SURVEY PAPER

NON-INSTANTANEOUS IMPULSES IN CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Recent modeling of real world phenomena give rise to Caputo type fractional order differential equations with non-instantaneous impulses. The main goal of the survey is to highlight some basic points in introducing non-instantaneous impulses in Caputo fractional differential equations. In the literature there are two approaches in interpretation of the solutions. Both approaches are compared and their advantages and disadvantages are illustrated with examples. Also some existence results are derived.

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1. Introduction

Fractional calculus is the theory of integrals and derivatives of arbitrary non-integer order, which unifies and generalizes the concepts of ordinary differentiation and integration. For more details on geometric and physical interpretations of fractional derivatives and for a general historical perspective, we refer the reader to the monographs [18, 31, 33], to the survey papers [37, 38], and the cited references therein.

Impulsive differential equations arise from real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are natural in biology, physics, engineering, etc.

In the literature there are two popular types of impulses:

- *instantaneous impulses* - the duration of these changes is relatively short compared to the overall duration of the whole process. For ordinary differential equations with impulses we refer the reader to the monographs [26], [34] and the cited references therein. There are also many recent contributions on fractional order differential equations with instantaneous impulses (see, for example, [4], [5], [9], [16], [17], [20], [24], [32], [40], [43]);
- *non-instantaneous impulses* - an impulsive action, starting abruptly at a fixed point and its action continues on a finite time interval. This kind of impulse is observed in lasers, and in the intravenous introduction of drugs in the bloodstream. Hernandez and O'Regan ([22]) introduced this new class of abstract differential equations where the impulses are not instantaneous, and they investigated the existence of mild and classical solutions. For recent works, we refer the reader to [6], [7], [8], [10], [12], [19], [23], [27], [29], [30], [35], [44].

The main goal of the survey is to present basic points in introducing non-instantaneous impulses in Caputo type fractional differential equations. In the literature there are two approaches in the interpretation of solutions. Both approaches are compared and their advantages/disadvantages are illustrated with examples. The existence of non-instantaneous impulsive fractional differential equations and the corresponding sufficient conditions are discussed using both approaches.

2. Preliminary notes on fractional derivatives and equations

Fractional calculus generalizes the derivatives and integrals of a function of a non-integer order [18, 31, 33].

In many applications in science and engineering, the fractional order q is often less than 1, **so we restrict $q \in (0,1)$ everywhere in the paper**.

1. The Riemann–Liouville (RL) fractional derivative of order $q \in (0,1)$ of $m(t)$ is given by ([18], [31], [33])

$$
{}_{t_0}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s)ds, \quad t \geq t_0,
$$

where $\Gamma(.)$ denotes the Gamma function.

2. The Caputo fractional derivative of order $q \in (0,1)$ is defined by ([18], [31])

$$
{}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s)ds, \quad t \geq t_0. \qquad (2.1)
$$

The Caputo and the Riemann-Liouville formulations coincide when the initial conditions are zero. Note, that the RL derivative is meaningful under weaker smoothness requirements, but the derivative in the Caputo sense allows an easier interpretation of conventional initial conditions.

3. Ordinary differential equations versus Caputo fractional differential equations

We compare some properties of the ordinary differential equations (ODE) and Caputo-type fractional differential equations (FrDE).

I. Ordinary differential equations. Consider the ODE

$$x'(t) = f(t, x) \quad \text{for} \quad t \geq \tau, \tag{3.1}$$

with the initial condition

$$x(\tau) = \tilde{x}_0, \tag{3.2}$$

where $\tilde{x}_0 \in \mathbb{R}^n$.

Denote the solution of the IVP for ODE (3.1), (3.2) by $x(t; \tau, \tilde{x}_0)$.

Now consider the same ODE (3.1) with different initial time $\tau_1 > \tau$, i.e. consider (3.1) with the following initial condition

$$x(\tau_1) = \tilde{u}_0, \tag{3.3}$$

where $\tilde{u}_0 \in \mathbb{R}^n$.

REMARK 3.1.   For the IVP for ODE (3.1), (3.3) note that the right side part $f(t, x)$ has to be defined only for $t \geq \tau_1$.

We can look at IVP for ODE (3.1), (3.3) in two different ways:

(*A1 for ODE*).   From the general solution $x(t; \tau, c)$ of (3.1) with initial condition $x(\tau) = c$ (c is an arbitrary constant) we choose the one $x(t; \tau, c_1)$ with $x(\tau_1; \tau, c_1) = \tilde{u}_0$. We call it a solution of the IVP (3.1), (3.3) for $t \geq \tau_1$ and denote it by $x(t; \tau_1, \tilde{u}_0)$. Then using $x(t_1; \tau, c_1) = c_1 + \int_\tau^{\tau_1} f(s, x(s; \tau, c_1))ds$ we obtain that the solution $x(t) = x(t; \tau_1, \tilde{u}_0)$ of (3.1), (3.3) will satisfy

$$\begin{aligned} x(t) &= c_1 + \int_\tau^t f(s, x(s; \tau, c_1))ds \\ &= \tilde{u}_0 - \int_\tau^{\tau_1} f(s, x(s; \tau, c_1))ds + \int_\tau^t f(s, x(s; \tau, c_1))ds, \quad t \geq \tau_1. \end{aligned} \tag{3.4}$$

(*A2 for ODE*).   Consider (3.1), (3.3) as a new IVP and its solution, defined for $t \geq \tau_1$, and we call this a solution of the IVP (3.1), (3.3). Then

the solution will satisfy the following integral equation

$$x(t) = \tilde{u}_0 + \int_{\tau_1}^{t} f(s, x(s))ds, \quad t \geq \tau_1. \tag{3.5}$$

REMARK 3.2.   In the general case for ODE's both points of view do not differ since $\int_{\tau}^{t} f(s, x(s))ds = \int_{\tau}^{\tau_1} f(s, x(s))ds + \int_{\tau_1}^{t} f(s, x(s))ds$, i.e. $x(t; \tau, \tilde{x}_0) \equiv x(t; \tau_1, \tilde{u}_0)$ for $t \geq \tau_1$ with $x(\tau_1; \tau, \tilde{x}_0) = \tilde{u}_0$.

II. Caputo-type fractional differential equations. Consider the fractional differential equation (FrDE) with Caputo fractional derivatives

$$_{\tau}^{c}D^q x(t) = f(t, x) \quad \text{for} \quad t \geq \tau \tag{3.6}$$

with initial condition

$$x(\tau) = \tilde{x}_0, \tag{3.7}$$

where $\tilde{x}_0 \in \mathbb{R}^n$.

Denote the solution of the IVP for FrDE (3.6), (3.7) by $x(t; \tau, \tilde{x}_0)$.

The solution $x(t) = x(t; \tau, \tilde{x}_0)$ of IVP for FrDE (3.6), (3.7) satisfies the fractional Volterra integral equation

$$x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t - s)^{q-1} f(s, x(s))ds, \quad t \geq \tau. \tag{3.8}$$

Now change the initial time to $\tau_1 > \tau$ and consider FrDE (3.6) with the initial condition (3.3). Then as in the ordinary case (see (A1 for ODE) and (A2 for ODE)), there are two approaches to define the solution of the new IVP for the Caputo-type FrDE:

(*A1 for FrDE*).   From the set of all solutions $x(t; \tau, c)$ of FrDE (3.6) with initial condition $x(\tau) = c$, (c is an arbitrary constant), we choose the one $x(t; \tau, c_1)$ with $x(\tau_1; \tau, c_1) = \tilde{u}_0$. We call it a solution of the IVP (3.6), (3.3) for $t \geq \tau_1$ and denote it by $x(t; \tau_1, \tilde{u}_0)$. Therefore, $x(t; \tau_1, \tilde{u}_0) \equiv x(t; \tau, c_1)$ for $t \geq \tau_1$. Then using Eq. (3.8) with $t = \tau_1$, $\tilde{x}_0 = c_1$, the solution $x(t) = x(t; \tau_1, \tilde{u}_0)$ of IVP for FrDE (3.6), (3.3) will satisfy the following integral equation

$$\begin{aligned} x(t) &= c_1 + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t - s)^{q-1} f(s, x(s; \tau, c_1))ds \\ &= \tilde{u}_0 - \frac{1}{\Gamma(q)} \int_{\tau}^{\tau_1} (\tau_1 - s)^{q-1} f(s, x(s; \tau, c_1))ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t - s)^{q-1} f(s, x(s; \tau, c_1))ds, \quad t \geq \tau_1. \end{aligned} \tag{3.9}$$

(compare Eq. (3.4) in the ordinary case $q = 1$ with Eq. (3.9) in the fractional case).

K. Diethelm pointed out that the problem consisting of Eqs. (3.6) and (3.3) is more closely related to a boundary value problem than to an initial value problem. This is a contrast to the situation observed for first-order differential equations (see Section 6 in [18]).

REMARK 3.3.   Using (*A1 for FrDE*) we keep one of the basic properties of ODEs, namely, $x(t; \tau_1, x(\tau_1; \tau, c)) = x(t; \tau, c)$ for $t \geq \tau_1$.

(*A2 for FrDE*).   Set up a new initial value problem for $t \geq \tau_1$ whose solution will satisfy the following fractional integral equation

$$x(t) = \tilde{u}_0 + \frac{1}{\Gamma(q)} \int_{\tau_1}^{t} (t - s)^{q-1} f(s, x(s)) ds, \quad t \geq \tau_1. \qquad (3.10)$$

(Compare Eq. (3.5) in the ordinary case $q = 1$ with Eq. (3.10) in the fractional case).

The fractional integral equation (3.10) is equivalent to the following Caputo fractional differential equation

$$^{c}_{\tau_1} D^q x(t) = f(t, x) \quad \text{for} \ \ t \geq \tau_1 \qquad (3.11)$$

with initial condition (3.3).

REMARK 3.4.   In the general case both points of view (*A1 for FrDE*) and (*A2 for FrDE*) differ, since

$$\int_{\tau}^{t} (t-s)^{q-1} f(s, x(s)) ds \neq \int_{\tau}^{\tau_1} (\tau_1-s)^{q-1} f(s, x(s)) ds + \int_{\tau_1}^{t} (t-s)^{q-1} f(s, x(s)) ds,$$

(compare with Remark 3.2).

REMARK 3.5.   Using (*A2 for FrDE*) we lose one of the basic properties of ODE's, namely, $x(t; \tau_1, x(\tau_1; \tau, c)) \neq x(t; \tau, c)$ for $t > \tau_1$.

REMARK 3.6.   In (*A2 for FrDE*) the right side part $f(t, x)$ of the IVP (3.6), (3.3) has to be defined only for $t \geq \tau_1$.

EXAMPLE 1. Consider FrDE(3.6) with $n = 1$, $\tau = 0$, $\tau_1 = 1$, $\tilde{u}_0 = 0$.
Case 1.   Let

$$f(t, x) \equiv h(t) = \begin{cases} 0 & t \in [0, 1], \\ 1 - t & t \geq 1. \end{cases}$$

Case 1.1. (Approach (*A1 for FrDE*)). According to formula (3.9) we get

$$
\begin{aligned}
x(t) &= 0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s) ds \\
&= \frac{1}{\Gamma(q)} \int_1^t (t-s)^{q-1} (1-s) ds, \quad t \geq 1.
\end{aligned}
\tag{3.12}
$$

Case 1.2. (Approach (*A2 for FrDE*)). According to (3.10) we get

$$
x(t) = 0 + \frac{1}{\Gamma(q)} \int_1^t (t-s)^{q-1} (1-s) ds, \quad t \geq 1.
\tag{3.13}
$$

In this particular case both solutions coincide.

Case 2.   Let $f(t,x) = 1 - t, \quad t \geq 0$.

Case 2.1. (Approach (*A1 for FrDE*)). According to formula (3.9) we get

$$
x(t) = -\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1}(1-s) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(1-s) ds, \quad t \geq 1. \tag{3.14}
$$

Case 2.2. (Approach (*A2 for FrDE*)). According to (3.10) the solution is given by (3.13).

In this particular case Eq. (3.14) differs from Eq. (3.13).

Therefore, the definition of the function $f(t,x)$ to the left of the initial point has no influence in (*A2 for FrDE*) (similar to the ODE situation) but it has a huge influence in (*A1 for FrDE*).

$\square$

REMARK 3.7.   Note that (*A1 for FrDE*) is similar in some sense to a boundary value problem, whereas (*A2 for FrDE*) is close to the idea of initial value problems defined and studied in the classical books [18], [31] (the initial time coincides with the lower limit of the Caputo fractional derivative).

EXAMPLE 2.   Consider FrDE (3.6) with $n = 1$, $q = 0.8$, $f(t,x) \equiv 1$, $\tau = 0$ and $\tau_1 > 0$.

The solution of IVP for FrDE (3.6), (3.7) is $x(t; 0, \tilde{x}_0) = \tilde{x}_0 + 1.25 \frac{t^{0.8}}{\Gamma(0.8)}$.

Using (*A1 for FrDE*) we get the solution of IVP for FrDE (3.6), (3.3), namely, $x(t; \tau_1, \tilde{u}_0) = \tilde{u}_0 + 1.25 \frac{t^{0.8} - \tau_1^{0.8}}{\Gamma(0.8)}$.

Using (*A2 for FrDE*) the solution of IVP for FrDE (3.6), (3.3) (or the equivalent (3.11), (3.3)) is $x(t; \tau_1, \tilde{u}_0) = \tilde{u}_0 + 1.25 \frac{(t-\tau_1)^{0.8}}{\Gamma(0.8)}$.

In this particular case both solutions differ.

$\square$

Now consider a case when $f(t,x)$ depends implicitly on x.

EXAMPLE 3. Consider the FrDE (3.6) with $n = 1$, $f(t, x) = x$, $\tau = 0$, $\tau_1 > 0$.

The solution of IVP for FrDE (3.6), (3.7) is $x(t; 0, \tilde{x}_0) = \tilde{x}_0 E_q(t^q)$.

Using (*A1 for FrDE*) and the general solution of IVP for FrDE (3.6), (3.7) we get $x(\tau_1; 0, c) = cE_q(\tau_1^q) = \tilde{u}_0$, or $c = \frac{\tilde{u}_0}{E_q(\tau_1^q)}$. Then from (3.9) we get the solution of IVP for FrDE (3.6), (3.3) $x(t; \tau_1, \tilde{u}_0) = \tilde{u}_0 \frac{E_q(t^q)}{E_q(\tau_1^q)}$.

Using (*A2 for FrDE*) the solution of IVP for FrDE (3.6), (3.3) (or the equivalent (3.11), (3.3)) is $x(t; \tau_1, \tilde{u}_0) = \tilde{u}_0 E_q((t - \tau_1)^q)$.

Solutions obtained by both approaches differ but in the ordinary case, ($q = 1$) the Mittag-Leffler function $E_1(t) = e^t$ and both solutions coincide.

$\square$

REMARK 3.8. Both approaches described above usually differ and give different solutions in the general case.

4. Non-instantaneous impulses in Caputo fractional differential equations

Now we set up the IVP for Caputo fractional differential equations with non-instantaneous impulses.

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_0 < t_i < s_i < t_{i+1}$, $i = 1, 2, \ldots$, and $t_0 \in \mathbb{R}_+$. Without loss of generality we assume $0 \leq t_0 < s_0$.

Consider the initial value problem (IVP) for the nonlinear *noninstantaneous impulsive fractional differential differential equation* (NIFrDE)

$$\begin{aligned}
&{}_{t_0}^{c}D^q x(t) = f(t, x) \quad \text{for} \quad t \in (t_k, s_k], \quad k = 0, 1, \ldots, \\
&x(t) = \phi_k(t, x(t), x(s_{k-1} - 0)) \quad \text{for} \quad t \in (s_{k-1}, t_k], \quad k = 1, 2, \ldots, \quad (4.1) \\
&x(t_0) = x_0,
\end{aligned}$$

where $x_0 \in \mathbb{R}^n$, $f : \cup_{k=0}^{\infty} [t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_{k-1}, t_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, ($k = 1, 2, 3, \ldots$).

DEFINITION 1. For NIFrDE (4.1) the intervals $(s_{k-1}, t_k]$, $k = 1, 2, \ldots$, are called intervals of non-instantaneous impulses, and $\phi_k(t, x, y)$, $k = 1, 2, \ldots$, are called non-instantaneous impulsive functions.

REMARK 4.1. If $t_k = s_{k-1}$, $k = 1, 2, \ldots$ then the IVP for NIFrDE (4.1) reduces to an IVP for impulsive fractional differential equations.

We give a brief description of the solution of IVP for NIFrDE (4.1). Based on the description of the solution of FrDE given in Section 3 we set up two approaches to the solutions of non-instantaneous fractional impulsive differential equations.

The definition of the solution $x(t; t_0, x_0)$ for $t > t_0$ depends on your point of view:

(*A1 for NIFrDE*). Let $f(t, x)$ be defined for $t \geq t_0$, $x \in \mathbb{R}^n$. Following approach (*A1 for FrDE*) and Eq. (3.9) with $\tau = t_0$, $\tau_1 = t_k, k = 1, 2, \ldots$ and $\tilde{u}_0 = x(t_k - 0; t_0, x_0) = \phi_k(t_k, x(t_k - 0; t_0, x_0), x(s_{k-1} - 0; t_0, x_0))$ given in Section 3, we get the solution of the IVP for NIFrDE (4.1) by the equalities (integral and algebraic)

$$x(t; t_0, x_0) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, x(s; t_0, x_0)) ds, & t \in (t_0, s_0] \\ \phi_k(t, x(t; t_0, x_0), x(s_{k-1} - 0; t_0, x_0)), \\ \qquad t \in (s_{k-1}, t_k], \ k = 1, 2, 3, \ldots, \\ \phi_k(t_k, x(t_k; t_0, x_0), x(s_{k-1} - 0; t_0, x_0)) \\ \qquad - \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} f(s, x(s; t_0, x_0)) ds \\ \qquad + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, x(s; t_0, x_0)) ds, \\ \qquad t \in (t_k, s_k], \ k = 1, 2, \ldots. \end{cases}$$

$$(4.2)$$

REMARK 4.2. In the special case $\phi_k(t, x(t), x(s_{k-1} - 0)) = g_k(t, x(t))$ the reduced formula of (4.2) is given in Section 9, [39], and Eq. (8), [44].

REMARK 4.3. The approach (*A1 for NIFrDE*) is applied in [19], [41] for studying periodic solutions, and in [20], [39], [44] for studying existence.

(*A2 for NIFrDE*). Let $f(t, x)$ be defined only for $t \in \cup_{k=0}^\infty [t_k, s_k]$ and $x \in \mathbb{R}^n$, i.e. it is defined only on the intervals without non-instantaneous impulses. Then using the approach (*A2 for FrDE*) and Eq. (3.10) with $\tau = t_0$, $\tau_1 = t_k$, $k = 1, 2, \ldots$ and $\tilde{u}_0 = x(t_k - 0; t_0, x_0) = \phi_k(t_k, x(t_k - 0; t_0, x_0), x(s_{k-1} - 0; t_0, x_0))$ given in Section 3 we get the solution of the IVP for NIFrDE (4.1) by the equalities (integral

and algebraic)

$$x(t;t_0,x_0) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s,x(s;t_0,x_0))ds, \\ \qquad \text{for } t \in [t_0,s_0], \\ \phi_k(t,x(t;t_0,x_0),x(s_{k-1}-0;t_0,x_0)) \\ \qquad \text{for } t \in (s_{k-1},t_k], \ k=1,2,\ldots, \\ \phi_k(t_k,x(t_k;t_0,x_0),x(s_{k-1}-0;t_0,x_0)) \\ \qquad +\frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s,x(s;t_0,x_0))ds \\ \qquad \text{for } t \in [t_k,s_k], \ k=1,2,\ldots. \end{cases} \qquad (4.3)$$

Following the approach (*A2 for FrDE*) the solution of the IVP for NIFrDE (4.1) is given by

$$x(t;t_0,x_0) = \begin{cases} X_k(t) \qquad \text{for } t \in (t_k,s_k], \ k=0,1,2,\ldots, \\ \phi_k(t,x(t;t_0,x_0),X_{k-1}(s_{k-1}-0)), \\ \qquad \text{for } t \in (s_{k-1},t_k], \quad k=1,2,\ldots, \end{cases} \qquad (4.4)$$

where
- the function $X_0(t)$ is the solution of IVP for FrDE (3.6), (3.7) with $\tau = t_0$ and $\tilde{x}_0 = x_0$;
- the function $X_k(t)$ is the solution of IVP for FrDE (3.6), (3.7) with $\tau = t_k$, and $\tilde{x}_0 = \phi_k(t_k,x(t_k;t_0,x_0),X_{k-1}(s_{k-1}-0))$, $k=1,2,\ldots$.

REMARK 4.4. The approach (*A2 for NIFrDE*) is applied in [1] for partial fractional differential equations, in [2], [3] for partial fractional inclusions, in [11] for stability, in [14] for abstract fractional differential equations, in [15], [25] for fractional integro-differential equations, in [21] for fractional functional differential equations of order $\alpha \in (1,2)$, in [27] for boundary value problems, and in [45] for existence.

Now we discuss the statement of the problem (4.1) and the type of impulsive functions. In some papers (see, for example, [25] [39], [43], [44]) the special case $\phi_k(t,x(t),x(s_{k-1}-0)) = g_k(t,x(t))$ is studied.

EXAMPLE 4. Consider the IVP for NIFrDE (4.1) with $n=1$, $t_0=0$, $q=0.8$, $f_k(t,x)=1$ for $t \in [t_k,s_k]$, $k=0,1,2,\ldots$.

We will discuss several cases w.r.t. the type of impulsive function $\phi_k(t,x,y)$.

Case 1. Let $\phi_k(t,x,y) = g_k(t)$ for $t \in [s_{k-1},t_k]$ $(k=1,2,3,\ldots)$.

(*A1 for NIFrDE*). According to Eq. (4.2) and Lemma 2.7, [44] the solution is

$$x(t; 0, x_0) = \begin{cases} x_0 + 1.25\frac{t^{0.8}}{\Gamma(0.8)}, & t \in (0, s_0], \\ g_k(t), & t \in (s_{k-1}, t_k], \ k = 1, 2, 3, \ldots, \\ g_k(t_k) + 1.25\frac{t^{0.8} - t_k^{0.8}}{\Gamma(0.8)}, & t \in (t_k, s_k], \ k = 1, 2, \ldots. \end{cases}$$

The solution depends on the initial value x_0 only on the interval $(0, s_0]$. Therefore, the solutions $x(t; 0, x_0)$ and $x(t; 0, \tilde{x}_0)$ with different initial values $x_0 \neq \tilde{x}_0$ will differ only on the first interval $(0, s_0]$ and $x(t; 0, x_0) \equiv x(t; 0, \tilde{x}_0)$ for all $t > s_0$.

(*A2 for NIFrDE*). According to Eq. (4.3) the solution is given by

$$x(t; 0, x_0) = \begin{cases} x_0 + 1.25\frac{t^{0.8}}{\Gamma(0.8)}, & t \in (0, s_0], \\ g_k(t), & t \in (s_{k-1}, t_k], \ k = 1, 2, 3, \ldots, \\ g_k(t_k) + 1.25\frac{(t - t_k)^{0.8}}{\Gamma(0.8)}, & t \in (t_k, s_k], \ k = 1, 2, \ldots. \end{cases}$$

Applying (*A2 for NIFrDE*) similarly to (*A1 for NIFrDE*) we obtain that the solutions $x(t; 0, x_0)$ and $x(t; 0, \tilde{x}_0)$ with $x_0 \neq \tilde{x}_0$ coincide for all $t > s_0$.

Case 2. Let $\phi_k(t, x, y) = g_k(t, y)$ for $t \in [s_{k-1}, t_k]$, ($k = 1, 2, 3, \ldots$), i.e. the impulsive conditions are $x(t) = g_k(t, x(s_{k-1} - 0))$, $t \in (s_{k-1}, t_k]$ and the impulsive functions depend on the value of the solution before the jump.

Then applying (*A1 for NIFrDE*) the solution is

$$x(t; 0, x_0) = \begin{cases} x_0 + 1.25\frac{t^{0.8}}{\Gamma(0.8)} & t \in (0, s_0] \\ g_1(t, x_0 + 1.25\frac{s_0^{0.8}}{\Gamma(0.8)}) & t \in (s_0, t_1] \\ g_1(t_1, x_0 + 1.25\frac{s_0^{0.8}}{\Gamma(0.8)}) + 1.25\frac{t^{0.8} - t_1^{0.8}}{\Gamma(0.8)} & t \in (t_1, s_1] \\ g_2\left(t, g_1(t_1, x_0 + 1.25\frac{s_0^{0.8}}{\Gamma(0.8)}) + 1.25\frac{s_1^{0.8} - t_1^{0.8}}{\Gamma(0.8)}\right) & t \in (s_1, t_2] \\ \ldots\ldots\ldots \end{cases}$$

Now the solution depends on the initial value x_0 for all $t \geq 0$. The same happens with the application of (A2 for NIFrDE).

Case 3. Let $\phi_k(t, x, y) = a_k x + b_k$ for $t \in [s_{k-1}, t_k]$, ($k = 1, 2, 3, \ldots$) where a_k, b_k are constants.

If $a_k = 1$, $b_k = 0$ then any function $x(t)$ will satisfy the impulsive condition $x(t) = x(t)$ for $t \in [s_{k-1}, t_k]$, ($k = 1, 2, 3, \ldots$) and obviously the IVP for NIFrDE (4.1) will have an infinite number of solutions.

If $a_k = 1$, $b_k \neq 0$ then no function $x(t)$ will satisfy the impulsive condition $x(t) = x(t) + b$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \dots)$ and obviously the IVP for NIFrDE (4.1) will have no solution.

If $a_k \neq 1$, $b_k = 0$ then the only function $x(t)$ that satisfies the impulsive condition $x(t) = ax(t)$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \dots)$ is the zero function, and therefore any solution of IVP for NIFrDE (4.1) will be zero on $(s_{k-1}, t_k]$, $(k = 1, 2, 3, \dots)$

If $a_k \neq 1$, $b_k \neq 0$ then there will be a unique function $x(t)$ that satisfies the impulsive condition $x(t) = ax(t) + b$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \dots)$ and we can talk about uniqueness of the solution IVP for NIFrDE (4.1).

Case 4. Let $\phi_k(t, x, y) = \arctan(x) + \cos(x) + y$ for $t \in [s_{k-1}, t_k]$ $(k = 1, 2, 3, \dots)$. Then the algebraic equation $x = \arctan(x) + \cos(x) + y$ could have more than one solution (for example if $y = 1$, then there are 5 constant solutions), i.e. we do not have uniqueness.

$\square$

REMARK 4.5.   In the general case the impulsive functions in (4.1) have to depend on the value of the solution before the impulse, i.e. the impulsive condition has to be given by the function $\phi_k(t, x(t), x(s_{k-1} - 0))$ for $t \in (s_{k-1}, t_k]$, $k = 1, 2, \dots$.

REMARK 4.6.   To discuss the existence and uniqueness of the solution of NIFrDE (4.1) we need the equation corresponding to the impulsive condition $x = \phi_k(t, x, y)$, $k = 1, 2, \dots$ to have a unique solution $x_k(t, y)$ for all $k = 1, 2, \dots$.

EXAMPLE 5.   Consider the IVP for NIFrDE (4.1) with $n = 1$, $t_0 = 0$ and $f(t, x) = Ax$, $t \geq 0$, i.e. consider

$$_{t_0}^{c} D^q x(t) = Ax \text{ for } t \in (t_k, s_k], \ k = 0, 1, 2, \dots,$$
$$[6pt]x(t) = \phi_k(t, x(t), x(s_{k-1} - 0)) \text{ for } \quad t \in (s_{k-1}, t_k], \ k = 1, 2, \dots, \quad (4.5)$$
$$x(0) = x_0,$$

where $x_0 \in \mathbb{R}$ and A is a constant.

(*A1 for NIFrDE*). According to Eq. (4.2) the solution $x(t; 0, x_0)$ of (4.5) is given by

$$
x(t;0,x_0) = \begin{cases}
x_0 E_q(At^q) & \text{for } t \in [0,s_0], \\
\phi_k(t,x(t),x(s_{k-1}-0)) \\
\quad \text{for } t \in (s_{k-1},t_k], \quad k=1,2,\dots, \\
\phi_k(t_k,x(t_k),x(s_{k-1}-0)) \\
\quad -A\frac{1}{\Gamma(q)}\int_0^{t_k}(t_k-s)^{q-1}x(s)ds \\
\quad +A\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}x(s)ds \\
\quad \text{for } t \in (t_k,s_k], \quad k=1,2,\dots.
\end{cases} \tag{4.6}
$$

(*A2 for NIFrDE*). Applying formulas (4.4) and (4.3), we obtain the solution $x(t;0,x_0)$ of (4.5):

$$
x(t;0,x_0) = \begin{cases}
x_0 E_q(At^q) & \text{for } t \in [0,s_0], \\
\phi_k(t,x(t),x(s_{k-1}-0)) \\
\quad \text{for } t \in (s_{k-1},t_k], \quad k=1,2,\dots, \\
\phi_k(t_k,x(t_k),x(s_{k-1}-0))E_q(A(t-t_k)^q) \\
\quad \times\left(\prod_{i=0}^{k-1}E_q(A(s_i-t_i)^q)\right) \\
\quad \text{for } t \in [t_k,s_k], \quad k=1,2,\dots.
\end{cases} \tag{4.7}
$$

Comparing (4.6) and (4.7) we can see the approach (A2 for NIFrDE) gives the solution in a closed form.

Case 1. Let $\phi_k(t,x,y) = a_k x$ for $t \in [s_{k-1},t_k]$ $(k=1,2,3,\dots)$, where $a_k \neq 1$ are constants. Since the solution of the equation $x = a_k x$ is the zero solution, then the non-instantaneous impulsive condition of (4.5) is reduced to $x(t) = 0$ for $t \in (s_{k-1},t_k]$, $k=1,2,\dots$.

According to (*A1 for NIFrDE*) and Eq. (4.6) the solution of (4.5) is

$$
x(t;0,x_0) = \begin{cases}
x_0 E_q(At^q) & \text{for } t \in [0,s_0], \\
0 & \text{for } t \in (s_{k-1},t_k], \quad k=1,2,\dots, \\
-A\frac{1}{\Gamma(q)}\int_0^{t_k}(t_k-s)^{q-1}x(s)ds \\
\quad +A\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}x(s)ds \\
\quad \text{for } t \in (t_k,s_k], \quad k=1,2,\dots.
\end{cases} \tag{4.8}
$$

According to (*A2 for NIFrDE*) and Eq. (4.7) the solution of (4.5) is

$$
x(t;0,x_0) = \begin{cases}
x_0 E_q(At^q) & \text{for } t \in [0,s_0], \\
0 & \text{for } t > s_0.
\end{cases} \tag{4.9}
$$

Consider the ordinary case $(q = 1)$ of (4.5), i.e. the non-instantaneous impulsive differential equation $x' = Ax$ for $t \in (t_k,s_k]$, $k = 0,1,2,\dots$ and $x(t) = 0$ for $t \in (s_{k-1}t_k]$, $k = 1,2,\dots$. Then the solution of the

corresponding IVP for non-instantaneous impulsive differential equation is

$$x(t; 0, x_0) = \begin{cases} x_0 e^{At} & \text{for } t \in [0, s_0], \\ 0 & \text{for } t > s_0. \end{cases} \qquad (4.10)$$

Eq. (4.9) is similar to Eq. (4.10), which shows the approach (A2 for NIFrDE) seems to be a natural generalization of the ordinary case.

Case 2. Let $\phi_k(t, x, y) = a_k y$, $a_k = \text{const}$, $k = 1, 2, 3, \ldots$.

Applying (*A1 for NIFrDE*) and Eq. (4.6), we obtain the solution of (4.5)

$$x(t; 0, x_0) = \begin{cases} x_0 E_q(At^q) & \text{for } t \in [0, s_0], \\ a_k x(s_{k-1} - 0) & \text{for } t \in (s_{k-1}, t_k], \quad k = 1, 2, \ldots, \\ a_k x(s_{k-1} - 0) & \\ \quad -A\frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} x(s) ds & \\ \quad +A\frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} x(s) ds & \\ \quad \text{for } t \in (t_k, s_k], \quad k = 1, 2, \ldots. \end{cases} \qquad (4.11)$$

Applying (*A2 for NIFrDE*) and Eq.(4.7), we get

$$x(t; 0, x_0) = \begin{cases} x_0 E_q(At^q) & \text{for } t \in [0, s_0], \\ x_0 \prod_{i=0}^{k-1} \left(a_{i+1} E_q(A(s_i - t_i)^q) \right) & \\ \quad \text{for } t \in (s_{k-1}, t_k], \quad k = 1, 2, \ldots, \\ x_0 E_q(A(t - t_k)^q) \prod_{i=0}^{k-1} \left(a_{i+1} E_q(A(s_i - t_i)^q) \right) & \\ \quad \text{for } t \in [t_k, s_k], \quad k = 1, 2, \ldots. \end{cases} \qquad (4.12)$$

The approach (*A2 for NIFrDE*) gives the explicit form for the solution.

Case 3. Let $A = 0$ and $\phi_k(t, x, y) = a_k(t)y$, $a_k : [t_k, s_k] \to \mathbb{R}$, $k = 1, 2, \ldots$.

Applying (*A1 for NIFrDE*) and Eq. (4.11) we obtain

$$x(t; 0, x_0) = \begin{cases} x_0 & \text{for } t \in [0, s_0], \\ x_0 a_k(t) \prod_{i=1}^{k-1} a_i(t_i) & \\ \quad \text{for } t \in (s_{k-1}, t_k], \quad k = 1, 2, \ldots, \\ x_0 \prod_{i=1}^{k} a_k(t_i) & \text{for } t \in (t_k, s_k], \quad k = 1, 2, \ldots. \end{cases} \qquad (4.13)$$

Applying (*A1 for NIFrDE*) and Eq. (4.11) we obtain (4.13), and therefore the formulas for the solutions, obtained by both approaches, coincide. □

EXAMPLE 6. Consider the IVP for the scalar NIFrDE (4.1) with $f(t, x) = \frac{1}{t - 0.5(t_k + s_{k-1})}$ for $t \geq t_0$. The function f is not defined on the whole interval $[s_{k-1}, t_k]$, $k = 1, 2, \ldots$.

Applying (*A1 for NIFrDE*) the integral $\int_{t_0}^{t}(t-s)^{q-1}f(t,s)ds$ is not convergent for all $t > s_0$, so the formula (4.2) is not applicable and this approach does not give a solution.

The application of (*A2 for NIFrDE*) and formula (4.3) causes no problem since we use the integral $\int_{t_k}^{t}(t-s)^{q-1}\frac{1}{s-0.5(t_k+s_{k-1})}ds$ for $t \in (t_k, s_k]$ which is convergent.

<div align="right">□</div>

REMARK 4.7. The approach (*A1 for NIFrDE*) and the application of formula (4.2) for the solution of NIFrDE (4.1) require the function $f(t,x)$ to be defined on the whole interval $[t_0, \infty)$ although this function is not used on $\bigcup_{k=1}^{\infty}(s_{k-1}, t_k]$. In [41] the conditions on the function f (such as the Lipschitz condition) are set up only on the intervals with no impulses $[t_k, s_k]$, $k = 1, 2, \ldots$ and this causes conflicts in the proofs (see Theorem 3.1-3.4 in [41]).

The approach (*A2 for NIFrDE*) requires the function $f(t,x)$ to be defined only on the interval $[t_k, s_k]$, $k = 1, 2, \ldots$ on which this function is applied.

5. Instantaneous impulses in Caputo fractional differential equations

Consider the special case when $s_{k-1} = t_k$, $k = 1, 2, \ldots$. Then any interval of non-instantaneous impulses is reduced to a point and any impulsive function ϕ_k is reduced to $\phi_k(t_k, x(t_k), x(t_k - 0))$ for $k = 1, 2, 3, \ldots$. Assume the equation $x = \phi_k(t_k, x, y)$, $k = 1, 2, \ldots$ has a unique solution w.r.t. x: $x = B_k(y)$ (see Remark 14). Then the impulsive condition could be presented as $x(t_k + 0) = B_k(x(t_k - 0))$ for $k = 1, 2, 3, \ldots$. Then problem (4.1) will be reduced to an IVP of Caputo-type impulsive fractional differential equation (IFrDE)

$$_{t_0}^{c}D^q x(t) = f(t,x) \text{ for } t \neq t_k, \ k = 1, \ldots,$$

$$x(t_k + 0) = x(t_k - 0) + I_k(x(t_k - 0)) \quad \text{for } k = 1, 2, \ldots, \qquad (5.1)$$

$$x(t_0) = x_0,$$

where $x_0 \in \mathbb{R}^n$, $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $(k = 1, 2, 3, \ldots)$ is defined by $I_k(y) = B_k(y) - y$, $k = 1, 2, \ldots$.

Then both approaches (*A1 for NIFrDE*) and (*A2 for NIFrDE*) given in Section 4 reduce to IFrDE (5.1):

(*A1 for IFrDE*). Eq. (4.2) is reduced and the formula for the solution of (5.1) is given by

$$x(t; t_0, x_0) = x(t_k - 0; t_0, x_0) + I_k(x(t_k - 0; t_0, x_0))$$
$$- \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} f(s, x(s; t_0, x_0)) ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x(s; t_0, x_0)) ds, \tag{5.2}$$
$$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots,$$

where $I_0(x) \equiv 0$ and $x(t_0 - 0; t_0, x_0) = x_0$.

Use induction and obtain that

$$x(t; t_0, x_0) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x(s; t_0, x_0)) ds$$
$$+ \sum_{j=1}^{k} I_j(x(t_j - 0; t_0, x_0)), \tag{5.3}$$
$$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots.$$

In the special case $t_0 = 0$, $I_k(x) = x + y_k$, $y_k = $ const and $f(t, x) = h(t)$ an implicit formula for the solution is (see Lemma 3.2, [20]):

$$x(t; 0, x_0) = x_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(q)} \int_{0}^{t} (t - s)^{q-1} h(s) ds$$
$$\text{for } t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots.$$

(*A2 for IFrDE*). Eq. (4.3) is reduced and the formula for the solution of (5.1) is given by

$$x(t; t_0, x_0) = x(t_k - 0; t_0, x_0) + I_k(x(t_k - 0; t_0, x_0))$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s; t_0, x_0)) ds, \tag{5.4}$$
$$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots.$$

where $I_0(x) \equiv 0$ and $x(t_0 - 0; t_0, x_0) = x_0$.
Use induction in (5.4) and obtain

$$x(t; t_0, x_0) = x_0 + \frac{1}{\Gamma(q)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} f(s, x(s; t_0, x_0)) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f_k(s, x(s; t_0, x_0)) ds \qquad (5.5)$$

$$+ \sum_{j=1}^{k} I_j(x(t_j - 0; t_0, x_0)),$$

$$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots.$$

REMARK 5.1.   Note that if the impulsive function is of the type $g_k(t, x(t))$ as in some published papers, the non-instantaneous impulsive condition in (4.1) reduces to $x(t_k + 0) = g_k(t_k, x(t_k + 0))$, $k = 1, 2, \ldots$ which does not give the amount of the jump at the impulsive point t_k for the unknown function.

EXAMPLE 7.   Consider the fractional comparison principle for FrDE (3.6) (see, for example, [28], [36]):

If $x, y \in C(\mathbb{R}_+)$ and $\,_0^C D^q x(t) \leq \,_0^C D^q y(t)$, then $x(0) \leq y(0)$ implies $x(t) \leq y(t)$, $t \geq 0$.

Now we will discuss the application of the comparison principle to the IFrDE (5.1).

Applying the approach (*A1 for IFrDE*) the solution of (5.1) will be (see, for example, p. 5, [36]) $x(t; t_0, x_0) = u_k(t; t_k, x_k^+)$ for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$, where $u_k(t; t_k, x_k^+)$ is the solution of the FrDE without impulses (3.6) with initial condition (3.3) with $\tau = t_0$, $\tau_1 = t_k$, $(\tilde{u}_0) = x_k^+ = x(t_k; t_0, x_0) + I_k(x(t_k; t_0, x_0))$. Therefore, the application of the above given fractional comparison principle on the interval $[t_k, t_{k+1}]$ is not allowed since $\,_{t_0}^C D^q x(t) \neq \,_{t_k}^C D^q x(t)$ (this was used in [36]).

Applying (*A2 for NIFrDE*), the solution of (5.1) will be $x(t; t_0, x_0) = u_k(t; t_k, x_k^+)$ for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$, where $u_k(t; t_k, x_k^+)$ is the solution of the FrDE without impulses (3.6) with initial condition (3.3) with $\tau = t_k$, $\tau_1 = t_k$, $\tilde{u}_0 = x_k^+ = x(t_k; t_0, x_0) + I_k(x(t_k; t_0, x_0))$. Therefore, the application of the above given fractional comparison principle on the interval $[t_k, t_{k+1}]$ is allowed since the lower limit of the fractional derivative and the initial time of the problem coincide.

$\square$

6. **Existence results**

In this section we consider IVP for NFrDE (4.1) when $n = 1$, i.e. the scalar case on the finite interval $J = [t_0, T]$, $T < \infty$ with $s_m = T$. We study existence for IVP for NIFrDE (4.1) using both approaches.

Introduce the following classes of functions

$$PC([t_0, T]) = \{u : [t_0, T] \to \mathbb{R} : \ u \in C([t_0, T]/\{s_k\}_{k=0}^m, \mathbb{R}) :$$
$$u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty, \ u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty,$$
$$k = 1, 2, \ldots, m\},$$
$$NPC^1([t_0, T]) = \{u : [t_0, T] \to \mathbb{R} : \ u \in C([t_0, T]/\{s_k\}_{k=0}^m, \mathbb{R}),$$
$$u \in C^1(\cup_{k=0}^m [t_k, s_k], \mathbb{R}) :$$
$$u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty, \ u'(s_k) = \lim_{t \uparrow s_k} u'(t) < \infty,$$
$$u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty, \ k = 1, 2, \ldots, m\}.$$

(*A1 for NIFrDE*). In [44] the NIFrDE (4.1) is studied when $f(t, x)$ is continuously defined on the whole interval J and the impulsive functions do not depend on the value of the solution before the impulse, i.e. $\phi_k(t, x, y) = g_k(t, x)$, $k = 1, 2, \ldots$ (see Examples 4, 5 and our comment in Remark 13).

Let $\Psi > 0$, $\varphi \in PC(J, \mathbb{R})$ and consider the fractional noninstantaneous differential inequalities

$$|_{t_0}^c D^q y(t) - f(t, y)| \leq \varphi(t) \text{ for } t \in (t_k, s_k], \ k = 0, 1, \ldots, m$$
$$|y(t) - \phi_k(t, y(t))| \leq \Psi \quad \text{for } t \in (s_{k-1}, t_k], \ k = 1, 2, \ldots, m. \tag{6.1}$$

THEOREM 6.1. (by *A1 for NIFrDE*, Theorem 4.2, [44]) *Let the following conditions be satisfied:*

1. *The function $f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a positive constant L_f such that $|f(t, x) - f(t, y)| \leq L_f |x - y|$ for each $t \in J$, $x, y \in \mathbb{R}$.*
2. *The functions $\phi_k(t, x) \in C([s_{k-1}, t_k] \times \mathbb{R}, \mathbb{R})$ and there exist constants L_k such that $|\phi_k(t, x_1) - \phi_k(t, x_2)| \leq L_k |x_1 - x_2|$ for each $t \in [s_{k-1}, t_k]$, $x_1, x_2 \in \mathbb{R}$.*
3. *The function $y(t)$ satisfies the fractional non-instantaneous differential inequalities (6.1) with $\Psi > 0$ is a constant, $\varphi \in C(J, \mathbb{R})$ is a nondecreasing function in $\bigcup_{i=0}^m [t_i, s_i]$ such that there exists a constant C_φ with*

$$\Big(\int_{t_0}^t (\varphi(s))^{\frac{1}{p}} ds \Big)^p \leq C_\varphi \varphi(t) \ \ \text{for} \ \ t \in J.$$

Then there exists a unique solution $\tilde{x}(t)$ of the IVP for NIFrDE (4.1) with $x_0 = y(t_0)$ such that it satisfies the integral-algebraic equations (4.2) and

$$|y(t) - \tilde{x}(t)| \leq \frac{\frac{2C_\varphi}{\Gamma(q)}\left(\frac{1-p}{q-p}\right)^{1-p}T^{q-p} + 1}{1 - M}(\varphi(t) + \Psi) \qquad (6.2)$$

for all $t \in J$ provided that $0 < p < q < 1$, where the constant $M = \max\{M_1, M_2\} < 1$, with

$$M_1 = \max\{L_k + \frac{L_f C_\varphi}{\Gamma(q)}\left(\frac{1-p}{q-p}\right)^{1-p}(s_k^{q-p} + t_k^{q-p}), \qquad (6.3)$$
$$k = 0, 1, 2 \ldots, m\} < 1$$

and

$$M_2 = \max\{L_k + \frac{L_f}{\Gamma(q+1)}(t_k^q + s_k^q)|, \ k = 1, 2 \ldots, m\} < 1. \qquad (6.4)$$

REMARK 6.1. The function φ in condition 3 of Theorem 6.1 and the fractional non-instantaneous differential inequalities (6.1) is used only on the intervals $[t_k, s_k], \ k = 0, 1, 2, \ldots, m$. However because of the application of (*A1 for NIFrDE*) this function has to be defined on the whole interval $J = [t_0, T]$.

REMARK 6.2. Note that the condition $M_1, M_2 < 1$ concerning constants M_1, M_2 given in (6.3) and (6.4) requires conditions on the impulsive points t_k, s_k and on the Lipschitz constants. Also this condition does not allow the result to be generalized to the infinite interval $[t_0, \infty)$.

REMARK 6.3. Note that in [44] the definition of Ulam–Hyers–Rassias stability w.r.t. (φ, Ψ) of NIFrDE (4.1) is given. However since the stability property is usually only meaningful for an infinite interval and Theorem 6.1 is true on a finite interval we will skip comments on this type of stability.

(*A2 for NIFrDE*). In our study we will use the result for FrDE (3.6).

LEMMA 6.1. (Theorem 3.1, [42]) *Let the following conditions be satisfied:*

1. *The function $f \in C(I, \mathbb{R}), \ I = [\tau, T]$ and there exists a positive constant L such that $|f(t, x) - f(t, y)| \leq L|x - y|, \ t \in I, \ x, y \in \mathbb{R}$.*
2. *The function $y \in C^1(I, \mathbb{R})$ satisfies the fractional differential equation*

$$|_\tau^c D^q y(t) - f(t, y(t)| \leq \varpi(t), \quad t \in I,$$

where the function $\varpi \in C(I, \mathbb{R})$ *is such that*

$$\frac{1}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} \varpi(s) ds \le K\varpi(t), \quad t \in I,$$

with $0 < KL < 1$.

Then there exists a unique function $x(t) \in C(I, \mathbb{R})$ *such that*

$$x(t) = y(\tau) + \frac{1}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} f(s, x(s)) ds, \quad t \in I \qquad (6.5)$$

and

$$|y(t) - x(t)| \le \frac{K}{1-KL} \varpi(t), \quad t \in I. \qquad (6.6)$$

Now we give sufficient conditions for existence of the NIFrDE (4.1) by an application of the approach (*A2 for NIFrDE*) for the solution.

Let $\Psi_k > 0$, $\varphi_k \in C([t_k, s_k], \mathbb{R})$, $k = 0, 1, \dots, m$ and consider the fractional non-instantaneous differential inequalities

$$
\begin{aligned}
&|{}_{t_k}^c D^q y(t) - f(t, y)| \le \varphi_k(t) \\
&\qquad \text{for } t \in (t_k, s_k], \ k = 0, 1, \dots, m \\
&|y(t) - \phi_k(t, y(t), y(s_{k-1} - 0))| \le \Psi_k \\
&\qquad \text{for } t \in (s_{k-1}, t_k], \ k = 1, 2, \dots, m.
\end{aligned}
\qquad (6.7)
$$

REMARK 6.4.   Note if $y(t)$ is a solution of the fractional non-instantaneous differential inequalities (6.7) then this solution satisfies the integral-algebraic inequalities

$$
\begin{cases}
|y(t) - y(t_k) + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s, y(s)) ds| \\
\quad \le \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \varphi_k(s) ds, \\
\qquad t \in (t_k, s_k], \ k = 0, 1, 2, \dots, m, \\
|y(t) - \phi_k(t, y(t), y(s_{k-1} - 0))| \le \Psi_k, \\
\qquad t \in (s_{k-1}, t_k], \ k = 1, 2, \dots, m.
\end{cases}
$$

THEOREM 6.2.   (by *A2 for NIFrDE*) *Let the following conditions be satisfied:*

1. *The function* $f \in C(\cup_{k=0}^m [t_k, s_k] \times \mathbb{R}, \mathbb{R})$ *and there exist positive constants* $L_k = L_k(f)$, $k = 0, 1, 2, \dots, m$, *such that* $|f(t, x) - f(t, y)| \le L_k|x - y|$ *for each* $t \in [t_k, s_k]$, $x, y \in \mathbb{R}$, $k = 0, 1, \dots, m$.
2. *The functions* $\phi_k(t, x, y) \in C([s_{k-1}, t_k] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$ *are such that for any* $t \in [s_{k-1}, t_k]$ *and* $y \in \mathbb{R}$ *there exists a unique solution* $x = \gamma_k(t, y)$ *of the algebraic equation* $x = \phi_k(t, x, y)$ *w.r.t.* x, *and there exist constants* $l_k = l_k(\phi_k) \in (0, 1)$, $k = 1, 2, \dots, m$

such that $|\phi_k(t, x_1, y_1) - \phi_k(t, x_2, y_2)| \leq l_k(|x_1 - x_2| + |y_1 - y_2|)$ for each $t \in [s_{k-1}, t_k]$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $k = 1, \ldots, m$.

3. The functions $\varphi_k \in C([t_k, s_k], \mathbb{R})$, $k = 0, 1, \ldots$ are nondecreasing functions and there exist constants $C_k = C_k(\varphi_k) > 0$, $L_k C_k < 1$, $k = 0, 1, \ldots, m$ such that

$$\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \varphi_k(s) ds \leq C_k \varphi_k(t), \quad t \in [t_k, s_k]. \tag{6.8}$$

Then for each solution $y(t) \in NPC^1([t_0, T], \mathbb{R})$ of the fractional differential inequality (6.7) there exists a solution $x(t)$ such that $x \in NPC^1([t_0, T], \mathbb{R})$ of the IVP for NIFrDE (4.1) with $x_0 = y(t_0)$ and it satisfies the integral-algebraic equations (4.3) and

$$|y(t) - x(t)| \leq$$
$$\begin{cases} \frac{C_0}{1 - C_0 L_0} \varphi_0(t) = F_0(s), & t \in (t_0, s_0], \\ \frac{C_k}{1 - C_k L_k} \varphi_k(t) + \frac{1}{1 - l_k} \frac{1}{1 - C_k L_k} \Big(\Psi_k + l_k F_{k-1}(s_{k-1}) \Big) = F_k(s), \\ \quad t \in (t_k, s_k], \ 1, 2, \ldots, m \\ \frac{1}{1 - l_k} \Big(\Psi_k + l_k F_{k-1}(s_{k-1}) \Big), & t \in (s_{k-1}, t_k], \ k = 1, 2, \ldots, m. \end{cases} \tag{6.9}$$

P r o o f. We will use induction.

Let $t \in [t_0, s_0]$. According to Lemma 6.1 with $\tau = t_0, T = s_0, L = L_0$, $K = C_0$ and $\varpi(t) = \varphi_0(t)$ there exists a solution $x_0(t) \in C([t_0, s_0], \mathbb{R})$ satisfying the integral equality

$$x_0(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x_0(s)) ds \tag{6.10}$$

and the inequality

$$|y(t) - x_0(t)| \leq \frac{C_0}{1 - C_0 L_0} \varphi_0(t) = F_0(t), \quad t \in [t_0, s_0]. \tag{6.11}$$

Let $t \in (s_0, t_1]$. Denote the solution of the algebraic equation $x = \phi_0(t, x, x_0(s_0 - 0))$ by $\tilde{x}_0(t)$ and

$$|y(t) - \tilde{x}_0(t)|$$
$$\leq |y(t) - \phi_1(t, y(t), y(s_0 - 0))|$$
$$\quad + |\phi_1(t, y(t), y(s_0 - 0)) - \phi_1(t, \tilde{x}_0(t), x_0(s_0 - 0))| \tag{6.12}$$
$$\leq \Psi_1 + l_1(|y(t) - \tilde{x}_0(t)| + |y(s_0 - 0) - x_0(s_0 - 0)|)$$
$$\leq \Psi_1 + l_1 |y(t) - \tilde{x}_0(t)| + l_1 F_0(s_0)$$

or

$$|y(t) - \tilde{x}_0(t)| \leq \frac{1}{1 - l_1} \Big(\Psi_1 + l_1 F_0(s_0) \Big), \quad t \in (s_0, t_1]. \tag{6.13}$$

Let $t \in (t_1, s_1]$. Define the function

$$\tilde{y}(t) = y(t) - y(t_1) + \phi_1(t_1, \tilde{x}_0(t_1), x_0(s_0)).$$

Then

$$
\begin{aligned}
&|{}_{t_1}^c D^q \tilde{y}(t) - f_1(t, \tilde{y}(t))| \\
&\leq |{}_{t_1}^c D^q y(t) - f_1(t, y(t))| + |f_1(t, \tilde{y}(t)) - f_1(t, y(t))| \\
&\leq \varphi_1(t) + L_1|\tilde{y}(t) - y(t)|.
\end{aligned}
\tag{6.14}
$$

From Remark 6.4, condition 2 and inequalities (6.11), (6.13) we obtain

$$
\begin{aligned}
|\tilde{y}(t) - y(t)| &\leq |y(t_1) - \phi_1(t_1, y(t_1), y(s_0 - 0))| \\
&\quad + |\phi_1(t_1, y(t_1), y(s_0 - 0)) - \phi_1(t_1, \tilde{x}_0(t_1), x_0(s_0))| \\
&\leq \Psi_1 + l_1|y(t_1) - \tilde{x}_0(t_1)| + l_1|y(s_0 - 0) - x_0(s_0)| \\
&\leq \Psi_1 + \frac{l_1}{1 - l_1}\Psi_1 + \frac{l_1 l_1}{(1 - l_1)}F_0(s_0) + l_1 F_0(s_0) \\
&= \frac{1}{1 - l_1}\Psi_1 + \frac{l_1}{1 - l_1}F_0(s_0).
\end{aligned}
\tag{6.15}
$$

From (6.14), (6.15) we get

$$|{}_{t_1}^c D^q \tilde{y}(t) - f_1(t, \tilde{y}(t))| \leq \varphi_1(t) + \frac{L_1}{1 - l_1}\Psi_1 + \frac{L_1 l_1}{1 - l_1}F_0(s_0). \tag{6.16}$$

According to Lemma 6.1 with $\tau = t_1, T = s_1$, $L = L_1$, $K = C_1$, $y(t) = \tilde{y}(t)$ and $\varpi(t) = \varphi_1(t) + \frac{L_1}{1-l_1}\Psi_1 + \frac{L_1 l_1}{1-l_1}F_0(s_0)$ there exists a solution $x_1(t) \in C([t_1, s_1], \mathbb{R})$ satisfying the integral equation

$$
\begin{aligned}
x_1(t) &= \phi_1(t_1, \tilde{x}_0(t_1), x_0(s_0)) \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_1}^t (t - s)^{q-1} f_1(s, x_1(s))ds, \quad t \in (t_1, s_1]
\end{aligned}
\tag{6.17}
$$

and

$$
\begin{aligned}
&|\tilde{y}(t) - x_1(t)| \leq \\
&\frac{C_1}{1 - C_1 L_1}\left(\varphi_1(t) + \frac{L_1}{1 - l_1}\Psi_1 + \frac{L_1 l_1}{1 - l_1}F_0(s_0)\right) \quad t \in (t_1, s_1].
\end{aligned}
\tag{6.18}
$$

Using inequalities (6.15) and (6.18), we get

$$
\begin{aligned}
|y(t) - x_1(t)| &\leq |\tilde{y}(t) - x_1(t)| + |\tilde{y}(t) - y(t)| \\
&\leq \frac{C_1}{1 - C_1 L_1}\left(\varphi_1(t) + \frac{L_1}{1 - l_1}\Psi_1 + \frac{L_1 l_1}{1 - l_1}F_0(s_0)\right) \\
&\quad + \frac{1}{1 - l_1}\Psi_1 + \frac{l_1}{1 - l_1}F_0(s_0) \\
&\leq \frac{C_1}{1 - C_1 L_1}\varphi_1(t) + \frac{1}{1 - l_1}\frac{1}{1 - C_1 L_1}\Psi_1 + \frac{l_1}{1 - l_1}\frac{1}{1 - C_1 L_1}F_0(s_0),
\end{aligned}
\tag{6.19}
$$

i.e.

$$|y(t) - x_1(t)|$$
$$\leq \frac{C_1}{1 - C_1 L_1} \varphi_1(t) + \frac{1}{1 - l_1} \frac{1}{1 - C_1 L_1} \Big(\Psi_1 + l_1 F_0(s_0) \Big) \qquad (6.20)$$
$$= F_1(s), \quad t \in (t_1, s_1].$$

Following this inductive process we construct the function

$$x(t) = \begin{cases} x_k(t), & t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, m \\ \tilde{x}_{k-1}(t), & t \in (s_{k-1}, t_k], \ k = 1, 2, \ldots, m \end{cases} \qquad (6.21)$$

which is a solution of IVP for the NIFrDE (4.1) with $x_0 = y(t_0)$ and satisfies (6.9).

$\square$

REMARK 6.5. Note in (6.9) for the solution in Theorem 6.2 the points t_k, s_k are not included (compare with (6.2) in Theorem 6.1). This allows the result of Theorem 6.2 to be generalized to the infinite interval $[t_0, \infty)$ for appropriate values of the constants L_k, l_k, C_k (for example, $l_k : \prod_{i=1}^{\infty}(1 - l_i) < K_1 < \infty$, $\prod_{i=1}^{\infty} \frac{l_i}{1-l_i} < K_2 < \infty$, $C_k : \prod_{i=0}^{\infty}(1 - C_i L_i) < K_3 < \infty$ and $\prod_{i=0}^{\infty} \frac{C_i}{1-C_i L_i} < K_4 < \infty$).

EXAMPLE 8. Let $0 = t_0 < s_0 = 1 < t_1 = 2 < s_1 = 4 < t_2 = 5 < s_2 = 7 < t_3 = 9 < s_3 = 10$. Consider the IVP for NIFrDE (4.1) with $n = 1$ and $q = 0.1$, i.e.

$$_0^c D^{0.1} x(t) = 0.2x \tan(t) \text{ for } t \in \cup_{k=0}^{3}(t_k, s_k],$$

$$x(t) = \frac{1}{k+1} \Big(x(t) + x(s_{k-1} - 0) \Big) \quad \text{for } t \in (s_{k-1}, t_k], \quad k = 1, 2, 3, \quad (6.22)$$

$$x(0) = 1.$$

The function $f(t, x) = 0.2x \tan(t)$ is not defined and continuous on the whole interval $[0, 10]$. Therefore the conditions of Theorem 1 are not satisfied for (6.22) and approach (*A1 for NIFrDE*) and Theorem 6.1 does not guarantee the existence.

The function $f \in C(\cup_{k=0}^{3}[t_k, s_k] \times \mathbb{R}, \mathbb{R})$ and there exist positive constants $L_0 = 0.312, L_1 = 0.44, L_2 = 0.68, L_3 = 0.13$, i.e. condition 1 of Theorem 6.2 is satisfied. Let $\phi_k(t, x, y) = \frac{1}{k+1}(x + y)$, $k = 1, 2, 3$. Then condition 2 of Theorem 6.2 is satisfied with $l_k = \frac{1}{k+1}$, $k = 1, 2, 3$.

Consider the function $y(t) \equiv 1, t \in [0, 10]$ which satisfies the inequalities (6.7) with $\varphi_k(t) \equiv L_k$, $t \in (t_k, s_k]$, $k = 0, 1, 2, 3$ and $\Psi_k = |\frac{k-1}{k+1}|$, $k = 1, 2, 3$. Then, $\frac{L_k}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} ds = \frac{(t-t_k)^q}{q\Gamma(q)} L_k \leq C_k \varphi_k(t)$, $t \in [t_k, s_k]$ with $C_k =$

$\frac{(s_k - t_k)^q}{q\Gamma(q)}$, $k = 0, 1, 2, 3$, i.e. $C_0 = 1.052$, $C_1 = 1.13$, $C_2 = 1.13$, $C_3 = 1.13$.
Condition 3 of Theorem 6.2 is satisfied.

According to Theorem 6.2 there exists a solution $x(t)$ of the IVP for
NIFrDE (6.22) for which the inequality (6.9) holds. In this case the solution
is defined by approach (*A2 for NIFrDE*).

<div align="right">□</div>

7. Conclusions

Initial value problems for Caputo fractional differential equations with
noninstantaneous impulses are discussed. We emphasize some basic points:

- the impulsive functions have to depend on not only the unknown
  function of the current argument but also on the value of the un-
  known function before the impulse, i.e. $\phi_k(t, x(t), x(s_{k-1} - 0))$ (see
  Example 4 and Remark 4.5);
- the application of approach *A1* requires the definition of the right
  side part of the Caputo fractional differential equation to be defined
  on the whole interval of consideration including the intervals of
  impulses. This is not the same as in approach *A2* (see Example 6
  and Remark 4.7);
- approach *A1* does not allow the application of the step by step
  method w.r.t. to the intervals of impulses to be used directly. This
  is not the same as in approach *A2* (see Example 7);
- approach *A1* can not be applied to Caputo fractional differential
  equations with switching right sides parts, i.e. when f is defined in
  different ways on each interval without impulses. This is not the
  same as in approach *A2* (see Example 1);
- in the application of approach *A2* the basic property for ODE
  $x(t; \tau_1, x(\tau_1; \tau, c)) = x(t; \tau, c)$, $t \geq \tau_1$ is lost. This is not the same
  as in approach *A1* (see Remark 3.4);
- impulsive fractional differential equations are a special case of frac-
  tional differential equations with noninstantaneous impulses (see
  Section 5);
- both approaches are used to study the existence of noninstanta-
  neous impulsive fractional differential equations and the advan-
  tages/disadvantages in the corresponding conditions are discussed
  (see Section 6).

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