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Introduction

Let S be a minimal surface of general type defined over $\mathbb{C}$. We call S a canonical surface if the rational map associated with $|K|$ is birational onto its image. Assume that S is a canonical surface with a non-linear pencil, and let $f : S \to B$ be the corresponding fibration. Since S is canonical, any general fibre of f is a non-hyperelliptic curve. A natural question is then: what is the genus of a general fibre? This leads us to studying the slope of non-hyperelliptic fibrations. For a hyperelliptic fibration of genus g, $4 - 4/g$ is the best possible lower bound of the slope by [P] and [H1]. Later, Xiao [X] showed that the slope is not less than $4 - 4/g$ even when non-hyperelliptic. But, for non-hyperelliptic fibrations, it may not be the best bound. In fact, we showed in [K2] that the slope is not less than 3 when $g = 3$ (see also [H2] and [R2]), and Xiao himself conjectured that the slope is strictly greater than $4 - 4/g$ for non-hyperelliptic fibrations ([X, Conjecture 1]).

At present, we have two methods for studying the slope. The first is Xiao's method [X] of relative projections and the second is *counting relative hyperquadrics* which is still at an experimental stage (see [R2] and [K2]). Combining these two, we show that the slope is not less than $24/7$ for $g = 4$ and give a bound $40/11$ for $g = 5$ (Theorems 4.1 and 5.1). We also answer affirmatively to Xiao's conjecture referred above (Proposition 2.6).

As an application, we show in Section 6 that, for an irregular canonical surface S (with a non-linear pencil), the canonical image cannot be cut out by quadrics when $K^2 \leq (10/3)\chi(\mathcal{O}_S)$. For irregular surfaces, Reid's conjecture [R1, p. 541] may be shown along the same line if we can sufficiently develop the second method.

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1. - Relative hyperquadrics

Let B be a non-singular projective curve of genus b, and let $\mathcal{E}$ be a locally free sheaf on B. We put $\mu(\mathcal{E}) = \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$. According to [HN], $\mathcal{E}$ has a uniquely determined filtration by its sub-bundles $\mathcal{E}_i$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$$

which satisfies

(i) $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable for $1 \leq i \leq \ell$,

(ii) $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ for $1 \leq i \leq \ell - 1$.

As usual, we call such a filtration the Harder-Narashimhan filtration of $\mathcal{E}$. Put $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ and $r_i = \mathrm{rk}(\mathcal{E}_i)$. Then

$$\deg(\mathcal{E}) = \sum_{i=1}^{\ell-1} r_i(\mu_i - \mu_{i+1}) + r_\ell \mu_\ell.$$

Let $\pi : \mathbb{P}(\mathcal{E}) \to B$ be the projective bundle associated with $\mathcal{E}$. We denote by $T_\mathcal{E}$ and F a tautological divisor such that $\pi_* \mathcal{O}(T_\mathcal{E}) = \mathcal{E}$ and a fibre of π, respectively. Note that for any $\mathbb{R}$-divisor D on $\mathbb{P}(\mathcal{E})$, there are real numbers x, y satisfying $D \equiv xT_\mathcal{E} + yF$, where the symbol $\equiv$ means numerical equivalence.

The following can be found in [N].

LEMMA 1.1. *An $\mathbb{R}$-divisor which is numerically equivalent to $T_\mathcal{E} - xF$ is pseudo-effective if and only if $x \leq \mu_1$. It is nef if and only if $x \leq \mu_\ell$.*

Assume that $\ell \geq 2$. For $1 \leq i \leq \ell - 1$ let

$$\rho_i : W_i \to \mathbb{P}(\mathcal{E})$$

denote the blowing-up along $B_i = \mathbb{P}(\mathcal{E}/\mathcal{E}_i)$. Then W_i has a projective space bundle structure $\pi_i : W_i \to \mathbb{P}(\mathcal{E}_i)$. We put $\mathbb{E}_i = \rho_i^{-1}(B_i)$. Then $\pi_i^* T_{\mathcal{E}_i}$ is linearly equivalent to $\rho_i^* T_\mathcal{E} - \mathbb{E}_i$. Furthermore, $\mathbb{E}_i$ is isomorphic to the fibre product $\mathbb{P}(\mathcal{E}_i) \times_B B_i$. Let $p_1 : \mathbb{E}_i \to \mathbb{P}(\mathcal{E}_i)$ be the projection map onto the first factor. Then $p_1 = \pi_i|_{\mathbb{E}_i}$. Similarly, if $p_2 : \mathbb{E}_i \to B_i$ denotes the projection to the second factor, then $p_2 = \rho_i|_{\mathbb{E}_i}$. In particular, $[-\mathbb{E}_i]|_{\mathbb{E}_i}$ is given by $p_1^* T_{\mathcal{E}_i} - p_2^* T_{\mathcal{E}/\mathcal{E}_i}$.

The following is essentially the same as [N, Claim (4.8)].

LEMMA 1.2. *Assume that an $\mathbb{R}$-divisor $Q \equiv p_1^* T_{\mathcal{E}_i} + p_2^* T_{\mathcal{E}/\mathcal{E}_i} - xF$ on $\mathbb{E}_i$ is pseudo-effective. Then $x \leq \mu_1 + \mu_\ell + \deg(\mathcal{E}_{\ell-1}/\mathcal{E}_i)$.*

PROOF. Since $T_{\mathcal{E}/\mathcal{E}_i} - \mu_\ell F$ is nef on B_i, $H_y = T_{\mathcal{E}/\mathcal{E}_i} - (\mu_\ell - y)F$ is ample for any positive rational number y. Let m be a sufficiently large positive integer such that mH_y is a very ample $\mathbb{Z}$-divisor, and choose $s - 1$ general members $H_j \in |mH_y|$ so that $C = \cap_j H_j$ is an irreducible non-singular curve, where $s = \mathrm{rk}(\mathcal{E}/\mathcal{E}_i)$. Let $\tau : C \to B$ denote the natural map. Then $\mathbb{P}(\mathcal{E}_i) \times_B C \simeq \mathbb{P}(\tau^*\mathcal{E}_i)$. Since the restriction of Q to this space is numerically equivalent to

$$T_{\tau^*\mathcal{E}_i} - \mu_1(\tau^*\mathcal{E}_i)F_C + \{(T_{\mathcal{E}/\mathcal{E}_i} + (\mu_1 - x)F) \cdot C\}F_C,$$

where F_C denotes a fibre of $\mathbb{P}(\tau^*\mathcal{E}_i) \to C$, and since it must be pseudo-effective, it follows from Lemma 1.1 that $(T_{\mathcal{E}/\mathcal{E}_i} + (\mu_1 - x)F) \cdot C \geq 0$, that is, $(T_{\mathcal{E}/\mathcal{E}_i} + (\mu_1 - x)F)H_y^{s-1} \geq 0$. Letting $y \downarrow 0$, we get

$$x \leq \deg(\mathcal{E}/\mathcal{E}_i) - s\mu_\ell + \mu_1 + \mu_\ell = \deg(\mathcal{E}_{\ell-1}/\mathcal{E}_i) + \mu_1 + \mu_\ell. \qquad \square$$

An effective divisor Q on $\mathbb{P}(\mathcal{E})$ is called a *relative hyperquadric* if it is numerically equivalent to $2T_\mathcal{E} - xF$ for some $x \in \mathbb{Z}$. It is said to be of rank r, $\mathrm{rk}(Q) = r$, if it induces a hyperquadric of rank r on a generic fibre of $\mathbb{P}(\mathcal{E})$.

LEMMA 1.3. *Assume that $\ell \geq 2$ and consider a relative hyperquadric $Q \equiv 2T_\mathcal{E} - xF$ on $\mathbb{P}(\mathcal{E})$. If Q is not singular along $B_{\ell-1}$, then $x \leq \mu_1 + \mu_\ell$.*

PROOF. We may assume that $x > 2\mu_\ell$. Then, by Lemma 1.1, Q vanishes on $B_{\ell-1}$, since $Q|_{B_{\ell-1}} \equiv 2T_{\mathcal{E}/\mathcal{E}_{\ell-1}} - xF$. However, since Q is not singular along $B_{\ell-1}$, it cannot vanish twice along $B_{\ell-1}$. Let $\tilde{Q}$ be the proper transform of Q by $\rho_{\ell-1}$. Then

$$\tilde{Q} \equiv \rho_{\ell-1}^*(2T_\mathcal{E} - xF) - \mathbb{E}_{\ell-1} = \rho_{\ell-1}^*T_\mathcal{E} + \pi_{\ell-1}^*T_{\mathcal{E}_{\ell-1}} - xF.$$

Hence $\tilde{Q}|_{\mathbb{E}_{\ell-1}} \equiv p_1^*T_{\mathcal{E}_{\ell-1}} + p_2^*T_{\mathcal{E}/\mathcal{E}_{\ell-1}} - xF$. Since it must be effective, we get $x \leq \mu_1 + \mu_\ell$ by Lemma 1.2. $\square$

LEMMA 1.4. *Let $Q \equiv 2T_\mathcal{E} - xF$ be a relative hyperquadric on $\mathbb{P}(\mathcal{E})$. If $x > \mu_1 + \mu_i$, then $\mathrm{rk}(Q) \leq r_{i-1}$ and Q is singular along B_{i-1}.*

PROOF. Since $x > \mu_1 + \mu_\ell$, it follows from Lemma 1.3 that Q is singular along $B_{\ell-1}$. Let $\tilde{Q}$ be the proper transform of Q by $\rho_{\ell-1}$. Then

$$\tilde{Q} \equiv \rho_{\ell-1}^*(2T_\mathcal{E} - xF) - 2\mathbb{E}_{\ell-1} = \pi_{\ell-1}^*(2T_{\mathcal{E}_{\ell-1}} - xF).$$

Hence there exists a relative hyperquadric $Q_{\ell-1} \equiv 2T_{\mathcal{E}_{\ell-1}} - xF$ on $\mathbb{P}(\mathcal{E}_{\ell-1})$ satisfying $\mathrm{rk}(Q) = \mathrm{rk}(Q_{\ell-1}) \leq r_{\ell-1}$. Now, the assertion can be shown by induction. $\square$

LEMMA 1.5. *Let $Q \equiv 2T_\mathcal{E} - xF$ be a relative hyperquadric on $\mathbb{P}(\mathcal{E})$. If $\mathrm{rk}(Q) \geq 3$, then the following hold.*

(1) *If $r_1 \geq 3$, then $x \leq 2\mu_1$.*

(2) *If $r_1 = 2$, then $x \leq \mu_1 + \mu_2$.*

(3) *If $r_1 = 1$ and $r_2 \geq 3$, then $x \leq 2\mu_2$.*

(4) *If* $r_1 = 1$ *and* $r_2 = 2$, *then* $x \leq \min\{2\mu_2, \mu_1 + \mu_3\}$.

PROOF. (1) follows from Lemma 1.1 applied to a $\mathbb{Q}$-divisor $Q/2$. We only have to show that $x \leq 2\mu_2$ in (3) and (4), since the other assertions follow from Lemma 1.4. Assume that $r_1 = 1$. Then B_1 is a relative hyperplane on $\mathbb{P}(\mathcal{E})$. Since $\mathrm{rk}(Q) \geq 3$, we see that Q cannot vanish identically on B_1. Note that $0 \subset \mathcal{E}_2/\mathcal{E}_1 \subset \cdots \subset \mathcal{E}/\mathcal{E}_1$ is the Harder-Narashimhan filtration of $\mathcal{E}/\mathcal{E}_1$. Since $Q|_{B_1} \equiv 2T_{\mathcal{E}/\mathcal{E}_1} - xF$, we get $x \leq 2\mu_2$ by Lemma 1.1. □

LEMMA 1.6. *Let* $Q \equiv 2T_{\mathcal{E}} - xF$ *be a relative hyperquadric on* $\mathbb{P}(\mathcal{E})$. *If* $\mathrm{rk}(Q) \geq 4$, *then the following hold.*

(1) *If* $r_1 \geq 4$, *then* $x \leq 2\mu_1$.

(2) *If* $r_1 = 3$, *then* $x \leq \mu_1 + \mu_2$.

(3) *If* $r_1 = 2$ *and* $r_2 \geq 4$, *then* $x \leq \mu_1 + \mu_2$.

(4) *If* $r_1 = 2$ *and* $r_2 = 3$, *then* $x \leq \mu_1 + \mu_3$.

(5) *If* $r_1 = 1$ *and* $r_2 \geq 4$, *then* $x \leq 2\mu_2$.

(6) *If* $r_1 = 1$ *and* $r_2 = 3$, *then* $x \leq \min\{2\mu_2, \mu_1 + \mu_3\}$.

(7) *If* $r_1 = 1$, $r_2 = 2$ *and* $r_3 \geq 4$, *then* $x \leq \mu_2 + \mu_3$.

(8) *If* $r_1 = 1$, $r_2 = 2$ *and* $r_3 = 3$, *then* $x \leq \min\{\mu_2 + \mu_3, \mu_1 + \mu_4\}$.

PROOF. We show that $x \leq \mu_2 + \mu_3$ in (7) and (8). Assume by contradiction that $x > \mu_2 + \mu_3$. Since $r_1 = 1$, B_1 is a relative hyperplane on $\mathbb{P}(\mathcal{E})$. We have $Q|_{B_1} \equiv 2T_{\mathcal{E}/\mathcal{E}_1} - xF$. Since $x > \mu_2 + \mu_3$, it follows from Lemma 1.4 that $Q|_{B_1}$ is singular along B_2 which is a relative hyperplane of B_1. This implies that, on $F \simeq \mathbb{P}^{r-1}$, Q is defined by $X_1 L(X_1, \ldots, X_r) + cX_2^2 = 0$ with a system of homogeneous coordinates $(X_1, \ldots, X_r)$ on F satisfying $B_1|_F = (X_1)$, where L is a linear form and c is a constant. In particular, Q cannot be of rank ≥ 4. Hence $x \leq \mu_2 + \mu_3$.

The other assertions can be shown similarly as in Lemma 1.5. □

REMARK 1.7. Put $\nu_j = \mu_i$ when $r_{i-1} < j \leq r_i$ $(1 \leq i \leq \ell)$. Then $\nu_1 \geq \cdots \geq \nu_r$, $r = \mathrm{rk}(\mathcal{E})$, and $\deg(\mathcal{E}) = \sum \nu_j$. With this notation, the conditions in Lemma 1.5 (resp. Lemma 1.6) can be written as $x \leq \min\{2\nu_2, \nu_1 + \nu_3\}$ (resp. $x \leq \min\{\nu_2 + \nu_3, \nu_1 + \nu_4\}$).

2. - Some inequalities

Let $f : S \to B$ be a surjective holomorphic map of a non-singular projective surface S onto a non-singular projective curve B with connected fibres. We always assume that f is relatively minimal, that is, no fibre of f contains a (-1)-curve. If a general fibre of f is a (non-)hyperelliptic curve of genus $g \geq 2$, we call f a (non-)hyperelliptic fibration of genus g. Let $K_{S/B}$ be the relative canonical bundle. It is nef by Arakelov's theorem [B].

LEMMA 2.1. *Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$, and put $b = g(B)$. Then $f_*\omega_{S/B}$ is a locally free sheaf of rank g and degree $\Delta(f) := \chi(\mathcal{O}_S) - (g-1)(b-1)$. Furthermore, the following hold.*

(1) $\Delta(f) > 0$ *unless f is locally trivial.*

(2) *Every locally free quotient of $f_*\omega_{S/B}$ has nonnegative degree.*

PROOF. $\mathrm{rk}(f_*\omega_{S/B})$ equals the genus of a fibre. The assertion about the degree follows from the Riemann-Roch theorem (on S and B) and the Leray spectral sequence, since we have $R^1f_*\omega_{S/B} = f_*\mathcal{O}_S$ by the relative duality theorem. (1) and (2) can be found in [B] and [F], respectively. $\square$

When f is not locally trivial, we put $\lambda(f) = K^2_{S/B}/\Delta(f)$ and call it the *slope* of f.

NOTATION 2.2. Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$. Put $\mathcal{E} = f_*\omega_{S/B}$ and let $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$ be its Harder-Narashimhan filtration. The natural sheaf homomorphism $f^*\mathcal{E} \to \omega_{S/B}$ induces a rational map $h : S \to \mathbb{P}(\mathcal{E})$. The image $V = h(S)$ is called the relative canonical image. To be more precise, let $\mathcal{A}$ be a sufficiently ample divisor on B, and put $L(\mathcal{A}) = K_{S/B} + f^*\mathcal{A}$. Let $\sigma : \tilde{S} \to S$ be a composition of blowing-ups such that the variable part $|M(\mathcal{A})|$ of $|\sigma^*L(\mathcal{A})|$ is free from base points. We assume that σ is the shortest among those with such a property. Let Z be the fixed part of $|\sigma^*L(\mathcal{A})|$ and let E be an exceptional divisor with $\tilde{K} = \sigma^*K + [E]$, where $\tilde{K}$ is the canonical bundle of $\tilde{S}$. Since $\mathcal{A}$ is sufficiently ample, we can assume that Z has no horizontal components. In particular, we see that $M(\mathcal{A})$ induces a canonical divisor on a general fibre D of the induced fibration $\tilde{f} : \tilde{S} \to B$. The holomorphic map associated with $M(\mathcal{A})$ factors thorugh $\mathbb{P}(\mathcal{E})$ and we have a holomorphic map $\tilde{h} : \tilde{S} \to \mathbb{P}(\mathcal{E})$ over h which satisfies $M(\mathcal{A}) = \tilde{h}^*(T_{\mathcal{E}} + \pi^*\mathcal{A})$. Then $V = \tilde{h}(\tilde{S})$. When f is non-hyperelliptic, V is birational to S and any general fibre of $V \to B$ can be identified with a canonical curve of genus g.

Put $M = \tilde{h}^*T_{\mathcal{E}}$. Since $M - \mu_\ell D$ is nef by Lemma 1.1 and since $\mu_\ell \geq 0$ by Lemma 2.1, (2), we see that M is nef.

We have (at least) two methods for studying the slope of non-hyperelliptic fibrations, which we recall below.

(I) Relative projections ([X])

Here we recall Xiao's method. For each $1 \leq i \leq \ell$, the natural sheaf homomorphism $f^*\mathcal{E}_i \subset f^*f_*\omega_{S/B} \to \omega_{S/B}$ induces a rational map $h_i : S \to \mathbb{P}(\mathcal{E}_i)$ over B. We let $\sigma_i : S_i \to S$ be a composition of blowing-ups which eliminates the indeterminacy of h_i. We choose a non-singular model S^* which dominates all the S_i's, and we denote by $\rho : S^* \to S$ the natural map. Let M_i be the pull-back to S^* of $T_{\mathcal{E}_i}$. Let D^* be a general fibre of the induced fibration

$S^* \to B$ and put $N_i = M_i - \mu_i D^*$, $Z_i = \rho^* K_{S/B} - M_i$. Then Z_i is effective and, by Lemma 1.1, N_i is a nef $\mathbb{Q}$-divisor. Note that, modulo exceptional curves, Z_ℓ corresponds to Z. In particular, we see that $Z_\ell D^* = 0$. Note also that $Z_i - Z_\ell$ corresponds to the inverse image of the center B_i of a relative projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}_i)$.

Put $d_i = N_i D^*(1 \leq i \leq \ell)$. Note that $d_\ell = 2g-2$. For $1 \leq i \leq \ell - 1$, d_i is the degree of an $r_i - 1$ dimensional linear system $|M_i||_{D^*}$ and hence Clifford's theorem shows that $d_i \geq 2r_i - 1$ unless $(d_1, r_1) = (0,1)$ when f is non-hyperelliptic. We recall two inequalities which follow from [X, Lemma 2].

$$K_{S/B}^2 \geq \sum_{i=1}^{\ell-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 4(g-1)\mu_\ell, \tag{2.1}$$

$$K_{S/B}^2 \geq (d_1 + 2g - 2)(\mu_1 - \mu_\ell) + 4(g-1)\mu_\ell. \tag{2.2}$$

(II) Counting relative hyperquadrics

Let $f : S \to B$ be a non-hyperelliptic fibration. We can assume that $\mathcal{A}$ is taken so that the holomorphic map associated with $|T_{\mathcal{E}} + \pi^*\mathcal{A}|$ gives a quadratically normal embedding of $\mathbb{P}(\mathcal{E})$. Then we have

$$h^0(2M(\mathcal{A})) \geq h^0(2T_{\mathcal{E}} + 2\pi^*\mathcal{A}) - h^0(\mathcal{I}_V(2T_{\mathcal{E}} + 2\pi^*\mathcal{A})) \tag{2.3}$$

where $\mathcal{I}_V$ denotes the ideal sheaf of V if $\mathbb{P}(\mathcal{E})$. Since the restriction map $H^0(M(\mathcal{A})) \to H^0(K_D)$ is surjective, we can lift all the quadric relations in $S^2H^0(K_D)$ to $S^2H^0(M(\mathcal{A}))$. Since $H^0(M(\mathcal{A})) \simeq H^0(T_{\mathcal{E}} + \pi^*\mathcal{A})$, it follows that $H^0(\mathcal{I}_V(2T_{\mathcal{E}} + \pi^*\mathcal{A})) \to H^0(\mathcal{I}_{D'}(2))$ is surjective, where $\mathcal{I}_{D'}$ is the ideal sheaf of $D' = \tilde{h}(D)$ in $F \simeq \mathbb{P}^{g-1}$. Since f is non-hyperelliptic, we have $h^0(\mathcal{I}_{D'}(2)) = (g-2)(g-3)/2$. Put

$$x_i = \max\{\deg\delta | \mathrm{rk}\{H^0(\mathcal{I}_V(2T_{\mathcal{E}} - \pi^*\delta)) \to H^0(\mathcal{I}_{D'}(2))\} \geq i\},$$

where δ ranges over $\mathrm{Pic}(B)$. Then $x_1 \geq x_2 \geq \cdots \geq x_k$, where $k = (g-2)(g-3)/2$. We can find a set of divisors $\{\delta_i\}$ with $\deg\delta_i = x_i (1 \leq i \leq k)$ and relative hyperquadrics Q_i linearly equivalent to $2T_{\mathcal{E}}\pi^*\delta_i$ such that they induce a basis for $H^0(\mathcal{I}_{D'}(2))$. Furthermore, we can assume that $H^0(\mathcal{I}_V(2T_{\mathcal{E}} + 2\pi^*\mathcal{A}))$ is generated by them in the sense that

$$H^0(\mathcal{I}_V(2T_{\mathcal{E}} + 2\pi^*\mathcal{A})) = \bigoplus_i H^0(2\mathcal{A} + \delta_i)Q_i.$$

Since $\mathcal{A}$ is sufficiently ample, $2\mathcal{A} + \delta_i$ cannot be a special divisor. Hence

$$h^0(\mathcal{I}_V(2T_{\mathcal{E}} + 2\pi^*\mathcal{A})) = \sum_i x_i + (g-2)(g-3)(2a + 1 - b)/2,$$

where $a = \deg \mathcal{A}$. We have

$$h^0(2T_{\mathcal{E}} + 2\pi^*\mathcal{A}) = (g+1)\Delta(f) + g(g+1)(2a+1b)/2$$

by the Riemann-Roch theorem. Therefore, we can re-write (2.3) as

$$h^0(2M(\mathcal{A})) \geq (g+1)\Delta(f) + 3(g-1)(2a+1-b) - \sum_i x_i. \tag{2.4}$$

LEMMA 2.3. *$h^1(E + Z - M(\tilde{\mathcal{A}})) \leq M(E+Z)/2$, where $\tilde{\mathcal{A}} = 2\mathcal{A} - K_B$.*

PROOF. Since $E + Z$ has no horizontal components with respect to $\tilde{f}$, we can find an effective divisor $\mathcal{A}_1$ on B satisfying $\tilde{f}^*\mathcal{A}_1 \geq E + Z$. We assume that $\deg \mathcal{A}_1$ is minimal among those divisors with such a property, and put $L_1 = \tilde{f}^*\mathcal{A}_1$. Since $\mathcal{A}$ is sufficiently ample, there exists an irreducible non-singular member $L_2 \in |M(\tilde{\mathcal{A}} - \mathcal{A}_1)|$. Put $L_3 = (L_1 - E - Z) + L_2$. Since $L_3 \geq L_2$, we can assume that $|L_3|$ induces a birational map of $\tilde{S}$ onto the image. Then, by Ramanujam's theorem, we get $h^1(-L_3) = h^0(\mathcal{O}_{L_3}) - 1$. Consider the cohomology long exact sequences for

$$0 \to \mathcal{O}_{L_3} \to \mathcal{O}_{L_1+L_2}(E+Z) \to \mathcal{O}_{E+Z}(E+Z) \to 0,$$

$$0 \to \mathcal{O}_{L_1}(E+Z-L_2) \to \mathcal{O}_{L_1+L_2}(E+Z) \to \mathcal{O}_{L_2}(E+Z) \to 0.$$

From these, we get

$$h^0(\mathcal{O}_{L_3}) \leq h^0(\mathcal{O}_{L_1+L_2}(E+Z)) \leq h^0(\mathcal{O}_{L_1}(E+Z-L_2)) + h^0(\mathcal{O}_{L_2}(E+Z)).$$

Since, on fibres, $[E + Z]$ is trivial and L_2 looks like a canonical divisor, we have that

$$h^0(\mathcal{O}_{L_1}(E+Z-L_2)) = h^0(\mathcal{O}_{L_1}(-L_2)) = 0.$$

Hence we get

$$h^1(-L_3) \leq h^0(\mathcal{O}_{L_2}(E+Z)) - 1 \leq L_2(E+Z)/2 = M(E+Z)/2$$

by Clifford's theorem. □

Since $\chi(2M(\mathcal{A})) = M^2 + \Delta(f) + 3(g-1)(2a+1-b) - M(E+Z)$ by the Riemann-Roch theorem, and since we have $h^i(2M(\mathcal{A})) = h^{2-i}(E + Z - M(\tilde{\mathcal{A}}))$, it follows from (2.4) and Lemma 2.3 that

$$M^2 \geq g\Delta(f) - \sum_{i=1}^{(g-2)(g-3)/2} x_i + \frac{1}{2}M(E+Z). \tag{2.5}$$

Since $K_{S/B}^2 = M^2 + (\sigma^* K_{S/B} + M)Z$, we have in particular

$$K_{S/B}^2 \geq g\Delta(f) - \sum_{i=1}^{(g-2)(g-3)/2} x_i. \tag{2.6}$$

REMARK 2.4. There is another version due to Reid [R2]. It is easy to see that $f_*(\omega_{S/B}^{\otimes 2})$ is a locally free sheaf of rank $3g-3$ and degree $K_{S/B}^2 + \Delta(f)$. If f is non-hyperelliptic, then the sheaf homomorphism $S^2(f_*\omega_{S/B}) \to f_*(\omega_{S/B}^{\otimes 2})$ is generically surjective by Max Noether's theorem. Hence we have an exact sequence of sheaves on B:

$$0 \to \mathcal{R} \to S^2(f_*\omega_{S/B}) \to f_*(\omega_{S/B}^{\otimes 2}) \to \mathcal{T} \to 0, \tag{2.7}$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{R}$ is a locally free sheaf of rank $(g-2)(g-3)/2$. Since $\deg S^2(f_*\omega_{S/B}) = (g+1)\Delta(f)$, it follows from (2.7) that

$$K_{S/B}^2 = g\Delta(f) - \deg \mathcal{R} + \operatorname{length} \mathcal{T} \geq g\Delta(f) - \deg \mathcal{R}. \tag{2.8}$$

We close the section giving an application of method (II).

LEMMA 2.5. *Let $f : S \to B$ be a non-hyperelliptic fibration of genus g. Suppose that $f_*\omega_{S/B}$ is semi-stable. Then*

$$K_{S/B}^2 \geq \left(5 - \frac{6}{g}\right)\Delta(f). \tag{2.9}$$

PROOF. We give here two proofs using (2.6) and (2.8), respectively.

(1) Since $Q_i \equiv 2T_{\mathcal{E}} - x_i F$ is effective, it follows from Lemma 1.1 that $x_i \leq 2\Delta(f)/g$ since $f_*\omega_{S/B}$ is semi-stable. Hence we get (2.9) from (2.6).

(2) Since $f_*\omega_{S/B}$ is semi-stable, so is $S^2(f_*\omega_{S/B})$ (see, e.g., [G]). Hence we have $\mu(\mathcal{R}) \leq \mu(S^2(f_*\omega_{S/B}))$, that is, $g \deg \mathcal{R} \leq (g-2)(g-3)\Delta(f)$. Substituting this in (2.8) we get (2.9). □

PROPOSITION 2.6. *Let $f : S \to B$ be a non-hyperelliptic fibration of genus g, and assume that it is not locally trivial. Then $\lambda(f) > 4 - 4/g$. Hence the conjecture of Xiao* [X, *Conjecture* 1] *is true.*

PROOF. Xiao [X, Theorem 2] showed that $\lambda(f) > 4 - 4/g$ when $f_*\omega_{S/B}$ is not semi-stable, by using (2.1) and (2.2). Hence we can assume that $f_*\omega_{S/B}$ is semi-stable. But then, we have a stronger inequality (2.9). □

LEMMA 2.7. *Let $f : S \to B$ be a non-hyperelliptic fibration of genus $g \geq 4$. Assume that the Harder-Narashimhan filtration of $f_*\omega_{S/B}$ is $0 \subset \mathcal{E}_1 \subset f_*\omega_{S/B}$ and* $\operatorname{rk}(\mathcal{E}_1) = 1$. *Then* (2.9) *holds without equality.*

PROOF. Since all the Q_i's have rank ≥ 3, we have $x_i \leq 2\mu_2 < 2\Delta(f)/g$ by Lemma 1.5. Hence (2.6) implies (2.9). □

3. - The case $g = 3$

In this section, we consider non-hyperelliptic fibrations of genus 3 in order to supplement [K2] and give a geometric interpretation of length $\mathcal{T}$ in (2.8). Some results here overlap with [H3].

Let $f : S \to B$ be a non-hyperelliptic fibration of genus 3 and let the notation be as in 2.2. The relative canonical image V is a divisor on $\mathbb{P}(\mathcal{E})$ linearly equivalent to $4T_{\mathcal{E}} - \pi^* \mathcal{A}_0$ for some divisor $\mathcal{A}_0$ on B. Put $a = \deg \mathcal{A}$ and $a_0 = \deg \mathcal{A}_0$. Since $\tilde{h}$ is a birational holomorphic map onto the image and since $M(\mathcal{A}) = \tilde{h}^*(T_{\mathcal{E}} + \pi^* \mathcal{A})$, we have

$$M(\mathcal{A})^2 = (T_{\mathcal{E}} + \pi^* \mathcal{A})^2 (4T_{\mathcal{E}} - \pi^* \mathcal{A}_0) = 4\Delta(f) + 8a - a_0.$$

Hence

$$M^2 - 3\Delta(f) = \Delta(f) - a_0. \tag{3.1}$$

Since $K_{S/B}^2 = M^2 + (\sigma^* K_{S/B} + M)Z$, (3.1) is equivalent to

$$K_{S/B}^2 - 3\Delta(f) = \Delta(f) - a_0 + (\sigma^* K_{S/B} + M)Z. \tag{3.2}$$

In view of (2.8), the right hand side of (3.2) is nothing but length $\mathcal{T}$ (since $\mathcal{R} = 0$).

Let C be a general member of $|M(\mathcal{A})|$. Then

$$\begin{aligned} 2g(C) - 2 &= M(\mathcal{A})(\tilde{K} + M(\mathcal{A})) \\ &= 8\Delta(f) + 12a - 2a_0 + 8(b-1) + M(E+Z). \end{aligned}$$

On the other hand, the arithmetic genus of $C' = \tilde{h}(C)$ is given by

$$\begin{aligned} 2p_a(C') - 2 &= (T_{\mathcal{E}} + \pi^* \mathcal{A})(4T_{\mathcal{E}} - \pi^* \mathcal{A}_0)(2T_{\mathcal{E}} + \pi^*(\det \mathcal{E} + \omega_B + \mathcal{A} - \mathcal{A}_0)) \\ &= 12\Delta(f) + 8(b-1) + 12a - 6a_0. \end{aligned}$$

Hence

$$p_a(C') - g(C) = 2\Delta(f) - 2a_0 - M(E+Z)/2 \geq 0. \tag{3.3}$$

Note further that the conductor of $C \to C'$ is given by

$$\tilde{h}^* \omega_{C'} - \omega_C = \tilde{f}^*(\det \mathcal{E} - \mathcal{A}_0)|_C - (E+Z)|_C. \tag{3.4}$$

The following is a refinement of [K2, Theorem 1.2].

Lemma 3.1. *Let the notation be as above. For a non-hyperelliptic fibration $f : S \to B$ of genus 3, $K_{S/B}^2 \geq M^2 \geq 3\Delta(f)$ holds. If $M^2 = 3\Delta(f)$, then $K_{S/B}^2 = 3\Delta(f)$.*

PROOF. It follows from (3.3) that $\Delta(f) \geq a_0$. Hence we have $M^2 \geq 3\Delta(f)$ by (3.1). Assume that $M^2 = 3\Delta(f)$, that is, $a_0 = \Delta(f)$. Then, by (3.3), we have $M(E+Z) = 0$. Since $0 \leq (\sigma^* K_{S/B})Z = MZ + Z^2 = Z^2$, Hodge's index theorem shows that $Z = 0$. Hence (3.2) implies that $K_{S/B}^2 = 3\Delta(f)$. □

The above equalities are sometimes useful in determining the singularity of V.

THEOREM 3.2. *When $K_{S/B}^2 = 3\Delta(f)$, V has at most rational double points, and it is linearly equivalent to $4T_{\mathcal{E}} - \pi^* \det \mathcal{E}$. When $K_{S/B}^2 > 3\Delta(f)$, V is non-normal. In particular, if $K_{S/B}^2 = 3\Delta(f) + 1$, V has at most rational double points except for a double conic curve described in* [K1, § 9].

PROOF. Assume first that $K_{S/B}^2 = 3\Delta(f)$. Then $a_0 = \Delta(f)$, and $|L(\mathcal{A})|$ has no base locus as we saw in the proof of Lemma 3.1. We have $p_a(C') = g(C)$ by (3.3). It follows that V has at most isolated singular points. We have

$$\begin{aligned}\chi(\mathcal{O}_V) &= \chi(\mathcal{O}_{\mathbb{P}(\mathcal{E})}) - \chi(-V) \\ &= 1 - b + \chi(T_{\mathcal{E}} + \pi^*(\det \mathcal{E} + K_B - \mathcal{A}_0)) \\ &= \Delta(f) + 2b - 2 = \chi(\mathcal{O}_S).\end{aligned}$$

Hence V has at most rational singular points. Since V is a hypersurface of a non-singular 3-fold $\mathbb{P}(\mathcal{E})$, it has at most rational double points. In particular, we have $\omega_{S/B} = h^*\omega_{V/B}$. Since $\omega_{V/B}$ is induced from $T_{\mathcal{E}} + \pi^*(\det \mathcal{E} - \mathcal{A}_0)$ and $K_{S/B} = h^* T_{\mathcal{E}}$, we see that $f^*(\det \mathcal{E} - \mathcal{A}_0)$ is linearly equivalent to zero. That is, $\mathcal{A}_0 = \det \mathcal{E}$.

It follows from (2.5), (3.1) and (3.3) that $p_a(C') - g(C) \geq M^2 - 3\Delta(f)$. Hence, by Lemma 3.1, we have $p_a(C') - g(C) > 0$ when $K_{S/B}^2 > 3\Delta(f)$. Since C' is obtained by cutting V by a general member of $|T_{\mathcal{E}} + \pi^* \mathcal{A}|$, it follows that V has more than isolated singular points.

Assume that $K_{S/B}^2 = 3\Delta(f) + 1$. By Lemma 3.1, we must have $M^2 = K_{S/B}^2$. It follows that $\Delta(f) = a_0 + 1$ and that $|L(\mathcal{A})|$ has no base locus. By (3.3) and (3.4), we have $p_a(C') - g(C) = 2$ and $h^*\omega_{C'} - \omega_C = f^*(\det \mathcal{E} - \mathcal{A}_0)|_C$. Hence C' has two double points contained in a unique fiber. Since V has no horizontal singular locus, we see that V has a double curve along a conic traced out by the singular points of C'. The rest follows from an argument in [K1, § 9]. □

REMARK 3.3. Horikawa [H2] announced that he classified degenerate fibres in genus 3 pencils. Though a part of it can be found in [H3], the whole body has not appeared yet.

4. - The case $g = 4$

In this section we show the following theorem with several lemmas.

THEOREM 4.1. *$f : S \to B$ be a non-hyperelliptic fibration of genus 4. Then*

$$K_{S/B}^2 \geq \frac{24}{7}\Delta(f). \tag{4.1}$$

If a general fibre of f has two distinct g_3^1's, then

$$K_{S/B}^2 \geq \frac{7}{2}\Delta(f). \tag{4.2}$$

For the proof of Theorem 4.1, we freely use the notation of the previous sections. In particular, we set $\mathcal{E} = f_*\omega_{S/B}$ and let $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$ be the Harder-Narashimhan filtration. By § 2, (II), there exists a relative hyperquadric $Q \equiv 2T_\mathcal{E} - xF$ through the relative canonical image V and

$$K_{S/B}^2 \geq 4\Delta(f) - x. \tag{4.3}$$

Since $\mathrm{rk}(Q) = 4$ if and only if a general fibre of f has two distinct g_3^1's, the second part of Theorem 4.1 is nothing but the following:

LEMMA 4.2. *If $\mathrm{rk}(Q) = 4$, then (4.2) holds.*

PROOF. In view of (4.3), we only have to check that $x \leq \Delta(f)/2$. But this is straightforward applying Lemma 1.6. Let $\nu_1, \dots, \nu_4$ be as in Remark 1.7. Then it follows from Lemma 1.6 that $x \leq \min\{\nu_2 + \nu_3, \nu_1 + \nu_4\}$. Hence $2x \leq \sum_{j=1}^{4} \nu_j = \Delta(f)$. □

LEMMA 4.3. *If $x \leq \mu_1 + \mu_\ell$, then (4.1) holds.*

PROOF. By (2.2), we have $K_{S/B}^2 \geq (d_1 + 6)(\mu_1 - \mu_\ell) + 12\mu_\ell \geq 6(\mu_1 + \mu_\ell)$. Hence (4.1) holds if $\mu_1 + \mu_\ell \geq (4/7)\Delta(f)$. Assume that $\mu_1 + \mu_\ell \leq (4/7)\Delta(f)$. Then $x \leq \mu_1 + \mu_\ell \leq (4/7)\Delta(f)$ and we get (4.1) from (4.3). □

Recall that a canonical curve of genus 4 cannot meet the vertex of the quadric through it, if the quadric is of rank 3.

LEMMA 4.4. *If $x > \mu_1 + \mu_\ell$, then $r_{\ell-1} = 3$ and $d_{\ell-1} = 6$.*

PROOF. If $x > \mu_1 + \mu_\ell$ then, by Lemma 1.3, Q is singular along $B_{\ell-1}$. Since $\mathrm{rk}(Q) \geq 3$ and $r_\ell = 4$, we must have $r_{\ell-1} = 3$ by Lemma 1.4.

We have $d_{\ell-1} = 6 - Z_{\ell-1}D^*$. Since $\mathrm{rk}(Q) = 3$ and since $B_{\ell-1}$ is the (relative) vertex of Q, we see that any general fibre of $V \to B$ cannot meet $B_{\ell-1}$. Since $Z_{\ell-1} - Z_\ell$ corresponds to $B_{\ell-1} \cap V$ as we remarked in § 2, (I), we have $(Z_{\ell-1} - Z_\ell)D^* = 0$. It follows that $d_{\ell-1} = 6$, since we always have $Z_\ell D^* = 0$. □

We complete the proof of Theorem 4.1 with the following:

LEMMA 4.5. *Even if* $x > \mu_1 + \mu_\ell$, (4.1) *holds.*

PROOF. We can assume that $r_{\ell-1} = 3$ and $d_{\ell-1} = 6$ by Lemma 4.4.

Assume that $\ell = 2$. Since $r_1 = 3$, we get $x \leq 2\mu_1$ by Lemma 1.5. On the other hand, since $d_1 = 6$, it follows from (2.1) that $K^2_{S/B} \geq 12(\mu_1 - \mu_2) + 12\mu_2 = 12\mu_1$. Hence, if $\mu_1 \geq (2/7)\Delta(f)$, we get (4.1). If $\mu_1 \leq (2/7)\Delta(f)$, then $x \leq (4/7)\Delta(f)$ and (4.1) follows from (4.3).

Assume that $\ell = 3$. Since $r_1 \leq 2$ and $r_2 = 3$, we have $x \leq \mu_1 + \mu_2$ by Lemma 1.5. Since $d_2 = 6$, it follows from (2.1) that

$$K^2_{S/B} \geq (d_1 + 6)(\mu_1 - \mu_2) + 12(\mu_2 - \mu_3) + 12\mu_3 \geq 6(\mu_1 + \mu_2).$$

Hence we can show (4.1) as we did in Lemma 4.3.

Assume that $\ell = 4$. By Lemma 1.5, we have $x \leq \min\{2\mu_2, \mu_1 + \mu_3\}$. Since $d_3 = 6$, it follows from (2.1) that

$$K^2_{S/B} \geq 3(\mu_1 - \mu_2) + 9(\mu_2 - \mu_3) + 12(\mu_3 - \mu_4) + 12\mu_4 = 3(\mu_1 + 2\mu_2 + \mu_3).$$

Hence $K^2_{S/B} \geq 6 \min\{2\mu_2, \mu_1 + \mu_3\}$ and we can show (4.1) as we did in Lemma 4.3. □

5. - The case $g = 5$

In this section we show the following theorem with several lemmas.

THEOREM 5.1. *Let $f : S \to B$ be a non-hyperelliptic fibration of genus 5. When a general fibre of f is non-trigonal we have:*

$$K^2_{S/B} \geq M^2 \geq 4\Delta(f). \tag{5.1}$$

When a general fibre is trigonal we have:

$$K^2_{S/B} \geq \frac{40}{11}\Delta(f). \tag{5.2}$$

By (II), there are three relative hyperquadrics $Q_i \equiv 2T_{\mathcal{E}} - x_i F$, $1 \leq i \leq 3$, through V satisfying $x_1 \geq x_2 \geq x_3$ and

$$K^2_{S/B} \geq 5\Delta(f) - x, \quad x = \sum_{i=1}^{3} x_i. \tag{5.3}$$

LEMMA 5.2. *Let $f : S \to B$ be a non-hyperelliptic, non-trigonal fibration of genus 5. Then $K^2_{S/B} \geq M^2 \geq 4\Delta(f)$. If $M^2 = 4\Delta(f)$ then $K^2_{S/B} = 4\Delta(f)$.*

PROOF. Since a general fibre of f is non-trigonal, the relative canonical image V is an irreducible component of $\bigcap_{i=1}^{3} Q_i$. Hence, comparing degrees, we get $M(\mathcal{A})^2 \le (T_{\mathcal{E}} + \pi^*\mathcal{A})^2\Pi_i(2T_{\mathcal{E}} - x_iF)$, that is, $M^2 \le 8\Delta(f) - 4x$. Eliminating x from (2.5) using this, we get

$$M^2 \ge 4\Delta(f) + \frac{2}{3}M(E+Z)$$

from which the assertion follows immediately. □

In the rest of the section, we assume that $f: S \to B$ is a trigonal fibration of genus 5. Recall that, for a suitable choice of homogeneous coordinates $(X_0, \dots, X_4)$ on $\mathbb{P}^4$, any quadric through a trigonal canonical curve of genus 5 can be written as $c_1(X_1^2 - X_0X_2) + c_2(X_0X_4 - X_1X_3) + c_3(X_2X_3 - X_1X_4)$. Hence there is only one quadric of rank 3, and the vertices of any two independent members cannot meet. Without loosing generality, we can assume that $\mathrm{rk}(Q_1) \ge 3$, $\mathrm{rk}(Q_3) \ge \mathrm{rk}(Q_2) \ge 4$.

LEMMA 5.3. *If $r_i = 2$ then $x_3 \le 2\mu_{i+1}$.*

PROOF. Assume contrarily that $x_3 > 2\mu_{i+1}$. Then all the Q_j's vanish identically on B_i which is a $\mathbb{P}^2$-bundle on B. This contradicts the fact that $\cap Q_j$ induces a Hirzebruch surface on a general fibre of $\mathbb{P}(\mathcal{E}) \to B$. □

LEMMA 5.4. *Assume that there are rational numbers y_1 and y_2 satisfying $x \le y_1$, $K_{S/B}^2 \ge y_2$ and $8y_1 \le 3y_2$. Then (5.2) holds. In particular, (5.2) holds when $x \le 3(\mu_1 + \mu_\ell)$.*

PROOF. It follows from (5.3) that $K_{S/B}^2 \ge 5\Delta(f) - y_1$. Hence (5.2) holds when $y_1 \le (15/11)\Delta(f)$. Assume that $y_1 \ge (15/11)\Delta(f)$. Since $3y_2 \ge 8y_1$, we have $K_{S/B}^2 \ge y_2 \ge (8/3)y_1$. Hence (5.2) holds. In particular, since we have $K_{S/B}^2 \ge 8(\mu_1 + \mu_\ell)$ by (2.2), we get (5.2) if $x \le 3(\mu_1 + \mu_\ell)$. □

We can assume that $x > 3(\mu_1 + \mu_\ell)$. Then $x_1 > \mu_1 + \mu_\ell$.

LEMMA 5.5. *Assume that $x_1 > \mu_1 + \mu_\ell$. Then $x_i \le \mu_1 + \mu_\ell$ for $i = 2, 3$ and $r_{\ell-1} \ge 3$. If $r_{\ell-1} = 3$ then $d_{\ell-1} = 6$. If $r_{\ell-1} = 4$ then $d_{\ell-1} \ge 7$.*

PROOF. Since $x_1 > \mu_1 + \mu_\ell$, Q_1 is singular along $B_{\ell-1}$ by Lemma 1.3. Since $\mathrm{rk}(Q_1) \ge 3$, we have $r_{\ell-1} \ge 3$. Furthermore, Q_2 and Q_3 cannot be singular along $B_{\ell-1}$ as we remarked just before Lemma 5.3. Hence x_2, $x_3 \le \mu_1 + \mu_\ell$ by Lemma 1.3 again. If $r_{\ell-1} = 3$, then $\mathrm{rk}(Q_1) = 3$. Since a trigonal curve of genus 5 meets the vertex of rank 3 quadric through it at two points, we get $d_{\ell-1} = 8 - 2 = 6$. If $r_{\ell-1} = 4$ then $d_{\ell-1} \ge 7$ by Clifford's theorem. □

LEMMA 5.6. *Assume that $\ell = 2$ and $x_1 > \mu_1 + \mu_2$. Then $K_{S/B}^2 \ge (15/4)\Delta(f)$.*

PROOF. Since we have $x_1 \leq 2\mu_1$ by lemma 1.5 and $x_i \leq \mu_1+\mu_2$ for $i = 2,3$ by Lemma 5.5, we get $x \leq 4\mu_1 + 2\mu_2$.

Assume that $r_1 = 3$. We have $K_{S/B}^2 \geq 5\Delta(f) - 2(2\mu_1 + \mu_2)$ by (5.3). On the other hand, it follows from (2.2) that $K_{S/B}^2 \geq 14\mu_1 + 2\mu_2$, since $d_1 = 6$ by Lemma 5.5. Since $\Delta(f) = 3\mu_1 + 2\mu_2$, these inequalities imply $K_{S/B}^2 \geq (15/4)\Delta(f)$.

Assume that $r_1 = 4$. Since $\Delta(f) = 4\mu_1 + \mu_2$, we have $x \leq \Delta(f) + \mu_2 < \Delta(f) + \Delta(f)/5$. Hence we get $K_{S/B}^2 > (19/5)\Delta(f)$ from (5.3). □

We assume that $\ell \geq 3$ in the sequel.

LEMMA 5.7. *Assume that* $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ *and* $r_{\ell-1} = 3$. *Then* (5.2) *holds.*

PROOF. We have $\ell = 3$ or 4. Note that $\mathrm{rk}(Q_1) = 3$ and $\mathrm{rk}(Q_i) \geq 4$ for $i = 2,3$.

We have $x_1 \leq \mu_1 + \mu_{\ell-1}$ by Lemma 1.5, $x_2 \leq \mu_1 + \mu_\ell$ by Lemma 5.5 and $x_3 \leq 2\mu_{\ell-1}$ by Lemmas 1.6 and 5.3. Hence $x \leq 2\mu_1 + 3\mu_{\ell-1} + \mu_\ell$. On the other hand, applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_{\ell-1}, \mu_\ell\}$, we get

$$K_{S/B}^2 \geq 6(\mu_1 - \mu_{\ell-1}) + 14(\mu_{\ell-1} - \mu_\ell) + 16\mu_\ell = 6\mu_1 + 8\mu_{\ell-1} + 2\mu_\ell,$$

since $d_1 \geq 0$, $d_{\ell-1} = 6$ and $d_\ell = 8$. We have $\mu_1 > \mu_\ell$. It follows that

$$8(2\mu_1 + 3\mu_{\ell-1} + \mu_\ell) < 3(6\mu_1 + 8\mu_{\ell-1} + 2\mu_\ell).$$

Applying Lemma 5.4, we see that (5.2) holds without equality. □

LEMMA 5.8. *Assume that* $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ *and* $r_{\ell-1} = 4$. *If* $r_{\ell-2} \leq 2$, *then* (5.2) *holds.*

PROOF. We have $\ell = 3$ or 4. Since $r_{\ell-2} \leq 2$, it follows from Lemma 1.4 that $x_1 \leq \mu_1 + \mu_{\ell-1}$. We have $x_2 \leq \mu_1 + \mu_\ell$ by Lemma 5.5. Furthermore, we can assume that $x_3 \leq 2\mu_{\ell-1}$ by Lemmas 1.6 and 5.3. Hence $x \leq 2\mu_1 + 3\mu_{\ell-1} + \mu_\ell$. On the other hand, applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_{\ell-1}, \mu_\ell\}$, we get

$$K_{S/B}^2 \geq 7(\mu_1 - \mu_{\ell-1}) + 15(\mu_{\ell-1} - \mu_\ell) + 16\mu_\ell = 7\mu_1 + 8\mu_{\ell-1} + \mu_\ell,$$

since $d_1 \geq 0$, $d_{\ell-1} \geq 7$ and $d_\ell = 8$. It follows from $\mu_1 > \mu_\ell$ that

$$8(2\mu_1 + 3\mu_{\ell-1} + \mu_\ell) < 3(7\mu_1 + 8\mu_{\ell-1} + \mu_\ell).$$

Hence, as in the the previous lemma, we see that (5.2) holds without equality. □

LEMMA 5.9. *Assume that* $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ *and* $r_{\ell-1} = 4$. *If* $r_{\ell-2} = 3$ *and* $x_1 > \mu_1 + \mu_{\ell-1}$, *then* (5.2) *holds.*

PROOF. Since $x_1 > \mu_1 + \mu_{\ell-1}$, $B_{\ell-2}$ is the relative vertex of Q_1 and it follows that $d_{\ell-2} = 6$.

Assume that $\ell = 3$. Since $d_1 = 6$, we have $K^2_{S/B} \geq 14\mu_1 + 2\mu_3$ by (2.2). By Lemmas 1.5 and 5.5, we have $x_1 \leq 2\mu_1$ and $x_2, x_3 \leq \mu_1 + \mu_3$. Hence $x \leq 4\mu_1 + 2\mu_3$. We can show that $K^2_{S/B} > (15/4)\Delta(f)$ using (5.3).

Assume that $\ell = 4$ or 5. We have $x_1 \leq \mu_1 + \mu_{\ell-2}$ and $x_2 \leq \mu_1 + \mu_\ell$ by Lemmas 1.5 and 5.5, respectively. Furthermore, we have $x_3 \leq 2\mu_{\ell-2}$ by Lemmas 1.6 and 5.3. Hence $x \leq 2\mu_1 + 3\mu_{\ell-2} + \mu_\ell$. On the other hand, applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_{\ell-2}, \mu_\ell\}$, we get

$$K^2_{S/B} \geq 6(\mu_1 - \mu_{\ell-2}) + 14(\mu_{\ell-2} - \mu_\ell) + 16\mu_\ell = 6\mu_1 + 8\mu_{\ell-2} + 2\mu_\ell,$$

since $d_1 \geq 0$, $d_{\ell-2} = 6$ and $d_\ell = 8$. Hence we see that (5.2) holds without equality as in the proof of Lemma 5.7. □

We finish the proof of Theorem 5.1 with the following:

LEMMA 5.10. *Assume that* $\ell \geq 3$, $x > 3(\mu_1 + \mu_\ell)$ *and* $r_{\ell-1} = 4$. *If* $r_{\ell-2} = 3$ *and* $x_1 \leq \mu_1 + \mu_{\ell-1}$, *then* (5.2) *holds.*

PROOF. Assume that $\ell = 3$. Since $x \leq (\mu_1 + \mu_2) + 2(\mu_1 + \mu_3) = 3\mu_1 + \mu_2 + 2\mu_3$ and $\Delta(f) = 3\mu_1 + \mu_2 + \mu_3$, it follows from (5.3) that $K^2_{S/B} > (19/5)\Delta(f)$, which is stronger than (5.2).

Assume that $\ell = 4$ and $r_1 = 1$. Then $x_1 \leq 2\mu_2$ and $x_2, x_3 \leq \mu_1 + \mu_4$ by Lemmas 1.5 and 5.5. Since $x_1 > \mu_1 + \mu_4$, we have in particular $\mu_1 + \mu_4 < 2\mu_2$. We have $x \leq 2(\mu_1 + \mu_2 + \mu_4)$. Applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_2, \mu_4\}$ we get

$$K^2_{S/B} \geq 5(\mu_1 - \mu_2) + 13(\mu_2 - \mu_4) + 16\mu_4 = 5\mu_1 + 8\mu_2 + 3\mu_4,$$

since $d_1 \geq 0$, $d_2 \geq 5$ and $d_4 = 8$. Since $6(\mu_2 - \mu_4) + (2\mu_2 - \mu_1 - \mu_4) > 0$, we have $3(5\mu_1 + 8\mu_2 + 3\mu_4) > 16(\mu_1 + \mu_2 + \mu_4)$ and therefore (5.2) holds without equality.

Assume that $\ell = 4$ and $r_1 = 2$. We get $x_1 \leq \mu_1 + \mu_3$ and $x_2, x_3 \leq \mu_1 + \mu_4$ by Lemma 5.5. Hence $x \leq 3\mu_1 + \mu_3 + 2\mu_4$. Applying [X, Lemma 2] for the sequence $\{\mu_1, \mu_3, \mu_4\}$, we get

$$K^2_{S/B} \geq 10(\mu_1 - \mu_3) + 15(\mu_3 - \mu_4) + 16\mu_4 > 8\mu_1 + 7\mu_3 + \mu_4,$$

since $d_1 \geq 3$, $d_3 \geq 7$ and $d_4 = 8$. Since $\mu_3 > \mu_4$, we have $3(8\mu_1 + 7\mu_3 + \mu_4) > 8(3\mu_1 + \mu_3 + 2\mu_4)$ and, therefore, (5.2) holds without equality.

Assume that $\ell = 5$. We have $x_1 \leq \min\{2\mu_2, \mu_1 + \mu_4\}$, $x_2 \leq \min\{\mu_2 + \mu_3, \mu_1 + \mu_5\}$ and $x_3 \leq \min\{2\mu_3, \mu_1 + \mu_5\}$ by Lemmas 1.5, 1.6, 5.3 and 5.5. If $\mu_2 + \mu_3 \leq \mu_1 + \mu_5$, then we get $x \leq 2\mu_2 + (\mu_1 + \mu_5) + 2\mu_3 \leq 3(\mu_1 + \mu_5)$ which contradicts the assumption of the lemma. Hence $\mu_2 + \mu_3 > \mu_1 + \mu_5$. Then we have $x \leq (\mu_1 + \mu_4) + (\mu_1 + \mu_5) + 2\mu_3 = 2\mu_1 + 2\mu_3 + \mu_4 + \mu_5$. Note that we have $11x \leq 15\Delta(f) = 15\sum \mu_i$ when $7(\mu_1 + \mu_3) \leq 15\mu_2 + 4(\mu_4 + \mu_5)$. In particular, (5.2) will follow from (5.3) if $2\mu_2 \geq \mu_1 + \mu_3$. So, we may assume that $2\mu_2 < \mu_1 + \mu_3$. Then, since $\mu_3 - \mu_5 > \mu_1\mu_2$ and $\mu_1 - \mu_2 > \mu_2 - \mu_3$, we get

$$3(\mu_3 - \mu_5) > (\mu_1 - \mu_2) + (\mu_2 - \mu_3) + \mu_3 - \mu_5 = \mu_1 - \mu_5 > \mu_1 - \mu_4.$$

We apply [X, Lemma 2] for the sequence $\{\mu_1, \mu_3, \mu_4, \mu_5\}$ to get

$$K_{S/B}^2 \geq 5(\mu_1 - \mu_3) + 12(\mu_3 - \mu_4) + 15(\mu_4 - \mu_5) + 16\mu_5 = 5\mu_1 + 7\mu_3 + 3\mu_4 + \mu_5,$$

since $d \geq 0$, $d_3 \geq 5$, $d_4 \geq 7$ and $d_5 = 8$. Note that we have

$$\begin{aligned} & 3(5\mu_1 + 7\mu_3 + 3\mu_4 + \mu_5) \\ = \ & 8(\mu_1 + \mu_4) + 8(\mu_1 + \mu_5) + 16\mu_3 + 5(\mu_3 - \mu_5) - (\mu_1 - \mu_4) \\ > \ & 8x + 2(\mu_3 - \mu_5). \end{aligned}$$

Hence (5.2) can be shown using Lemma 5.4. □

Inequality (5.1) gives us a hope that the following holds.

CONJECTURE. $K_{S/B}^2 \geq 4\Delta(f)$ holds for a Petri general fibration.

6. - Application

Let S be a canonical surface and X its canonical image. The intersection of all hyperquadrics through X is called the quadric hull of X and denoted by $Q(X)$. The dimension of the irreducible component of $Q(X)$ containing X is called the *quadric dimension* of S. A conjecture of Miles Reid [R1] states that every canonical surface with $K^2 < 4p_g - 12$ has quadric dimension 3.

THEOREM 6.1. *Let S be an irregular canonical surface and assume that the image of the Albanese map of S is a curve. Then $K^2 \geq 3\chi(\mathcal{O}_S) + 10(q-1)$. When $K^2 \leq (10/3)\chi(\mathcal{O}_S) + (122/7)(q-1)$, the Albanese pencil is a non-hyperelliptic fibration of genus* 3. *When $K^2 \leq \min\{(10/3)\chi(\mathcal{O}_S) + (122/7)(q-1),\ 4p_g - 12 + q\}$, the quadric dimension of S is* 3 *and the irreducible component of $Q(X)$ containing the canonical image X is birationally a threefold scroll over a curve.*

PROOF. The first inequality was remarked in [K2]. By the assumption, the Albanese map induces a non-hyperelliptic fibration $f: S \to B$, where B is the Albanese image and hence $g(B) = q$. If f has genus g, then it follows from Proposition 2.6 that $K_{S/B}^2 > (4 - 4/g)\Delta(f)$, that is, $K^2 > (4 - 4/g)(\chi(\mathcal{O}_S) + (g+1)(q-1))$. We have $g \leq 5$ when $K^2 \leq (10/3)\chi(\mathcal{O}_S) + (122/7)(q-1)$. The cases $g = 4$ and $g = 5$ can be excluded by Theorems 4.1 and 5.1, respectively. Hence we have $g = 3$. As for the last assertion, we remark that the restriction map

$H^0(K) \to H^0(K_D)$ is surjective and, therefore, X is contained in a threefold scroll over a curve (possibly a cone). Then [K4, Theorem 8.3] applies. □

LEMMA 6.2. *Let S be a minimal surface of general type with a non-linear pencil. If $K^2 < 4\chi(\mathcal{O}_S)$ then the base of the pencil is a curve of genus $q(S)$. If S is a canonical surface with a non-linear pencil, then*

$$K^2 \geq \min\{4\chi(\mathcal{O}_S), 3\chi(\mathcal{O}_S) + 10(q-1)\} \tag{6.1}$$

PROOF. Let $f : S \to B$ be the fibration associated with the non-linear pencil. If $q > b = g(B)$, then it follows from [X, Theorem 1] that $K_{S/B}^2 \geq 4\Delta(f)$ which implies that $K^2 \geq 4\chi(\mathcal{O}_S)$ since $b > 0$. Hence we have $b = q$ when $K^2 < 4\chi(\mathcal{O}_S)$.

Assume that S is a canonical surface. Then f is non-hyperelliptic. Hence we have $K_{S/B}^2 \geq 3\Delta(f)$ by Corollary 2.6 and Lemma 3.1. When $K^2 < 4\chi(\mathcal{O}_S)$, this implies that $K^2 \geq 3\chi(\mathcal{O}_S) + 10(q-1)$, since $b = q$ and $g \geq 3$. □

THEOREM 6.3. *Let S be a canonical surface with a non-linear pencil. If $K^2 \leq \min\{(10/3)\chi(\mathcal{O}_S), 4p_g - 12 + q\}$ then S has quadric dimension 3.*

PROOF. Let $f : S \to B$ be the fibration associated with the non-linear pencil. By Lemma 6.2, we have $g(B) = q$. Since $K^2 \leq (10/3)\chi(\mathcal{O}_S)$, one can show that f is a non-hyperelliptic fibration of genus 3 as in Theorem 6.1. The rest follows from [K4, Theorem 8.3]. □

COROLLARY 6.4. *Let S be a canonical surface with $q = 1$ and $K^2 \leq (10/3)\chi(\mathcal{O}_S)$. Then the Albanese map gives a non-hyperelliptic fibration of genus 3. If $K^2 \leq \min\{(10/3)\chi, 4\chi - 11\}$ then S has quadric dimension 3.*

This and Theorem 3.2 give a picture of canonical surfaces with $q = 1$ and $K^2 = 3\chi$ or $3\chi+1$, which is quite similar to the regular case (see [AK] and [K1]): they have a pencil of non-hyperelliptic curves of genus 3. Another "similar" result is the following theorem which will be shown in the next section (see [K3] for the regular case).

THEOREM 6.5. *The moduli space of even canonical surfaces with $K^2 = 3\chi(\mathcal{O}_S) + 1$ and $q = 1$ is non-reduced.*

REMARK 6.6. Ashikaga [A] constructed a series of canonical surfaces with a non-hyperelliptic fibration of genus 3. See also [K2].

7. - Proof of Theorem 6.5

In this section we show Theorem 6.5. Though the proof is essentially the same as in [K3], there is one point which is unclear: a vector bundle on an elliptic curve is not necessarily decomposable.

Let S be a canonical surface with $K^2 = 3\chi(\mathcal{O}_S) + 1$, $q(S) = 1$ and let $f : S \to B = \mathrm{Alb}(S)$ be the Albanese map. By Corollary 6.4, any general fibre D of f is a non-hyperelliptic curve of genus 3. Assume further that S is an even surface, that is, there is a line bundle L with $K = 2L$. Since L^2 is even and $K^2 = 4L^2$, there exists a non-negative integer n satisfying

$$\chi = 8n + 5, \quad L^2 = 6n + 4. \tag{7.1}$$

By the Riemann-Roch theorem, we have

$$2h^0(L) - h^1(L) = -L^2/2 + \chi = 5n + 3. \tag{7.2}$$

Since D is of genus 3 we have $LD = 2$. Since D is non-hyperelliptic, we have $h^0(L|_D) = 1$ by Clifford's theorem. It follows that the rational map Φ_L associated with $|L|$ factors through $f : S \to B$. Hence there is a divisor $\mathcal{L}$ on B such that $L = [f^*\mathcal{L} + Z_L]$, where Z_L is the fixed part of $|L|$. We have $h^0(\mathcal{L}) \geq h^0(L) \geq (5n + 3)/2$ by (7.2). Hence $\deg \mathcal{L} \geq (5n + 3)/2$. Since $LD = 2$, we have $L^2 = 2 \deg \mathcal{L} + LZ_L$, that is,

$$LZ_L = 6n + 4 - 2 \deg \mathcal{L}. \tag{7.3}$$

Put $\mathcal{E} = f_*\omega_{S/B} = f_*\omega_S$ and let $\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$ be the Harder-Narashimhan filtration of $\mathcal{E}$ as usual. Let $\pi : \mathbb{P}(\mathcal{E}) \to B$ be the associated projective bundle. As we have seen in Section 3, we have a holomorphic map $h : S \to \mathbb{P}(\mathcal{E})$ satisfying $K = h^*T_\mathcal{E}$, and $V = h(S)$ is linearly equivalent to $4T_\mathcal{E} - \pi^*\mathfrak{A}_0$, $\deg \mathfrak{A}_0 = \chi - 1$.

LEMMA 7.1. *The vector bundle $f_*\omega_S$ splits as a direct sum of line bundles. More precisely, there are three line bundles $\mathcal{L}_i (0 \leq i \leq 2)$ on B satisfying $f_*\omega_S = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ and* $\deg \mathcal{L}_0 \leq n + 1$, $\deg \mathcal{L}_1 \geq 2n + 1$, $\deg \mathcal{L}_2 \geq 5n + 3$.

PROOF. Since $K = 2L = [2f^*\mathcal{L} + 2Z_L]$, we see that $|K - 2f^*\mathcal{L}|$ contains an effective divisor. Since $H^0(K) \simeq H^0(T_\mathcal{E})$, it follows that $H^0(T_\mathcal{E} - 2\pi^*\mathcal{L}) \neq 0$. Then, by Lemma 1.1, we get

$$\mu_1 \geq 2 \deg \mathcal{L} \geq \begin{cases} 5n + 3 & \text{if } n \text{ is odd,} \\ 5n + 4 & \text{if } n \text{ is even.} \end{cases}$$

Since $\deg \mathcal{E} = \chi = 8n + 5$ and since $\deg \mathcal{E} \geq \deg \mathcal{E}_1 = r_1\mu_1$, we must have $r_1 = 1$. Recall that V is numerically equivalent to

$$4T_\mathcal{E} - (\chi - 1)F = 4(T_\mathcal{E} - (2n + 1)F).$$

Since V cannot vanish identically on $\mathbb{P}(\mathcal{E}/\mathcal{E}_1)$, it follows from Lemma 1.1 that $\mu_1(\mathcal{E}/\mathcal{E}_1) \geq 2n + 1$. We have

$$\deg(\mathcal{E}/\mathcal{E}_1) = 8n + 5 - \deg \mathcal{E}_1 = 8n + 5 - \mu_1.$$

Hence $\deg(\mathcal{E}/\mathcal{E}_1) \leq 3n+2$ if n is odd, and $\deg \mathcal{E}/\mathcal{E}_1 \leq 3n+1$ if n is even. Since $\mu(\mathcal{E}/\mathcal{E}_1) < \mu_1(\mathcal{E}/\mathcal{E}_1)$, we see in particular that $\mathcal{E}/\mathcal{E}_1$ is not semi-stable. Let $0 \subset \mathcal{F}_1 \subset \mathcal{E}/\mathcal{E}_1$ be the Harder-Narashimhan filtration of $\mathcal{E}/\mathcal{E}_1$, and put $\mathcal{F}_2 = (\mathcal{E}/\mathcal{E}_1)/\mathcal{F}_1$. Then $\deg \mathcal{F}_1 \geq 2n+1$ and we have $\deg \mathcal{F}_2 \leq n+1$ if n is odd, and $\deg \mathcal{F}_2 \leq n$ if n is even. Hence $\deg \mathcal{F}_1 - \deg \mathcal{F}_2 > 0$ and $H^1(\mathcal{F}_1 - \mathcal{F}_2) = 0$. This implies that $\mathcal{E}/\mathcal{E}_1 = \mathcal{F}_1 \oplus \mathcal{F}_2$.

Since $\mathcal{E}_1$ and $\mathcal{F}_1$ are of positive degree, we have $h^1(\mathcal{E}) = h^1(\mathcal{E}/\mathcal{E}_1) = h^1(\mathcal{F}_2)$ from the cohomology long exact sequence for

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_1 \to 0.$$

On the other hand, since $\mathcal{E} = f_*\omega_S$, we have $h^1(\mathcal{E}) = 0$. Hence $h^1(\mathcal{F}_2) = 0$ and we have $\deg \mathcal{F}_2 \geq 0$. Then

$$\deg \mathcal{E}_1 - \deg \mathcal{F}_1 \geq \deg \mathcal{E}_1 - \deg \mathcal{E}/\mathcal{E}_1 \geq 2n+1.$$

It follows that $H^1((\mathcal{E}/\mathcal{E}_1)^* \otimes \mathcal{E}_1) = 0$. This implies that $\mathcal{E} = \mathcal{E}_1 \oplus (\mathcal{E}/\mathcal{E}_1)$. Now, put $\mathcal{L}_0 = \mathcal{F}_2$, $\mathcal{L}_1 = \mathcal{F}_1$ and $\mathcal{L}_2 = \mathcal{E}_1$. □

LEMMA 7.2. *Let the notation be as in Lemma* 7.1. *Then n is odd,* $\deg \mathcal{L}_0 = n+1$, $\deg \mathcal{L}_1 = 2n+1$ *and* $\deg \mathcal{L}_2 = 5n+3$. *Furthermore, V is linearly equivalent to* $4T_{\mathcal{E}} - 4\pi^*\mathcal{L}_1$.

PROOF. We can find sections X_i of $[T_{\mathcal{E}} - \pi^*\mathcal{L}_i]$ such that (X_0, X_1, X_2) forms a system of homogeneous coordinates on fibres of π. Assume that V is linearly equivalent to $4T_{\mathcal{E}} - \pi^*\mathcal{A}_0$ as in Section 3, and recall that $\deg \mathcal{A}_0 = \chi - 1 = 8n+4$. Then the equation of V can be written as

$$\sum \phi_{ij} X_0^{4-i-j} X_1^i X_2^j = 0,$$

where ϕ_{ij} is a section of $L_{ij} = (4-i-j)\mathcal{L}_0 + i\mathcal{L}_1 + j\mathcal{L}_2 - \mathcal{A}_0$. If $\deg L_{01} < 0$, then V has a multiple curve along $X_1 = X_2 = 0$. Hence $\deg L_{01} \geq 0$, that is, $3\deg \mathcal{L}_0 + \deg \mathcal{L}_2 \geq 8n+4$. Since $\deg \mathcal{L}_0 + \deg \mathcal{L}_1 + \deg \mathcal{L}_2 = 8n+5$, we get $2\deg \mathcal{L}_0 \geq \deg \mathcal{L}_1 - 1$. Since $\deg \mathcal{L}_0 \leq n+1$ and $\deg \mathcal{L}_1 \geq 2n+1$, we have either

(i) $\deg \mathcal{L}_0 = n$, $\deg \mathcal{L}_1 = 2n+1$, $\deg \mathcal{L}_2 = 5n+4$, or

(ii) $\deg \mathcal{L}_0 = n+1$, $\deg \mathcal{L}_1 = 2n+1$, $\deg \mathcal{L}_2 = 5n+3$.

We show that (i) is impossible. Assume by contradiction that (i) is the case. Note that V contains an elliptic curve B' defined by $X_1 = X_2 = 0$. We have $\deg L_{01} = 0$. If $\phi_{01} = 0$, then V would have a multiple curve along B', which is impossible. Hence L_{01} must be trivial and ϕ_{01} is a non-zero constant. But then V is non-singular in a neighbourhood of B'. This is impossible, since V is singular along a fibre which meets B'.

Hence we have (ii). In particular, it follows from the proof of Lemma 7.1 that n is odd. We know that V is defined by an equation of the form

$$\phi_{40}X_1^4 + X_2(\phi_{01}X_0^3 + \cdots + \phi_{04}X_2^3) = 0. \tag{7.4}$$

Since $\deg L_{40} = 0$ and ϕ_{40} cannot be zero, L_{40} is a trivial bundle, which means that $\mathcal{A}_0$ is linearly equivalent to $4\mathcal{L}_1$. □

Put $n = 2k - 1$.

LEMMA 7.3. *$\mathcal{L}_2 = 2\mathcal{L}$, $LZ_L = 2k$, $DZ_L = 2$ and $Z_L^2 = -8k + 2$.*

PROOF. In the proof of Lemma 7.1, we have

$$\deg \mathcal{L}_2 = \mu_1 \geq 2 \deg \mathcal{L} = 5n + 3.$$

Since $\deg \mathcal{L}_2 = 5n + 3 = 10k - 2$, we get $\deg \mathcal{L} = 5k - 1$. Recall that $H^0(T_{\mathcal{E}} - 2\pi^*\mathcal{L}) \neq 0$. Since any element of $H^0(T_{\mathcal{E}} - 2\pi^*\mathcal{L})$ can be written as ψX_2 with $\psi \in H^0(\mathcal{L}_2 - 2\mathcal{L})$, and since $\mathcal{L}_2 - 2\mathcal{L}$ is of degree 0, we see that $\mathcal{L}_2 = 2\mathcal{L}$.

Since $\deg \mathcal{L} = 5k - 1$, it follows from (7.3) that $LZ_L = n + 1 = 2k$. Since $LD = 2$, we have $DZ_L = 2$. We have $2k = LZ_L = (\deg \mathcal{L})DZ_L + Z_L^2$. Hence $Z_L^2 = -8k + 2$. □

Note that we have $K = h^*((X_2) + \pi^*\mathcal{L}_2) = h^*(X_2) + 2f^*\mathcal{L}$. Hence (X_2) corresponds $2Z_L$. We can show the following as in [K3, Lemma 2.3] using (7.4).

LEMMA 7.4. *$Z_L = 2G_0 + G_1$, where G_0 is a non-singular elliptic curve and G_1 is a (-2)-curve.*

Since every even canonical surface with $K^2 = 3\chi + 1$ and $q = 1$ has a (-2)-curve G_1, we have Theorem 6.5 by a result of Burns-Wahl [BW] (see [K3, Proof of Theorem 1.5]).

EXAMPLE. Let $\mathcal{M}$ be a line bundle of degree 2 on an elliptic curve B which induces the double covering $B \to \mathbb{P}^1$. Choose a point $P \in B$ with $2P \in |\mathcal{M}|$. Put $\mathcal{L}_0 = k\mathcal{M}$, $\mathcal{L}_1 = (2k-1)\mathcal{M} + [P]$, $\mathcal{L}_2 = (5k-1)\mathcal{M}$ and $\mathcal{E} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$. Let $\xi \in H^0([P])$ define P, and choose sufficiently general members $\Phi_0 \in H^0(2T_{\mathcal{E}} - 2\pi^*\mathcal{L}_1)$ and $\Phi_1 \in H^0(3T_{\mathcal{E}}\pi^*(4\mathcal{L}_1 - \mathcal{L}_2 + 2[P]))$. We consider a surface defined in the total space of $[2T_{\mathcal{E}} - \pi^*(2\mathcal{L}_1 + [P])] \to \mathbb{P}(\mathcal{E})$ by

$$\xi w - \Phi_0 = w^2 - X_2\Phi_1 = 0.$$

where w is a fibre coordinate. It is easy to see that it has only one rational double point of type A_1 and the minimal resolution is an even canonical surface with $K^2 = 3\chi + 1$, $q = 1$ and $\chi = 16k - 3$ (see [K3]).

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