

NON-HOMOGENEOUS STRESSES IN VISCO-ELASTIC MEDIA*

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1. Introduction. The purpose of this paper is the extension of the theory of elasticity to include visco-elastic media. The materials considered in this paper are *isotropic* and *incompressible*, and are characterized by *linear* relations between the components of stress, strain, and their derivatives with respect to time. As in the classical theory of elasticity, only *small* strains will be considered. Body forces, in particular inertia forces, will be neglected.

In the following, σ_{ik} ($i, k = 1, 2, 3$) and ϵ_{ik} denote the components of the tensors of stress and strain with respect to a system of rectangular axes x_i . σ_{11} , σ_{22} , σ_{33} are the normal stresses, $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{31} = \sigma_{13}$ the shearing stresses. Similarly, ϵ_{11} , ϵ_{22} , ϵ_{33} are the normal strains, $\epsilon_{12} = \epsilon_{21}$, $\epsilon_{23} = \epsilon_{32}$, $\epsilon_{31} = \epsilon_{13}$ the shearing strains. If u_i are the components of the displacement vector,

$$\epsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}), \quad (1)$$

where the index after a comma denotes differentiation with respect to the corresponding coordinate x , i.e., $u_{i,k} = \partial u_i / \partial x_k$; $u_{k,i} = \partial u_k / \partial x_i$.

Irrespective of the mechanical properties of the material, the stresses must satisfy the equilibrium conditions

$$\sigma_{ik,k} = 0, \quad (2)$$

where the summation convention of tensor calculus has been used.¹ Similarly, the strain components must satisfy the conditions of compatibility,

$$\epsilon_{ik,lm} + \epsilon_{lm,ik} = \epsilon_{il,km} + \epsilon_{km,il}, \quad (3)$$

where $\epsilon_{ik,lm} = \partial^2 \epsilon_{ik} / \partial x_l \partial x_m$, etc. While there are obviously three equations of equilibrium (corresponding to the three values which the subscript i in (2) can assume), it may at first glance appear that there are 3^4 equations of compatibility. On account of the high degree of symmetry in (3), the number of equations of compatibility reduces, however, to six; three equations of the type obtained from (3) when e.g., $i = k = 1$ and $l = m = 2$, and three equations of the type obtained from (3) when e.g., $i = k = 1$, $l = 2$, $m = 3$.

By themselves, Eqs. (2) and (3) are not sufficient to determine the states of stress and strain in a body subject to given surface stresses. A further necessary set of equations are those relating the stress components to the strain components in the general

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¹ According to this convention $\sigma_{ik,k}$ stands for the sum of all the terms obtained by giving k the values 1, 2, 3. In general, whenever a subscript appears twice in the same monomial, this subscript is to be given the values 1, 2, 3 and the resulting terms are to be added. Such a repeated subscript is called a *dummy subscript*.

case of combined stresses. It is through these *stress-strain* relations that the properties of the material enter the problem.

In the case of an incompressible material, $\epsilon_{ii} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0$. The stress-strain relations are most easily discussed when the following decomposition of the stress tensor is introduced. Define the *mean normal stress* as

$$\sigma = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}), \quad (4)$$

and the *deviatoric part* of the stress tensor as

$$s_{ik} = \sigma_{ik} - \sigma\delta_{ik}, \quad (5)$$

where

$$\delta_{ik} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

The stress-strain relations of an isotropic, incompressible elastic material can then be written in the form

$$s_{ik} = 2G\epsilon_{ik}, \quad (6)$$

where G denotes the modulus of rigidity.

In view of (5), the equilibrium condition (2) yields

$$s_{ik,k} + \sigma_{,k}\delta_{ik} = s_{ik,k} + \sigma_{,i} = 0. \quad (7)$$

But, according to (6) and (1),

$$s_{ik,k} = 2G\epsilon_{ik,k} = G(u_{i,kk} + u_{k,ik}) = Gu_{i,kk}, \quad (8)$$

since for an incompressible material $u_{k,k} = 0$ and, consequently, $u_{k,ik} = u_{k,ki} = 0$. Comparing (7) and (8), we find

$$\sigma_{,i} = -Gu_{i,kk}. \quad (9)$$

Hence

$$\sigma_{,ii} = -Gu_{i,kki} = 0, \quad (10)$$

on account of the incompressibility of the material.

According to (5),

$$\sigma_{ik,il} = s_{ik,il} + \sigma_{,il}\delta_{ik} = s_{ik,il},$$

on account of (10). Making use of (6), (1) and (9), we transform this in the following manner:

$$\sigma_{ik,il} = 2G\epsilon_{ik,il} = G(u_{i,kil} + u_{k,ili}) = -2\sigma_{,ik}.$$

Thus

$$\sigma_{ik,il} + 2\sigma_{,ik} = 0. \quad (11)$$

In the case of an incompressible elastic body in equilibrium the boundary conditions may be given in the form of three functions $f_i(x)$ which define the components of the forces (per unit area) applied to the surface of the body. The forces f_i must, of course, be in equilibrium, i.e., the surface integral of $f_i(x)$ must vanish for $i = 1, 2, 3$. If n_k denotes the unit vector directed along the exterior normal of the surface of the body, the stress components at the surface must then satisfy the conditions

$$\sigma_{ik}n_k = f_i. \quad (12)$$

The values of the surface stresses, in conjunction with Eqs. (2) and (11), define the stress distribution in the body and, consequently, also the strain distribution and, to within a rigid body displacement of the entire body, the displacement components.

On the other hand, the displacement components may be given on the surface of the body. These given surface displacements must, of course, be compatible with the assumed incompressibility of the material, i.e., the surface integral of the normal displacement component $u_i n_i$ must vanish. Elimination of σ from (9) furnishes $u_{i, kkl} - u_{l, kki} = 0$, or, after a change of subscripts,

$$u_{i, kll} - u_{k, ill} = 0. \quad (12a)$$

Eqs. (12) in conjunction with the condition of incompressibility, $u_{i, i} = 0$, and the given surface values determine the displacement components.

2. Stress-strain relations of visco-elastic materials. Equations similar to (11) and (12a) may be derived for visco-elastic materials characterized by linear relations between the components of stress, strain and their derivatives with respect to time.

In the case of an incompressible material of the type considered by Voigt² we have the stress-strain relations

$$s_{ik} = 2G\epsilon_{ik} + 2\mu\dot{\epsilon}_{ik}, \quad (13)$$

where μ is the *coefficient of viscosity*.

In the case of an incompressible material of the Maxwell type, we have

$$\dot{\epsilon}_{ik} = \frac{1}{2G} \dot{s}_{ik} + \frac{1}{2G\tau} s_{ik}, \quad (14)$$

where dots denote differentiation with respect to time, and τ is the *relaxation time*.³

Generalizing, we may consider incompressible materials characterized by stress-strain relations of the form

$$\left(\frac{\partial^m}{\partial t^m} + a_{m-1} \frac{\partial^{m-1}}{\partial t^{m-1}} + \cdots + a_0 \right) s_{ik} = \left(b_n \frac{\partial^n}{\partial t^n} + b_{n-1} \frac{\partial^{n-1}}{\partial t^{n-1}} + \cdots + b_0 \right) \epsilon_{ik}, \quad (15)$$

where $a_{m-1}, \dots, a_0, b_n, b_{n-1}, \dots, b_0$ are constants characteristic of the material.

For such materials two types of boundary value problems may be considered. In the first case the surface forces $f_i(x, t)$ are given as functions of the position x and the time t ; for $t=0$ these surface forces and their $m-1$ first derivatives are supposed to vanish as well as all stress components and their derivatives up to the order $m-1$. Moreover, at any given time the forces f_i must be in equilibrium. If, for $t \geq 0$, the forces are analytic functions of time, this implies that the surface integral of any derivative $\partial^p f_i / \partial t^p$ must vanish for, say, $t=0$. The *first boundary value problem* calls for the determination of the stress distribution $\sigma_{ik}(x, t)$ fulfilling these boundary conditions and initial conditions.

In the second case the surface displacements $u_i(x, t)$ are given as functions of the position x and the time t ; for $t=0$ these surface displacements and their $n-1$ first derivatives are supposed to vanish, as well as the displacements in the interior of the

² W. Voigt, Abh. Göttingen Ges. Wiss. **36** (1899), 47 pp.

³ R. Simha has recently used the stress-strain relations which are obtained from Eqs. (14) by substituting the stress tensor σ_{ik} for its deviatoric part s_{ik} [J. Appl. Phys. **13**, 201 (1942)]. Such stress-strain relations imply that, at constant strain, the stress decays exponentially with a relaxation time τ which is independent of the geometrical nature of the stress. This treatment ignores the fact that viscous flow, which is the cause of relaxation, is a response to *shearing* stresses only. In an incompressible material a uniform hydrostatic pressure does not produce viscous flow, and, hence, does not tend to relax. Contrary to Simha's stress-strain relations, our Eqs. (14) reflect this behavior.

body and their derivatives up to the order $n-1$. Moreover, on account of the assumed incompressibility of the material, the surface integral of the normal displacement component $u_i n_i$ must vanish for any time. If, for $t \geq 0$, the displacements are analytic functions of time, this means that the surface integral of all expressions of the form $(\partial^p u_i / \partial t^p) n_i$ must vanish for, say, $t=0$. The *second boundary value problem* calls for the determination of the displacements $u_i(x, t)$ in the interior of the body, fulfilling these boundary conditions and initial conditions.

Let us rewrite the stress-strain relation (15) in the form

$$P \sigma_{ik} = 2Q \epsilon_{ik}, \quad (16)$$

where P and Q denote the linear differential operators

$$\begin{aligned} P &= \frac{\partial^m}{\partial t^m} + a_{m-1} \frac{\partial^{m-1}}{\partial t^{m-1}} + \cdots + a_1 \frac{\partial}{\partial t} + a_0, \\ Q &= b_n \frac{\partial^n}{\partial t^n} + b_{n-1} \frac{\partial^{n-1}}{\partial t^{n-1}} + \cdots + b_1 \frac{\partial}{\partial t} + b_0. \end{aligned} \quad (16a)$$

Starting from the stress-strain relations (16) and repeating the various steps which led to the Eqs. (11) and (12a), we obtain

$$P(\sigma_{ik,ii} + 2\sigma_{,ikk}) = 0 \quad (17)$$

and

$$Q(u_{i,kil} - u_{k,ili}) = 0 \quad (18)$$

as the equations governing the solution of the first and second boundary value problem, respectively.

For example, consider the first boundary value problem for an incompressible material of the Voigt type. Comparing Eqs. (13), (16) and (16a), we see that for this material

$$P = 1, \quad Q = \mu \frac{\partial}{\partial t} + G.$$

Eq. (17) consequently takes the same form as for an incompressible elastic material (see Eq. (11)). This means that, *in the case of the first boundary value problem, the stress distribution in an incompressible material of the Voigt type is identical with that in an incompressible elastic material under the same instantaneous surface forces*. This stress distribution does not depend on the past stressing history, although, of course, the displacements do.

This result is readily extended to the case of an incompressible visco-elastic material characterized by a stress-strain relation (16). Consider, for instance, the first boundary value problem for a given set of surface forces $f_i(x, t)$ which, in addition to fulfilling the conditions stipulated above, are supposed to be analytic functions of time for $t \geq 0$. If $\bar{\sigma}_{ik}(x, t)$ denotes the *static*⁴ stress distribution in an incompressible elastic body of the same shape which is subjected to the surface forces $f_i(x, t)$, the required stress distribution in the visco-elastic body is given by

$$\sigma_{ik}(x, t) = \bar{\sigma}_{ik}(x, t).$$

⁴ The term "static" is used here to indicate that, though the stresses $\bar{\sigma}_{ik}$ depend on t as do the forces f_i , no inertia effects should be taken into account in computing these stresses. In fact, as far as this elastic body is concerned, t plays the role of a parameter which need by no means be identified with the time.

Indeed, by definition, the stresses $\bar{\sigma}_{ik}$ satisfy the conditions (2), (11) and (12) for any value of t . Since, like the surface forces, these stresses are analytic functions of time, this means that they also satisfy the condition (17). *The result formulated above for the first boundary value problem of an incompressible material of the Voigt type applies, therefore, to any visco-elastic material characterized by stress-strain relations of the form (16).*

A similar result is obtained in the case of the second boundary value problem for an incompressible visco-elastic material obeying stress-strain relations of the form (16), if the prescribed surface displacements $u_i(x, t)$ fulfill the conditions formulated above and, in addition, are analytic functions of time for $t \geq 0$. The displacements $u_i(x, t)$ then equal the *static* displacements $\bar{u}_i(x, t)$ of an incompressible elastic body of the same shape, subjected to the given surface displacements $u_i(x, t)$.

3. Determination of the displacements in the first boundary value problem of visco-elasticity. Let us first consider the particularly simple case, where the given surface forces can be factored into the form:

$$f_i = f_i(x)g(t). \quad (19)$$

According to what has been said above, the stress distribution which these surface forces produce in the visco-elastic body has then the form

$$\sigma_{ik}(x, t) = \bar{\sigma}_{ik}(x)g(t), \quad (20)$$

where $\bar{\sigma}_{ik}(x)$ denotes the stresses which the surface forces $f_i(x)$ produce in an incompressible elastic body of the same shape. Introducing the stresses (20) into the stress-strain relation (16), we see that the strains in the visco-elastic body can be written in the form

$$\epsilon_{ik}(x, t) = \bar{\epsilon}_{ik}(x)h(t), \quad (21)$$

where $h(t)$ satisfies the differential equation

$$\mathbf{Q}h = Pg, \quad (22)$$

while h and its derivatives up to the order $n-1$ vanish for $t=0$. As regards the quantities $\bar{\epsilon}_{ik}(x)$, they are related to the stresses $\bar{\sigma}_{ik}(x)$ by

$$\bar{s}_{ik} = 2\bar{\epsilon}_{ik}, \quad (23)$$

where \bar{s}_{ik} denotes the deviatoric part of the stress tensor $\bar{\sigma}_{ik}$. In other terms, the quantities $\bar{\epsilon}_{ik}$ are the strains in an incompressible elastic body of the same shape and of unit modulus of rigidity, which is subjected to the surface forces $f_i(x)$. We shall call these strains the *equivalent elastic strains*. In order to obtain the function $h(t)$, all we have to do is to consider the *response* of the visco-elastic material under consideration to a simple shearing stress s varying according to $s = 2g(t)$. The shearing strain produced by this stress equals $h(t)$. The strains produced in the visco-elastic body by the surface forces $f_i(x)g(t)$ are then obtained by multiplying the equivalent elastic strains by the response function $h(t)$.

Since the differential equations for stresses and strains are linear, solutions of this type may be superimposed on each other. Let us, now, assume that our result holds good even if, contrary to the assumption made above, the surface forces are not analytical functions of time for $t \geq 0$. In particular consider the case when $f_i = f_i(x)g(\xi, t)$, where $g(\xi, t)$ is Heaviside's unit step function defined by

$$g(\xi, t) = \begin{cases} 0, & \text{if } t < \xi, \\ 1, & \text{if } t \geq \xi. \end{cases}$$

Let $h(\xi, t)$ denote the response of the visco-elastic material under consideration to a simple shearing stress $s = 2g(\xi, t)$. Since the surface forces $f_i(x, t)$ can be represented in the form

$$f_i(x, t) = \int_0^\infty \dot{f}_i(x, \xi) g(\xi, t) d\xi, \quad (24)$$

the following formal integral representation of the strains produced by these surface forces in the visco-elastic body suggests itself:

$$\epsilon_{ik}(x, t) = \int_0^\infty \bar{\epsilon}_{ik}(x, \xi) h(\xi, t) d\xi, \quad (25)$$

where $\bar{\epsilon}_{ik}(x, \xi) d\xi$ are the equivalent elastic strains corresponding to the surface forces $\dot{f}_i(x, \xi) d\xi$. It can be shown that (25) indeed furnishes the strains of the visco-elastic body whenever the surface forces can be represented in the form (24). Moreover, to within a rigid body displacement the displacements of the visco-elastic body are given by

$$u_i(x, t) = \int_0^\infty \bar{u}_i(x, \xi) h(\xi, t) d\xi, \quad (26)$$

where $\bar{u}_i(x, \xi) d\xi$ are *equivalent elastic displacements* produced in an incompressible elastic body of the same shape and of unit modulus of rigidity, by the surface forces $\dot{f}_i(x, \xi) d\xi$.

Let us consider the following example: A thin cantilever beam of length L and cross sectional moment of inertia I is clamped rigidly at the end $x=0$. The beam consists of an incompressible visco-elastic material of the Voigt type (stress-strain relations (13)), and is subjected to the transverse load

$$f(x, t) = c \left(1 - \frac{x}{L} \right) t^2,$$

per unit of length, c being a constant. At first sight, it may seem that the problem of determining the bending moments and transverse displacements of the beam is outside the scope of our theory, since at the clamped end we have prescribed deformations rather than prescribed forces. However, the system being statically determinate, the transverse reaction and the bending moment at the clamped end are completely determined by the given loads. Consequently, the problem may be considered as a first boundary value problem, if we make the usual assumption that the distribution of stresses over the end section is irrelevant as long as it leads to the resultant and the resultant moment required by the equilibrium of the beam. The displacements of an incompressible elastic cantilever beam of unit modulus of rigidity, loaded by $f(x) = c(1 - x/L)$, are

$$\bar{u}(x) = \frac{cx^2}{360IL} (10L^3 - 10L^2x + 5Lx^2 - x^3),$$

where account has already been taken of the fact that the Young's modulus equals $3G$ for an incompressible elastic material.

Now, in accordance with (13), the response $h(t)$ of Voigt's material to a simple shearing stress $s = 2t^2$ is found from

$$2t^2 = 2Gh + 2\mu\dot{h}, \quad h(0) = 0.$$

One obtains

$$h(t) = \frac{t^2}{G} - 2\frac{\mu}{G^2}t + 2\frac{\mu^2}{G^3}[1 - e^{-Gt/\mu}].$$

The deflection of the visco-elastic beam is, therefore, given by

$$u(x, t) = \frac{cx^2}{360GIL} [10L^3 - 10L^2x + 5Lx^2 - x^3] \cdot \left[t^2 - 2\frac{\mu}{G}t + 2\frac{\mu^2}{G^2}(1 - e^{-Gt/\mu}) \right].$$

The statically determinate bending moments are completely determined by the given loads.

4. Determination of the stresses in the second boundary value problem of visco-elasticity. A similar procedure leads to the determination of the stresses in the second boundary value problem of visco-elasticity. Consider first the case when the given surface displacements can be factored into the form $u_i = u_i(x)g(t)$, and denote by $\bar{\sigma}_{ik}(x)$ the *equivalent elastic stresses*, i.e., the static stresses set up in an incompressible elastic body by the surface displacements $u_i(x)$. Furthermore, determine the *response function* $h(t)$, i.e., half the shearing stress produced in the visco-elastic material under consideration by a simple shearing strain $g(t)$. The required stress distribution in the visco-elastic body is then given by $\sigma_{ik}(x, t) = \bar{\sigma}_i(x)h(t)$.

In the general case, the stresses in the second boundary value problem may be represented in the form

$$\sigma_{ik}(x, t) = \int_0^\infty \bar{\sigma}_{ik}(x, \xi) h(\xi, t) d\xi, \quad (27)$$

where $\bar{\sigma}_{ki}(x, \xi)d\xi$ are the equivalent elastic stresses corresponding to the surface displacements $\dot{u}_i(x, \xi)d\xi$, and $2h(\xi, t)$ is the response of the visco-elastic material to a simple shearing strain

$$g(\xi, t) = \begin{cases} 0, & \text{if } t < \xi, \\ 1, & \text{if } t \geq \xi. \end{cases}$$

5. Summary. The solution of the first and second boundary value problems of visco-elasticity is reduced to the solution of equivalent boundary value problems of elasticity, and the determination of the response of the visco-elastic material under consideration to a simple shearing stress or a simple shearing strain. It remains to be seen in how far the technique developed here can be applied to the solution of the third (mixed) boundary value problem where the surface forces are prescribed on part of the surface of the body, and the surface displacement on the rest of this surface.