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Non-homogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: a local existence theorem

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Abstract An initial-boundary value problem for 1-D flow of a compressible viscous heat-conducting micropolar fluid is considered; the fluid is assumed thermodynamically perfect and polytropic. The original problem is transformed into homogeneous one and studied the Faedo-Galerkin method. A local-in-time existence of generalized solution is proved.

Keywords Micropolar fluid · Generalized solution · Weak and strong convergences

Mathematics Subject Classification (2000) 35K55 · 35Q40 · 76N10 · 46E35

1 Introduction

In this paper we consider nonstationary 1-D flow of a compressible viscous and heat-conducting micropolar fluid, being in a thermodynamical sense perfect and polytropic. In [3,4] we considered the problem with homogeneous boundary conditions.

Here we study the case of non-homogeneous boundary conditions for velocity and microrotation ("piston problem", see [6] for classical fluid). Introducing auxiliary functions we transform the original system in a system with homogeneous boundary conditions which we analyse, as in [3], using the Faedo-Galerkin method and get a local-in-time existence of generalized solution. In our proof we use some ideas of S.N. Antontsev, A.V. Kazhykhov, A.V. Monakhov [1] applied to the case of classical fluid and results from [3] as well.

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2 Statement of the problem and the main result

Let ρ, v, ω and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangian description.Then the problem which we consider has the formulation as follows [3]:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - K \frac{\partial}{\partial x}(\rho\theta), \tag{2.2}$$

$$\rho \frac{\partial \omega}{\partial t} = A\left[\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \omega}{\partial x}\right) - \omega\right], \tag{2.3}$$

$$\rho \frac{\partial \theta}{\partial t} = -K\rho^2\theta \frac{\partial v}{\partial x} + \rho^2 \left(\frac{\partial v}{\partial x}\right)^2 + \rho^2 \left(\frac{\partial \omega}{\partial x}\right)^2 + \omega^2 + D\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) \tag{2.4}$$

in $]0, 1[\times]0, T[$, $T > 0$, where K, A and D are positive constants. Equations (2.1)–(2.4) are, respectively, local forms of the conservations laws for the mass, momentum, momentum moment and energy. We take the following non-homogeneous initial and boundary conditions:

$$\rho(x, 0) = \rho_0(x), \tag{2.5}$$

$$v(x, 0) = v_0(x), \tag{2.6}$$

$$\omega(x, 0) = \omega_0(x), \tag{2.7}$$

$$\theta(x, 0) = \theta_0(x), \tag{2.8}$$

$$v(0, t) = \mu_0(t), \quad v(1, t) = \mu_1(t), \tag{2.9}$$

$$\omega(0, t) = \nu_0(t), \quad \omega(1, t) = \nu_1(t), \tag{2.10}$$

$$\frac{\partial \theta}{\partial x}(0, t) = \frac{\partial \theta}{\partial x}(1, t) = 0 \tag{2.11}$$

for $x \in]0, 1[$, $t \in]0, T[$. Here ρ_0, v_0, ω_0, θ_0, μ_0, μ_1, ν_0 and ν_1 are given functions. We assume the compatibility conditions

$$v_0(0) = \mu_0(0), \quad v_0(1) = \mu_1(0), \tag{2.12}$$

$$\omega_0(0) = \nu_0(0), \quad \omega_0(1) = \nu_1(0) \tag{2.13}$$

and the inequalities

$$0 < m \le \rho_0(x) \le M, \quad m \le \theta_0(x) \le M \quad \text{for } x \in]0, 1[, \tag{2.14}$$

where $m, M \in R^+$. We assume also that there exists a constant $\delta > 0$ such that

$$l(t) = \int_0^1 \frac{1}{\rho_0(x)} dx + \int_0^t [\mu_1(\tau) - \mu_0(\tau)]\, d\tau \ge \delta, \quad t \in]0, T[. \tag{2.15}$$

Definition 2.1 A generalized solution of the problem (2.1)–(2.11) in the domain $Q_T =]0,1[\times]0,T[$ is a function

$$(x,t) \to (\rho, v, \omega, \theta)(x,t) \quad (x,t) \in Q_T, \tag{2.16}$$

where

$$\rho \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T),\ \inf_{Q_T} \rho > 0, \tag{2.17}$$

$$v, \omega, \theta \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T) \cap L^2(0,T;H^2(]0,1[)), \tag{2.18}$$

that satisfies equations (2.1)–(2.4) a.e. in Q_T and conditions (2.5)–(2.11) in the sense of traces.

Remark 2.1 From embedding properties (e.g. [2]) one can conclude that from (2.17) and (2.18) it follows:

$$\rho \in L^\infty(0,T;C([0,1])) \cap C\left([0,T], L^2(]0,1[)\right), \tag{2.19}$$

$$v, \omega, \theta \in L^2\left(0,T;C^{(1)}([0,1])\right) \cap C([0,T], H^1(]0,1[)), \tag{2.20}$$

$$v, \omega, \theta \in C\left(\bar{Q}_T\right). \tag{2.21}$$

In the same way as in [3] we can prove that the problem (2.1)–(2.11) has at most one generalized solution in Q_T.

The aim of this paper is to prove the following result.

Theorem 2.1 *Let the functions*

$$\mu_0, \mu_1, \nu_0, \nu_1 \in H^2(]0,T[), \tag{2.22}$$

$$\rho_0, v_0, \omega_0, \theta_0 \in H^1(]0,1[) \tag{2.23}$$

satisfy conditions (2.12)–(2.15)*. Then there exists* T_0*,* $0 < T_0 \le T$*, such that the problem* (2.1)–(2.11) *has a generalized solution in* $Q_0 = Q_{T_0}$*, having the property*

$$\theta > 0 \ \ in\ \bar{Q}_0. \tag{2.24}$$

3 An equivalent setting of the problem (2.1)–(2.11)

Instead of the velocity v and microrotation ω we introduce new functions V and W in order to obtain a problem with the homogeneous boundary conditions.

Writing the equation (2.1) in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial v}{\partial x} \tag{3.1}$$

and using (2.9) we get

$$\int_0^1 \frac{dx}{\rho(x,t)} = l(t), \quad t \in]0, T[, \tag{3.2}$$

where the function l is defined by (2.15). Let be

$$v_1(x,t) = \frac{\mu(t)}{l(t)} \int_0^x \frac{d\xi}{\rho(\xi,t)} + \mu_0(t), \tag{3.3}$$

$$\omega_1(x,t) = \frac{\nu(t)}{l(t)} \int_0^x \frac{d\xi}{\rho(\xi,t)} + \nu_0(t) \text{ on } Q_T, \tag{3.4}$$

where $\mu(t) = \mu_1(t) - \mu_0(t)$ and $\nu(t) = \nu_1(t) - \nu_0(t)$. It is evident that

$$v_1(0,t) = \mu_0(t), \; v_1(1,t) = \mu_1(t), \tag{3.5}$$

$$\omega_1(0,t) = \nu_0(t), \; \omega_1(1,t) = \nu_1(t), \; t \in]0, T[. \tag{3.6}$$

Inserting

$$V(x,t) = v(x,t) - v_1(x,t), \quad W(x,t) = \omega(x,t) - \omega_1(x,t) \tag{3.7}$$

in to (2.1)–(2.4) we get the following equivalent system:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial V}{\partial x} + \frac{\mu}{l}\rho = 0, \tag{3.8}$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial V}{\partial x}\right) - K\frac{\partial}{\partial x}(\rho\theta) - \frac{\partial v_1}{\partial t}, \tag{3.9}$$

$$\rho \frac{\partial W}{\partial t} = A\left[\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial W}{\partial x}\right) - \omega_1 - W\right] - \rho \frac{\partial \omega_1}{\partial t}, \tag{3.10}$$

$$\begin{aligned}\rho \frac{\partial \theta}{\partial t} &= -K\rho^2\theta \frac{\partial V}{\partial x} - K\rho\theta \frac{\mu}{l} + \rho^2 \left(\frac{\partial V}{\partial x}\right)^2 + 2\rho \frac{\partial V}{\partial x}\frac{\mu}{l} + \left(\frac{\mu}{l}\right)^2 \\ &\quad + \rho^2 \left(\frac{\partial W}{\partial x}\right)^2 + 2\rho \frac{\partial W}{\partial x}\frac{\nu}{l} + \left(\frac{\nu}{l}\right)^2 + (W+\omega_1)^2 + D\rho \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right),\end{aligned} \tag{3.11}$$

with the homogeneous boundary conditions

$$V(0,t) = V(1,t) = 0, \; W(0,t) = W(1,t) = 0, \tag{3.12}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \tag{3.13}$$

for $t \in]0, T[$ and initial conditions

$$\rho(x,0) = \rho_0(x), \ V(x,0) = V_0(x), \tag{3.14}$$

$$W(x,0) = W_0(x), \ \theta(x,0) = \theta_0(x) \tag{3.15}$$

for $x \in]0, 1[$, where

$$V_0(x) = v_0(x) - \frac{\mu(0)}{l(0)} \int_0^x \frac{1}{\rho_0(\xi)} d\xi - \mu_0(0), \tag{3.16}$$

$$W_0(x) = \omega_0(x) - \frac{\nu(0)}{l(0)} \int_0^x \frac{1}{\rho_0(\xi)} d\xi - \nu_0(0). \tag{3.17}$$

Notice that because of (2.12), (2.13), (2.22) and (2.23) we have

$$V_0, W_0 \in H_0^1(]0, 1[). \tag{3.18}$$

We intend to prove the following result.

Theorem 3.1 *Under the assumptions of* Theorem 2.1, *there exists* T_0, $0 < T_0 \leq T$, *such that the problem* (3.8)–(3.15) *has a generalized solution* (ρ, V, W, θ) *in* Q_{T_0}. *Moreover,*

$$\theta > 0 \ \ in \ \bar{Q}_{T_0}. \tag{3.19}$$

4 Approximate solutions

A local generalized solution to the problem (3.8)–(3.15) we shall find as a limit of approximated solutions

$$\left(\rho^n, V^n, W^n, \theta^n\right), \quad n \in \mathbf{N}, \tag{4.1}$$

where

$$V^n(x,t) = \sum_{i=1}^{n} V_i^n(t) \sin(\pi i x), \tag{4.2}$$

$$W^n(x,t) = \sum_{j=1}^{n} W_j^n(t) \sin(\pi j x), \tag{4.3}$$

$$\theta^n(x,t) = \sum_{k=0}^{n} \theta_k^n(t) \cos(\pi k x); \tag{4.4}$$

here V_i^n, W_j^n, θ_k^n are unknown functions defined and smooth on an interval $[0, T_n]$, $T_n \le T$. Evidently, the boundary conditions

$$V^n(0,t) = V^n(1,t) = W^n(0,t) = W^n(1,t) = \frac{\partial \theta^n}{\partial x}(0,t) = \frac{\partial \theta^n}{\partial x}(1,t) = 0 \tag{4.5}$$

are satisfied. According to the Faedo-Galerkin method, we take the following conditions:

$$\frac{\partial \rho^n}{\partial t} + \left(\rho^n\right)^2 \frac{\partial V^n}{\partial x} + \frac{\mu}{l}\rho^n = 0, \quad \rho^n(x,0) = \rho_0(x), \tag{4.6}$$

$$\int_0^1 \left[\frac{\partial V^n}{\partial t} - \frac{\partial}{\partial x}\left(\rho^n \frac{\partial V^n}{\partial x}\right) + K\frac{\partial}{\partial x}\left(\rho^n\theta^n\right) + \frac{\partial v_1^n}{\partial t}\right]\sin(\pi i x)dx = 0, \tag{4.7}$$

$$\int_0^1 \left[\frac{\partial W^n}{\partial t} - A\frac{\partial}{\partial x}\left(\rho^n \frac{\partial W^n}{\partial x}\right) + A\frac{\omega_1^n}{\rho^n} + A\frac{W^n}{\rho^n} + \frac{\partial \omega_1^n}{\partial t}\right]\sin(\pi j x)dx = 0, \tag{4.8}$$

$$\begin{aligned}\int_0^1 &\left[\frac{\partial \theta^n}{\partial t} + K\rho^n\theta^n\frac{\partial V^n}{\partial x} + K\theta^n\frac{\mu}{l} - \rho^n\left(\frac{\partial V^n}{\partial x}\right)^2 - 2\frac{\partial V^n}{\partial x}\frac{\mu}{l}\right.\\ &-\left(\frac{\mu}{l}\right)^2\frac{1}{\rho^n} - \rho^n\left(\frac{\partial W^n}{\partial x}\right)^2 - 2\frac{\partial W^n}{\partial x}\frac{\nu}{l} - \left(\frac{\nu}{l}\right)^2\frac{1}{\rho^n} - (W^n+\omega_1^n)^2\frac{1}{\rho^n}\\ &\left.-D\frac{\partial}{\partial x}\left(\rho^n\frac{\partial \theta^n}{\partial x}\right)\right]\cos(\pi k x)dx = 0\end{aligned} \tag{4.9}$$

$(i, j = 1, \ldots, n, k = 0, 1, \ldots, n)$ where

$$v_1^n(x,t) = \frac{\mu(t)}{l(t)}\int_0^x \frac{d\xi}{\rho^n(\xi,t)} + \mu_0(t), \tag{4.10}$$

$$\omega_1^n(x,t) = \frac{\nu(t)}{l(t)}\int_0^x \frac{d\xi}{\rho^n(\xi,t)} + \nu_0(t). \tag{4.11}$$

From (4.6) and (4.2) it follows

$$\begin{aligned}\rho^n(x,t) &= \rho_0(x)\exp\left\{-\int_0^t \frac{\mu}{l}d\tau\right\}\left(1+\rho_0(x)\int_0^t \frac{\partial V^n}{\partial x}\exp\left\{-\int_0^\tau \frac{\mu}{l}ds\right\}d\tau\right)^{-1}\\ &= \rho_0(x)\exp\left\{-\int_0^t \frac{\mu}{l}d\tau\right\}\left(1+\rho_0(x)\sum_{i=1}^n(\pi i)\cos(\pi i x)\right.\\ &\qquad\left.\int_0^t V_i^n(\tau)\exp\left\{-\int_0^\tau \frac{\mu}{l}ds\right\}d\tau\right)^{-1}.\end{aligned} \tag{4.12}$$

Taking into account (2.14), (2.15) and (2.22) we conclude that there exists T_n, $0 < T_n \leq T$, such that

$$\rho^n(x,t) > 0 \text{ for } (x,t) \in]0,1[\times]0,T_n[$$

and therefore conditions (4.8)–(4.11) have a sense.

Let V_{0i}, W_{0j}, $(i, j = 1, \ldots, n)$ and $\theta_{0k} (k = 0, \ldots, n)$ be the Fourier coefficients of the functions V_0, W_0, and θ_0, respectively:

$$V_{0i} = 2\int_0^1 V_0(x)\sin(\pi i x)dx, \quad i = 1, \ldots, n$$

$$W_{0i} = 2\int_0^1 W_0(x)\sin(\pi j x)dx, \quad j = 1, \ldots, n$$

$$\theta_{00} = \int_0^1 \theta_0(x)dx, \quad \theta_{0k} = 2\int_0^1 \theta_0(x)\cos(\pi k x)dx, \quad k = 1, \ldots, n;$$

let

$$V_0^n(x) = \sum_{i=1}^n V_{0i}\sin(\pi i x), \tag{4.13}$$

$$W_0^n(x) = \sum_{j=1}^n W_{0j}\sin(\pi j x), \tag{4.14}$$

$$\theta_0^n(x) = \sum_{k=0}^n \theta_{0k}\cos(\pi k x). \tag{4.15}$$

The initial conditions for V^n, W^n and θ^n we take in the form:

$$V^n(x,0) = V_0^n(x), \tag{4.16}$$

$$W^n(x,0) = W_0^n(x), \tag{4.17}$$

$$\theta^n(x,0) = \theta_0^n(x). \tag{4.18}$$

Let

$$z_r^n(t) = \int_0^t V_r^n(\tau)\exp\left\{-\int_0^\tau \frac{\mu}{l}ds\right\}d\tau, \quad r = 1, \ldots, n. \tag{4.19}$$

Inserting (4.2)–(4.4), (4.10)–(4.12) and (4.19) in to (4.7)–(4.9), for $\Big\{(V_i^n, W_j^n, \theta_k^n, z_r^n):$ $i, j, r = 1, \ldots, n, k = 0, \ldots, n\Big\}$ we obtain the following Cauchy problem:

$$\dot{V}_i^n(t) = \phi_i^n\left(t, V_1^n, \ldots, V_n^n, W_1^n, \ldots, W_n^n, \theta_0^n, \theta_1^n, \ldots, \theta_n^n, z_1^n, \ldots, z_n^n\right), \tag{4.20}$$

$$\dot{W}_j^n(t) = \Psi_j^n\left(t, V_1^n, \ldots, V_n^n, W_1^n, \ldots, W_n^n, \theta_0^n, \theta_1^n, \ldots, \theta_n^n, z_1^n, \ldots, z_n^n\right), \tag{4.21}$$

$$\dot{\theta}_k^n(t) = \lambda_k \Pi_k^n(t, V_1^n, \ldots, V_n^n, W_1^n, \ldots, W_n^n, \theta_0^n, \theta_1^n, \ldots, \theta_n^n, z_1^n, \ldots, z_n^n), \tag{4.22}$$

$$\dot{z}_r^n(t) = V_r^n(t) \exp\left\{-\int_0^t \frac{\mu}{l} d\tau\right\}, \tag{4.23}$$

$$V_i^n(0) = V_{0i},\ W_j^n(0) = W_{0j},\ \theta_k^n(0) = \theta_{0k},\ z_r^n(0) = 0, \tag{4.24}$$

where $\lambda_0 = 1, \lambda_k = 2$ for $k = 1, 2, \ldots, n$ and

$$\Phi_i^n = 2\int_0^1 \left[\frac{\partial}{\partial x}\left(\rho^n \frac{\partial V^n}{\partial x}\right) - K\frac{\partial}{\partial x}\left(\rho^n\theta^n\right) - \frac{\partial v_1^n}{\partial t}\right] \sin(\pi i x)\, dx, \tag{4.25}$$

$$\Psi_j^n = 2\int_0^1 \left[A\left(\frac{\partial}{\partial x}\left(\rho^n \frac{\partial W^n}{\partial x}\right) - \frac{\omega_1^n}{\rho^n} - \frac{W^n}{\rho^n}\right) - \frac{\partial \omega_1^n}{\partial t}\right] \sin(\pi j x) dx, \tag{4.26}$$

$$\Pi_k^n = \int_0^1 \left[\frac{\mu^2+\nu^2}{l^2\rho^n} - K\rho^n\theta^n\frac{\partial V^n}{\partial x} - K\theta^n\frac{\mu}{l} + \rho^n\left(\frac{\partial V^n}{\partial x}\right)^2 + 2\frac{\partial V^n}{\partial x}\frac{\mu}{l}\right.$$
$$\left. + \rho^n\left(\frac{\partial W^n}{\partial x}\right)^2 + 2\frac{\partial W^n}{\partial x}\frac{\nu}{l} + \frac{(W^n+\omega_1^n)^2}{\rho^n} + D\frac{\partial}{\partial x}\left(\rho^n\frac{\partial \theta^n}{\partial x}\right)\right]\cos(\pi k x)dx. \tag{4.27}$$

Taking into account (4.6) from (4.10) and (4.11) we find that

$$\frac{\partial v_1^n}{\partial t} = \left[\left(\frac{\mu}{l}\right)' + \left(\frac{\mu}{l}\right)^2\right]\int_0^x \frac{1}{\rho^n} d\xi + \frac{\mu}{l}V^n + \mu_0', \tag{4.28}$$

$$\frac{\partial \omega_1^n}{\partial t} = \left[\left(\frac{\nu}{l}\right)' + \frac{\nu\mu}{l^2}\right]\int_0^x \frac{1}{\rho^n} d\xi + \frac{\nu}{l}V^n + \nu_0'; \tag{4.29}$$

because of the embedding $H^2(]0,T[) \subset C^{(1)}([0,T])$ it holds $\left(\frac{\mu}{l}\right)', (\frac{\nu}{l})', \mu_0', \nu_0' \in C([0,T])$.

The functions on the right-hand side of (4.20)–(4.23) satisfy the conditions of the Cauchy-Picard theorem (e.g. [5]) and one can easily conclude that the following statements are valid.

Lemma 4.1 *For each $n \in N$ there exists $T_n,\ 0 < T_n \leq T$, such that the Cauchy problem (4.20)–(4.24) has a unique solution , defined on $[0, T_n]$. The functions V^n, W^n and θ^n defined by the formulas (4.2)–(4.4) belong to $C^{(1)}(\bar{Q}_n)$, $Q_n =]0,1[\times]0,T_n[$ and satisfy conditions (4.16)–(4.18).*

Lemma 4.2 *There exists T_n, $0 < T_n \leq T$, such that the function ρ_n, defined by (4.12), satisfies the condition*

$$m_1 \leq \rho^n(x,t) \leq M_1 \quad on \quad \bar{Q}_n, \tag{4.30}$$

where

$$m_1 = \frac{m}{2}\exp\left\{-\int_0^T |\frac{\mu}{l}|d\tau\right\}, \quad M_1 = 2M\exp\left\{\int_0^T |\frac{\mu}{l}|d\tau\right\}. \tag{4.31}$$

5 A priori estimates

Our purpose is to find out T_0, $0 < T_0 \leq T$, such that for each $n \in N$ there exists a solution of the problem (4.20)–(4.24), defined on $[0, T_0]$. It will be sufficient to find out uniform (in $n \in N$) a priori estimates for a solution $(\rho^n, V^n, W^n, \theta^n)$ defined through Lemmas 4.1 and 4.2. In that what follows we denote by $C > 0$ or $C_i > 0$ $(i = 1, 2, \ldots)$ a generic constant, not depending on $n \in N$ and having possibly different values at different places. We also use the notation

$$\|f\| = \|f\|_{L^2(]0,1[)}.$$

Lemma 5.1 *The following inequalities hold true:*

$$|v_1^n(x,t)| \leq C, \quad |\omega_1^n(x,t)| \leq C, \quad (x,t) \in \bar{Q}_n, \tag{5.1}$$

$$\|\tfrac{\partial v_1^n}{\partial t}(t)\|^2 \leq C\left(1 + \|V^n(t)\|^2\right), \tag{5.2}$$

$$\|\tfrac{\partial \omega_1^n}{\partial t}(t)\|^2 \leq C\left(1 + \|V^n(t)\|^2\right), \quad t \in [0, T_n]. \tag{5.3}$$

Proof The conclusion follows directly from (4.10)–(4.11) and (4.28)–(4.29). □

Lemma 5.2 *For $t \in [0, T_n]$ it holds the inequality*

$$\|W^n(t)\|^2 + \int_0^t [\|\frac{\partial W^n}{\partial x}(\tau)\|^2 + \|W^n(\tau)\|^2]d\tau \leq C(1 + \int_0^t \|V^n(\tau)\|^2 d\tau). \tag{5.4}$$

Proof Multiplying (4.8) by W_j^n and summing over $j = 1, \ldots, n$, after integration by parts we obtain

$$\frac{1}{2A}\frac{d}{dt}\|W^n(t)\|^2 + \int_0^1 \rho^n(\frac{\partial W^n}{\partial x})^2 dx + \int_0^1 \frac{(W^n)^2}{\rho^n}dx$$
$$= -\int_0^1 \frac{\omega_1^n W^n}{\rho^n}dx - \frac{1}{A}\int_0^1 \frac{\partial \omega_1^n}{\partial t}W^n dx.$$

With the help of the Young inequality and taking into account (5.1), (5.3) and the inequality $\|W_0^n\| \leq \|W_0\|$, after integration over $]0, t[$, $t \leq T_n$ we obtain

$$\|W^n(t)\|^2 + \int_0^t \left[\left\| \frac{\partial W^n}{\partial x}(\tau) \right\|^2 + \|W^n(\tau)\|^2 \right] d\tau \leq C \left(1 + \int_0^t \|V^n(\tau)\|^2 d\tau \right) + \|W_0\|^2$$

and hence (5.4). □

Lemma 5.3 *For $t \in [0, T_n]$ it holds*

$$\left| \int_0^1 \theta^n (x, t)\, dx \right| \leq C \left(1 + \|V^n(t)\|^2 + \int_0^t \|V^n(\tau)\|^2 d\tau \right). \tag{5.5}$$

Proof Multiplying (4.7) by V_i^n and summing over $i = 1, \ldots, n$, after integration by parts and using (4.9) for $k = 0$, we have

$$\frac{d}{dt} y_n(t) + K \frac{\mu}{l} y_n(t) = \frac{K}{2} \frac{\mu}{l} \|V^n\|^2 + \left(\frac{\mu^2}{l^2} + \frac{\nu^2}{l^2} \right) \int_0^1 \frac{1}{\rho^n} dx$$
$$- \int_0^1 \frac{\partial v_1^n}{\partial t} V^n dx + \int_0^1 \rho^n \left(\frac{\partial W^n}{\partial x} \right)^2 dx + \int_0^1 \frac{\left(W^n + \omega_1^n \right)^2}{\rho^n} dx,$$

where

$$y_n(t) = \frac{1}{2} \|V^n(t)\|^2 + \int_0^1 \theta^n(x, t) dx.$$

Applying (2.22), (4.30), (5.1), (5.2) and the Young inequality we obtain

$$\frac{d}{dt} y_n(t) + K \frac{\mu}{l} y_n(t) \leq C \left(1 + \|V^n(t)\|^2 + \left\| \frac{\partial W^n}{\partial x}(t) \right\|^2 + \|W^n(t)\|^2 \right). \tag{5.6}$$

Multiplying (5.6) by $\exp \left\{ K \int_0^t \frac{\mu}{l} d\tau \right\}$ and integrating over $]0, t[$, $t \leq T_n$ and taking into account (5.4) and the inequality

$$y^n(0) = \frac{1}{2} \|V_0^n\|^2 + \int_0^1 \theta_0^n dx \leq \frac{1}{2} \|V_0\|^2 + \|\theta_0\| \leq C,$$

we get

$$y_n(t) \le C\left(1+\int_0^t \|V^n(\tau)\|^2 d\tau\right),$$

i.e.

$$\int_0^1 \theta^n(x,t)dx \le \left(1+\int_0^t \|V^n(\tau)\|^2 d\tau\right) \le C\left(1+\|V^n(t)\|^2+\int_0^t \|V^n(\tau)\|^2 d\tau\right).$$

In the same way we conclude that

$$y_n\,(t) \ge -C\left(1+\int_0^t \|V^n(\tau)\|^2 d\tau\right)$$

or,

$$\int_0^1 \theta^n(x,t)dx \ge -C\left(1+\|V^n(t)\|^2+\int_0^t \|V^n(\tau)\|^2 d\tau\right).$$

□

Lemma 5.4 *For* $(x,t) \in \bar{Q}_n$ *it holds the inequality*

$$|\theta^n\,(x,t)\,| \le C\left(1+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|+\|V^n(t)\|^2+\int_0^t \|V^n(\tau)\|^2 d\tau\right). \tag{5.7}$$

Proof In the same way as in [3, Lemma 5.3] we verify that

$$|\theta^n(x,t)| \le \left\|\frac{\partial\theta^n}{\partial x}(t)\right\|+\left|\int_0^1 \theta^n(x,t)dx\right|,$$

for $(x,t) \in \bar{Q}_n$. Applying (5.5) we obtain (5.7). □

Lemma 5.5 *For* $t \in [0,T_n]$ *it holds the inequality*

$$\left\|\frac{\partial\rho^n}{\partial x}(t)\right\|^2 \le C\left(1+\int_0^t \left\|\frac{\partial^2 V^n}{\partial x^2}(\tau)\right\|^2 d\tau\right). \tag{5.8}$$

Proof The conclusion follows immediately from (4.12). □

Lemma 5.6 *For $t \in [0, T_n]$ it holds the inequality*

$$\frac{d}{dt}\left(\left\|\frac{\partial V^n}{\partial x}(t)\right\|^2+\left\|\frac{\partial W^n}{\partial x}(t)\right\|^2+\left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2\right)$$
$$+\left(\left\|\frac{\partial^2 V^n}{\partial x^2}(t)\right\|^2+\left\|\frac{\partial^2 W^n}{\partial x^2}(t)\right\|^2+\left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2\right)$$
$$\leq C\left(1+\left\|\frac{\partial V^n}{\partial x}(t)\right\|^8+\left\|\frac{\partial W^n}{\partial x}(t)\right\|^8+\left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^8+\left(\int_0^t\left\|\frac{\partial^2 V^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right). \quad (5.9)$$

Proof As in [3], multiplying (4.7), (4.8) and (4.9) respectively by $(\pi i)^2 V_i^n$, $(\pi j)^2 W_j^n$ and $(\pi k)^2 \theta_k^n$ and taking into account (4.2)–(4.4), after summation over $i, j, k = 1, 2, \ldots, n$ and addition of the obtained equations, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\frac{\partial V^n}{\partial x}\|^2+\|\frac{\partial W^n}{\partial x}\|^2+\|\frac{\partial \theta^n}{\partial x}\|^2\right)+\int_0^1 \rho^n\left[\left(\frac{\partial^2 V^n}{\partial x^2}\right)^2+A\left(\frac{\partial^2 W^n}{\partial x^2}\right)^2\right.$$
$$\left.+D\left(\frac{\partial^2 \theta^n}{\partial x^2}\right)^2\right]dx=-\int_0^1\frac{\partial \rho^n}{\partial x}\frac{\partial V^n}{\partial x}\frac{\partial^2 V^n}{\partial x^2}dx-A\int_0^1\frac{\partial \rho^n}{\partial x}\frac{\partial W^n}{\partial x}\frac{\partial^2 W^n}{\partial x^2}dx$$
$$-D\int_0^1\frac{\partial \rho^n}{\partial x}\frac{\partial \theta^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx+K\int_0^1\frac{\partial \rho^n}{\partial x}\theta^n\frac{\partial^2 V^n}{\partial x^2}dx+K\int_0^1\rho^n\frac{\partial \theta^n}{\partial x}\frac{\partial^2 V^n}{\partial x^2}dx$$
$$+\int_0^1\frac{\partial v_1^n}{\partial t}\frac{\partial^2 V^n}{\partial x^2}dx+A\int_0^1\frac{\omega_1^n}{\rho^n}\frac{\partial^2 W^n}{\partial x^2}dx+A\int_0^1\frac{W^n}{\rho^n}\frac{\partial^2 W^n}{\partial x^2}dx+\int_0^1\frac{\partial \omega_1^n}{\partial t}\frac{\partial^2 W^n}{\partial x^2}dx$$
$$+K\int_0^1\rho^n\theta^n\frac{\partial V^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx+K\frac{\mu}{l}\int_0^1\theta^n\frac{\partial^2 \theta^n}{\partial x^2}dx-\int_0^1\rho^n\left(\frac{\partial V^n}{\partial x}\right)^2\frac{\partial^2 \theta^n}{\partial x^2}dx$$
$$-2\frac{\mu}{l}\int_0^1\frac{\partial V^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx-\left(\frac{\mu^2}{l^2}+\frac{\nu^2}{l^2}\right)\int_0^1\frac{1}{\rho^n}\frac{\partial^2 \theta^n}{\partial x^2}dx-\int_0^1\rho^n\left(\frac{\partial W^n}{\partial x}\right)^2\frac{\partial^2 \theta^n}{\partial x^2}dx$$
$$-2\frac{\nu}{l}\int_0^1\frac{\partial W^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}dx-\int_0^1\frac{\left(W^n+\omega_1^n\right)^2}{\rho^n}\frac{\partial^2 \theta^n}{\partial x^2}dx. \quad (5.10)$$

Taking into account (4.30), (5.1)–(5.5), (5.7), (5.8) and the inequalities

$$|f|^2 \leq 2\|f\|\|\frac{\partial f}{\partial x}\|,\ \ |\frac{\partial f}{\partial x}|^2 \leq 2\|\frac{\partial f}{\partial x}\|\|\frac{\partial^2 f}{\partial x^2}\|, \quad (5.11)$$

$$\|f\| \leq 2\|\frac{\partial f}{\partial x}\|, \ \|\frac{\partial f}{\partial x}\| \leq 2\|\frac{\partial^2 f}{\partial x^2}\| \tag{5.12}$$

(for a function f vanishing at $x = 0$ and $x = 1$ or with the first derivative vanishing at the same points) we estimate the terms on the right-hand side of (5.10). For instance,

$$\left|\int_0^1 \frac{\left(W^n + \omega_1^n\right)^2}{\rho^n} \frac{\partial^2 \theta^n}{\partial x^2} dx\right| \leq C \left(\int_0^1 (W^n)^2 |\frac{\partial^2 \theta^n}{\partial x^2}| dx + \int_0^1 |\frac{\partial^2 \theta^n}{\partial x^2}| dx\right)$$
$$\leq C \left(\max_{x\in[0,1]} |W^n(x,t)| \|W^n(t)\| \|\frac{\partial^2 \theta^n}{\partial x^2}(t)\| + \|\frac{\partial^2 \theta^n}{\partial x^2}(t)\|\right)$$
$$\leq C(\|W^n(t)\|^{3/2} \|\frac{\partial W^n}{\partial x}(t)\|^{1/2} \|\frac{\partial^2 \theta^n}{\partial x^2}(t)\| + \|\frac{\partial^2 \theta^n}{\partial x^2}(t)\|);$$

applying the Young inequality, we get

$$\left|\int_0^1 \frac{(W^n + \omega_1^n)^2}{\rho^n} \frac{\partial^2 \theta^n}{\partial x^2} dx\right| \leq \varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\|W^n(t)\|^3 \left\|\frac{\partial W^n}{\partial x}(t)\right\| + \varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C$$
$$\leq 2\,\varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\left(\|W^n(t)\|^4 + \left\|\frac{\partial W^n}{\partial x}(t)\right\|^4 + 1\right)$$
$$\leq 2\,\varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial W^n}{\partial x}(t)\right\|^8 + \left(\int_0^t \left\|\frac{\partial^2 V^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right);$$

where $\varepsilon > 0$ is arbitrary. In an analogous way we obtain the following inequalities:

$$\left|\left(\frac{\mu^2}{l^2} + \frac{\nu^2}{l^2}\right) \int_0^1 \frac{1}{\rho^n} \frac{\partial^2 \theta^n}{\partial x^2} dx\right| \leq \varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C,$$

$$\left|2\frac{\mu}{l} \int_0^1 \frac{\partial W^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx\right| \leq \varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial W^n}{\partial x}(t)\right\|^8\right),$$

$$\left|\int_0^1 \rho^n (\frac{\partial W^n}{\partial x})^2 \frac{\partial^2 \theta^n}{\partial x^2} dx\right| \leq \varepsilon \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + \varepsilon \left\|\frac{\partial^2 W^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial W^n}{\partial x}(t)\right\|^8\right),$$

$$\left|2\frac{\nu}{l}\int\limits_0^1 \frac{\partial V^n}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2\theta^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial V^n}{\partial x}(t)\right\|^8\right),$$

$$\left|\int\limits_0^1 \rho^n(\frac{\partial V^n}{\partial x})^2\frac{\partial^2\theta^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2\theta^n}{\partial x^2}(t)\right\|^2 + \varepsilon\left\|\frac{\partial^2 V^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial V^n}{\partial x}(t)\right\|^8\right),$$

$$\left|K\frac{\mu}{l}\int\limits_0^1 \theta^n\frac{\partial^2\theta^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2\theta^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^8 + \left\|\frac{\partial V^n}{\partial x}(t)\right\|^8 + \left(\int\limits_0^t\left\|\frac{\partial^2 V^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right),$$

$$\left|K\int\limits_0^1 \rho^n\theta^n\frac{\partial V^n}{\partial x}\frac{\partial^2\theta^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2\theta^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^8 + \left\|\frac{\partial V^n}{\partial x}(t)\right\|^8 + \left(\int\limits_0^t\left\|\frac{\partial^2 V^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right),$$

$$\left|\int\limits_0^1 \frac{\partial\omega_1^n}{\partial t}\frac{\partial^2 W^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2 W^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial V^n}{\partial x}(t)\right\|^8\right),$$

$$\left|A\int\limits_0^1 \frac{W^n}{\rho^n}\frac{\partial^2 W^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2 W^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left(\int\limits_0^t\left\|\frac{\partial^2 V^n}{\partial x^2}(\tau)d\tau\right\|^2\right)^4\right),$$

$$\left|A\int\limits_0^1 \frac{\omega_1^n}{\rho^n}\frac{\partial^2 W^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2 W^n}{\partial x^2}(t)\right\|^2 + C,$$

$$\left|\int\limits_0^1 \frac{\partial v_1^n}{\partial t}\frac{\partial^2 V^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2 V^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial V^n}{\partial x}(t)\right\|^8\right),$$

$$\left|K\int\limits_0^1 \rho^n\frac{\partial\theta^n}{\partial x}\frac{\partial^2 V^n}{\partial x^2}dx\right| \le \varepsilon\left\|\frac{\partial^2 V^n}{\partial x^2}(t)\right\|^2 + C\left(1+\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^8\right),$$

$$\left| K \int_0^1 \frac{\partial \rho^n}{\partial x} \theta^n \frac{\partial^2 V^n}{\partial x^2} dx \right| \le \varepsilon \left\| \frac{\partial^2 V^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^8 + \left\| \frac{\partial V^n}{\partial x}(t) \right\|^8 + \left(\int_0^t \left\| \frac{\partial^2 V^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right),$$

$$\left| D \int_0^1 \frac{\partial \rho^n}{\partial x} \frac{\partial \theta^n}{\partial x} \frac{\partial^2 \theta^n}{\partial x^2} dx \right| \le \varepsilon \left\| \frac{\partial^2 \theta^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^8 + \left(\int_0^t \left\| \frac{\partial^2 V^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right),$$

$$\left| A \int_0^1 \frac{\partial \rho^n}{\partial x} \frac{\partial W^n}{\partial x} \frac{\partial^2 W^n}{\partial x^2} dx \right| \le \varepsilon \left\| \frac{\partial^2 W^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial W^n}{\partial x}(t) \right\|^8 + \left(\int_0^t \left\| \frac{\partial^2 V^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right),$$

$$\left| \int_0^1 \frac{\partial \rho^n}{\partial x} \frac{\partial V^n}{\partial x} \frac{\partial^2 V^n}{\partial x^2} dx \right| \le \varepsilon \left\| \frac{\partial^2 V^n}{\partial x^2}(t) \right\|^2 + C \left(1 + \left\| \frac{\partial V^n}{\partial x}(t) \right\|^8 + \left(\int_0^t \left\| \frac{\partial^2 V^n}{\partial x^2}(\tau) \right\|^2 d\tau \right)^4 \right).$$

Application of the above inequalities to (5.10) leads to (5.9). □

Lemma 5.7 *There exists $T_0 \in \mathbf{R}^+$ ($T_0 \le T$) such that for each $n \in \mathbf{N}$ the functions V^n, W^n, θ^n and ρ^n satisfy the inequalities*

$$\max_{t\in[0,T_0]} \left(\left\| \frac{\partial V^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial W^n}{\partial x}(t) \right\|^2 + \left\| \frac{\partial \theta^n}{\partial x}(t) \right\|^2 \right) + \int_0^{T_0} \left(\left\| \frac{\partial^2 V^n}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 W^n}{\partial x^2}(\tau) \right\|^2 + \left\| \frac{\partial^2 \theta^n}{\partial x^2}(\tau) \right\|^2 \right) d\tau \le C, \quad (5.13)$$

$$\max_{t\in[0,T_0]} \left\| \frac{\partial \rho^n}{\partial x}(t) \right\|^2 \le C, \quad (5.14)$$

$$m_1 \le \rho^n(x,t) \le M_1, \quad (x,t) \in \bar{Q}_0 \quad (Q_0 = Q_{T_0}) \quad (5.15)$$

(m_1 *and* M_1 *are defined by* (4.31)),

$$\max_{t\in[0,T_0]} \|V^n(t)\|^2 \le C, \quad \max_{t\in[0,T_0]} \|W^n(t)\|^2 \le C, \quad \max_{t\in[0,T_0]} \|\theta^n(t)\|^2 \le C. \tag{5.16}$$

Proof In the same way as in [3, Lemma 5.6] from (5.9) we find out that there exists $T_0 (0 < T_0 \le T)$ such that (5.13)–(5.15) are satisfied for each $n \in \mathbf{N}$. Taking into account that

$$\left|\theta_0^n(t)\right| = \left|\int_0^1 \theta^n(x,t)dx\right|, \quad \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2 = \sum_{k=1}^n \left(\theta_k^n(t)\right)^2 \frac{(\pi k)^2}{2},$$

$$\|V^n(t)\| \le 2\left\|\frac{\partial V^n}{\partial x}(t)\right\| \le C, \ \|W^n(t)\| \le 2\left\|\frac{\partial W^n}{\partial x}(t)\right\| \le C, \quad t \in [0, T_0]$$

and using (5.5) we obtain (5.16). □

From the above estimates we conclude that for each $n \in \mathbf{N}$ the Cauchy problem (4.20)–(4.24) has a unique solution defined on $[0, T_0]$.

Lemma 5.8 *Let* T_0 *be defined by* Lemma 5.7. *Then for each* $n \in \mathbf{N}$ *the following estimates hold true:*

$$\max_{(x,t)\in\bar{Q}_0} |v_1^n(x,t)| \le C, \quad \max_{(x,t)\in\bar{Q}_0} |\omega_1^n(x,t)| \le C, \tag{5.17}$$

$$\max_{t\in[0,T_0]} \left\|\frac{\partial v_1^n}{\partial x}(t)\right\| \le C, \quad \max_{t\in[0,T_0]} \left\|\frac{\partial \omega_1^n}{\partial x}(t)\right\| \le C, \tag{5.18}$$

$$\max_{t\in[0,T_0]} \left\|\frac{\partial^2 v_1^n}{\partial x^2}(t)\right\| \le C, \quad \max_{t\in[0,T_0]} \left\|\frac{\partial^2 \omega_1^n}{\partial x^2}(t)\right\| \le C, \tag{5.19}$$

$$\max_{t\in[0,T_0]} \left\|\frac{\partial v_1^n}{\partial t}(t)\right\| \le C, \quad \max_{t\in[0,T_0]} \left\|\frac{\partial \omega_1^n}{\partial t}(t)\right\| \le C. \tag{5.20}$$

Proof Estimates (5.17)–(5.19) we get directly from (4.10) and (4.11). From (5.2) and (5.3) it follows (5.20). □

Lemma 5.9 *For* T_0 *defined by* Lemma 5.7 *and for each* $n \in \mathbf{N}$ *the inequalities*

$$\int_0^{T_0} \left(\left\|\frac{\partial V^n}{\partial t}(\tau)\right\|^2 + \left\|\frac{\partial W^n}{\partial t}(\tau)\right\|^2 + \left\|\frac{\partial \theta^n}{\partial t}(\tau)\right\|^2\right) d\tau \le C, \tag{5.21}$$

$$\max_{t\in[0,T_0]} \left\|\frac{\partial \rho^n}{\partial t}(t)\right\| \le C \tag{5.22}$$

hold true.

Proof In the same way as in [3, Lemma 5.7] from (4.7), (4.8) and (4.9) we get (5.21). The estimate for (5.22) follows directly from (4.6). □

From Lemmas 5.7, 5.8 and 5.9 we obtain immediately the next result.

Proposition 5.1 *Let T_0 be defined by* Lemma 5.7. *Then for the sequence* $\{(\rho^n, V^n, W^n, \theta^n, v_1^n, \omega_1^n) : n \in \mathbf{N}\}$ *the following statements hold true:*

i) $\{\rho^n\}$ *is bounded in* $L^\infty(Q_0)$, $L^\infty(0, T_0; H^1(]0, 1[))$ *and* $H^1(Q_0)$*;*
ii) $\{V^n\}$, $\{W^n\}$, $\{\theta^n\}$, $\{v_1^n\}$, $\{\omega_1^n\}$ *are bounded in* $L^\infty(0, T_0; H^1(]0, 1[))$, $H^1(Q_0)$ *and* $L^2(0, T_0; H^2(]0, 1[))$.

6 The proofs of Theorems 3.1 and 2.1

The proof of Theorem 3.1 is very similar to that of Theorem 2.2 in [3].

Let T_0, $0 < T_0 \le T$, be defined by Lemma 5.7.

Lemma 6.1 *There exists a function*

$$\rho \in L^\infty\left(0, T_0; H^1(]0, 1[)\right) \cap H^1(Q_0) \cap C(\bar{Q}_0)$$

and a subsequence of $\{\rho^n\}$ (*for simplicity denoted again as* $\{\rho^n\}$), *such that*

$$\rho^n \to \rho \text{ weakly-}* \text{ in } L^\infty(0, T_0; H^1(]0, 1[)), \tag{6.1}$$
$$\rho^n \to \rho \text{ weakly in } H^1(Q_0), \tag{6.2}$$
$$\rho^n \to \rho \text{ strongly in } C(\bar{Q}_0). \tag{6.3}$$

The function ρ *satisfies the conditions*

$$m_1 \le \rho(x, t) \le M_1 \text{ in } \bar{Q}_0, \tag{6.4}$$
$$\rho(x, 0) = \rho_0(x), \quad x \in [0, 1]. \tag{6.5}$$

Proof Conclusions (6.1) and (6.2) follow from Proposition 5.1. Let $(x, t), (x', t') \in \bar{Q}_0$; then

$$|\rho^n(x, t) - \rho^n(x', t')| \le |\rho^n(x, t) - \rho^n(x', t)| + |\rho^n(x', t) - \rho^n(x', t')|.$$

Using (4.6) and Proposition 5.1 we obtain

$$|\rho^n(x, t) - \rho^n(x', t)| \le \int_{x'}^{x} \left|\frac{\partial \rho^n}{\partial x}(\xi, t)\right| d\xi \le C|x - x'|^{1/2},$$

$$|\rho^n(x',t)-\rho^n(x',t')| \le \int_{t'}^{t} \left|\frac{\partial \rho^n}{\partial t}(x',\tau)\right| d\tau = \int_{t'}^{t} \left|(\rho^n)^2 \frac{\partial V^n}{\partial x} + \frac{\mu}{l}\rho^n\right| d\tau$$

$$\le C\left(\int_{t'}^{t} \|V^n(\tau)\|_{H^2(]0,1[)} d\tau + \int_{t'}^{t} \left|\frac{\mu}{l}\right| d\tau\right) \le C|t-t'|^{1/2}.$$

The statement (6.3) follows now from the Arzela'-Ascoli theorem; conditions (6.4) and (6.5) follow from (5.15) and (4.6). □

Lemma 6.2 *There exists a subsequence of* $\{v_1^n, \omega_1^n\}$ *(denoted again as* $\{v_1^n, \omega_1^n\}$*), such that*

$$(v_1^n, \omega_1^n) \to (v_1, \omega_1) \text{ weakly-}* \text{ in } (L^\infty(0,T_0;H^1(]0,1[)))^2, \tag{6.6}$$

$$(v_1^n, \omega_1^n) \to (v_1, \omega_1) \text{ weakly in } (H^1(Q_0))^2, \tag{6.7}$$

$$(v_1^n, \omega_1^n) \to (v_1, \omega_1) \text{ weakly in } \left(L^2\left(0,T_0;H^2(]0,1[)\right)\right)^2, \tag{6.8}$$

$$(v_1^n, \omega_1^n) \to (v_1, \omega_1) \text{ strongly in } C(\bar{Q}_0), \tag{6.9}$$

where the functions v_1 *and* ω_1 *are defined by* (3.3) *and* (3.4).

Proof The conclusions follow from (6.3), (4.10), (4.11) and Proposition 5.1. □

Lemma 6.3 *There exist functions*

$$V, W, \theta \in L^\infty(0,T_0;H^1(]0,1[)) \cap H^1(Q_0) \cap L^2(0,T_0;H^2(]0,1[))$$

and a subsequence of $\{V^n, W^n, \theta^n\}$ *(denoted again as* $\{V^n, W^n, \theta^n\}$*), such that*

$$(V^n, W^n, \theta^n) \to (V, W, \theta) \text{ weakly-}* \text{ in } (L^\infty(0,T_0;H^1(]0,1[)))^3, \tag{6.10}$$

$$(V^n, W^n, \theta^n) \to (V, W, \theta) \text{ weakly in } (H^1(Q_0))^3, \tag{6.11}$$

$$(V^n, W^n, \theta^n) \to (V, W, \theta) \text{ strongly in } (L^2(Q_0))^3, \tag{6.12}$$

$$(V^n, W^n, \theta^n) \to (V, W, \theta) \text{ weakly in } (L^2(0,T_0;H^2(]0,1[)))^3. \tag{6.13}$$

The functions V, W *and* θ *satisfy the conditions*

$$V(0,t) = V(1,t) = W(0,t) = W(1,t) = 0, \quad t \in [0,T_0], \tag{6.14}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \text{ a.e. in }]0,T_0[, \tag{6.15}$$

$$V(x,0) = V_0(x),\ W(x,0) = W_0(x),\ \theta(x,0) = \theta_0(x), \quad x \in [0,1], \tag{6.16}$$

$$\theta > 0 \text{ in } \bar{Q}_0. \tag{6.17}$$

Proof The conclusions follow from Proposition 5.1, embedding properties (see Remark 2.1) and Green's formula. In the same way as in [3, Lemma 6.4] we get (6.17). □

Applying the Lemmas 6.1–6.3 to the equations (4.6)–(4.9) we get the following result.

Lemma 6.4 *The functions* $\rho, V, W, \theta, v_1, \omega_1$*, defined by* Lemmas 6.1– 6.3 *and formulas* (3.3) *and* (3.4)*, satisfy the equations* (3.8)–(3.11) *a.e. in* Q_0.

From Lemma 6.4 it follows immediately that the function $(\rho, V + v_1, W + \omega_1, \theta)$ is a generalized solution of the problem (2.1)–(2.11) in Q_0.

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