcal{F})$  and  $S$  is the antipode, then standard integration on  $(\Omega^*[[\hbar]], \wedge_\star, d)$  is graded 2-cyclic. This is always true for abelian twists but does not hold in general; however, in our case, the twisted antipode  $S_{\mathcal{F}}$  coincides with  $S$ . It is then straightforward to demonstrate (4.22) on  $\mathcal{S}(\mathcal{M})$  by using the representation of primitive elements of  $H$  on functions in (3.22) and antisymmetry of  $R^{ijk}$ . Hence, we conclude that for the nonassociative exterior differential calculus  $(\Omega^*[[\hbar]], \wedge_\star, d)$ , the graded 2-cyclicity condition

$$\int_{\mathcal{M}} \omega \wedge_\star \omega' = (-1)^{\text{deg}(\omega) \text{deg}(\omega')} \int_{\mathcal{M}} \omega' \wedge_\star \omega = \int_{\mathcal{M}} \omega \wedge \omega' \tag{4.23}$$

is indeed satisfied.

The 3-cyclicity condition (4.17) can also be generalized by noticing that similarly to  $\partial_i$  being a derivation for the nonassociative star product by (3.28), the Lie derivative  $\mathcal{L}_i$  is a derivation of the deformed exterior product  $\wedge_\star$  by (3.25) and the discussion that followed (3.28), and since  $[\mathcal{L}_i, \mathcal{L}_j] = \mathcal{L}_{[\partial_i, \partial_j]} = 0$  by (4.7), one has

$$\omega \wedge_\star (\omega' \wedge_\star \omega'') = (\omega \wedge_\star \omega') \wedge_\star \omega'' + \text{total Lie derivative.} \tag{4.24}$$

Since  $\int_{\mathcal{M}} \mathcal{L}_i(\omega) = 0$  for all  $\omega \in \Omega^\bullet$ , we thus get the generic 3-cyclicity condition

$$\int_{\mathcal{M}} \omega \wedge_\star (\omega' \wedge_\star \omega'') = \int_{\mathcal{M}} (\omega \wedge_\star \omega') \wedge_\star \omega'' =: \int_{\mathcal{M}} \omega \wedge_\star \omega' \wedge_\star \omega'' \tag{4.25}$$

for all  $\omega, \omega', \omega'' \in \Omega^\bullet$ , generalizing (4.17) which is the  $\Omega^0$  case.

## V. NONASSOCIATIVE QUANTUM MECHANICS

The standard formulation of quantum mechanics is based on linear operators acting on a separable Hilbert space and the corresponding operator algebras are by construction associative. Nevertheless, it turns out that the mathematical tools and structures that we have developed in this paper do in fact allow for a direct quantitative discussion of nonassociativity in quantum mechanics, adding to the more qualitative arguments that can already be found in the literature. The lack of associativity alters the theory of quantum mechanics drastically, but against all odds a consistent formulation is apparently indeed possible. To the best of our knowledge, the bulk of the material presented in this section is new.

### A. Nambu-Heisenberg brackets, star products, and compositions

The task of formulating a nonassociative version of quantum mechanics is closely related to the quest of quantizing Nambu-Poisson brackets (see, e.g., Refs. 27 and 49 and references therein). A natural choice would be the Jacobiator of operators, but it obviously vanishes for associative operator algebras. As a work-around, a Nambu-Heisenberg bracket was introduced by Nambu as half of a Jacobiator<sup>47</sup>

$$[A, B, C]_{\text{NH}} = ABC + CAB + BCA - BAC - ACB - CBA. \tag{5.1}$$

It is straightforward to evaluate the Nambu-Heisenberg bracket on coordinate functions for any of our star products. (Nambu-Heisenberg brackets have been previously investigated in the context of phase space quantum mechanics based on *associative* star products in Ref. 53.) For instance, for the associative constant  $\bar{p}$  star product (3.14), we find

$$[x^i, x^j, x^k]_{\star, \text{NH}} = i \hbar (R^{ijl} \bar{p}_l x^k + R^{jkl} \bar{p}_l x^i + R^{kil} \bar{p}_l x^j). \tag{5.2}$$

Nambu suggested to consider nonassociative algebras for the quantization of his bracket. We do have the tools now to study this proposal. In the nonassociative case, we need to specify which

operators are multiplied first. We choose by default the first pair and write

$$[A, B, C]_{\text{NH}} = [A, B]C + [C, A]B + [B, C]A, \tag{5.3}$$

where  $[A, B]C := (AB)C - (BA)C$ . For the nonassociative star product (3.26), evaluated on a triple of coordinate functions, this gives

$$[x^i, x^j, x^k]_{\star\text{-NH}} = i\hbar(R^{ijl} p_l \star x^k + R^{jkl} p_l \star x^i + R^{kil} p_l \star x^j). \tag{5.4}$$

The opposite Nambu-Heisenberg bracket

$$[A, B, C]'_{\text{NH}} = C[B, A] + B[A, C] + A[C, B] \tag{5.5}$$

is in general no longer equal to minus the original Nambu-Heisenberg bracket. Their sum gives the Jacobiator

$$[[A, B, C]] := [[A, B], C] + [[C, A], B] + [[B, C], A] = [A, B, C]_{\text{NH}} + [A, B, C]'_{\text{NH}}. \tag{5.6}$$

For the nonassociative star product (3.26), evaluated on a triple of coordinate functions, we obtain the non-zero Jacobiator (c.f. (1.2))

$$[[x^i, x^j, x^k]]_{\star} = i\hbar(R^{ijl} [p_l, x^k]_{\star} + R^{jkl} [p_l, x^i]_{\star} + R^{kil} [p_l, x^j]_{\star}) = 3\hbar^2 R^{ijk} \tag{5.7}$$

as a more convincing candidate than (5.2) or (5.4) for a quantized Nambu-Poisson bracket.

An indirect approach to nonassociative quantum mechanics can be based on the family of associative star products  $\bar{\star}$  for constant  $\bar{p}$ -slices and the mappings that link them, and to  $\star$  by twists, very much in the spirit of describing a general manifold in terms of Euclidean spaces by local coordinate charts and transition functions. A regular operator/Hilbert space approach to nonassociative quantum mechanics can in fact be based on standard canonical quantization and the twist (3.19) from the Moyal-Weyl product (3.4) to the nonassociative product (3.26); after quantization, the twist is expressed in terms of operators acting on a suitable Hilbert space.

Instead of these solid but indirect approaches to nonassociative quantum mechanics, we shall pursue a more direct approach: The phase space formulation of quantum mechanics<sup>45</sup> is powerful enough to study nonassociative quantum mechanics *in situ* (see, e.g., Ref. 52 and references therein). Observables are implemented as real functions on phase space, states are represented by pseudo-probability Wigner-type density functions, and noncommutativity of operators enters via a star product of functions, which is the deformation quantization of a classical Poisson structure.

Let us start by introducing some convenient notation and conventions. We introduce the compositions  $\circ$  and  $\bar{\circ}$  by

$$(A \circ B) \star C := A \star (B \star C), \quad C \star (A \bar{\circ} B) := (C \star A) \star B \tag{5.8}$$

for all  $A, B, C \in C^\infty(\mathcal{M})[[\hbar]]$ . The compositions are related by complex conjugation:  $(A \circ B)^* = B^* \bar{\circ} A^*$  and *vice versa*. We choose the convention that  $\bar{\circ}$  is evaluated before  $\circ$  in all expressions that involve both compositions, so that

$$(A \circ B) \star (C \bar{\circ} D) := ((A \circ B) \star C) \star D = (A \star (B \star C)) \star D. \tag{5.9}$$

This convention looks asymmetric, but as long as we are just computing expectation values, it gives the same results as the alternative convention as a consequence of 3-cyclicity. Physically this is a remnant of operator-state duality. In the context of time-evolution and similar transformations, this duality is, however, no longer a symmetry in the nonassociative setting. The compositions can be extended to an arbitrary number of functions and are by construction associative. Like the star products that we are considering in this paper, the compositions are noncommutative and unital:  $1 \circ A = A = A \circ 1$ . For an associative algebra,  $\circ$  would just be the product in that algebra. However, in the nonassociative case,  $A \circ B$  cannot even generally be replaced by some suitable element of the algebra  $(C^\infty(\mathcal{M})[[\hbar]], \star)$ , because if this were possible then  $(A \circ B) \star 1 = A \star (B \star 1) = A \star B$  would imply  $A \circ B = A \star B$  and thus  $(A \star B) \star C = (A \circ B) \star C = A \star (B \star C)$  for all  $C \in C^\infty(\mathcal{M})[[\hbar]]$ . In a nonassociative algebra, this is obviously not true for all  $A, B \in C^\infty(\mathcal{M})[[\hbar]]$ . There are, however, some notable exceptions, e.g.,  $x^i \circ x^i = x^i \star x^i = (x^i)^2$  and  $p_i \circ p_i = p_i \star p_i = (p_i)^2$ .

**B. States, operators, and eigenvalues**

States map observables to numbers, which are interpreted as expectation values and link theory to experiment. For this purpose, one requires convexity, reality, unit trace, and positivity properties. The latter property is particularly difficult to implement in a nonassociative setting. A definition that ultimately fulfills all these requirements is as follows. A *state*  $\rho$  is an expression of the form

$$\rho = \sum_{\alpha=1}^n \lambda_{\alpha} \psi_{\alpha} \bar{\circ} \psi_{\alpha}^*, \tag{5.10}$$

where  $n \geq 1$ ,  $\lambda_{\alpha} > 0$ ,  $\sum_{\alpha=1}^n \lambda_{\alpha} = 1$ , and  $\psi_{\alpha}$  are complex-valued *phase space wave functions*, which are normalized as

$$\int_{\mathcal{M}} d^{2d}x |\psi_{\alpha}|^2 = 1 \tag{5.11}$$

but are not necessarily orthogonal. (Using the familiar language of quantum mechanics, we refer to complex-valued functions on phase space that are multiplied by star products as “operators” and to real-valued functions on phase space that are associated to something that can in principle be measured as “observables”. The phase space wave functions  $\psi_{\alpha}$  and their complex conjugates  $\psi_{\alpha}^*$  should not be confused with state vector kets or bras. The corresponding objects in ordinary quantum mechanics are normalized but otherwise arbitrary operators that are not necessarily related to rank one projectors.) For any two states  $\rho_1$  and  $\rho_2$ , the convex linear combination  $\rho_3 = \lambda \rho_1 + (1 - \lambda) \rho_2$  with  $\lambda \in [0, 1]$  is again a state. The space of states is thus a convex set, whose extrema we call pure states. A necessary (but not sufficient) condition for a state to be pure is that it is of the form  $\rho = \psi \bar{\circ} \psi^*$ . Given a state  $\rho$ , the expectation value of a function on phase space (“operator”)  $A$  is obtained by the phase space integral

$$\begin{aligned} \langle A \rangle &:= \int_{\mathcal{M}} d^{2d}x A \star \rho \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x A \star (\psi_{\alpha} \bar{\circ} \psi_{\alpha}^*) \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (A \star \psi_{\alpha}) \star \psi_{\alpha}^* \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x \psi_{\alpha}^* \star (A \star \psi_{\alpha}) = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x \psi_{\alpha}^* (A \star \psi_{\alpha}), \end{aligned} \tag{5.12}$$

where we have used 2-cyclicity. Using 3-cyclicity and the fact that complex conjugation acts anti-involutively on the star product,  $(A \star \psi)^* = \psi^* \star A^*$ , we find

$$\langle A \rangle^* = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (A \star \psi_{\alpha})^* \star \psi_{\alpha} = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x \psi_{\alpha}^* \star (A^* \star \psi_{\alpha}) = \langle A^* \rangle. \tag{5.13}$$

Observables (i.e., real-valued functions on phase space  $A^* = A$ ) therefore have real expectation values as desired. We will later show that 3-cyclicity also ensures reality of eigenvalues. Thanks to 3-cyclicity, our approach to nonassociative quantum mechanics is thus not affected by a previously proposed no-go theorem.<sup>30</sup>

Expectation values can also be computed for star products of functions (because star products of functions are again functions). The definition of expectation value can be further extended to compositions of operators as

$$\begin{aligned} \langle A_1 \circ A_2 \circ \dots \circ A_k \rangle &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (A_1 \circ A_2 \circ \dots \circ A_k) \star \psi_{\alpha} \bar{\circ} \psi_{\alpha}^* \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x [A_1 \star (A_2 \star (\dots \star (A_k \star \psi_{\alpha}) \dots))] \star \psi_{\alpha}^*. \end{aligned} \tag{5.14}$$

Positivity is a tricky concept in the nonassociative setting. In terms of our definition of a state, it is realized for any state  $\rho$  and any function on phase space  $A$  as

$$\begin{aligned} \langle A^* \circ A \rangle &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x \psi_{\alpha}^* \star (A^* \star (A \star \psi_{\alpha})) \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (\psi_{\alpha}^* \star A^*) \star (A \star \psi_{\alpha}) \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (A \star \psi_{\alpha})^* (A \star \psi_{\alpha}) \\ &= \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x |A \star \psi_{\alpha}|^2 \geq 0, \end{aligned} \tag{5.15}$$

where we have used 2-cyclicity, 3-cyclicity, and anti-involutivity with respect to complex conjugation. With a similar computation, we see that

$$(A, B) := \langle A^* \circ B \rangle = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (A \star \psi_{\alpha})^* (B \star \psi_{\alpha}) \tag{5.16}$$

defines a semi-definite sesquilinear form for any given state  $\rho$ . This will be the basis of the derivation of uncertainty relations below because it implies the Cauchy-Schwarz inequality

$$|(A, B)|^2 \leq (A, A) (B, B). \tag{5.17}$$

Using 3-cyclicity, the expectation value (5.12) of a single operator (no compositions) can be rewritten in terms of a *state function*  $S_{\rho} = \sum_{\alpha=1}^n \lambda_{\alpha} \psi_{\alpha} \star \psi_{\alpha}^*$  as

$$\langle A \rangle = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x (A \star \psi_{\alpha}) \star \psi_{\alpha}^* = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x A \star (\psi_{\alpha} \star \psi_{\alpha}^*) = \int_{\mathcal{M}} d^{2d}x A S_{\rho}. \tag{5.18}$$

The state function  $S_{\rho}$  is a real function on phase space that is normalized as

$$\langle 1 \rangle = \int_{\mathcal{M}} d^{2d}x S_{\rho} = \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x |\psi_{\alpha}|^2 = 1 \tag{5.19}$$

but is not necessarily non-negative everywhere. It plays the role of a quasi-probability distribution function, like the Wigner function in the associative case. However, unlike the associative case, we cannot formulate the theory entirely in terms of the state function  $S_{\rho}$ , but rather we also need to frequently refer to the phase space wave functions  $\psi_{\alpha}$ .

A function (“operator”)  $A$  can have eigenfunctions  $f$  (with respect to  $\star$ -multiplication) with eigenvalues  $\lambda \in \mathbb{C}$ :  $A \star f = \lambda f$ . Complex conjugation implies  $f^* \star A^* = \lambda^* f^*$ . We can show that real functions  $A = A^*$  have real eigenvalues, but this fact is not quite as straightforward as in the associative case. We have

$$f^* \star (A \star f) - (f^* \star A) \star f = (\lambda - \lambda^*) (f^* \star f). \tag{5.20}$$

The left-hand side of this equation is non-zero in general but it vanishes after integrating over phase space and using 3-cyclicity. We obtain

$$(\lambda - \lambda^*) \int_{\mathcal{M}} d^{2d}x f^* \star f = (\lambda - \lambda^*) \int_{\mathcal{M}} d^{2d}x |f|^2 = 0. \tag{5.21}$$

The integral is non-zero unless  $f$  is identically equal to zero and therefore  $\lambda = \lambda^*$  as desired. Using similar manipulations, we can show that eigenfunctions are orthogonal if they correspond to distinct eigenvalues. In the nonassociative case, we need to distinguish eigen-state functions and eigen-wave functions (unless we integrate and use 3-cyclicity):  $A \star \psi = \lambda \psi$  does not necessarily imply  $A \star S_{\rho} = \lambda S_{\rho}$ , where  $\rho = \psi \bar{\circ} \psi^*$  and  $S_{\rho} = \psi \star \psi^*$ , because  $(A \star \psi) \star \psi^* \neq A \star (\psi \star \psi^*)$  in general.

All definitions here and in the following are consistent with the associative limit of phase space quantum mechanics. The nonassociative case is more restrictive and in a way it teaches us also something about ordinary phase space quantum mechanics. We have attempted to keep all definitions as general as possible. Depending on the intended application, further restrictions may be necessary; for example, it is natural to require states to be symmetric:  $\rho = \rho'$ , where  $\rho$  is as given in (5.10) and  $\rho' = \sum_{\alpha=1}^n \lambda_{\alpha} \psi_{\alpha}^* \circ \psi_{\alpha}$ .

**C. Uncertainty relations, area and volume operators**

A pair of operators that do not commute cannot have a complete set of common eigenstates; a pair of operators with a central non-zero commutator do not have any simultaneous eigenstates. These well-known facts of quantum mechanics are important for measurements and can also be verified for nonassociative phase space quantum mechanics. A new feature is that analogous statements hold for any triple of operators that do not associate. Let us illustrate this for phase space coordinate functions  $x^I \in \{x^1, \dots, x^d, p_1, \dots, p_d\}$  with commutator and associator

$$x^I \star x^J - x^J \star x^I = i \hbar \Theta^{IJ}, \quad (x^I \star x^J) \star x^K - x^I \star (x^J \star x^K) = \frac{\hbar^2}{2} R^{IJK}, \quad (5.22)$$

respectively, where  $R^{IJK} := \partial_K \Theta^{IJ}$  is constant and non-zero (and then equal to  $R^{ijk}$ ) only for (selected) configuration space coordinates, c.f. (3.34) and (3.27). Let us assume that a pair of phase space coordinates  $x^I$  and  $x^J$  with  $I \neq J$  have a common (normalized) eigen-state function  $S$ :  $x^I \star S = \lambda^I S$  and  $x^J \star S = \lambda^J S$ . Using 3-cyclicity, this implies

$$\langle [x^I, x^J]_{\star} \rangle = \int_{\mathcal{M}} d^{2d}x (x^I \star (x^J \star S) - x^J \star (x^I \star S)) = \lambda^I \lambda^J - \lambda^J \lambda^I = 0, \quad (5.23)$$

and hence  $x^I$  and  $x^J$  with  $\langle \Theta^{IJ} \rangle \neq 0$  cannot have a common eigen-state function  $S$ . Let us now assume that a triple of phase space coordinates  $x^I, x^J$ , and  $x^K$  have a common eigen-state function  $S$  with eigenvalues  $\lambda^I, \lambda^J$ , and  $\lambda^K$ . Using 3-cyclicity repeatedly, we find

$$\begin{aligned} \int_{\mathcal{M}} d^{2d}x ((x^I \star x^J) \star x^K) \star S &= \int_{\mathcal{M}} d^{2d}x (x^I \star x^J) \star (x^K \star S) \\ &= \lambda^K \int_{\mathcal{M}} d^{2d}x (x^I \star x^J) \star S \\ &= \lambda^K \int_{\mathcal{M}} d^{2d}x x^I \star (x^J \star S) = \lambda^K \lambda^J \lambda^I, \end{aligned} \quad (5.24)$$

while using 2-cyclicity, 3-cyclicity, and the fact that  $\lambda^I$  must be real we, find similarly

$$\int_{\mathcal{M}} d^{2d}x (x^I \star (x^J \star x^K)) \star S = \int_{\mathcal{M}} d^{2d}x (x^J \star x^K) \star (S \star x^I) = \lambda^I \lambda^K \lambda^J. \quad (5.25)$$

Taking the difference of the two expressions implies

$$\frac{\hbar^2}{2} R^{IJK} = \lambda^K \lambda^J \lambda^I - \lambda^I \lambda^K \lambda^J = 0, \quad (5.26)$$

and we arrive at the striking result that coordinates  $x^i, x^j$ , and  $x^k$  which do not associate, i.e., for which  $R^{ijk} \neq 0$ , cannot have a common eigen-state function  $S$ ; hence they cannot be measured simultaneously with arbitrary precision. This is a clear sign of a coarse-graining (quantization) of space in the presence of  $R$ -flux.

Let us now turn to the study of uncertainties. In the definition of the uncertainties, we *a priori* face the problem of having to make a choice between using expectation values based on phase space wave functions (with the advantage of the availability of inequalities) and state functions (with computational advantages). In the computation of uncertainties for phase space coordinates, this luckily does not play a role because  $x^I \circ x^I = x^I \star x^I$  and thus

$$0 \leq \sum_{\alpha=1}^n \lambda_{\alpha} \int_{\mathcal{M}} d^{2d}x \psi_{\alpha}^* \star (x^I \star (x^I \star \psi_{\alpha})) = \int_{\mathcal{M}} d^{2d}x (x^I \star x^I) \star S_{\rho} = \langle (x^I)^{\star 2} \rangle. \quad (5.27)$$



Without ambiguity, we can therefore define the uncertainty as usual in terms of the expectation value of the square of the shifted coordinate  $\tilde{x}^I := x^I - \langle x^I \rangle$  as

$$\Delta x^I := \sqrt{\langle (\tilde{x}^I)^{\star 2} \rangle} = \sqrt{\langle (x^I)^{\star 2} \rangle - \langle x^I \rangle^2}. \tag{5.28}$$

This uncertainty is zero for eigen-state functions ( $x^I \star S = \lambda S$ ) as well as for eigen-wave functions ( $x^I \star \psi = \lambda \psi$ ). The function  $x^I$  is real and the uncertainty can thus be rewritten as

$$(\Delta x^I)^2 = \langle \tilde{x}^I \star \tilde{x}^I \rangle = \langle \tilde{x}^I \circ \tilde{x}^I \rangle = (\tilde{x}^I, \tilde{x}^I). \tag{5.29}$$

Using the Cauchy-Schwarz inequality (5.17) and decomposing into imaginary and real parts, we get

$$(\Delta x^I)^2 (\Delta x^J)^2 \geq |(\tilde{x}^I, \tilde{x}^J)|^2 = \frac{1}{4} |\langle [x^I, x^J]_\circ \rangle|^2 + \frac{1}{4} |\langle \{\tilde{x}^I, \tilde{x}^J\}_\circ \rangle|^2, \tag{5.30}$$

where  $[A, B]_\circ := A \circ B - B \circ A$  and  $\{A, B\}_\circ := A \circ B + B \circ A$ . Ignoring the last term yields a Born-Jordan-Heisenberg-type uncertainty relation

$$\Delta x^I \Delta x^J \geq \frac{1}{2} |\langle [x^I, x^J]_\circ \rangle|. \tag{5.31}$$

To proceed from here, we need to distinguish several cases: Whenever one of the phase space coordinates is a momentum  $p_i$ , nonassociativity does not play a role in the sense that then “ $\circ = \star$ ,” i.e.,  $[p_i, p_j]_\circ = [p_i, p_j]_\star = 0$  and  $[p_i, x^j]_\circ = [p_i, x^j]_\star = -i \hbar \delta_i^j$ , and therefore

$$\Delta p_i \Delta p_j \geq 0, \quad \Delta x^i \Delta p_j \geq \frac{\hbar}{2} \delta_i^j \tag{5.32}$$

as in ordinary quantum mechanics. The non-trivial uncertainty relation for a pair of coordinates  $x^i$  and  $x^j$  is new and requires a more complicated computation. With the help of the associator (3.27), we can express  $[x^i, x^j]_\circ$  in terms of a star commutator and obtain the surprising result

$$\begin{aligned} [x^i, x^j]_\circ \star \psi &:= x^i \star (x^j \star \psi) - x^j \star (x^i \star \psi) \\ &= [x^i, x^j]_\star \star \psi - \hbar^2 R^{ijk} \partial_k \psi \\ &= i \hbar R^{ijk} (p_k \star \psi + i \hbar \partial_k \psi) \\ &= i \hbar R^{ijk} (p_k \psi - \frac{i \hbar}{2} \partial_k \psi + i \hbar \partial_k \psi) = i \hbar R^{ijk} \psi \star p_k, \end{aligned} \tag{5.33}$$

that is, the momentum operator ends up on the “wrong” side of  $\psi$ . Using this result, we obtain an uncertainty relation for position measurements

$$\Delta x^i \Delta x^j \geq \frac{\hbar}{2} |R^{ijk} \langle p_k \rangle'|, \tag{5.34}$$

where  $\langle p_k \rangle'$  is the expectation value of  $p_k$  computed with respect to the *opposite* state  $\rho'$ . Only for symmetric or antisymmetric states, i.e.,  $\rho' = \pm \rho$ , will thus be equal to the standard expectation value  $\langle p_k \rangle$ , and one should consider adding this requirement to the definition of a state. The new uncertainty relation (5.34) features uncertainties for position measurements in directions transverse to momentum, while the usual Heisenberg uncertainty relation relates uncertainties of position and momentum in the same direction.

It is tempting to interpret the left-hand side of (5.34) as an area uncertainty that grows linearly with transverse momentum, but this is misleading: The position uncertainty relation makes a prediction for the average outcome of many identically prepared experiments in which *either*  $x^i$  *or*  $x^j$  is measured. In none of these (Gedanken-)experiments are positions in two different directions measured simultaneously (or one shortly after the other), but this would be required for a genuine area uncertainty. An analogous criticism applies to the superficial interpretation of Heisenberg uncertainty as an uncertainty of phase space areas in ordinary quantum mechanics. To remedy the situation, we shall define an area operator whose expectation value can be computed and interpreted as fundamental area measurement uncertainty (or minimal area). The approach generalizes to higher dimensional objects and we will also derive a fundamental volume measurement uncertainty, which results from the nonassociativity of coordinate functions. (See also Ref. 10 for area and volume operators in an associative setting.)

The oriented area spanned by two segments  $\delta \vec{r}_1$  and  $\delta \vec{r}_2$  in three-dimensional Euclidean space is given by the vector product  $\delta \vec{r}_1 \times \delta \vec{r}_2$ , while the volume spanned by three segments  $\delta \vec{r}_1$ ,  $\delta \vec{r}_2$  and

$\delta\vec{r}_3$  is the triple scalar product  $\delta\vec{r}_1 \cdot (\delta\vec{r}_2 \times \delta\vec{r}_3)$ . This can easily be generalized to higher dimensional parallelepipeds and to embedding spaces of arbitrary dimension. The most convenient description of these areas and (higher dimensional) volumes for our purposes is in terms of antisymmetrized sums of products of components of displacement vectors  $\delta\vec{r}$ . For the sake of generality, we might as well consider displacements in phase space. For the description of quantum uncertainties, we replace all of the displacement vectors by the single displacement vector (of operators)  $\vec{x} = \vec{x} - \langle \vec{x} \rangle$ , and promote commutative pointwise multiplication to the noncommutative and nonassociative star product  $\star$ . Furthermore, we would like to construct observables, i.e., real functions on phase space. Taking all this into consideration, the appropriate area (uncertainty) operator in directions  $x^I, x^J$  is

$$A^{IJ} = \Im([ \tilde{x}^I, \tilde{x}^J ]_\star) = -i (\tilde{x}^I \star \tilde{x}^J - \tilde{x}^J \star \tilde{x}^I) \tag{5.35}$$

and the volume (uncertainty) operator in directions  $x^I, x^J, x^K$  is (c.f. (5.4) and (5.7))

$$V^{IJK} = \Re([ \tilde{x}^I, \tilde{x}^J, \tilde{x}^K ]_{\star\text{-NH}}) = \frac{1}{2} [[ \tilde{x}^I, \tilde{x}^J, \tilde{x}^K ]]_\star. \tag{5.36}$$

The expectation values of these (oriented) area and volume operators are easily computed to be

$$\langle A^{IJ} \rangle = \hbar \langle \Theta^{IJ} \rangle, \quad \langle V^{IJK} \rangle = \frac{3}{2} \hbar^2 R^{IJK}, \tag{5.37}$$

with three interesting special cases

$$\langle A^{x^i, p_j} \rangle = \hbar \delta^i_j, \quad \langle A^{ij} \rangle = \hbar R^{ijk} \langle p_k \rangle, \quad \langle V^{ijk} \rangle = \frac{3}{2} \hbar^2 R^{ijk}. \tag{5.38}$$

The first expression describes phase space cells with area  $\hbar$ . The second expression illustrates an area uncertainty proportional to the magnitude of the transverse momentum. The third expression indicates a minimal resolvable volume of order  $\frac{3}{2} \hbar^2 |R|$  due to nonassociativity-induced position measurement uncertainties (here,  $|R|$  is a generalized determinant of the antisymmetric 3-tensor  $R^{ijk}$ ). Uncertainties similar to (5.38) have appeared previously in Refs. 42 and 15, and here we have provided a concrete and rigorous derivation of them as expectation values of area and volume operators.

### D. Dynamics and transformations

Let us close this section with some remarks on dynamics, and similar state and operator transformations in nonassociative quantum mechanics. Time evolution and other transformations should leave the structure of the theory intact. In particular notions of positivity, normalization of probabilities and reality should be preserved. Observables (i.e., real functions on phase space) should be mapped to observables and (pure) states to (pure) states. As in ordinary quantum mechanics, there are two approaches that fulfill all these requirements. In the nonassociative case, the two approaches are, however, no longer equivalent.

A Schrödinger-type approach focuses on evolution equations for the phase space wave functions. The starting point is the phase space Schrödinger equation

$$i \hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \star \psi, \tag{5.39}$$

which applies to all  $\psi_\alpha$  and  $\psi_\alpha^*$  in the state  $\rho$  (c.f. (5.10)), and where the Hamiltonian  $\mathcal{H}$  is a real function on phase space. Observationally, only the time evolution  $\frac{\partial}{\partial t} \langle A \rangle$  of expectation values is relevant. It can be computed either from the Schrödinger equation (5.39) or equivalently from the time evolution equation for operators and compositions of operators given by

$$\frac{\partial \alpha}{\partial t} = \frac{i}{\hbar} [\mathcal{H}, \alpha]_\circ, \tag{5.40}$$

where  $\alpha = A$  or  $A \circ B$  or  $A \circ B \circ C$ , etc. The  $\circ$ -commutator in (5.40) is a  $\circ$ -derivation

$$[\mathcal{H}, A \circ B]_\circ = [\mathcal{H}, A]_\circ \circ B + A \circ [\mathcal{H}, B]_\circ, \tag{5.41}$$

and thus  $\frac{\partial}{\partial t} (A \circ B) = \frac{\partial A}{\partial t} \circ B + A \circ \frac{\partial B}{\partial t}$ . For stationary states, wave functions simply change by a time-dependent phase and we can study energy eigenvalues  $E$  via the time-independent Schrödinger

equation

$$\mathcal{H} \star \psi = E \psi . \quad (5.42)$$

A Heisenberg-type approach focuses on  $\star$ -commutator based evolution equations for operators given by

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar} [H, A]_{\star} . \quad (5.43)$$

This time evolution equation again fulfills all our requirements. It can be applied to single functions (“operators”) as well as to star products of functions, *but it is not a derivation of  $\star$*  since

$$\begin{aligned} [H, A]_{\star} \star B + A \star [H, B]_{\star} &= (H \star A) \star B - (A \star H) \star B + A \star (H \star B) - A \star (B \star H) \\ &\neq [H, (A \star B)]_{\star} = H \star (A \star B) - (A \star B) \star H . \end{aligned} \quad (5.44)$$

This surprising fact should be seen as an interesting feature of the theory, not as a mistake. We can still compute the time-dependence of any operator that we are interested in but we cannot determine it from the time-dependence of its constituent parts. Similarly to the Schrödinger-type approach, there is an alternative equivalent way to compute the time-dependence of expectation values, in this case by the evolution equation for phase space state functions

$$\frac{\partial S}{\partial t} = \frac{1}{i \hbar} [H, S]_{\star} . \quad (5.45)$$

Stationary state functions  $\star$ -commute with the Hamiltonian function  $H$  and we can study energy eigenvalues  $E$  via

$$H \star S = E S . \quad (5.46)$$

The only major difference between this equation and (5.42) is that  $S$  should be a real function while there is *a priori* no such requirement for  $\psi$ .

We have found two sets of inequivalent but equally consistent transformation equations. Which approach should be used for what is ultimately a question of the physics that we would like to describe. The evolution equations of the Heisenberg-type approach close in the algebra of operators and appear therefore predestined to define active transformations like time evolution (i.e., dynamics), while the Schrödinger-type expressions could then still be useful to describe certain symmetries of the theory. In this section, we have focused on single Hamiltonian dynamics instead of the also very interesting possibility of Nambu multi-Hamiltonian dynamics.<sup>47</sup>

## VI. GENERAL $R$ -FLUXES

In this final section, we briefly discuss some preliminary steps towards extending the analysis of the present paper to more complicated non-geometric  $R$ -flux compactifications. We consider, in particular, the two separate cases in turn where the constant 3-tensor  $R^{ijk}$  is replaced with a general function of the position coordinates  $x \in M$  and where the 2-tensor  $\theta^{\tilde{i}}(p) = R^{ijk} p_k$  is replaced by a general function of the conjugate momenta  $p \in (\mathbb{R}^d)^*$ . The former type of generalisation has been discussed recently in the context of double field theory in Ref. 17, while the latter type of generalisations arise in Refs. 42 and 25.

### A. Nambu-Poisson structures

The extension of our results to non-constant  $R$ -fluxes is closely related to the problem of quantization of generic Nambu-Poisson structures (see, e.g., Refs. 27 and 49 and references therein). A Nambu-Poisson 3-bracket (there is a more general notion of Nambu-Poisson  $p$ -brackets for  $p \geq 2$  which however we do not need in this paper) is a skew-symmetric ternary bracket defined on the space of smooth functions  $C^{\infty}(M)$  on a manifold  $M$ , which generalizes the Poisson 2-bracket and can be expressed in terms of a trivector field  $\Pi \in C^{\infty}(M, \wedge^3 TM)$  as  $\{f, g, h\} = \Pi(df, dg, dh)$ . The

bracket is used to define a Nambu multi-Hamiltonian flow

$$\frac{df}{dt} = X_{H_1, H_2} f := \{f, H_1, H_2\} \quad (6.1)$$

with Nambu Hamiltonian vector field  $X_{H_1, H_2}$  for any two smooth functions  $H_1$  and  $H_2$ . For a Nambu-Poisson structure, one requires that the vector fields  $X_{H_1, H_2}$  act as a derivation on the bracket, so that

$$X_{H_1, H_2} \{f, g, h\} = \{X_{H_1, H_2} f, g, h\} + \{f, X_{H_1, H_2} g, h\} + \{f, g, X_{H_1, H_2} h\}. \quad (6.2)$$

This implies that the linear span of Nambu Hamiltonian vector fields defines a Lie algebra with Lie bracket

$$[X_{f, g}, X_{f', g'}] = X_{X_{f, g} f', g'} + X_{f', X_{f, g} g'}. \quad (6.3)$$

The condition (6.2), when expressed solely in terms of brackets, is known as the fundamental identity.<sup>47</sup> It is the generalization of the Jacobi identity for Poisson brackets, which is a differential condition on a Poisson bivector. For 3-brackets, the fundamental identity is a differential as well as an algebraic condition on the 3-vector field  $\Pi$ . The algebraic condition implies<sup>2</sup> that  $\Pi$  is a decomposable trivector

$$\Pi = X_1 \wedge X_2 \wedge X_3. \quad (6.4)$$

The vector fields  $X_1$ ,  $X_2$ , and  $X_3$  are linearly independent (unless  $\Pi = 0$ ) and in view of (6.3) they define an involutive distribution. This implies that the local as well as the global Frobenius theorem applies and in particular that around each point of the manifold  $M$  there exists a coordinate chart  $(U; x^1, x^2, x^3)$  such that

$$\Pi = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}. \quad (6.5)$$

This expression can be multiplied with a scalar pre-factor (e.g., a constant) without spoiling the properties of a Nambu-Poisson structure.

The central object of interest of this paper is a constant trivector  $R$ -flux  $R = \frac{1}{3!} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$ . For appropriately chosen coordinates, the decomposition (6.5) implies that a Nambu-Poisson tensor  $\Pi$  is in fact such a constant trivector, at least locally, and most of the results of the present paper thus apply, including the formalism of twist deformation quantization. Conversely, if  $R$  extends locally in a three-dimensional submanifold of  $M$ , then  $R$  is a Nambu-Poisson tensor. The parts of this paper dealing with integration apply to the particular class of Nambu-Poisson structures for which (6.5) holds globally.

## B. Non-parabolic monodromies

When a three-torus  $\mathbb{T}^3$  in the  $Q$ -flux duality frame is viewed as a  $\mathbb{T}^2$ -fibration over  $S^1$  as discussed in Sec. I, a periodic translation along the base must act on the local complex structure modulus  $\tau$  of a fibre  $\mathbb{T}^2$  as an  $SL(2, \mathbb{Z})$  Möbius transformation, in order to end up with an automorphic fibre. These transformations define the monodromy properties of the fibration and fall into conjugacy classes of  $SL(2, \mathbb{Z})$ ;<sup>37</sup> the case of trivial monodromies corresponds to geometric spaces (manifolds), while non-trivial classes correspond to non-geometric spaces (T-folds). Parabolic monodromies are of infinite order and act as discrete shifts  $\tau \mapsto \tau + n$ , where  $n \in \mathbb{Z}$ . As discussed in Sec. I, under T-duality, the T-fold is mapped to the parabolic  $R$ -flux backgrounds characterized by the phase space relations (1.1); this algebra provides one of the simplest examples of non-associativity and may be regarded as the analog of the Moyal-Weyl background that arises in open string theory with a constant  $B$ -field (see, e.g., Ref. 51). The case of elliptic monodromies, which is of finite order and acts as  $\mathbb{Z}_N$ -transformations on the  $\mathbb{T}^2$  coordinates, was also considered in Refs. 42 and 25 where it was shown that the position coordinate commutator in (1.1) is generalised to a particular non-linear function  $\vartheta^{ij}(p)$  of momentum and  $R$ -flux. We briefly describe here how to extend the setting of Sec. III to allow for twist deformations that corresponds to a class of quasi-Poisson structures  $\Theta_e$  which are generic functions of momentum.

The class of generalizations of (3.34) that we are interested in is obtained by substituting (3.11) with a generic function of momentum  $\vartheta^{ij}(\bar{p})$  to get a bivector  $\Theta_e = \frac{1}{2} \Theta_e^{IJ} \partial_I \wedge \partial_J$  on phase space  $\mathcal{M}$  given by

$$\Theta_e = (\Theta_e^{IJ}) = \begin{pmatrix} \vartheta^{ij}(p) & \delta^i_j \\ -\delta^j_i & 0 \end{pmatrix}. \tag{6.6}$$

As in the case of the parabolic  $R$ -flux model,<sup>46</sup> a computation of the Schouten-Nijenhuis bracket of this bivector with itself reveals that it defines an  $H$ -twisted Poisson structure on  $\mathcal{M}$ , where  $H$  is the 3-form

$$H = \frac{1}{6} \tilde{\delta}^i \vartheta^{jk}(p) dp_i \wedge dp_j \wedge dp_k \tag{6.7}$$

which is the curvature of a twisting  $U(1)$  gerbe on momentum space. The corresponding Jacobiator  $J = \wedge^3 \Theta_e^\sharp(H)$  is the 3-vector whose only non-vanishing components are given by

$$J^{ijk}(p) = \frac{1}{3} (\tilde{\delta}^i \vartheta^{jk}(p) + \tilde{\delta}^j \vartheta^{ki}(p) + \tilde{\delta}^k \vartheta^{ij}(p)). \tag{6.8}$$

Kontsevich’s deformation quantization of a generic (quasi-)Poisson structure is *a priori* quite involved as the number of weights that has to be calculated at each order of the diagrammatic expansion of the star product increases geometrically. A nonassociative star product up to third order in a derivative expansion of a generic  $B$ -field was calculated in Ref. 34 by using a twisted Poisson sigma-model to determine the weights of Kontsevich graphs; from the topological sigma-model formalism, the Kontsevich formula inherits an invariance under the involution which interchanges functions and maps  $\Theta_e \mapsto -\Theta_e$ . By applying the open/closed string duality argument of Ref. 46, we can transport their results to our closed string case and quantize the  $R$ -flux background via the star product

$$\begin{aligned} f \star g &= [f \bar{\star} g]_{\bar{p} \rightarrow p} - \frac{\hbar^2}{12} \tilde{\delta}^k \vartheta^{ij} ([\partial_k \partial_i f \bar{\star} \partial_j g]_{\bar{p} \rightarrow p} + [\partial_j f \bar{\star} \partial_k \partial_i g]_{\bar{p} \rightarrow p}) \\ &\quad - \frac{i \hbar^3}{48} \tilde{\delta}^l \tilde{\delta}^k \vartheta^{ij} ([\partial_l \partial_k \partial_i f \bar{\star} \partial_j g]_{\bar{p} \rightarrow p} - [\partial_j f \bar{\star} \partial_l \partial_k \partial_i g]_{\bar{p} \rightarrow p}) \\ &\quad + \frac{\hbar^4}{288} (\tilde{\delta}^l \vartheta^{mn}) (\tilde{\delta}^k \vartheta^{ij}) ([\partial_l \partial_m \partial_k \partial_i f \bar{\star} \partial_n \partial_j g]_{\bar{p} \rightarrow p} \\ &\quad \quad \quad + 2 [\partial_l \partial_m \partial_j f \bar{\star} \partial_n \partial_k \partial_i g]_{\bar{p} \rightarrow p} + [\partial_n \partial_j f \bar{\star} \partial_l \partial_m \partial_k \partial_i g]_{\bar{p} \rightarrow p}) \\ &\quad + \mathcal{O}(\tilde{\delta}^3 \vartheta, (\tilde{\delta} \vartheta)^3) \end{aligned} \tag{6.9}$$

for  $f, g \in C^\infty(\mathcal{M})$ , where as before the operation  $[-]_{\bar{p} \rightarrow p}$  denotes the change from constant to dynamical momentum and

$$f \bar{\star} g = \mu_A \left( \exp \left[ \frac{i \hbar}{2} (\vartheta^{ij}(\bar{p}) \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\delta}^i - \tilde{\delta}^i \otimes \partial_i) \right] (f \otimes g) \right) \tag{6.10}$$

is an associative Moyal-Weyl type product on  $C^\infty(\mathcal{M})$ . One can check that this star product is nonassociative and that it reduces to the star product (3.26) in the parabolic case  $\vartheta^{ij}(p) = \theta^{ij}(p) = R^{ijk} p_k$ , by antisymmetry of the  $R$ -flux components. In particular, by substituting  $f$  and  $g$  with phase space coordinates, we find the quantum phase space relations

$$[x^i, x^j]_\star = i \hbar \vartheta^{ij}(p), \quad [x^i, p_j]_\star = i \hbar \delta^i_j, \quad [p_i, p_j]_\star = 0, \tag{6.11}$$

while the quantized Jacobiator is

$$[[x^i, x^j, x^k]]_\star = 3 \hbar^2 J^{ijk}(p). \tag{6.12}$$

We may now construct the pertinent Hopf algebra of symmetries of the closed string background. Consider the non-abelian Lie algebra  $\mathfrak{g}_e$  generated by  $\tilde{P}^i$  and  $P_i^{(f)} := f(p) P_i$ , where  $f(p) \in C^\infty(\mathbb{R}^d)$ , with the only non-trivial commutation relations given by

$$[\tilde{P}^i, P_j^{(f)}] = P_j^{(\tilde{\delta}^i f)}. \tag{6.13}$$

The generators of  $\mathfrak{g}_e$  are realised on phase space  $\mathcal{M}$  by the action (3.1) on  $C^\infty(\mathcal{M})$ ; they, respectively, generate momentum translations and position translations together with a momentum-dependent scaling by  $f(p)$ . In particular, this infinite-dimensional Lie algebra contains the Lie subalgebra of translations and Bopp shifts in phase space  $\mathfrak{g}$  that we used in Sec. III. The pertinent Hopf algebra  $H_e$  is the universal enveloping algebra  $U(\mathfrak{g}_e)$  equipped with the coalgebra structure (2.7).

A suitable (but not unique) twist  $\mathcal{F}_e \in H_e[[\hbar]] \otimes H_e[[\hbar]]$  that reproduces the star product (6.9) is given by

$$\begin{aligned} \mathcal{F}_e = & [\bar{\mathcal{F}}_e]_{\bar{p} \rightarrow p} - \frac{\hbar^2}{12} \bar{\delta}^k \vartheta^{ij}(p) [\bar{\mathcal{F}}_e]_{\bar{p} \rightarrow p} (P_k P_i \otimes P_j + P_j \otimes P_k P_i) \\ & - \frac{i\hbar^3}{48} \bar{\delta}^l \bar{\delta}^k \vartheta^{ij}(p) [\bar{\mathcal{F}}_e]_{\bar{p} \rightarrow p} (P_l P_k P_i \otimes P_j - P_j \otimes P_l P_k P_i) \\ & + \frac{\hbar^4}{288} (\bar{\delta}^l \vartheta^{mn}(p)) (\bar{\delta}^k \vartheta^{ij}(p)) [\bar{\mathcal{F}}_e]_{\bar{p} \rightarrow p} (P_l P_m P_k P_i \otimes P_n P_j \\ & \quad + 2 P_l P_m P_j \otimes P_n P_k P_i + P_n P_j \otimes P_l P_m P_k P_i) \\ & + \mathcal{O}(\bar{\delta}^3 \vartheta, (\bar{\delta} \vartheta)^3), \end{aligned} \tag{6.14}$$

where

$$\bar{\mathcal{F}}_e = \exp\left[-\frac{i\hbar}{2} (\vartheta^{ij}(\bar{p}) P_i \otimes P_j + P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_j)\right]. \tag{6.15}$$

The expression (6.14) defines a 2-cochain on  $H_e[[\hbar]]$  with coboundary  $\phi_e = \partial^* \mathcal{F}_e$  given by

$$\begin{aligned} \phi_e = & 1 \otimes 1 \otimes 1 + \frac{\hbar^2}{2} J^{ijk}(p) [\bar{\mathcal{F}}_{12} \bar{\mathcal{F}}_{23}]_{\bar{p} \rightarrow p} (P_i \otimes P_j \otimes P_k) \\ & + \frac{i\hbar^3}{8} \bar{\delta}^l J^{ijk}(p) [\bar{\mathcal{F}}_{12} \bar{\mathcal{F}}_{23}]_{\bar{p} \rightarrow p} (P_l P_i \otimes P_j \otimes P_k - P_j \otimes P_k \otimes P_l P_i) \\ & + \frac{\hbar^4}{8} (\bar{\delta}^l \vartheta^{mn}(p)) J^{ijk}(p) [\bar{\mathcal{F}}_{12} \bar{\mathcal{F}}_{23}]_{\bar{p} \rightarrow p} \\ & \quad \times (P_l P_m P_i \otimes P_k \otimes P_n P_j + P_l P_m P_i \otimes P_n P_k \otimes P_j + P_i P_n \otimes P_l P_m P_k \otimes P_j \\ & \quad + P_i \otimes P_l P_m P_k \otimes P_n P_j + 2 P_l P_i \otimes P_m P_k \otimes P_n P_j + 2 P_n P_i \otimes P_m P_k \otimes P_l P_j \\ & \quad + P_i \otimes P_n P_k \otimes P_l P_m P_j + P_i P_n \otimes P_k \otimes P_l P_m P_j) \\ & + \mathcal{O}(\bar{\delta}^3 \vartheta, (\bar{\delta} \vartheta)^3). \end{aligned} \tag{6.16}$$

It is straightforward to check that the expression (6.16) satisfies the cocycle condition (2.20) order by order in  $\hbar$ , and hence yields the counital associator 3-cocycle for the quasi-Hopf algebra obtained from twisting  $H_e$  by  $\mathcal{F}_e$ . Note that each term in (6.16) involves the classical Jacobiator (6.8).

The generality of this setting now allows for deformation quantization of the geometry of elliptic  $R$ -flux backgrounds up to third order in the  $R$ -flux. However, in this case, cyclicity of the nonassociative star product (6.9) is a more delicate issue and requires a more sophisticated definition of integration on non-parabolic  $R$ -spaces; see Ref. 34 for a detailed analysis of this problem in the context of open string theory and Ref. 17 for an investigation in the context of double field theory.

### ACKNOWLEDGMENTS

We thank P. Aschieri, R. Blumenhagen, A. Chatzistavrakidis, C.-S. Chu, J. Figueroa-O’Farrill, O. Lechtenfeld, F. Lizzi, D. Lüst, R. Nest, and C. Sämann for helpful discussions. A preliminary version of this paper was presented by D.M. at the Workshop on “Noncommutative Field Theory and Gravity” which took part from 8–15 September 2013 as part of the Corfu Summer Institute on Elementary Particle Physics and Gravity. The work of D.M. is supported by the Greek National Scholarship Foundation. The work of D.M. and R.J.S. was supported in part by the Consolidated Grant ST/J000310/1 from the UK Science and Technology Facilities Council. P.S. thanks the Deutsche Forschungsgemeinschaft for support within the RTG 1620 “Models of Gravity”, and P.-M. Ho and the CTS at NTU, Taipei, for helpful discussions, hospitality, and support.

<sup>1</sup> G. Aldazabal, D. Marqués, and C. Núñez, “Double field theory: A pedagogical review,” *Classical Quantum Gravity* **30**, 163001 (2013); e-print [arXiv:1305.1907](https://arxiv.org/abs/1305.1907) [hep-th].

<sup>2</sup> D. Alekseevsky and P. Guha, “On decomposability of Nambu-Poisson tensor,” *Acta Math. Univ. Comenianae* **65**, 1–9 (1996).

- <sup>3</sup> D. Andriot and A. Betz, “ $\beta$ -supergravity: A ten-dimensional theory with non-geometric fluxes and its geometric framework,” *J. High Energy Phys.* **1312**, 083 (2013); e-print [arXiv:1306.4381](#) [hep-th].
- <sup>4</sup> D. Andriot, M. Larfors, D. Lüüst, and P. Patalong, “(Non)commutative closed string on T-dual toroidal backgrounds,” *J. High Energy Phys.* **1306**, 021 (2013); e-print [arXiv:1211.6437](#) [hep-th].
- <sup>5</sup> D. Andriot, O. Hohm, M. Larfors, D. Lüüst, and P. Patalong, “Non-geometric fluxes in supergravity and double field theory,” *Fortsch. Phys.* **60**, 1150–1186 (2012); e-print [arXiv:1204.1979](#) [hep-th].
- <sup>6</sup> P. Aschieri and L. Castellani, “Noncommutative  $D = 4$  gravity coupled to fermions,” *J. High Energy Phys.* **0906**, 086 (2009); e-print [arXiv:0902.3817](#) [hep-th].
- <sup>7</sup> P. Aschieri and A. Schenkel, “Noncommutative connections on bimodules and Drinfel’d twist deformation,” *Adv. Theor. Math. Phys.* **18**, 513–612 (2014); e-print [arXiv:1210.0241](#) [math.QA].
- <sup>8</sup> P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml, and J. Wess, “Twisted gauge theories,” *Lett. Math. Phys.* **78**, 61–71 (2006); e-print [arXiv:hep-th/0603024](#).
- <sup>9</sup> P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, and J. Wess, “A gravity theory on noncommutative spaces,” *Class. Quant. Grav.* **22**, 3511–3532 (2005); e-print [arXiv:hep-th/0504183](#).
- <sup>10</sup> D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, “Quantum geometry on quantum spacetime: Distance, area and volume operators,” *Commun. Math. Phys.* **308**, 567–589 (2011); e-print [arXiv:1005.2130](#) [hep-th].
- <sup>11</sup> I. Bakas and D. Lüüst, “3-cocycles, nonassociative star products and the magnetic paradigm of  $R$ -flux string vacua,” *JHEP* **1401**, 171 (2014); e-print [arXiv:1309.3172](#) [hep-th].
- <sup>12</sup> E. Beggs and S. Majid, “Quantization by cochain twists and nonassociative differentials,” *J. Math. Phys.* **51**, 053522 (2010); e-print [arXiv:math.QA/0506450](#).
- <sup>13</sup> E. Beggs and S. Majid, “Nonassociative Riemannian geometry by twisting,” *J. Phys.: Conf. Ser.* **254**, 012002 (2010); e-print [arXiv:0912.1553](#)[math.QA].
- <sup>14</sup> D. S. Berman and D. C. Thompson, “Duality symmetric string and M-theory,” e-print [arXiv:1306.2643](#) [hep-th].
- <sup>15</sup> R. Blumenhagen and E. Plauschinn, “Nonassociative gravity in string theory?,” *J. Phys. A* **44**, 015401 (2011); e-print [arXiv:1010.1263](#)[hep-th].
- <sup>16</sup> R. Blumenhagen, A. Deser, E. Plauschinn, and F. Rennecke, “Non-geometric strings, symplectic gravity and differential geometry of Lie algebroids,” *J. High Energy Phys.* **1302**, 122 (2013); e-print [arXiv:1211.0030](#) [hep-th].
- <sup>17</sup> R. Blumenhagen, M. Fuchs, F. Hassler, D. Lüüst, and R. Sun, “Nonassociative deformations of geometry in double field theory,” *J. High Energy Phys.* **1404**, 141 (2014); e-print [arXiv:1312.0719](#) [hep-th].
- <sup>18</sup> R. Blumenhagen, B. Körs, D. Lüüst, and S. Stieberger, “Four-dimensional string compactifications with D-branes, orientifolds and fluxes,” *Phys. Rept.* **445**, 1–193 (2007); e-print [arXiv:hep-th/0610327](#).
- <sup>19</sup> R. Blumenhagen, A. Deser, D. Lüüst, E. Plauschinn, and F. Rennecke, “Non-geometric fluxes, asymmetric strings and non-associative geometry,” *J. Phys. A* **44**, 385401 (2011); e-print [arXiv:1106.0316](#) [hep-th].
- <sup>20</sup> R. Blumenhagen, A. Deser, E. Plauschinn, F. Rennecke, and C. Schmid, “The intriguing structure of non-geometric frames in string theory,” *Fortsch. Phys.* **61**, 893–925 (2013); e-print [arXiv:1304.2784](#) [hep-th].
- <sup>21</sup> P. Bouwknegt, K. Hannabuss, and V. Mathai, “Nonassociative tori and applications to T-duality,” *Commun. Math. Phys.* **264**, 41–69 (2006); e-print [arXiv:hep-th/0412092](#).
- <sup>22</sup> L. Castellani, “Higher form gauge fields and their nonassociative symmetry algebras,” *J. High Energy Phys.* **1409**, 055 (2014); e-print [arXiv:1310.7185](#) [hep-th].
- <sup>23</sup> A. S. Cattaneo, G. Felder, and L. Tomassini, “From local to global deformation quantization of Poisson manifolds,” *Duke Math. J.* **115**, 329–352 (2002); e-print [arXiv:math.QA/0012228](#).
- <sup>24</sup> A. Chatzistavarakidis and L. Jonke, “Matrix theory origins of non-geometric fluxes,” *J. High Energy Phys.* **1302**, 040 (2013); e-print [arXiv:1207.6412](#) [hep-th].
- <sup>25</sup> C. Condeescu, I. Florakis, and D. Lüüst, “Asymmetric orbifolds, non-geometric fluxes and noncommutativity in closed string theory,” *J. High Energy Phys.* **1204**, 121 (2012); e-print [arXiv:1202.6366](#) [hep-th].
- <sup>26</sup> C. Condeescu, I. Florakis, C. Kounnas, and D. Lüüst, “Gauged supergravities and non-geometric  $Q/R$ -fluxes from asymmetric orbifold CFT’s,” *J. High Energy Phys.* **1310**, 057 (2013); e-print [arXiv:1307.0999](#) [hep-th].
- <sup>27</sup> J. DeBellis, C. Saemann, and R. J. Szabo, “Quantized Nambu-Poisson manifolds and  $n$ -Lie algebras,” *J. Math. Phys.* **51**, 122303 (2010); e-print [arXiv:1001.3275](#) [hep-th].
- <sup>28</sup> M. R. Douglas and S. Kachru, “Flux compactification,” *Rev. Mod. Phys.* **79**, 733–796 (2007); e-print [arXiv:hep-th/0610102](#).
- <sup>29</sup> V. G. Drinfeld, “Quasi-Hopf algebras,” *Leningrad Math. J.* **1**, 1419–1457 (1990).
- <sup>30</sup> V. Dzhunushaliev, “Observables and unobservables in a nonassociative quantum theory,” *J. General. Lie Theor. Appl.* **2**, 269–272 (2008); e-print [arXiv:quant-ph/0702263](#).
- <sup>31</sup> M. Graña, “Flux compactifications in string theory: A comprehensive review,” *Phys. Rep.* **423**, 91–158 (2006); e-print [arXiv:hep-th/0509003](#).
- <sup>32</sup> M. Graña, R. Minasian, M. Petrini, and D. Waldram, “T-duality, generalized geometry and non-geometric backgrounds,” *J. High Energy Phys.* **0904**, 075 (2009); e-print [arXiv:0807.4527](#) [hep-th].
- <sup>33</sup> F. Hassler and D. Lüüst, “Noncommutative/nonassociative IIA (IIB) geometries from  $Q$ - and  $R$ -branes and their intersections,” *J. High Energy Phys.* **1307**, 048 (2013); e-print [arXiv:1303.1413](#) [hep-th].
- <sup>34</sup> M. Herbst, A. Kling, and M. Kreuzer, “Cyclicity of nonassociative products on D-branes,” *J. High Energy Phys.* **0403**, 003 (2004); e-print [arXiv:hep-th/0312043](#).
- <sup>35</sup> O. Hohm, D. Lüüst, and B. Zwiebach, “The spacetime of double field theory: Review, remarks, and outlook,” *Fortsch. Phys.* **61**, 926–966 (2013); e-print [arXiv:1309.2977](#) [hep-th].
- <sup>36</sup> C. M. Hull, “A geometry for non-geometric string backgrounds,” *J. High Energy Phys.* **0510**, 065 (2005); e-print [arXiv:hep-th/0406102](#).
- <sup>37</sup> C. M. Hull and R. A. Reid-Edwards, “Flux compactifications of string theory on twisted tori,” *Fortsch. Phys.* **57**, 862–894 (2009); e-print [arXiv:hep-th/0503114](#).

- <sup>38</sup> C. M. Hull and B. Zwiebach, “Double field theory,” *J. High Energy Phys.* **0909**, 099 (2009); e-print [arXiv:0904.4664](https://arxiv.org/abs/0904.4664) [hep-th].
- <sup>39</sup> R. Jackiw, “3-cocycle in mathematics and physics,” *Phys. Rev. Lett.* **54**, 159–162 (1985).
- <sup>40</sup> V. Kathotia, “Kontsevich’s universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula,” *Int. J. Math.* **11**, 523–551 (2000); e-print [arXiv:math.QA/9811174](https://arxiv.org/abs/math/9811174).
- <sup>41</sup> M. Kontsevich, “Deformation quantization of Poisson manifolds,” *Lett. Math. Phys.* **66**, 157–216 (2003); e-print [arXiv:q-alg/9709040](https://arxiv.org/abs/q-alg/9709040).
- <sup>42</sup> D. Lüst, “T-duality and closed string noncommutative (doubled) geometry,” *J. High Energy Phys.* **1012**, 084 (2010); e-print [arXiv:1010.1361](https://arxiv.org/abs/1010.1361) [hep-th].
- <sup>43</sup> S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, 1995).
- <sup>44</sup> S. Majid, “Gauge theory on nonassociative spaces,” *J. Math. Phys.* **46**, 103519 (2005); e-print [arXiv:math.QA/0506453](https://arxiv.org/abs/math/0506453).
- <sup>45</sup> J. E. Moyal, “Quantum mechanics as a statistical theory,” *Proc. Cambridge Philos. Soc.* **45**, 99–124 (1949).
- <sup>46</sup> D. Mylonas, P. Schupp, and R. J. Szabo, “Membrane sigma-models and quantization of non-geometric flux backgrounds,” *J. High Energy Phys.* **1209**, 012 (2012); e-print [arXiv:1207.0926](https://arxiv.org/abs/1207.0926) [hep-th].
- <sup>47</sup> Y. Nambu, “Generalized Hamiltonian dynamics,” *Phys. Rev. D* **7**, 2405–2412 (1973).
- <sup>48</sup> E. Plauschinn, “T-duality revisited,” *J. High Energy Phys.* **1401**, 131 (2014); e-print [arXiv:1310.4194](https://arxiv.org/abs/1310.4194) [hep-th].
- <sup>49</sup> C. Saemann and R. J. Szabo, “Groupoids, loop spaces and quantization of 2-plectic manifolds,” *Rev. Math. Phys.* **25**, 1330005 (2013); e-print [arXiv:1211.0395](https://arxiv.org/abs/1211.0395) [hep-th].
- <sup>50</sup> J. Shelton, W. Taylor, and B. Wecht, “Non-geometric flux compactifications,” *J. High Energy Phys.* **0510**, 085 (2005); e-print [arXiv:hep-th/0508133](https://arxiv.org/abs/hep-th/0508133).
- <sup>51</sup> R. J. Szabo, “Quantum field theory on noncommutative spaces,” *Phys. Rept.* **378**, 207–299 (2003); e-print [arXiv:hep-th/0109162](https://arxiv.org/abs/hep-th/0109162).
- <sup>52</sup> C. K. Zachos, “Deformation quantization: Quantum mechanics lives and works in phase space,” *Int. J. Mod. Phys. A* **17**, 297–316 (2002); e-print [arXiv:hep-th/0110114](https://arxiv.org/abs/hep-th/0110114).
- <sup>53</sup> C. K. Zachos and T. L. Curtright, “Deformation quantization of Nambu mechanics,” *AIP Conf. Proc.* **672**, 183–196 (2003); e-print [arXiv:quant-ph/0302106](https://arxiv.org/abs/quant-ph/0302106).