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NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES

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16. Abstract

THE NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES WHICH RESULTS FROM LAMINA NON-LINEAR CHARACTERISTICS WAS EXAMINED. THE ANALYSIS USES A RAMBERG-OSGOOD REPRESENTATION OF THE LAMINA TRANSVERSE AND SHEAR STRESS STRAIN CURVES IN CONJUNCTION WITH DEFORMATION THEORY TO DESCRIBE THE RESULTANT LAMINATE NON-LINEAR BEHAVIOR.

A LAMINATE HAVING AN ARBITRARY NUMBER OF ORIENTED LAYERS AND SUBJECTED TO A GENERAL STATE OF MEMBRANE STRESS WAS TREATED. PARAMETRIC RESULTS AND COMPARISON WITH EXPERIMENTAL DATA AND PRIOR THEORETICAL RESULTS ARE PRESENTED.

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LIST OF SYMBOLS

2a, 2b	- Laminate dimension;
c	- Fiber volume fraction;
C^*_{ijkl}	- Stiffness matrix;
E_1	- Elastic Young's modulus;
E_A	- Young's modulus in fiber direction;
E_T	- Young's modulus in transverse direction;
F	- Function of stress σ_{ij};
G_1, G	- Elastic shear modulus of matrix;
G_A	- Axial shear modulus;
G^s_A	- Effective secant shear modulus of composite;
ijkl	- Subscripts ranging from 1 to 3;
k,m,n	- Ramberg-Osgood parameters for the matrix;
$I_{1,2,3,4,5}$	- Stress invariants;
J_2	- $s_{ij}s_{ij}/2$;
K	- Number of laminae in a laminate;
L, $\underline{L}$	- Matrices, also quadratic function of stresses;
M,N	- Ramberg-Osgood parameters for the composite;
r, θ, z	- Cylindrical coordinate system;
s_{ij}	- Stress deviator;
S_{ij}	- Compliance matrix;
S'_{ij}	- Elastic compliance matrix;
S''_{ij}	- Inelastic compliance matrix;
t	- Laminate thickness;
T_i	- Surface tractions;
$T_{\alpha\beta}$	- Constant edge forces per unit length;
$u_{1,2,3}$	- Displacements in $x_{1,2,3}$ directions, respectively;
$\tilde{u}_i$	- Admissible displacement field;
U_ε	- Strain energy;
U_c	- Complimentary energy;
U_p	- Potential energy;

SYMBOLS CONTINUED

V	- Volume;
W^{ε}	- Strain energy density;
W^{σ}	- Complimentary energy density;
$x_{1,2,3}$	- Fixed coordinate directions;
$x_{1,2,3}^{(k)}$	- Local coordinate system for the kth lamina;
α,β	- Subscripts ranging from 1 to 2;
γ	- Shear strain;
ε	- Strain;
$\bar{\varepsilon}_{ij}$	- Average strain tensor;
$\bar{\varepsilon}'_{\alpha\beta}$	- Elastic strains;
$\bar{\varepsilon}''_{\alpha\beta}$	- Inelastic strains;
σ	- Stress;
σ_y	- Nominal composite yield stress;
σ'	- Nominal matrix yield stress;
$\underline{\sigma}^{\circ}$	- Stress $\sigma^{\circ}_{\alpha\beta}$ at the edges;
$\bar{\sigma}_{ij}$	- Average stress tensor;
σ°_{ij}	- Applied stress;
τ	- Shear stress;
τ_y	- Nominal composite yield shear stress;
τ_o	- Average shear stress in composite;
τ'_y	- Nominal matrix yield shear stress;
ν_A	- Poisson's ratio; and
θ_k	- Reinforcement angle.

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SUMMARY

The non-linear stress-strain behavior of fiber composite laminates has been analyzed to define the relationship between laminate behavior and the non-linear stress-strain characteristics of unidirectional composites. The resulting analysis has been programmed to yield an efficient computerized design and analysis tool.

The approach utilized herein was to adopt a Ramberg-Osgood representation of the non-linear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention was directed toward axial shear behavior, and an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix non-linear stress-strain curve. Second level of approach is to treat the interelationship between the properties of the unidirectional layers and those of the laminate. For this case, the starting point is a non-linear stress-strain curve for transverse stress and for axial shear and a linear stress-strain relation for stress in the fiber direction. The non-linear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters. In the present study, with this as a starting point, an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress.

A laminate having an arbitrary number of oriented layers and subjected to a general state of membrane stress was treated. Parametric results and comparison with experimental data and prior theoretical results are presented.

1. INTRODUCTION

A basic requirement for the engineer designing with fiber composite materials is a definition of the stiffness and strength of these materials under a variety of loading conditions, including cases for which experimental materials properties data are not available. For this purpose, it is necessary that he have at his disposal reasonably accurate procedures to predict these mechanical properties. Existing analyses can predict the elastic behavior of a laminated composite quite well when the elastic properties of the unidirectional materials from which it is made are known. However, the situation has been much more complicated and much less satisfactory with regard to the inelastic stiffness and strength of a laminate. The present program was undertaken to develop a computerized analysis of the inelastic behavior of fiber composite laminates which could be used as a design tool. The results of this study and comparisons of these results with experimental data are presented in this report.

It is essential to recognize that the utilization of fiber composite materails in structural design involves the incorporation of material design into the structural design process. This is illustrated clearly by the fact that the gross material properties of a fiber composite laminate change when any change is made in the laminate ply orientations. Even when the designer considers a material formed from a particular combination of fiber and matrix materials, there remains a large number of geometric variables associated with the laminate design. Thus, in the preliminary design phase, experimental material properties data will generally be too limited. In the case of elastic properties, sufficient capability to synthesize the necessary properties exists. This procedure generally starts with the definition of the elastic properties of unidirectional fiber composite materials. These can, of course, be determined experimentally. Also, when such data are not available, they can be estimated using a variety of analytical techniques. These latter are generally referred to as micromechanics analyses. For example, a set of relatively simple relations for predicting the moduli of unidirectional reinforced composites are presented in [1]. Alternate micromechanics approaches are described in [2] to [4]. A review of these methods is presented in [5]. With these properties available, it is assumed that the individual laminae are homogeneous and anisotropic. A laminate analysis is carried out in a straight forward fashion following methods originally developed for such materials as plywood, and more recently extended to the more general cases associated with fiber composite laminates (e.g., [6] to [8]).

However, contemporary fiber composite materials generally consist of elastic brittle fibers such as glass, boron or graphite in relatively soft matrix materials such as epoxy or aluminum. For these matrix materials it is reasonable to anticipate that at a certain loading state the matrix will begin to exhibit inelastic effects. This results in non-linear relations between structural loads and deformations. These inelastic effects can, of course, be expected to have a significant effect upon failure of the laminate. It is quite clear that adequate definition of these failure conditions are essential to achieve structural designs of high reliability.

In the present study, a non-linear laminate analysis has been developed which can provide realistic assessments of the stresses and strains in the various laminae and of the inelastic stiffnesses of the laminate at any stress level. This information can be used for assessment of such effects as structural stability or structural stress distributions. The stress distributions in the laminae and the laminates can also be utilized for the development of more realistic failure criteria.

Inelastic matrix behavior can be classified broadly as either time dependent or time independent. Time dependent behavior is called viscoelastic if linear and creep if non-linear. Polymeric matrices such as epoxy do exhibit such behavior. In the case of metallic matrix materials, such as aluminum, time dependent effects are generally negligible unless elevated temperature conditions are considered. The present study is concerned with time independent non-linear matrix behavior which is of significance for both polymeric and metallic matrices. Throughout this paper the expression "inelastic" is used to describe this time independent mechanical behavior. The method of approach to these problems is similar to that of the elastic analysis. Thus, it is necessary to determine, first, the inelastic properties of the unidirectional fiber composite materials. This can be done experimentally or by micromechanics methods. Given this information, a method to determine stresses and strains in an inelastic laminate is then devised. The problem is complicated by the fact that the inelastic stress-strain relations are non-linear.

A limited number of pertinent investigations can be found in the literature. Hill [4] considered, in approximate fashion, a limited aspect of inelastic behavior of a uniaxially reinforced material: the case of stress in fiber direction combined with isotropic transverse stress. Petit and Waddoups [6] devised an incremental method for laminate analysis in which it was assumed that in single laminae there is no interaction of stress components in different directions as far as lamina deformation is concerned. This assumption is restrictive, and also their incremental laminate analysis scheme is unduly complicated. Adams [7] used a finite element technique for numerical analysis of unidirectional materials in the form of periodic fiber arrays under conditions of plane strain. Huang [8] gave an approximate analysis for transverse inelastic behavior for a unidirectional material in plane strain, but it is diffucult to assess the validity of the approximations introduced.

A detailed analysis of the inelastic laminate problem has been given by Foye and Baker [9]. Using finite element methods, they computed the inelastic effective properties of unidirectional rectangular and square arrays of elastic fibers in inelastic matrix. These properties were then used in an inelastic laminate analysis. The analysis is based on incremental plasticity theory and is, unfortunately, very complicated and requires a great deal of computer time. The results obtained are, however, of great importance for comparison with results predicted by more simplified theories, such as the one which will be given in the present work.

The body of this report is divided into four major sections. In the first, consideration is given to the behavior of unidirectional fiber composite materials. This requires: a definition of the appropriate form of the inelastic stress-strain relations; some consideration of the relationship between composite properties and constituent properties; and a definition of the appropriate form of the interaction between various stress components. The basic objective in this phase of the report was to define appropriate constitutive relations for the individual lamina which can be used in the non-linear laminate analysis. Further, there is a desire to gain some insight into the influence of the particular constituent properties upon the lamina stress-strain relations. In this phase of the study, it is found useful to characterize the unidirectional material with the aid of Ramberg-Osgood stress-strain relations.

In the next section of the report, the analysis of the inelastic behavior of laminates is described. Here, a procedure for incorporating the non-linear constitutive relations into an analysis which defines the state of stress in the individual laminae under an arbitrary set of external loads, is defined. Analyses are developed for the case of symetric laminates subjected to membrane loading. The equations which are developed uniquely define the desired laminate internal average stress distribution under a given set of membrane loads. Governing equations, however, are non-linear and require numerical solution procedures. An efficient algorithm has been defined which enables computer solution to be achieved for arbitrary laminates at minimal cost. The solution is obtained by application of the Newton-Raphson method.

In the final section, the computerized analysis which has been developed is applied to series of problems. The first group presents comparisons with various analytical results from the more complex analyses of Ref. [6] and [9]. The second group of numerical results presents comparisons between theoretical results from the present model and available experimental data. The third group of results provides several parametric studies to gain insight into those factors which contribute significantly to the non-linear behavior of fiber composite laminates. Also, computations have been made to provide a preliminary assessment of combined load effects including comparisons with limited experimental data.

Details of the various analytical developements, as well as descriptions of the computer program, are presented in appendices to the report.

The principal result of the present program is a computer program which provides a simple engineering tool which can be used for the parametric study of the influence of material properties upon laminate performance. This laminate analysis capability can be used by the structural designer to define design allowable stresses and to aid in the selection of fiber composite materials for structural applications. A comparison of the present results with the limited amount of available experimental data shows good agreement. There are, however certain cases in which the agreement is not good, par ticularly as the laminate loading approaches failure. The results of the present analytical method agree well with the results for those problems for which more exact and more complex analytical results exist.

2. NON-LINEAR STRESS-STRAIN RELATIONS OF UNIAXIAL FIBER REINFORCED MATERIALS

2.1 General Form of Stress-Strain Relations

An effective stress-strain relation of a composite material is defined as a relation between average stress $\bar{\sigma}_{ij}$ and average strain $\bar{\varepsilon}_{ij}$. Here and in the following latin indices range over 1, 2, and 3. If the composite is elastic the general effective stress-strain relation takes the form

$$\bar{\sigma}_{ij} = C^*_{ijkl}\ \bar{\varepsilon}_{kl} \qquad (2.1.1)$$

where C^*_{ijkl} are the effective elastic moduli which are material constants and are thus independent of stress or strain. Thus, (2.1.1) is a linear relation between average stress and strain.

If the composite is subject to symmetries the form of (2.1.1) simplifies. For a uniaxial FRM the most important cases of symmetry are transverse isotropy, around fiber direction, and square array (square symmetry). In these cases the stress-strain relations (2.1.1) for transverse isotropy assume the form:

$$\begin{aligned}
\bar{\sigma}_{11} &= C^*_{11}\ \bar{\varepsilon}_{11} + C^*_{12}\ \bar{\varepsilon}_{22} + C^*_{12}\ \bar{\varepsilon}_{33} \\
\bar{\sigma}_{22} &= C^*_{12}\ \bar{\varepsilon}_{11} + C^*_{22}\ \bar{\varepsilon}_{22} + C^*_{23}\ \bar{\varepsilon}_{33} \\
\bar{\sigma}_{33} &= C^*_{12}\ \bar{\varepsilon}_{11} + C^*_{23}\ \bar{\varepsilon}_{22} + C^*_{22}\ \bar{\varepsilon}_{33} \\
\bar{\sigma}_{12} &= 2C^*_{44}\ \bar{\varepsilon}_{12} \\
\bar{\sigma}_{23} &= 2C^*_{55}\ \bar{\varepsilon}_{23} \\
\bar{\sigma}_{31} &= 2C^*_{44}\ \bar{\varepsilon}_{31}
\end{aligned} \qquad (2.1.2)$$

and

$$C^*_{55} = (C^*_{22} - C^*_{23})/2 \qquad (2.1.3)$$

In (2.1.2-3) 1 indicates direction and 2, 3 perpendicular directions transverse to 1.

In the event of inelastic matrix and elastic fibers, the situation is much more complicated since the stress-strain relation are nonlinearity and history dependent. In no case is stress proportional to strain so that superposition of effects is not valid, and in order to determine current strain it is not sufficient to know current stress but it is necessary to know precisely the variation of stress which preceded its current value. Thus, for a material in a known state of combined shear and uniaxial tension, the state of strain is different if: (a) tension is first applied and then the shear, (b) shear is first applied and then the tension- (c) tension and shear are applied simultaneously. For this reason stress-strain relations must be presented in incremental form. That is, strain increment is related to stress and stress increment. This complicates matters enormously. However, it is known that in the case of proportional loading, that is, all stresses at a point grow simultaneously in a fixed ratio to one another, incremental theory can be integrated into the much simpler total or deformation theory for which current strain is completely determined by current stress.

Deformation theories have a wider range of validity than proportional loading. Comparison of numerous detailed solutions carried out both incrementally and by much simpler deformation theory show surprising agreement in many cases, and Budiansky [10] has shown that deformation theory can also be valid for "neighboring" loading paths.

In the present work, we are concerned with composites which are subjected to some external load. If it is supposed that the various external load components grow proportionally, this does not necessarily imply that the components of stress at a typical internal point also grow proportionally. It is, however, felt that the manner of growth of these internal stress components cannot deviate severely from proportional loading if external loading is proportional. Consequently, deformation type stress-strain relations are assumed for the matrix.

This assumption results in considerable simplification. It will be seen that it yields results which are extremely close to the ones obtained in [9] on the basis of the much more complicated incremental theory.

It is shown in Appendix A that for elastic fibers and an inelastic matrix described by deformation type theory, the effective stress-strain relations for a transversely isotropic or square symmetric FRM are:

$$\begin{aligned}
\bar{\varepsilon}_{11} &= S_{11}\,\bar{\sigma}_{11} + S_{12}\,\bar{\sigma}_{22} + S_{12}\,\bar{\sigma}_{33} \\
\bar{\varepsilon}_{22} &= S_{12}\,\bar{\sigma}_{11} + S_{22}\,\bar{\sigma}_{22} + S_{23}\,\bar{\sigma}_{33} \\
\bar{\varepsilon}_{33} &= S_{12}\,\bar{\sigma}_{11} + S_{23}\,\bar{\sigma}_{22} + S_{22}\,\bar{\sigma}_{33} \\
\bar{\varepsilon}_{12} &= 2S_{44}\,\bar{\sigma}_{12} \\
\bar{\varepsilon}_{23} &= 2S_{55}\,\bar{\sigma}_{23} \\
\bar{\varepsilon}_{13} &= 2S_{44}\,\bar{\sigma}_{13}
\end{aligned} \tag{2.1.4}$$

and

$$S_{55} = (S_{22}-S_{23})/2 \tag{2.1.5}$$

The coefficients S_{11}, S_{12}, etc. are the effective inelastic compliances of the material and are <u>functions of the average stresses</u>, or rather of certain invariants of the average stress tensor.

We are here primarily concerned with thin uniaxially reinforced laminae which are in a state of plane stress. Let x_1 denote fiber direction, x_2 direction transverse to fibers in lamina plane, and x_3 direction perpendicular to lamina, Figure 1. Then the plane stress condition is expressed by:

$$\bar{\sigma}_{13} = \bar{\sigma}_{23} = \bar{\sigma}_{33} = 0 \tag{2.1.6}$$

Equs. (2.1.4) then assume the form:

$$\begin{aligned}
\bar{\varepsilon}_{11} &= S_{11}\,\bar{\sigma}_{11} + S_{12}\,\bar{\sigma}_{22} \\
\bar{\varepsilon}_{22} &= S_{12}\,\bar{\sigma}_{11} + S_{22}\,\bar{\sigma}_{22} \\
\bar{\varepsilon}_{12} &= 2S_{44}\,\bar{\sigma}_{12}
\end{aligned} \tag{2.1.7}$$

Note that $\bar{\varepsilon}_{33}$ does not vanish. It is however of no interest for present purposes.

The inelastic compliances in (2.1.7) are functions of the stresses $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$, $\bar{\sigma}_{12}$.

It is convenient to split the strains in (2.1.7) into elastic strains $\bar{\varepsilon}'_{\alpha\beta}$, and inelastic strains $\bar{\varepsilon}''_{\alpha\beta}$. Thus:

$$\bar{\varepsilon}_{\alpha\beta} = \bar{\varepsilon}'_{\alpha\beta} + \bar{\varepsilon}''_{\alpha\beta} \tag{2.1.8}$$

where here and in the following greek indices range over 1, 2. The elastic strains are recovered after unloading of the composite and are related to the stresses by elastic stress-strain relations. Thus:

$$\begin{aligned} \bar{\varepsilon}'_{11} &= S'_{11}\,\bar{\sigma}_{11} + S'_{12}\,\bar{\sigma}_{22} \\ \bar{\varepsilon}'_{22} &= S'_{12}\,\bar{\sigma}_{11} + S'_{22}\,\bar{\sigma}_{22} \\ \bar{\varepsilon}'_{12} &= 2S'_{44}\,\bar{\sigma}_{12} \end{aligned} \tag{2.1.9}$$

where

$$\begin{aligned} S'_{11} &= \frac{1}{E_A} \qquad & S'_{12} &= -\frac{\nu_A}{E_A} \\ S'_{22} &= \frac{1}{E_T} \qquad & S'_{44} &= \frac{1}{4G_A} \end{aligned} \tag{2.1.10}$$

Here E_A is the effective Young's modulus in fiber direction, ν_A - the associated effective Poisson's ration, E_T - the effective Young's modulus transverse to fibers and G_A - axial effective shear modulus, related to 1-2 shear.

The inelastic, permanent, strains then have the form:

$$\begin{aligned} \bar{\varepsilon}''_{11} &= S''_{11}\,\bar{\sigma}_{11} + S''_{12}\,\bar{\sigma}_{22} \\ \bar{\varepsilon}''_{22} &= S''_{12}\,\bar{\sigma}_{11} + S''_{22}\,\bar{\sigma}_{22} \\ \bar{\varepsilon}''_{12} &= 2S''_{44}\,\bar{\sigma}_{12} \end{aligned} \tag{2.1.11}$$

where

$$S''_{2\beta} = S''_{2\beta}\,(\bar{\sigma}_{11},\ \bar{\sigma}_{22},\ \bar{\sigma}_{12}) \tag{2.1.12}$$

In order to further simplify the stress-strain relations (2.1.11-.12), some specific features of FRM will be taken into account. In such materials, the fibers are by an order of magnitude stiffer than the matrix (for the case of boron and/or graphite fiber in an epoxy matrix the ratio of fiber to matrix Young's modulus can be in excess of 100). The stiffness ratio becomes larger in the inelastic range since the matrix loses stiffness (i.e., flows) while the fibers retain their stiffness. It is, therefore, clear that the stress $\bar{\sigma}_{11}$ in fiber direction is practically carried by the fibers alone, with insignificant matrix contribution.

On the other hand, the transverse stress $\bar{\sigma}_{22}$ and the shear stress $\bar{\sigma}_{12}$ are primarily carried by the matrix with little fiber contribution.

It follows that inelastic behavior of the FRM is produced primarily by $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$ while inelastic behavior for $\bar{\sigma}_{11}$ load can be neglected.

The foregoing comments are summarized into two basic assumptions:

(a) the inelastic strains $\bar{\varepsilon}''_{22}$ and $\bar{\varepsilon}''_{12}$ are not functions of $\bar{\sigma}_{11}$

(b) the inelastic strain $\bar{\varepsilon}''_{11}$ always vanishes.

On the basis of these assumptions, the stress-strain relations (2.1.11-.12) simplify to:

$$\begin{aligned} \bar{\varepsilon}''_{11} &= 0 \\ \bar{\varepsilon}''_{22} &= S''_{22}\,(\bar{\sigma}_{22},\ \bar{\sigma}_{12})\ \bar{\sigma}_{22} \\ \bar{\varepsilon}''_{12} &= 2S''_{44}\,(\bar{\sigma}_{22},\ \bar{\sigma}_{12})\ \bar{\sigma}_{12} \end{aligned} \qquad (2.1.13)$$

2.2 Plane Stress-Strain Relations in Ramberg-Osgood Form

A convenient representation of non-linear one dimensional stress-strain relations has been given by Ramberg and Osgood [11]. For uniaxial stress, for example:

$$\varepsilon = \frac{\sigma}{E_1} [1 + k \ (\frac{\sigma}{\sigma'})^{m-1}] \tag{2.2.1}$$

where E_1 represents the elastic Young's modulus, and k, σ', and m are three parameters to be obtained by curve fitting. The parameter σ' is sometimes called nominal yield stress. Equation (2.2.1) represents a family of curves with initial slope E_1, and monotonically decreasing slope with increasing σ. The curves flatten out with increasing m (Fig. 2). Without loss of generality (2.2.1) can be written in the form:

$$\varepsilon = \frac{\sigma}{E_1} \ [1 + (\frac{\sigma}{\sigma'_y})^{m-1}] \tag{2.2.2}$$

which will be used from now on. Similarly, a stress-strain curve in shear can be represented in the form:

$$\gamma = \frac{\tau}{G_1} [1 + (\frac{\tau}{\tau'_y})^{n-1}] \tag{2.2.3}$$

where G_1 is the elastic shear modulus.

It should be emphasized that (2.2.2-.3) are valid only for one dimensional cases. The question of the generalization to general states of stress and strain has no unique answer. One common used form is isotropic J_2 deformation theory [12].

Next, we consider the case of effective or macroscopic stress-strain relations for the special case of a uniaxially reinforced material in which the matrix in non-linear, with stress-strain relations in Ramberg-Osgood form.

Consider, for example, the case of uniaxial average stress $\bar{\sigma}_{22}$ in direction transverse to fibers, all other average stresses vanish. It then follows from (2.1.7) that:

$$\bar{\varepsilon}_{22} = S_{22} \ (\bar{\sigma}_{22}) \ \bar{\sigma}_{22} \tag{2.2.4}$$

Similarly, if the only nonvanishing average stress is $\bar{\sigma}_{12}$, the shear stress-strain relation of the composite is:

$$\bar{\varepsilon}_{12} = 2S_{44}(\bar{\sigma}_{12})\ \bar{\sigma}_{12} \tag{2.2.5}$$

Evidently the inelastic effective compliances S_{22} and S_{44} are functions of the parameters of the inelastic Ramberg-Osgood stress-strain relations of the matrix, of the elastic properties of the fibers and of the internal geometry of the composite. Actual prediction is a very difficult problem. Such problems will be cnsidered in limited fashion in the next paragraph.

Just as matrix stress-strain relations are represented in Ramberg-Osgood form, the same type of curve fitting can also be applied for the effective stress-strain relation of the composite. Thus (2.2.2-.3) are written in the form:

$$\bar{\varepsilon}_{22} = \frac{\bar{\sigma}_{22}}{E_T}\left[1 + \left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^{M-1}\right] \qquad \text{(a)}$$

$$\bar{\varepsilon}_{12} = \frac{\bar{\sigma}_{12}}{2G_A}\left[1 + \left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^{N-1}\right] \qquad \text{(b)} \tag{2.2.6}$$

Where E_T is the effective transverse elastic Young's modulus G_A - effective axial elastic shear modulus and σ_y, τ_y, M and N are curve fitting parameters which are in general quite different from the corresponding Ramberg-Osgood matrix parameters.

A question of fundamental and of practical importance is the form of the stress-strain relations for the case of plane stress, taking into account interaction among the various stress components. It should be noted in this repsect that (2.2.6) are special stress-strain relations when $\bar{\sigma}_{22}$ or $\bar{\sigma}_{12}$ act only by themselves.

It is recalled that equations (2.1.13) represent the inelastic parts of the strains for plane stress-strain relations for FRM with stiff fibers. It is shown in Appendix B that the Ramberg-Osgood form of such plane stress-strain relations is as follows:

$$\begin{aligned}
\bar{\varepsilon}''_{11} &= 0 \\
\bar{\varepsilon}''_{22} &= \frac{\bar{\sigma}_{22}}{E_T}\left[\left(\frac{\bar{\sigma}_{22}}{\sigma_Y}\right)^2 + \left(\frac{\bar{\sigma}_{12}}{\tau_Y}\right)^2\right]^{\frac{M-1}{2}} \\
\bar{\varepsilon}''_{12} &= \frac{\bar{\sigma}_{12}}{2G_A}\left[\left(\frac{\bar{\sigma}_{22}}{\sigma_Y}\right)^2 + \left(\frac{\bar{\sigma}_{12}}{\tau_Y}\right)^2\right]^{\frac{N-1}{2}}
\end{aligned} \qquad (2.2.7)$$

The parameters E_T, G_A, σ_Y, τ_Y, M, N in (2.2.7) are those of the one dimensional stress-strain relations (2.2.6) which may be regarded as experimentally (or perhaps theoretically) known.

The inelastic parts of the strains are given by (2.1.9-.10), and the total strains are then given by adding equations (2.2.7) and (2.1.9).

Equations (2.2.7) have been compared with computed numerical results given in [9]. Reasonable agreement was obtained. Comparisons for the interaction cases of transverse stress, $\bar{\sigma}_{22}$, versus transverse strain, $\bar{\varepsilon}_{22}$, in the presence of axial shear stress, $\bar{\sigma}_{12}$, and axial shear stress, $\bar{\sigma}_{12}$, versus axial shear strain, $\bar{\gamma}_{12}$, are shown in Figures 3 and 4 respectively (in both cases $\bar{\sigma}_{22}/\bar{\sigma}_{12} = 8/3$). It is seen that the agreement is fair for transverse stress-strain relations (Fig. 3) and very good for the shear stress-strain relations (Fig. 4).

Figures 3 and 4 also show the stress-strain relations obtained from Eqs. (2.2.7) for one dimensional transverse tension $\bar{\sigma}_{22}$, and axial shear, $\bar{\sigma}_{12}$, respectively.

2.3 Axial Shear Stress-Strain Relation

This paragraph is concerned with the problem of prediction of a one dimensional effective axial shear stress-strain relation of a uniaxial FRM in terms of matrix and fiber properties and the internal geometry of the composite.

The main reason for concentrating on the axial shear problem is that the inelastic effect is predominant in axial shear for which significant nonlinearity of the stress-strain response is obtained (e.g., Figure 4). The effect in fiber direction is practically non-existent as has indeed been assumed above, and is relatively small in transverse stress which is shown by the small curvature of the stress-strain relation in this case (e.g., Figure 3).

On the basis of all this, it can indeed be assumed as first approximation that the nonlinearity of the uniaxial FRM is limited to axial shear alone.

Consider a uniaxially reinforced lamina which is subjected to pure axial shear, Figure 5, on its surface. The boundary conditions are:

$$\begin{array}{lll} x_3 = \pm t/2 & \sigma_{31} = \sigma_{32} = \sigma_{33} = 0 & \\ x_2 = \pm b & \sigma_{12} = \tau_0 \quad \sigma_{22} = \sigma_{23} = 0 & \\ x_1 = \pm a & \sigma_{12} = \tau_0 \quad \sigma_{11} = \sigma_{13} = 0 & \end{array} \tag{2.3.1}$$

It may be shown that under such load the only nonvanishing average stress in the composite is:

$$\bar{\sigma}_{12} = \tau_0 \tag{2.3.2}$$

It would seem at first that, given the complexity of the internal geometry of the composite, the state of stress at any interior matrix or fiber point is generally three dimensional. Surprisingly enough, however, this is not so and the only nonvanishing stress components in the interior of the composites are the shear stresses σ_{12} and σ_{13}, which are moreover functions of x_2 and x_3 only. Thus the interior state of stress is:

$$
\begin{aligned}
\sigma_{12} &= \sigma_{12}\ (x_2,\ x_3) \\
\sigma_{13} &= \sigma_{13}\ (x_2,\ x_3) \\
\sigma_{11} &= \sigma_{22} = \sigma_{33} = \sigma_{23} = 0
\end{aligned}
\tag{2.3.3}
$$

The validity of equations (2.3.3) for the case of an elastic composite has been proved in [5]. Their validity for the present much more general inelastic case will be shown elsewhere.

The effective stress-strain relation of the composite in axial shear is defined by:

$$
\begin{aligned}
\bar{\varepsilon}_{12} &= \frac{\sigma_{12}}{2G_A^s} = \frac{\tau_o}{2G_A^s} \\
G_A^s &= G_A^s(\bar{\sigma}_{12}) = G_A^s(\tau_o)
\end{aligned}
\tag{2.3.4}
$$

where G_A^s is the <u>effective secant shear modulus</u> of the material. The nonlinearity of the stress-strain relation is expressed by the fact that G_A^s function of the applied stress.

It is seen that in order to determine G_A^s it is necessary to compute the average shear strain $\bar{\varepsilon}_{12}$ for given applied shear stress. This is a formidable problem even with the simplification (2.3.3) and we shall content ourselves with a brief outline of its formulation. To simplify matters, the fibers shall be assumed to be ideally rigid relative to the matrix. This is a very accurate assumption for the case of Boron and Graphite Fibers. There is no difficulty to extend the formulation to the case of non-rigid elastic fibers.

In view of (2.3.3) the problem is two dimensional and need only be considered in a typical x_2, x_3 section. In the matrix domain:

$$\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \qquad (2.3.5a)$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}\left[1 + \left(\frac{\tau}{\tau_y'}\right)^{n-1}\right] \qquad (2.3.5b)$$

$$\varepsilon_{13} = \frac{\sigma_{13}}{2G}\left[1 + \left(\frac{\tau}{\tau_y'}\right)^{n-1}\right] \qquad (2.3.5c)$$

$$\tau = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \qquad (2.3.6)$$

$$\varepsilon_{12} = \frac{1}{2}\frac{\partial u_1}{\partial x_2} \qquad (2.3.7a)$$

$$\varepsilon_{13} = \frac{1}{2}\frac{\partial u_1}{\partial x_3} \qquad (2.3.7b)$$

$$u_1 = u_1(x_2, x_3) \qquad (2.3.8)$$

and, $u_1 = 0$ at fiber/matrix interface.

Here equ. (2.3.5) is the only surviving equilibrium equation, (2.3.6) are Ramberg-Osgood stress-strain relations for isotropic J_2 theory (2.3.7) are usual strain-displacement relations in which u_2 and u_3 do not enter since it may be shown that they are not functions of x_1 and (2.3.8) expresses the ideal rigidity of the fibers.

Equs. (2.3.5-.8) must be solved subject to boundary condition (2.3.1). If this is done the strain ε_{12} is known everywhere and can be averaged to obtain G_A^S from (2.3.4).

The problem is exceedingly difficult because of the nonlinearity introduced by the stress-strain relations (2.3.6). There is very little hope to solve it analytically for any kind of fiber geometry. It should therefore be handled by numerical methods for fiber arrangements and fiber shapes of engineering interest.

Another way to approach an analytically intractable problem such as the present one is by variational techniques.

In this fashion, approximations or bounds for quantities of interest are obtained by methods which are much simpler than bonafide solution of the problem. Such variational methods have been extensively used for determination of effective elastic moduli of FRM (e.g., [5]).

In the course of the present work, it has been found that variational methods can also be used for inelastic problems such as the present one to obtain bounds on effective secant moduli. The main ingredients of the method are:

(a) Construction of an extremum principle in terms of an energy integral such that the true energy is the minimum of the integral.

(b) Expression of the true energy in terms of effective secant modulus.

(c) Establishment of admissible fields to obtain a value of the energy integral which is larger than the true energy, thus obtaining a bound for G_A^s.

The work involves complicated developments and derivations which are given in Appendix C. Here only the end result for a lower bound on G_A^s will be given for a special geometry of FRM which is known as composite cylinder assemblage. This geometry has been described in detail in [1, 5] and consists of an assemblage of composite cylinders of variable sizes which are joined together so as to fill the whole volume of the composite. In order to fill the whole volume, composite cylinders vary from finite to infinitesimal size. This geometry has been used to advantage for elastic FRM to obtain simple expressions for effective elastic moduli which are well verified by experiment [1, 5]. In the present case only a lower bound on G_A^s has been obtained for the case in which the exponent n in matrix stress-strain relations is n=3.

It has been found that with this exponent and proper choice of τ_Y, epoxy shear stress-strain relations can be well described. The result for the lower bound is:

$$G_A^s \geq G_{A(-)}^s = \frac{G\,\frac{1+c}{1-c}}{1 + \left(\frac{\tau_o}{\tau_y'}\right)^2 \frac{3+13c+c^2+c^3}{3(1+c)^3}} \tag{2.3.9}$$

where

c - volume fraction of fibers

G - elastic (initial) matrix shear modulus

τ_y' - Ramberg-Osgood matrix stress parameter, and

τ_o - applied shear stress.

It follows from (2.3.4) that:

$$\bar{\varepsilon}_{12} \leq \frac{\tau_o}{2G_{A(-)}^s} \tag{2.3.10}$$

In other words, with the lower bound on G_A^s an upper bound on $\bar{\varepsilon}_{12}$ variation with τ_o is obtained.

If (2.3.10) is explicitly written in terms of (2.3.9) it assumes the form:

$$\bar{\varepsilon}_{12} \leq \frac{\tau_o}{2G\frac{1+c}{1-c}} \left[1 + \left(\frac{\tau_o}{\tau_y'}\right)^2 \frac{3+13c+c^2+c^3}{3(1+c)^3}\right] \tag{2.3.11}$$

Recalling that for the composite cylinder assemblage with rigid fibers the axial elastic shear modulus G_A is given in [1, 5] as:

$$G_A = G\,\frac{1+c}{1-c} \tag{2.3.12}$$

and comparing (2.3.11) with (2.2.6) with choice of exponent N=3 (which is the same as matrix exponent), it is seen that:

$$\tau_Y^2 \geq \tau_Y'^2 \frac{3(1+c)^3}{3+13c+c^2+c^3} \tag{2.3.13}$$

The prediction of (2.3.11) has been compared with numerical results obtained in [9]. Figure 6 shows the variation of the right side of (2.3.11) in comparison with the results obtained in [9] for a fiber volume fraction, c=0.5. Since results of [9] were for boron fibers in epoxy matrix, the rigid fiber approximation is accurately valid. It is seen that the results are reasonably close. It should be noted that the geometry of [9] is a rectangular fiber array which is quite different from the composite cylinder assemblage geometry.

The results defined by (2.3.12) and (2.3.13) used in equation (2.2.6) yield the result plotted in non-dimensional form in Fig. 7. The shear strains are normalized with respect to the matrix elastic strain, γ_{ye}, at the yield stress, τ_Y:

$$\gamma_{ye} = \frac{\tau_Y}{G} \tag{2.3.14}$$

It is natural to also consider the establishment of an upper bound on G_A^s. Unfortunately, however, this is a matter of formidable difficulty for the reason that inversion of (2.3.6) to express stresses in terms of strains leads very complicated expressions. Further discussion of this difficulty is given in Appendix C.

3. ANALYSIS OF NON-LINEAR LAMINATES

3.1 Formulation

The general problem to be investigated in the present chapter is as follows: given the inelastic stress-strain relations of uniaxially reinforced laminae determined theoretically or experimentally, and a laminate composed of such laminae and loaded on its edges by uniformly distributed loads in the plate of the laminate:

(a) What are the stresses in the various laminae?

(b) What is the macroscopic strain response of the laminate to the loads?

This problem has been extensively investigated for elastic laminates, and the results obtained will serve as important guidelines for the present much more complicated problem. It is therefore very helpful to first briefly review the theory of elastic laminates.

Let the laminate be referred to a fixed system of coordinates x_1, x_2, x_3 as shown in Figure 8. This will henceforth be referred to as the laminate coordinate system.

Any lamina, kth say, in the laminate will be referred to its material system of coordinates $x_1^{(k)}$, $x_2^{(k)}$, x_3 where $x_1^{(k)}$ is in fiber direction, $x_2^{(k)}$ perpendicular to fiber direction and x_3 is the same as the laminate x_3, Figure 8. The reinforcement angle θ_k is defined by:

$$\theta = \sphericalangle\ (x_1,\ x_1^{(k)}) = \sphericalangle\ (x_2,\ x_2^{(k)}) \tag{3.1.1}$$

Let it be assumed that the laminae are in states of plane stress. It will be later explained under what conditions this is true. Then the stress-strain relations of a single lamina referred to its material coordinate system are written in the forms:

$$\varepsilon_{\alpha\beta}^{(k)} = S_{\alpha\beta\gamma\delta}^{(k)}\ \sigma_{\gamma\delta}^{(k)} \qquad \text{(a)}$$

$$(3.1.2)$$

$$\underline{\varepsilon}^{(k)} = \underline{S}^{(k)}\ \underline{\sigma}^{(k)} \qquad \text{(b)}$$

where (3.1.2a) is in tensor notation with range of subscripts 1, 2 and (3.1.2b) is in matrix notation. It should be noted that (3.1.2) represent the stress-strain relations (2.1.9 - .10), i.e.,

$$
\begin{aligned}
\varepsilon_{11}^{(k)} &= \frac{\sigma_{11}^{(k)}}{E_A^{(k)}} - \frac{\nu_A^{(k)}}{E_A^{(k)}} \sigma_{22}^{(k)} \\
\varepsilon_{22}^{(k)} &= -\frac{\nu_A^{(k)}}{E_A^{(k)}} \sigma_{11}^{(k)} + \frac{\sigma_{22}^{(k)}}{E_T^{(k)}} \\
\varepsilon_{12}^{(k)} &= \frac{\sigma_{12}^{(k)}}{2G_A^{(k)}}
\end{aligned}
\tag{3.1.3}
$$

Let a laminate of rectangular form, Figure 8, be loaded by a uniform edge stress:

$$
\begin{aligned}
\sigma_{11}(\pm a, x_2) &= \sigma_{11}^{\circ} \\
\sigma_{12}(\pm a, x_2) &= \sigma_{12}^{\circ} \\
\sigma_{12}(x_1, \pm b) &= \sigma_{12}^{\circ} \\
\sigma_{22}(x_1, \pm b) &= \sigma_{22}^{\circ}
\end{aligned}
\tag{3.1.4}
$$

The elasticity solution of the laminate must satisfy the following requirements:

(a) Equilibrium of stresses,

(b) Traction continuity at laminae interfaces,

(c) Boundary conditions (3.1.4), and

(d) Displacement continuity at laminae interfaces.

It is assumed that the stresses in any lamina are constant, but different in the different laminae. The condition (a) is satisfied within any lamina. Since the assumed lamina stresses are plane there are no traction components on laminae interfaces. Therefore (b) is satisfied.

The boundary conditions (3.1.4) cannot be strictly satisfied in each lamina but only in an average sense. To do this lamina stresses $\sigma_{\alpha\beta}^{(k)}$ referred to lamina material coordinates are transformed to laminate axes. The stresses in the kth lamina referred to laminate axes are denoted ${}^{(k)}\sigma_{\alpha\beta}$. The transformation is given by:

$$
\begin{aligned}
{}^{(k)}\sigma_{11} &= \sigma_{11}^{(k)} \cos^2\theta_k + \sigma_{22}^{(k)} \sin^2\theta_k - 2\sigma_{12}^{(k)} \cos\theta_k \sin\theta_k \\
{}^{(k)}\sigma_{22} &= \sigma_{22}^{(k)} \sin^2\theta_k + \sigma_{22}^{(k)} \cos^2\theta_k + 2\sigma_{12}^{(k)} \cos\theta_k \sin\theta_k \\
{}^{(k)}\sigma_{12} &= (\sigma_{11}^{(k)} - \sigma_{22}^{(k)}) \sin\theta_k \cos\theta_k + \sigma_{12}^{(k)} (\cos^2\theta_k - \sin^2\theta_k)
\end{aligned}
\tag{3.1.5}
$$

or in matrix notation:

$$
{}^{(k)}\underline{\sigma} = \underline{\theta}^{(k)} \underline{\sigma}^{(k)} \tag{3.1.6}
$$

Let the edges of the laminate be loaded by constant forces per unit length T_{11}, T_{22}, T_{12} and define the stresses (3.1.4) as edge averages over the laminate thickness h:

$$
\begin{aligned}
\sigma_{11}^{\circ} &= T_{11}/h \\
\sigma_{22}^{\circ} &= T_{22}/h \\
\sigma_{12}^{\circ} &= T_{12}/h
\end{aligned}
\tag{3.1.7}
$$

Equilibrium requires that:

$$
\begin{aligned}
\sum_{k=1}^{K} \sigma_{11}^{(k)} &= \sigma_{11}^{\circ} \\
\sum_{k=1}^{K} \sigma_{22}^{(k)} &= \sigma_{22}^{\circ} \\
\sum_{k=1}^{K} \sigma_{12}^{(k)} &= \sigma_{12}^{\circ}
\end{aligned}
\tag{3.1.8}
$$

where K is the number of laminae. Written in terms of stresses $\sigma_{\alpha\beta}^{(k)}$ using (3.1.6), we have:

$$
\sum_{k=1}^{K} \underline{\theta}^{(k)} \underline{\sigma}^{(k)} = \underline{\sigma}^{\circ} \tag{3.1.9}
$$

where $\underline{\sigma}^{\circ}$ denotes the stresses $\sigma_{\alpha\beta}^{\circ}$ at the edges.

Replacement of the boundary conditions (3.1.4) by (3.1.6) is an approximation of Saint Venant type. Thus, there must be expected edge perturbations (among them interlaminar shear) on the stresses predicted by laminate theory.

Equations (3.1.8) are three equations for the 3K stresses $\sigma_{\alpha\beta}^{(1)}$, $\sigma_{\alpha\beta}^{(2)} \ldots \sigma_{\alpha\beta}^{(K)}$ in the laminae. There are needed an additional 3(K-1) equations which are provided by displacement continuity at lamina interfaces, requirement (d).

Since the stresses in each laminae are by hypothesis uniform, so are the strains. Therefore, displacement continuity is ensured if the lamine strains in adjacent laminae, referred to laminae coordinate system are the same. Thus:

$$\begin{aligned} {}^{(k)}\varepsilon_{11} &= {}^{(k+1)}\varepsilon_{11} \\ {}^{(k)}\varepsilon_{22} &= {}^{(k+1)}\varepsilon_{22} \qquad k=1,2\ldots\ldots k \\ {}^{(k)}\varepsilon_{12} &= {}^{(k+1)}\varepsilon_{12} \end{aligned} \tag{3.1.10}$$

Equations (3.1.10) are the additional required 3(K-1) equations. They will be written in terms of laminae stresses $\sigma_{\alpha\beta}^{(k)}$ referred to laminae material axes. To do this it is noted that:

$$ {}^{(k)}\underline{\varepsilon} = \underline{\theta}^{(k)} \; \underline{\varepsilon}^{(k)} $$

which is just a transformation of (3.1.6). From (3.1.2b):

$$ {}^{(k)}\underline{\varepsilon} = \underline{\theta}^{(k)} \; \underline{S}^{(k)} \; \underline{\sigma}^{(k)} \tag{3.1.11}$$

and inserting the last result in (3.1.10):

$$ \underline{\theta}^{(k)} \; \underline{S}^{(k)} \underline{\sigma}^{(k)} = \underline{\theta}^{(k+1)} \underline{S}^{(k+1)} \; \underline{\sigma}^{(k+1)} \qquad k=1,2\ldots\ldots k \tag{3.1.12}$$

Equations (3.1.9) and (3.1.12) are 3K linear equations for the 3K stresses in an elastic laminate, with K layers.

It should be carefully noted that the analysis given above is based on plane stress conditions in individual laminae. This is a valid assumption if:

(a) The loads on the laminate are statically equivalent to in-plane forces (membrane forces) and produce neither bending nor twisting moments, and

(b) The laminate has a certain stacking sequence of laminae which defines a so called balanced or symmetric laminate.

This stacking sequence is an arrangement in which the laminate has a middle plane of geometrical and of material symmetry. The laminae are arranged in paris with respect to the plane of symmetry. The laminae of such pair have equal thicknesses, same distances from middle plane, and are of the same material with same angles of reinforcement.

In a non-symmetric laminate application of membrane forces will in general produce bending and twisting of laminae and thus a plane state of stress will not be realized. The symmetric laminate is, however, sufficiently versatile to cover most cases of practical interest.

Let it now be assumed that the laminate is inelastic but still fulfills the conditions of symmetry and pure membrane loading. In this case the only equations which necessarily change in the preceding development are the stress-strain relations of the laminae, (3.1.2), which must be replaced by inelastic laminae stress-strain relations are given by (2.1.7) where the compliances are now functions of the stresses. These compliances now replace the elastic compliances in (3.1.2) which thus become non-linear.

It is convenient for later purposes to rewrite (3.1.2) in the inelastic case in different form. To do this the strains $\varepsilon_{\alpha\beta}^{(k)}$ are first split into elastic strains (2.1.9) and inelastic strains (2.1.11). Preceeding to (3.1.12) this equation assumes the form:

$$\begin{aligned}
&\underline{\theta}^{(k+1)}\underline{S}^{1(k+1)}\underline{\sigma}^{(k+1)} - \underline{\theta}^{(k)}\underline{S}^{1(k)}\underline{\sigma}^{(k)} = \\
&- \underline{\theta}^{(k+1)}\underline{S}^{11(k+1)}\underline{\sigma}^{(k+1)} + \underline{\theta}^{(k)}\underline{S}^{11(k)}\underline{\sigma}^{(k)}
\end{aligned} \tag{3.1.13}$$

$k = 1, 2 \ldots\ldots k$

where

$\underline{S}^{1(k)}$ - elastic compliance matrix of kth layer

$\underline{S}^{11(k)}$- inelastic part of compliance matrix of kth layer

$$\underline{S}^{(k)} = \underline{S}^{1(k)} + \underline{S}^{11(k)} \tag{3.1.14}$$

Equations (3.1.13) are now written out in component form with notation (2.1.10), (2.1.12) for compliances:

$$\sigma_{11}^{(k+1)}(S_{11}^{1\ (k+1)}\cos^2\theta_{k+1}+S_{12}^{1(k+1)}\sin^2\theta_{k+1})$$

$$+\sigma_{22}^{(k+1)}(S_{12}^{1(k+1)}\cos^2\theta_{k+1}+S_{22}^{1(k+1)}\ \sin^2\theta_{k+1})$$

$$-4\sigma_{12}^{(k+1)}S_{44}^{1(k+1)}\cos\ \theta_{k+1}\sin\theta_{k+1}\ -\ \sigma_{11}^{(k)}(S_{11}^{1(k)}\cos^2\theta_k\ +S_{12}^{1(k)}\ \sin^2\theta_k)$$

$$+\ \sigma_{22}^{(k)}\ (S_{12}^{1(k)}\cos^2\theta_k+S_{22}^{1}\ \sin^2\theta_k)\ =\ 4\sigma_{12}^{(k)}\ S_{44}^{1(k)}\ \cos\ \theta_k\ \sin\theta_k$$

$$=\ -\ \ \sigma_{11}^{\ (k+1)}(S_{11}^{11(k+1)}\cos^2\theta_{k+1}+S_{12}^{11(k+1)}\sin^2\theta_{k+1})$$

$$+\ \sigma_{22}^{(k+1)}(S_{12}^{11(k+1)}\cos^2\theta_{k+1}+S_{22}^{11(k+1)}\sin^2\theta_{k+1}\ -\ 4\sigma_{12}^{(k+1)}S_{44}^{11(k+1)}\cos\theta_{k+1}\sin\theta_{k+1}$$

$$+\sigma_{11}^{(k)}(S_{11}^{11(k)}\cos^2\theta_k+S_{12}^{11(k)}\sin^2\theta_k)\ \ +\ \sigma_{22}^{(k)}(S_{12}^{11(k)}\cos^2\theta_k+S_{22}^{11}\sin^2\theta_k)$$

$$-\ 4\sigma_{12}^{(k)}S_{44}^{11(k)}\cos\theta_k\sin\theta_k \tag{3.1.15}$$

$$\sigma_{11}^{(k+1)}(S_{11}^{1(k+1)}\sin^2\theta_{k+1}+S_{12}^{1(k+1)}\cos^2\theta_{k+1}$$

$$+\sigma_{22}^{(k+1)}(S_{12}^{1(k+1)}\sin^2\theta_{k+1} + S_{22}^{1(k+1)}\cos^2\theta_{k+1})$$

$$+4\sigma_{12}^{(k+1)}S_{44}^{1(k+1)}\cos\theta_{k+1}\sin\theta_{k+1}$$

$$\sigma_{11}^{(k)}(S_{11}^{1(k)}\sin^2\theta_k + S_{12}^{1(k)}\cos^2\theta_k$$

$$+\sigma_{22}^{(k)}(S_{12}^{1(k)}\sin^2\theta_k + S_{22}^{1(k)}\cos^2\theta_k \qquad (3.1.16)$$

$$+4\sigma_{12}^{(k)}S_{12}^{1(k)}S_{44}^{1(k)}\cos\theta_k\sin\theta_k$$

$$= -\sigma_{11}^{(k+1)}(S_{11}^{11(k+1)}\sin^2\theta_{k+1} + S_{12}^{11(k+1)}\cos^2\theta_{k+1})$$

$$+\sigma_{22}^{(k+1}(S_{12}^{11(k+1)}\sin^2\theta_{k+1} + S_{22}^{11(k+1)}\cos^2\theta_{k+1}$$

$$+4\sigma_{12}^{(k+1)}S_{44}^{11(k+1)}\cos_{k+1}\sin\theta_{k+1} + \sigma_{11}^{(k)}(S_{11}^{11(k)}\sin\theta_k+S_{12}^{11(k)}\cos^2\theta_k)$$

$$+\sigma_{22}^{(k)}(S_{12}^{11(k)}\sin^2\theta_k + S_{22}^{11(k)}\cos^2\theta_k+4\sigma_{12}^{(k)}S_{44}^{11(k+1)}\cos_{k+1}\sin\theta_{k+1}$$

$$\sigma_{11}^{(k+1)}(S_{11}^{1(k+1)}-S_{22}^{1(k+1)}\sin\theta_{k+1}\cos\theta_{k+1}$$

$$+\sigma_{22}^{(k+1)}(S_{12}^{1(k+1)}-S_{22}^{1(k+1)})\sin\theta_{k+1}\cos\theta_{k+1}$$

$$+2\sigma_{12}^{(k+1)}S_{44}^{1(k+1)}(\cos^2\theta_{k+1} - \sin^2\theta_{k+1})$$

$$-\sigma_{11}^{(k)}(S_{11}^{1(k)} - S_{12}^{1(k)})\sin\theta_k\cos\theta_k$$

$$+\sigma_{22}^{(k)}(S_{12}^{11(k)} - S_{22}^{1(k)})\sin\theta_k\cos\theta_k+2\sigma_{12}^{(k)}S_{44}^{1(k)}(\cos^2\theta_k-\sin^2\theta_k)$$

$$= -\sigma_{11}^{(k+1)}(S_{11}^{11(k+1)} - S_{12}^{11(k+1)})\sin\theta_{k+1}\cos\theta_{k+1}$$

$$+\sigma_{22}^{(k+1)}(S_{12}^{11(k+1)} - S_{22}^{11(k+1)})\sin\theta_k\cos\theta_k$$

$$+2\sigma_{12}^{(k+1)}S_{44}^{11(k+1)}(\cos^2\theta_{k+1} - \sin^2\theta_{k+1})$$

$$+\sigma_{11}^{(k)}(S_{11}^{11(k)} - S_{12}^{11(k)})\sin\theta_k\cos\theta_k$$

$$+\sigma_{22}^{(k)}(S_{12}^{11(k)} - S_{22}^{11(k)})\sin\theta_k\cos\theta_k$$

$$+2\sigma_{12}^{(k)}S_{44}^{11(k)}(\cos^2\theta_k - \sin^2\theta_k) \qquad (3.1.17)$$

$k = 1,2 \ldots\ldots k-1$

To these must be adjoined equations (3.1.9) which are written here in components:

$$\sum_{k=1}^{k}(\sigma_{11}^{(k)}\cos^2\theta_k+\sigma_{22}^{(k)}\sin^2\theta_k-2\sigma_{12}^{(k)}\cos\theta_k\ \sin\theta_k)t_k = \sigma_{11}^{\circ}\ h \qquad \text{(a)}$$

$$\sum_{k=1}^{k}(\sigma_{11}^{(k)}\sin^2\theta_k+\sigma_{22}^{(k)}\cos^2\theta_k+2\sigma_{12}^{(k)}\cos\theta_k\ \sin\theta_k)\ t_k = \sigma_{22}^{\circ}\ h \qquad \text{(b)}$$

$$\sum_{k=1}^{k}(\sigma_{11}^{(k)}-\sigma_{22}^{(k)})\cos\theta_k\sin\theta_k+\sigma_{12}^{(k)}(\cos^2\theta_k-\sin^2\theta_k)t_k = \sigma_{12}^{\circ}\ h \qquad \text{(c)}$$

(3.1.18)

We now consider special cases of interest. In the first case the inelastic laminae strains have the form (2.1.13). Then the right side of (3.1.15-.17) simplifies by setting:

$$S_{11}^{11(k)} = S_{11}^{11(k+1)} = S_{12}^{11(k)} = S_{12}^{11(k+1)} = 0$$

$$S_{22}^{11(k)} = S_{22}^{11(k)}\ (\sigma_{22}^{(k)},\ \sigma_{12}^{(k)})$$

$$S_{22}^{11(k+1)} = S_{22}^{11(k+1)}(\sigma_{22}^{(k+1)},\ \sigma_{12}^{(k+1)})$$

$$S_{44}^{11(k)} = S_{44}^{11(k)}\ (\sigma_{22}^{(k)},\ \sigma_{12}^{(k)})$$

$$S_{44}^{11(k+1)} = S_{44}^{11(k+1)}(\sigma_{22}^{(k+1)},\ \sigma_{12}^{(k+1)}) \qquad (3.1.19)$$

Once the stresses in the laminae have been obtained the strains in the laminae, referred to laminate axes, are determined from (3.1.11). Since the strains in all laminae are the same when referred to the laminate coordinate system, these are also the average laminate strains and thus determine the inealstic response of the laminate.

In the simplest case the lamina material is assumed to be inelastic in shear only. In that event we have in addition to (3.1.19):

$$S''^{(k)}_{22} = S''^{(k+1)}_{22} = 0 \qquad (3.1.20)$$

and for Ramberg-Osgood presentation of inelastic part of shear compliance:

$$2S^{11(k)}_{44} = \frac{1}{2G_A^{(k)}} \left(\frac{\sigma_{12}^{(k)}}{\tau_y^{(k)}}\right)^{N_k - 1}$$

$$2S^{11(k+1)}_{44} = \frac{1}{2G_A^{(k+1)}} \left(\frac{\sigma_{12}^{(k+1)}}{\tau_y^{(k+1)}}\right)^{N_{k+1} - 1} \qquad (3.1.21)$$

In Ramberg-Osgood representation (2.2.1) the inelastic parts of the compliances assume forms such as:

$$S^{11(k)}_{22} = \frac{1}{E_T^{(k)}} \left[\left(\frac{\sigma_{22}^{(k)}}{\sigma_y^{(k)}}\right)^2 + \left(\frac{\sigma_{12}^{(k)}}{\tau_y^{(k)}}\right)^2 \right]^{1/2(M_k - 1)}$$

$$2S^{11(k)}_{44} = \frac{1}{2G_A^{(k)}} \left[\left(\frac{\sigma_{22}^{(k)}}{\sigma_y^{(k)}}\right)^2 + \left(\frac{\sigma_{12}^{(k)}}{\tau_y^{(k)}}\right)^2 \right]^{1/2(N_k - 1)}$$

$$(3.1.22)$$

$$S^{11(k+1)}_{22} = \frac{1}{E_T^{(k+1)}} \left[\left(\frac{\sigma_{22}^{(k+1)}}{\sigma_y^{(k+1)}}\right)^2 \left(\frac{\sigma_{12}^{(k+1)}}{\tau_y^{(k+1)}}\right)^2 \right]^{1/2(M_{k+1} - 1)}$$

$$2S^{11(k+1)}_{44} = \frac{1}{2G_A^{(k+1)}} \left[\left(\frac{\sigma_{22}^{(k+1)}}{\sigma_y^{(k+1)}}\right)^2 + \left(\frac{\sigma_{12}^{(k+1)}}{\tau_y^{(k+1)}}\right)^2 \right]^{1/2(N_k - 1)}$$

3.2 Method of Solution

The equations which define the laminae stresses are (3.1.9) and (3.1.13) in condensed form, or equivalently, (3.1.15- 3.1.17), (3.1.18) in full form. To explain the solution method it is simpler to write in terms of the condensed form.

Define the matrices:

$$\begin{aligned}
\underline{L}^{1(k+1)} &= \underline{\theta}^{(k+1)}\underline{S}^{1(k+1)} \\
\underline{L}^{11(k+1)} &= \underline{\theta}^{(k+1)}\underline{S}^{11(k+1)} \\
\underline{L}^{1(k)} &= \underline{\theta}^{(k)}\underline{S}^{1(k)} \\
\underline{L}^{11(k)} &= \underline{\theta}^{(k)}\underline{S}^{11(k)}
\end{aligned} \qquad (3.2.1)$$

Then equs. (3.1.13) assume the form:

$$\underline{L}^{1(k+1)}\underline{\sigma}^{(k+1)}-\underline{L}^{1(k)}\underline{\sigma}^{(k)} = -\underline{L}^{11(k+1)}\underline{\sigma}^{(k+1)}+\underline{L}^{11(k)}\underline{\sigma}^{(k)} \qquad (3.2.2)$$

to which are adjoined equs. (3.1.9) which are here rewritten:

$$\sum_{k=1}^{K} \underline{\theta}^{(k)}\underline{\sigma}^{(k)} = \underline{\sigma}^{\circ} \qquad (3.2.3)$$

The equations may be solved numerically by an iteration method which proceeds as follows: Consider equs. (3.2.2-3) with the right side of (3.2.2) zero. This defines a set of stresses $\underline{\sigma}_{\circ}^{(k)}$ given by:

$$\underline{L}^{1(k+1)}\underline{\sigma}_{\circ}^{(k+1)} - \underline{L}^{1(k)}\underline{\sigma}_{\circ}^{(k)} = 0 \qquad (a)$$

$$k = 1,2 \;.....k=1 \qquad (3.2.4)$$

$$\sum_{k=1}^{K} \underline{\theta}^{(k)}\,\underline{\sigma}_{\circ}^{(k)} = \underline{\sigma}^{\circ} \qquad (b)$$

Since (3.2.4a) contains only elastic compliances $S'^{(k)}$ it is seen that the equations are linear and define the stresses in an elastic laminate. Now insert the stresses $\underline{\sigma}_{\circ}^{(k)}$ into the right side of (3.2.2) and define the stresses $\underline{\sigma}_{1}^{(k)}$ by:

$$\underline{L}^{1(k+1)}\underline{\sigma}_1^{(k+1)} - \underline{L}^{1(k)}\underline{\sigma}_1^{(k)} = -\underline{L}^{11(k+1)}[\underline{\sigma}_\circ^{(k+1)}]\underline{\sigma}_\circ^{(k+1)} \quad \text{(a)}$$

$$+\underline{L}^{11(k)}[\underline{\sigma}_\circ^{(k)}]\underline{\sigma}_\circ^{(k)} \qquad (3.2.5)$$

$$\sum_{k=1}^{K} \underline{\theta}^{(k)}\underline{\sigma}_1^{(k)} = \underline{\sigma}^\circ \qquad \text{(b)}$$

Equs. (3.2.5) defines (hopefully) a new approximation $\underline{\sigma}_1^{(k)}$ which is the solution of a set of linear equations. The stresses in square brackets in the right side of (3.2.5) are to emphasize the stress dependence of the non-linear parts of the compliances.

The procedure just initiated can be repeated indefinitely. In general:

$$\underline{L}^{1(k+1)}\underline{\sigma}_{\ell+1}^{(k+1)} - \underline{L}^{1(k)}\underline{\sigma}_{\ell+1}^{(k)} = -\underline{L}^{11(k+1)}[\underline{\sigma}_\ell^{(k+1)}]\,\underline{\sigma}_\ell^{(k+1)}$$

$$+\underline{L}^{11(k)}[\underline{\sigma}_\ell^{(k)}]\,\underline{\sigma}_\ell^{(k)} \qquad (3.2.6)$$

$$\sum_{k=1}^{K} \underline{\theta}^{(k)}\underline{\sigma}_{\ell+1}^{(k)} = \underline{\sigma}^\circ$$

This iteration procedure is quite easy to carry out with aid of a computer. It replaces the solution of a set of non-linear equations by solution of a sequence of linear equations, provided of course, that convergence is obtained.

It should be noted that the first iteration step does not necessarily have to start with equs. (3.2.4a), i.e., with zero right side of (3.2.2). Any stresses $\sigma_\circ^{(k)}$ which fulfill (3.2.4b) can be used to start the iteration with (3.2.5) and continuing with the general iteration relation (3.2.6).

It is desired to obtain a laminate solution for only one load system $\underline{\sigma}^\circ$ then it would seem most logical to start with (3.2.4). But suppose there is a sequence of loadings $\Delta\underline{\sigma}^\circ$, $2\Delta\underline{\sigma}^\circ \ldots n\Delta\underline{\sigma}_\circ$. Suppose that a solution for (n-1) $\Delta\underline{\sigma}^\circ$ has been obtained and that a solution for $n\Delta\underline{\sigma}^\circ$ is desired. One possibility is to multiply all stresses due to the load (n-1) $\Delta\underline{\sigma}^\circ$ by the factor n/(n-1). The stresses thus obtained certainly also satisfy (3.2.6b) because of the linearity of these equations. They will generally be reasonable starting values $\underline{\sigma}_o^{(k)}$ for the iteration.

This method of iteration to obtain a solution was found to work well for many sample problems; however, there were cases in which the solution did not converge. Attempts to modify the recurrence relations to overcome this problem met with only partial success. Thus, an alternate procedure for solution was defined. The solution was obtained by application of the Newton-Raphson method.

The set of 3K nonlinear equations represented by equs. (3.2.2-.3) may be presented in the form:

$$F_n\ (\sigma_{ij}{}^k) = 0 \qquad n = 1,2 \ldots 3K \tag{3.2.7}$$

The function F_i is expanded in a Taylor series about an arbitrary set of initial stresses which may be taken as the solutions of the elasticity problem. Considering only two terms of the series, it is found that

$$F_i = \overset{\circ}{F}_i + \frac{\partial \overset{\circ}{F}_i}{\partial \sigma_{mn}{}^k}\ \Delta\sigma_{mn}{}^k = 0 \tag{3.2.8}$$

or

$$\sigma_{ij}{}^k = {}^\circ\sigma_{ij}{}^k - [\frac{\partial \overset{\circ}{F}_m}{\partial \sigma_{ij}{}^k}]^{-1}\ \overset{\circ}{F}_m \tag{3.2.9}$$

where σ_{ij}^k is the corrected solution obtained from the assumed solution ${}^\circ\sigma_{ij}^k$. Using σ_{ij}^k as the initial guess, the process is repeated until the result is obtained within a desired accuracy. A recurrence form of equation (3.2.9) to obtain the stresses at t+1 cycle from t cycle can be constructed as follows:

$$(\sigma_{ij}{}^{k})_{t+1} = (\sigma_{ij}{}^{k})_{t} - [\frac{\partial F_m}{\partial \sigma_{ij}{}^{k}}]^{-1} (F_m)_t \tag{3.2.10}$$

After the stresses σ^k_{ij} are obtained for all layers of the laminate, strains for any layer k in terms of laminae axes can be computed using equs. (3.1.3). Strains in terms of the laminate axes can be obtained using the strain transformation law.

This analysis has been developed into an efficient computer program. A description of the program including a listing, is presented in Appendix E.

3.3 Numerical Results

The computer program which has been developed under the present study has been utilized in the analysis of a variety of different composite laminates. The initial studies using the computerized analysis were directed at presenting a comparison between the results of the present analysis and those of previous analyses, notably that of Ref. 9. (The present results were also compared to available experimental data, primarily those of Ref. 6 which had also been used for comparison with the analytical results in Ref. 9.) The objective of this phase of the numerical study was to determine whether the present results, which can be obtained with minimal computer usage, compare well with those of the more exact and complex analytical results in Ref. 9. The results of this comparison are highly encouraging, as will be shown below, and support the utilization of the present analysis as an efficient design tool.

In the second phase of the design numerical studies, consideration was given to examining the sensitivity of laminate results to individual properties of the layers. These parametric studies are presented for several classes of typical laminates.

A series of laminates of boron/epoxy composites for which experimental data had been obtained in Ref. 6 were examined analytically in Ref. 9. In Figures 9 to 15, results of the present analytical method are added to the comparison of experimental results of [6] and analytical results of [9]. For example, in Fig. 9, the experimental stress-strain curve for a 0-90 boron/epoxy laminate is compared to the analytical results obtained in Ref. 9 and in the present analysis. Both analytical results coincide; both show slightly less inelastic strain than the experiment. The solid point on the curve indicates the stress level at which fiber fracture is computed to occur in one of the layers of the laminate.

The shear stress-strain curve used in the present analysis was the best fit Ramberg-Osgood curve having an exponent n=3.

The values of modulus and yield stress obtained from the least squares fit are shown on the figure. A similar result is shown for the unidirectional tension ±45° laminate in Fig. 10. Here it is seen that the two analytical curves are similar, although the agreement is not as close as in Fig. 9. Experimental data reflect a substantially higher degree of inelasticity than either analytical result. The present analysis shows a higher degree of inelastic strain at the higher stress level than that of Ref. 9. However, the reverse is true in the comparison of the two analytical results shown in Fig. 11 for a ±30° laminate. The present results were obtained with a linear stress-strain curve in the transverse direction within each of the layers. The computations were made in this fashion because the transverse stress-strain curve of Ref. 9 does not show a significant degree of inelasticity.

Figure 12 presents results for the case of a quasi-isotropic laminate (0/±45/90) of boron/epoxy. Both the present result and that of Ref. 9 show a relatively insignificant amount of inelasticity. Again, the experimental data show a greater inelastic effect. Here the predicted failure strain level is in good agreement with the experimental failure strain level; however, there is a significant difference in the failure stress level. A similar result is presented in Fig. 13 for the quasi-isotropic laminate formed from the 0/±60° configuration.

Computations performed for the present study for laminates having fibers in several directions, including the loading direction, for a simple unidirectional load have shown a relatively small amount of inelastic strain. Another example of this is presented in Fig. 14 for a 0/±45° laminate. Here, however, the agreement of all the analytical methods and the experimental method is very good.

The final comparison taken from Ref. 9 is presented in Fig. 15 for a laminate having fibers in three different directions and a tensile load applied at some intermediate angle. The present analysis agrees reasonably well with the results

Ref. 9. The discrepancy between the failure load predicted on the basis of fiber failure and the experimentally observed failure stress is quite substantial. It is possible that fialure in laminate of this type caould result from shearing or transverse stresses within the individual layers, and thus, not be a result of tension in the fiber failure. This mode of failure has not been treated in the present computer program. The mode of failure observed experimentally is not known to the authors.

The experimentally measured response of a multidirectional laminate to an applied shear stress has been reported in Ref. 13. Comparison of the experimental result with the theory of Ref. 9 was presented in Ref. 14. Computations for this case, made using the present analysis and the prior analytical result (Ref. 14), are compared to the experimental result in Fig. 16. Again, correlation between the two analytical results is good, agreement between analytical and theoretical results is reasonably good with the experimental observation showing higher inelastic strains and lower tangent shear moduli at the very high stress levels.

The conclusion of these comparisons with analytical and experimental data seem to justify the adoption of the present computer program as a useful engineering tool for the design and analysis of composite laminates. However, it appears that further study of the failure region is required.

Parametric study of the influence of various laminate geometric and mechanical properties has also been explored. Fig. 17 shows the results obtained for a 0/±45° laminate indicating that the inelastic response in the transverse direction can become significant at higher stress levels. Failure due to fiber fracture under a transverse stress applied to the laminate occurs at strain levels larger than those plotted in Fig. 17. In the quasi-isotropic laminate having four fiber directions, (0/±45/90) the degree of inelasticity in the longitudinal and transverse directions is of course the same and is in both cases very small. It is to be anticipated, on an intuitive basis, that the maximum degree of inelastic response would be observed for a stress applied midway between two of the fiber directions on this quasi-isotropic laminate. The stress-strain curve for this latter case is also shown in Fig. 18. Although the inelastic strains for this case are not significant there is a large difference in the predicted failure stress levels based on stress in the fiber direction for the two cases. It is worthwhile to emphasize that the quasi-isotropic laminate need not be isotropic in its strength characteristics.

Because of the directional strength characteristics interesting effects may be expected for combined stress cases. Some results of the exploration of this question are presented in Fig. 19 where the four direction quasi-isotropic laminate is subjected to combined stress state with respect to a 22-1/2° axis of symmetry. This laminate shows high strength under both the unidirectional load and shear load by itself. The combined stress case for equal values of applied shear stress and axial stress results in fiber failure, and therefore, laminate failure, at a substantially lower stress. The stress-strain curve prior to failure is not affected significantly by the presence of combined stress. The quasi-isotropic laminate having fibers in three directions ($0/\pm60$) is examined in Fig. 20. The sensitivity of this laminate to the Ramberg-Osgood parameters for the individual ply had little effect upon the stress-strain result. Indeed as an extreme example of this variation all lamiates stiffnesses except the axial stiffness were equal to zero. Enforcement of the Kirchhoff-Love plate assumptions for this case results in the so-called netting analysis. The response for this netting case, which is linear, is shown by the dashed curve in Fig. 20. Even with this extreme assumption, matrix inelasticity does not introduce a significant amount of inelastic strain. Experimental data for comparison with this result are not easily available, however Ref. 17 does present a stress-strain curve for this case which shows a transverse failure stress for the quasi-isotropic 0/$\pm$60° laminate which is about 60% of the failure stress in the axial direction. Also, the inelastic strain at failure is approximately 30% larger than the elastic strain associated with the failure stress level. The netting analysis result presented here suggests that in order to obtain such a strain, one might have to consider that the axial stiffness, either in tension, compression or both; or that other effects not considered in the conventional laminate analysis, such as interlaminar or transverse shear deformations, might contribute significantly to the overall laminate deformation.

The influence of the characteristic stress levels for transverse stress and axial shear of the unidirectional layer of a boron/epoxy material is examined in Fig. 21. The measure of this effect is taken to be the influence upon the stress-strain curve for the unidirectional tension of $\pm$30° laminate. The strong sensitivity to the characteristic axial shear stress τ_y and the relative insensitivity to the transverse characteristic stress σ_y for the R-O representations is illustrated in the figure. A similar comparison made for a boron/aluminum laminate of the same geometry subjected to uniaxial applied stress is shown in Fig. 22. Similar sensitivities are observed for this case. Boron/aluminum laminate response under transverse applied stress with the same values of the Ramberg-Osgood parameters is shown in Fig. 23. Here the fiber failure criterion did not come into play and thus the computations were extended to rather large strains in matrix. It is clear, that for this case, the failure criterion based on other stress-strain components is required. The examination of the computer print-out permits one to terminate the stress-strain curves at some stress level prior to fiber fracture depending upon the choice of the failure criterion. This can be done rather readily. The choice of the failure criterion is discussed in Appendix D.

The lamina properties for boron/aluminum are used to analyze a $0°/\pm 30°$ laminate under combined loading. These results are shown in Fig. 24. Axial stress-strain curves are presented for varying ratios of axial shear stress to axial tensile stress.

4. CONCLUDING REMARKS

Current approaches to the definition of design allowable stress for advanced fiber composite laminates are based upon the utilization of extremely conservative criteria. These limit the laminate to stress levels below which no significant damage of any kind occurs. The utilization of overly conservative design criteria can negate much of the potential for effective design utilizing advanced composite materials. The heterogeneous nature of these materials is such that a variety of possible damage modes exist. Thus, matrix cracking or yielding, fiber fracture, debonding, and other inelastic effects can all occur in local regions at relatively low average stress levels. These nonuniform and nonlinear effects greatly complicate the problem of establishing reliable design allowables. In the present program, the problem of nonlinear laminate behavior resulting from nonlinearities in the behavior of the matrix material was studied. The objective of the program was to develop an understanding of the inelastic behavior of composite laminates and to develop a computer program which will be used as an engineering tool in the design of fiber composite laminated structures.

The method of approach utilized herein was to adopt a Ramberg-Osgood representation of the nonlinear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention herein was directed toward the axial shear behavior, in as much as experimental data had indicated that it is this type of load which results in the most significant nonlinearities in material behavior. For this case, an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix nonlinear stress-strain curve. This expression, which was developed as a lower bound, was found to give good agreement with the more exact results obtained by applying incremental plasticity theory and using a numerical finite element analysis to the assessment of the material behavior (Ref. 9).

The second level of approach treats the interelationship between the properties of the unidirectional layers and those of the laminate. For this case, one may consider that the starting point is a nonlinear stress-strain curve for transverse stress, and for axial shear stress, alone, and a linear stress-strain relation for stress in the fiber direction. The nonlinear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters.

In the present study, unlike other formulations an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress. A laminate having an arbitrary number of oriented layers, and subjected to a general state of membrane stress, was treated. The results of this analysis were programmed into an efficient computer routine for numerical evaluation of arbitrary laminates. Results obtained show good agreement with those of previous complex numerical methods utilizing incremental plasticity theory.

Certain limitations connected with this program should also be discussed. First, deformation type stress-strain relations have been used; hence, it is implicit in this result that the stress and strain values obtained for any given set of loads are functions only of those loads and not of the loading history. On the other hand, if points are computed for intermediate values of loads, following different load paths, then different intermediate conditions will be obtained. Thus, the question is raised as to what is the accuracy of the results obtained for paths which do not yield proportional loading. It is known that for local proportional loading, the deformation theory result is the same as that for the incremental theory. In the laminate, local proportional loading does not exist, in general, even when the external loading is proportional. However, the assumption is made that the deformation theory will yield an approximation which is satisfactory to generate a rational engineering tool. This can only be assessed by comparison with an exact analysis, or since this does not exist for the case of arbitrary loading paths, perhaps by comparison with experimental data.

Comparisons of the present results with experimental data tend to show moderately good agreement. There are, however, cases in which experimental results show a higher degree of inelastic strain than predicted by the present analysis. These experimental data are quite limited and may be insufficient for drawing conclusions in this regard.

The question of failure criteria incorporated into the present analysis required further consideration. The present analysis obtains more accurate representations of the stress components in the individual layers than have been obtained from elastic analyses. Hence, the use of these stress components in any failure criteria should represent an improvement in failure prediction

In addition to a description of the methods of analysis, and of the numerical comparisons which have been carried out, the present report also presents a description of the computer program for study of nonlinear behavior of laminates in sufficient detail to permit the utilization of this program by others.

APPENDIX A

SYMMETRY SIMPLIFICATION OF NON-LINEAR STRESS-STRAIN RELATIONS

The most general inelastic stress-strain relations of the deformation type are of the form

$$\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} \tag{1}$$

where S_{ijkl} are functions of the stresses. Let it be assumed that the material is transversely isotropic with x_1 axis of symmetry. Any rotation about x_1 changes ε_{ij} and σ_{ij} into ε'_{ij} and σ'_{ij}. Then the condition of transverse isotropy demands that

$$\varepsilon'_{ij} = S_{ijkl}\,\sigma'_{kl} \tag{2}$$

where S_{ijkl} in (1) and (2) are the same. To fulfill this last requirement it is necessary that S_{ijkl} be functions of stresses only through stress expressions which are invariant for rotations about the x_1 axis. There are five such invariants and they are given by, [15]

$$I_1 = \sigma_{11} \qquad I_2 = \sigma_{22}+\sigma_{33} \qquad I_3 = \sigma_{12}^2+\sigma_{13}^2 \tag{3}$$

$$I_4 = 1/2(\sigma_{22}-\sigma_{33})^2+2\sigma_{23}^2 \qquad I_5 = 1/2(\sigma_{22}-\sigma_{33})(\sigma_{12}^2-\sigma_{13}^2)+2\sigma_{12}\sigma_{13}\sigma_{23}$$

Thus

$$S_{ijkl} = S_{ijkl}\ (I_1, I_2, I_3, I_4, I_5) \tag{4}$$

It follows that for rotations around the x_1 axis of symmetry the S_{ijkl} behave as constants. Consequently, the symmetry reduction of (1) to transverse isotropy is just as in elasticity.

The reduction may be performed in following fashion: For rotation of angle θ about the x_1 axis, the stress tensor σ_{ij} transforms into σ'_{ij} in the following fashion

$$\begin{aligned}
\sigma'_{11} &= \sigma_{11} \\
\sigma'_{22} &= 1/2\ (\sigma_{22} + \sigma_{33}) + 1/2\ (\sigma_{22} - \sigma_{33}) \cos 2\theta + \sigma_{23} \sin 2\theta \\
\sigma'_{33} &= 1/2\ (\sigma_{22} + \sigma_{33}) - 1/2\ (\sigma_{22} - \sigma_{33}) \cos 2\theta - \sigma_{23} \sin 2\theta \\
\sigma'_{23} &= 1/2\ (\sigma_{33} - \sigma_{22}) \sin 2\theta + \sigma_{23} \cos 2\theta \\
\sigma'_{12} &= \sigma_{12} \cos \theta + \sigma_{13} \sin \theta \\
\sigma'_{13} &= -\sigma_{12} \sin \theta + \sigma_{13} \cos \theta
\end{aligned} \tag{5}$$

The same transformation relations obviously also hold for strains. If the transformed stresses and strains are introduced into (2) then coefficients of cos 2θ, sin 2θ, cos θ and sin θ and remaining terms independent of θ must be equal. These equalities result in relations among the various components which reduce the stress-strain law to the form (2.1.4- 5) from Chapter 2 of this report. (Average stresses and strains appear in the latter but this obviously makes no differences in the derivation.)

APPENDIX B

PLANE STRESS-STRAIN RELATIONS OF FIBER REINFORCED MATERIAL IN GENERALIZED RAMBERG-OSGOOD FORM

The purpose of the present appendix is to arrive at equs. (2.2.7). For convenience in writing, overbars on stresses and strains will be omitted.

The present development is guided by isotropic J_2 theory for deformation type plastic stress-strain relations. The basic assumption of this theory in the isotropic case is that the plastic strains have the form

$$\varepsilon''_{ij} = f(J_2)\, s_{ij} \tag{1}$$

where s_{ij} is the stress deviator and

$$J_2 = 1/2\ s_{ij}\ s_{ij} \tag{2}$$

is its second invariant.

It is instructive to work out the form of (1) for Ramberg-Osgood type stress-strain relations. Suppose that in pure shear the stress-strain relation is

$$\varepsilon''_{12} = \frac{\sigma_{12}}{2G} \left[1 + \left(\frac{\sigma_{12}}{\tau_y}\right)^{n-1}\right] \tag{3}$$

Now in pure shear it follows from (2) that

$$J_2 = \sigma_{12}^{\ 2}$$

Therefore (3) can be written in the form

$$\varepsilon''_{12} = \frac{\sigma_{12}}{2G} \left[1 + \left(\frac{\sqrt{J_2}}{\tau_y}\right)^{n-1}\right] \tag{4}$$

which is in the form(1). Consequently, in the general case of three dimensional stress and strain

$$\varepsilon''_{ij} = \frac{s_{ij}}{2G} \left[1 + \left(\frac{\sqrt{J_2}}{\tau_y}\right)^{n-1}\right] \tag{5}$$

It should be emphasized that there is nothing fundamental about (1). It is an assumption which states that the plastic strains can be represented by the stress deviator components multiplied by a function of a quadratic expression in the stresses which is J_2. The choice of J_2 for a quadratic expression is not arbitrary but may be arrived at by isotropy arguments.

In an anisotropic material it may be assumed by generalization that plastic strains are given by

$$\varepsilon''_{ij} = s_{ij} \; f \; (L) \tag{6}$$

Where L is some general quadratic function of the stresses. This assumption will form the basis of the present development.

Consider the stress-strain relations (2.1.13). It is assumed that s''_{22} and s''_{44} functions of the most general quadratic form in $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$.

Thus

$$\begin{aligned} s''_{22} &= s''_{22} \quad (A\bar{\sigma}_{22}{}^2 + B\bar{\sigma}_{22}\,\bar{\sigma}_{12} + C\bar{\sigma}_{12}{}^2) \\ s''_{44} &= s''_{44} \quad (A\bar{\sigma}_{22}{}^2 + B\bar{\sigma}_{22}\,\bar{\sigma}_{12} + C\bar{\sigma}_{12}{}^2) \end{aligned} \tag{7}$$

It should be noted that the material reacts in same fashion to positive or negative shear stress, therefore also in same fashion to some $\bar{\sigma}_{22}$ together with positive or negative shear stress. However, the middle term in the quadratic changes sign with shear stress. Therefore, this term should be omitted.

Now rewrite (7) in form

$$\begin{aligned} s''_{22} &= \frac{1}{E_T}\, f_{22} \; (\alpha^2 \, \bar{\sigma}_{22}{}^2 + \beta^2 \, \bar{\sigma}_{12}{}^2) \\ s''_{44} &= \frac{1}{2G_T}\, f_{44} \; (\alpha^2 \, \bar{\sigma}_{22}{}^2 + \beta^2 \, \bar{\sigma}_{12}{}^2) \end{aligned} \tag{8}$$

where f_{22} and f_{44} are nondimensional functions and α and β have dimensions of reciprocal of stress. If $\bar{\sigma}_{12}=0$ the first of (8) assumes the form

$$s''_{22} = \frac{1}{E_T} f_{22} (\alpha^2 \bar{\sigma}_{22}{}^2) \tag{9}$$

For one dimensional $\bar{\sigma}_{22}$, from the Ramberg-Osgood stress-strain relation (2.2.6a)

$$s''_{22} = \frac{1}{E_T} \left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^{M-1}$$

which can be written as

$$s''_{22} = \frac{1}{E_T} \left[\left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^2\right]^{\frac{M-1}{2}} \tag{10}$$

It follows from (8) and (10) that

$$\alpha^2 = \frac{1}{\sigma_y{}^2} \tag{11}$$

and the function of f_{22} is determined as (M-1)/2 power.
In similar fashion, when $\bar{\sigma}_{22}=0$, the second of (8) assumes the form

$$s''_{44} = \frac{1}{2G_T} f_{44} (\beta^2 \bar{\sigma}_{12}{}^2) \tag{12}$$

From the Ramberg-Osgood relation (2.2.6b) for one dimensional $\bar{\sigma}_{12}$

$$s''_{44} = \frac{1}{2G_T} \left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^{N-1}$$

which can be written as

$$s''_{44} = \frac{1}{2G_T} \left[\left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^2\right]^{\frac{N-1}{2}} \tag{13}$$

It follows from (12) and (13) that

$$\beta^2 = \frac{1}{\tau_y{}^2} \tag{14}$$

and the function f_{44} is determined as (N-1)/2 power.

Consequently (8) now assumes the form

$$
\begin{aligned}
s''_{22} &= \frac{1}{E_T}\left[\left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^2 + \left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^2\right]^{\frac{M-1}{2}} \\
s''_{44} &= \frac{1}{2G_T}\left[\left(\frac{\bar{\sigma}_{22}}{\sigma_y}\right)^2 + \left(\frac{\bar{\sigma}_{12}}{\tau_y}\right)^2\right]^{\frac{N-1}{2}}
\end{aligned}
\qquad (15)
$$

Then (2.2.7) follows from (15) and (2.1.13).

APPENDIX C

1. EXTREMUM PRINCIPLES OF DEFORMATION THEORY OF PLASTICITY

i. Principle of Minimum Potential Energy

Let

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \tag{1.1}$$

where C_{ijkl} are functions of the strains. The strain energy density is defined by the path dependent integral

$$W^{\varepsilon} = \int_{\underline{\varepsilon}=0}^{\underline{\varepsilon}} \sigma_{ij}(\underline{\varepsilon})\,d\varepsilon_{ij} \tag{1.2}$$

where $\underline{\varepsilon}$ is a concise notation for ε_{ij} . The strain energy U^{ε} of a body of volume V is defined by

$$U^{\varepsilon} = \int_V W^{\varepsilon} dV \tag{1.3}$$

Let the surface of the body be subjected to the boundary conditions

$$u_i(S) = u_i^{\circ} \text{ on } S_u \tag{1.4}$$

$$T_i(S) = T^{\circ}_i \text{ on } S_T$$

and let the body forces vanish. The potential energy U_p is defined by

$$U_p = \int_V W^{\varepsilon} dV - \int_{S_T} T^{\circ}_i u_i \tag{1.5}$$

Define an admissible displacement field $\tilde{u}_i(\underline{x})$ by

$$\tilde{u}_i = u^{\circ}_i \text{ on } S_u$$

$$\tilde{u}_i(\underline{x}) \text{ continuous everywhere} \tag{1.6}$$

Associated with $\tilde{u}_i$ are the strains $\tilde{\varepsilon}_{ij}$ derived from it by the usual relations.

Define $\tilde{W}^{\varepsilon}$ by

$$\tilde{W}^{\varepsilon} = \int_{\tilde{\varepsilon}=0}^{\tilde{\varepsilon}} \tilde{\sigma}_{ij}\, d\tilde{\varepsilon}_{ij} \tag{1.7}$$

where

$$\tilde{\sigma}_{ij} = C_{ijkl}(\tilde{\underline{\varepsilon}})\ \tilde{\varepsilon}_{kl} \tag{1.8}$$

Define

$$\tilde{U}_p = \int_V \tilde{W}^{\varepsilon} dV - \int_{S_T} T^{\circ}{}_i \tilde{u}_i dS \tag{1.9}$$

The principle of minimum potential energy for the present case then states that

$$\tilde{U}_p \geq U_p \tag{1.10}$$

equality taking place if and only if

$$\tilde{u}_i = u_i$$

In the event that displacements are prescribed over the entire surface, the surface integral in (1.9) vanishes. Then the principle reduces to that of minimum strain energy

$$\tilde{U}^{\varepsilon} \geq U^{\varepsilon} \tag{1.11}$$

ii. Principle of Minimum Complementary Energy

Let

$$\varepsilon_{ij} = S_{ijkl}\ (\underline{\sigma})\ \sigma_{kl} \tag{1.12}$$

where S_{ijkl} are stress dependent compliances

Define the complementary energy density W^σ by the path dependent integral

$$W^\sigma = \int_{\underset{\sim}{\sigma}=0}^{\underset{\sim}{\sigma}} \varepsilon_{ij} d\sigma_{ij} \tag{1.13}$$

Let the surface of the body be subjected to the boundary conditions (1.4) and let the body forces vanish. The complementary energy U_C is defined by

$$U_C = \int_V W^\sigma dV - \int_{S_u} T_i u^\circ_i \, dS \tag{1.14}$$

Define an admissible stress field $\tilde{\sigma}_{ij}$ by the following requirements

$$\tilde{\sigma}_{ij,j} = 0$$

$$\tilde{T}_i = \tilde{\sigma}_{ij} n_j \qquad \text{continuous everywhere} \tag{1.15}$$

$$T_i(S) = T^\circ_i \text{ on } S_T$$

Define the complementary energy functional $\tilde{U}_C$ by

$$\tilde{U}_C = \int_V \tilde{W}^\sigma dV - \int_{S_u} \tilde{T}_i u^\circ_i dS \tag{1.16}$$

where

$$\tilde{W}^\sigma = \int_{\underset{\sim}{\tilde{\sigma}}=0}^{\underset{\sim}{\tilde{\sigma}}} \tilde{\varepsilon}_{ij} d\tilde{\sigma}_{ij} \tag{1.17}$$

$$\tilde{\varepsilon}_{ij} = S_{ijkl}(\underset{\sim}{\tilde{\sigma}}) \, \tilde{\sigma}_{kl}$$

Then the principle of minimum complementary energy states that

$$\tilde{U}_C \geq U_C \tag{1.18}$$

equality occurring if and only if

$$\tilde{\sigma}_{ij} = \sigma_{ij}$$

If tractions are prescribed over the entire surface, $S_u=0$, the principle reduces to

$$\tilde{U}^\sigma \geq U^\sigma \tag{1.19}$$

For proof of these principles see e.g. [16]. An interesting application to obtain approximate solutions has been given in [17].

iii. Specialization of the Principles to Axial Shear with Ramberg-Osgood Stress-Strain Relations

In the case of axial shear of a uniaxially fiber reinforced material the only surviving stresses are

$$\sigma_{12} = \tau_2 \qquad \sigma_{13} = \tau_3 \tag{1.20}$$

where 1 indicates fiber direction. Denote the associated shear strains by

$$\varepsilon_{12} = \varepsilon_2 \qquad \varepsilon_{13} = \varepsilon_3 \tag{1.21}$$

Then the generalized Ramberg-Osgood stress-strain relations, Appendix B, (5) assume in the present case the form

$$\begin{aligned} \varepsilon_2 &= \frac{\tau_2}{2G}\left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1}\right] \\ \varepsilon_3 &= \frac{\tau_3}{2G}\left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1}\right] \\ \tau &= \sqrt{\tau_2{}^2 + \tau_3{}^2} \qquad \sqrt{J_2} \end{aligned} \tag{1.22}$$

In the present case

$$\sigma_{ij}d\varepsilon_{ij} = 2(\tau_2 d\varepsilon_2 + \tau_3 d\varepsilon_3) \tag{1.23}$$

Inserting (1.22) into (1.23) and using the relation

$$\tau d\tau = \tau_2 d\,\tau_2 + \tau_3 d\,\tau_3$$

it is easily shown that

$$\sigma_{ij} d\varepsilon_{ij} = \frac{\tau}{G}\left[1 + n\left(\frac{\tau}{\tau_y}\right)^{n-1}\right] d\tau \qquad (1.24)$$

To compute W^ε as defined by (1.2) it is necessary to integrate (1.24) from zero to some state of strain ε_2, ε_3. But it should be noted that (1.24) is expressed in terms of the variable τ only. Now τ can be expressed in terms of strains in following fashion. Define

$$\hat{\varepsilon} = \sqrt{\varepsilon_2^{\,2} + \varepsilon_3^{\,2}} \qquad (1.25)$$

It follows at once from (1.22) that

$$\hat{\varepsilon} = \frac{\tau}{2G}\left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1}\right] \qquad (1.26)$$

This relation defines τ as a function of $\hat{\varepsilon}$. Consequently, W^ε assumes the form

$$W^\varepsilon = \frac{1}{G}\int_0^{\tau(\hat{\varepsilon})} \tau\left[1 + n\left(\frac{\tau}{\tau_y}\right)^{n-1}\right] d\tau$$

which is easily integrated to yield

$$W^\varepsilon = \frac{\tau^2}{2G}\left[1 + \frac{2n}{n+1}\left(\frac{\tau}{\tau_y}\right)^{n-1}\right] \qquad (1.27)$$

$$\tau = \tau(\hat{\varepsilon})$$

According to (1.3) the strain energy U^ε is then given by the volume integral of (1.27). Note however that it is very difficult to express U^ε in terms of strains since this requires the solution of (1.26) for τ in terms of $\hat{\varepsilon}$. In general it is not possible to do this analytically. This places a severe limitation on the use of the principle of minimum potential energy or of minimum strain-energy with Ramberg-Osgood stress-strain relations.

Next we consider the principle of minimum complementary energy for axial shear. Since there are only shear stresses τ_2, τ_3 and shear strains ε_2, ε_3 the integrand in W^σ, (1.13), is given by

$$\varepsilon_{ij} d\sigma_{ij} = 2(\varepsilon_2 d\tau_2 + \varepsilon_3 d\tau_3) \tag{1.28}$$

It follows from (1.22-.23) **that** (1.28) is given by

$$\varepsilon_{ij} d\sigma_{ij} = \frac{\tau}{G} \left[1 + \left(\frac{\tau}{\tau_y}\right)^{n-1}\right] d\tau$$

Integration of this expression from 0 to τ yields

$$W^\sigma = \frac{\tau^2}{2G} \left[1 + \frac{2}{n+1} \left(\frac{\tau}{\tau_y}\right)^{n-1} \right] \tag{1.29}$$

Expression (1.29) now enters as the integral into the volume integral of U_C, (1.14).

We now examine the meaning of an admissible stress field $\tilde{\tau}_2$, $\tilde{\tau}_3$ in the present case. The only surviving equilibrium equation is

$$\frac{\partial \tilde{\tau}_2}{\partial x_2} + \frac{\partial \tilde{\tau}_3}{\partial x_3} = 0 \tag{1.30}$$

The traction components are

$$\begin{aligned} \tilde{T}_1 &= \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3 \\ \tilde{T}_2 &= \tilde{\tau}_2 n_1 \\ \tilde{T}_3 &= \tilde{\tau}_3 n_1 \end{aligned} \tag{1.31}$$

We shall be concerned with cylindrical boundaries in fiber reinforced materials whose generator is in x_1, direction. On such a surface $n_1=0$. Therefore the only surviving traction component on such a surface is

$$\tilde{T}_1 = \tilde{\tau}_n = \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3 \tag{1.32}$$

Consequently an admissible stress system $\tilde{\tau}_3$, $\tilde{\tau}_3$ must satisfy (1.30) and the value τ°_n of $\tilde{\tau}_n$ wherever prescribed on the boundary.

The complementary energy functional (1.16) assumes the form

$$\tilde{U}_C = \int_V \tilde{W}^\sigma dV - \int_{S_u} \tilde{T}_1 u^\circ_1 dS \qquad \text{(a)}$$

$$\tilde{W}^\sigma = \frac{\tilde{\tau}^2}{2G} \left[1 + \frac{2}{n+1} \left(\frac{\tilde{\tau}}{\tau_y}\right)^{n-1}\right] \qquad \text{(b)} \qquad (1.33)$$

$$\tilde{\tau} = \sqrt{\tilde{\tau}_2^{\,2} + \tilde{\tau}_3^{\,2}} \qquad \text{(c)}$$

2. LOWER BOUND FOR AXIAL SHEAR MODULUS

Consider a uniaxially reinforced lamina which is subjected to axial shear τ_o in the 1-2 plane on its boundary, fig. 5.

By the average stress theorem, of Ref. 5.

$$\bar{\sigma}_{12} = \tau_o \tag{2.1}$$

and all other average stresses vanish.

By the average theorem of virtual work, of Ref. 5,

$$\int_V \varepsilon_{ij} d\sigma_{ij} = \bar{\varepsilon}_{ij} d\,\sigma_{ij} \tag{2.2}$$

Since the only nonvanishing average stress in the present case is (2.1) we have

$$\bar{\varepsilon}_{ij} d\bar{\sigma}_{ij} = 2\bar{\varepsilon}_{12}\, d\tau_o \tag{2.3}$$

The complementary energy of the body is given by (14) of Appendix A. The surface integral vanishes however in the present case since no displacements are prescribed on the boundary. Now

$$\begin{aligned} U_C &= \int_V W^{\sigma} dV = \int_V \int_{\sigma=0}^{\sigma} \bar{\varepsilon}_{ij} d\bar{\sigma}_{ij} dV \\ &= \int_{\sigma=0}^{\sigma} \int \bar{\varepsilon}_{ij} d\bar{\sigma}_{ij} dV \quad \int_o^{\tau_o} \bar{\varepsilon}_{12}\, d\tau_o \end{aligned} \tag{2.4}$$

The last equality following from (2.2, 3).

By definition the effective secant modulus G_A^s is given by

$$\bar{\varepsilon}_{12} = \frac{\bar{\sigma}_{12}}{G_A^s(\bar{\sigma}_{12})} = \frac{\tau_o}{2G_A^s(\tau_o)} \tag{2.5}$$

Hence (2.4) assumes the form

$$U_C = V \int_o^{\tau_o} \frac{\tau_o d\tau_o}{G_A^s(\tau_o)} \tag{2.6}$$

In order to find a bound on G_A^s it will be necessary to find a bound on (2.6) by use of the principle of minimum complementary energy.

It is assumed that the fibers are infinitely rigid in comparison to the matrix. Therefore at fiber/matrix interface

$$u_1 = 0 \tag{2.7}$$

and the only contribution to the complementary energy is from the matrix. Thus, the surface integral in (1.33a) vanishes and it can be written as

$$\tilde{U}_C = \int_{V_m} \tilde{W}^\sigma \, dV \tag{2.8}$$

where V_m is the matrix volume.

Furthermore, by (2.3.3) the actual stresses are functions of x_2, x_3 only. It is therefore natural to also choose admissible stresses as functions of x_2, x_3 . Thus $\tilde{W}^\sigma$ in (1.33) becomes a function of x_2, x_3 only and therefore without loss of generality (1.33a) can be taken over unit length in fiber direction. Thus it can be written

$$\tilde{U}_C = \int_{A_m} \tilde{W}^\sigma \; (x_2, x_3) \; dx_2 \; dx_3 \tag{2.9}$$

In order to construct an admissible stress system it is necessary to devise a geometrical model for a uniaxially reinforced material . In past analyses of FRM two kinds of models have been successfully treated: Periodic arrays of identical circular fibers have been analyzed numerically with the aid of computers and the composite cylinder assemblage model has been treated analytically [1,5] yielding simple closed results. Since the present treatment is to be analytical the composite cylinder assemblage model will be used. A detailed description of the model has been given in[5]. Suffice it to say here that the model represents a cylindrical specimen of a fiber reinforced material as an assemblage of composite cylinders of different sizes which fill the space in the limit. In each composite cylinder the inner cylinder is a fiber and the outer shell is matrix material.

In all cylinders the ratios of fiber to matrix shell radius are the same, (figure 26).

It is recalled that an admissible stress system must satisfy equilibrium and boundary conditions. An obvious possibility for such an admissible field are the stresses of the elastic solution since they certainly satisfy the required conditions. These stresses are the same in any composite cylinder of the assemblage and are given in cylindrical coordinates by (see [5])

$$\begin{aligned} \tilde{\sigma}_{rz} &= \tilde{\tau}_r = \frac{\tau_o}{1+c}\left(1 + \frac{a^2}{r^2}\right)\cos\theta \\ \tilde{\sigma}_{\theta z} &= \tilde{\tau}_\theta = -\frac{\tau_o}{1+c}\left(1 - \frac{a^2}{r^2}\right)\sin\theta \end{aligned} \tag{2.10}$$

where c is the volume fraction of fibers, a is the radius of any fiber and r, θ are polar coordinates, fig. 26.

Since $\tilde{\tau}$ as expressed by (1.33c) is an invariant with respect to rotations about $x_1 = z$ we have also

$$\tilde{\tau}^2 = \tilde{\tau}_r^{\,2} + \tilde{\tau}_\theta^{\,2} \tag{2.11}$$

Substituting (2.10) into (2.11) yields

$$\tilde{\tau}^2 = p^2 \left(1 + \frac{1}{\rho^4} + \frac{2}{\rho^2}\cos\theta\right) \tag{2.12}$$

where

$$p = \frac{\tau_o}{1+c} \qquad \rho = \frac{r}{a} \tag{2.13}$$

To simplify the analysis the exponent n in (1.22) will be assigned the value

$$n = 3 \tag{2.14}$$

It has been found that with this value of n, experimentally obtained shear stress-strain relations of epoxy can be quite accurately represented with proper choice of τ_y. Recalling (1.33), (2.9) then assumes the form

$$\tilde{U}_C = \frac{1}{2G}\int_{A_m} \tilde{\tau}^2 \left[1 + \frac{1}{2}\left(\frac{\tilde{\tau}^2}{y}\right)^2\right] dA \tag{2.15}$$

where G is the matrix elastic shear modulus. Let the assemblage consist of K composite cylinder. Define $\tilde{U}_C^{\,k}$ for the kth composite cylinder by

$$\tilde{U}_C^{\,k} = \frac{1}{2G} \int_{A_{mk}} \tilde{\tau}^2 \left[1 + \frac{1}{2} \left(\frac{\tilde{\tau}}{\tau_y}\right)^2\right] dA \tag{2.16}$$

where A_{mk} is the matrix area $a_k \le r \le b_k$ in the kth composite cylinder. Then

$$\tilde{U}_C = \sum_{k=1}^{K} \tilde{U}_C^{\,k} \tag{2.17}$$

Since $\tilde{\tau}^2$ has been expressed in polar coordinates, (2.12), it is convenient to also evaluate (2.16) in the same coordinates. Using the variable ρ we have

$$\tilde{U}_C^{\,k} = \frac{1}{2G} \int_1^{\beta} \int_0^{2\pi} \tilde{\tau}^2 \left[1 + \left(\frac{\tilde{\tau}}{\tau_y}\right)^2\right] \rho d\rho d\theta \tag{2.18}$$

where

$$\beta = b_k/a_k \tag{2.19}$$

which by construction has the same value in all composite cylinders. Note also that the volume fraction of fibers c is given by

$$c = \left(\frac{a_k}{b_k}\right)^2 = \frac{1}{\beta^2} \tag{2.20}$$

Substituting (2.12) into (2.18) and carrying out the integration we have

$$\tilde{U}_C^{\,k} = \frac{\pi b_k^2}{2G} \tau_o^2 \left[\frac{1-c}{1+c} + \left(\frac{\tau_o}{\tau_y}\right)^2 \frac{3+10c-12c^2-c^4}{6(1+c)^4}\right] \tag{2.21}$$

where (2.20) has been used. It is seen that πb^2_k is the area of the cross section of the kth composite cylinder and the parenthesis has the same value for all composite cylinders. Therefore, if (2.21) is inserted into (2.17) we find

$$\tilde{U}_C = \frac{A}{2G} \tau_o^2 \left[\frac{1-c}{1+c} + \left(\frac{\tau_o}{\tau_y}\right)^2 \frac{3+10c-12c^2-c^4}{6(1+c)^4}\right] \tag{2.22}$$

Let (2.22) be written

$$\tilde{U}_C = A \int_o^{\tau_o} \frac{1}{A} \frac{d\tilde{U}_C}{d\tau_o} d\tau_o \tag{2.23}$$

Without loss of generality (2.6) can be evaluated for unit height of cylindrical specimen. Thus

$$U_C = A \int_o^{\tau_o} \frac{\tau_o d\tau_o}{G_A^S(\tau_o)} \tag{2.24}$$

Now introduce (2.23) and (2.24) into the minimum complementary inequality (1.18). Thus

$$\int_o^{\tau_o} \left[\frac{1}{A} \frac{d\tilde{U}_C}{d\tau_o} - \frac{\tau_o}{G_A^S(\tau_o)}\right] d\tau_o \geq 0 \tag{2.25}$$

Since the integral is positive for all values of τ_o the integrand must also be positive for all values of τ_o. It follows that

$$G_A^S(\tau_o) \geq \frac{A\tau_o}{d\tilde{U}_C/d\tau_o} = G^S_{A(-)} \tag{2.26}$$

where the extreme right denotes lower bound on the secant modulus G_A^S. Substituting (2.22) into (2.26) and rearranging we find the lower bound (2.3.9) of Chapter 2.

There naturally arises the question of the establishment of an upper bound. The difficulties involved have been discussed above: It is not in general possible to solve Ramberg-Osgood relations for stresses in terms of strains. It is therefore not possible to analytically express the potential energy functional in terms of admissible strains.

A possibility to resolve the difficulty is to write inelastic stress-strain relations of type (1.22) in the form

$$
\begin{aligned}
\tau_2 &= 2G\varepsilon_2 \left[1 - \left(\frac{\hat{\varepsilon}}{\varepsilon_y}\right)^{\alpha-1}\right] \\
\tau_3 &= 3G\varepsilon_3 \left[1 - \left(\frac{\hat{\varepsilon}}{\varepsilon_y}\right)^{\alpha-1}\right] \qquad (2.27) \\
\hat{\varepsilon} &= \sqrt{\varepsilon_2{}^2+\varepsilon_3{}^2}
\end{aligned}
$$

where α and ε_y are to be determined by curve fitting. The minus sign in the parenthesis is due to the fact that the stress-strain curve is below a straight line with the initial slope.

It should be noted that (2.27) are not an inversion of (1.22). They are merely another form of approximation of actual stress-strain curves.

In principle the representation (2.27) can now be used in conjunction with the principle of minimum potential energy to establish an upper bound on G_A^s in same fashion as a lower bound has been established. It has however been found that in attempting to fit (2.27) to actual epoxy stress-strain curves a fractional exponent α was needed. This led to integrals of formidable difficulty in the evaluation of potential energy functionals. Therefore this approach has not been continued here.

APPENDIX D

FAILURE OF NON-LINEAR LAMINATES

It is expedient to separate the problem of the establishment of failure criteria of laminates into two separate problems:

(a) Establishment of failure criteria for uniaxially fiber reinforced material, i.e., laminae.

(b) Establishment of failure criteria of the laminate on the basis of laminae failure criteria.

A great deal of wrok has been done on problem (a). The problem has been approached in micro as well as macro-fashion. In micro-approach, it is attempted to predict failure on the basis of local analysis of the interior of the composite. Such an approach evidently encounters extreme difficulties. Although important work of fundamental nature has been done in this area, we shall not be concerned with it here since the work has not advanced to the stage of prediction of failure criteria under states of combined stress.

In the macro-approach, a failure criterion is heuristically postulated as some function of pertinent state variables (generally average stresses) which also contains undetermined parameters. These parameters are then to be determined in terms of experimentally accessible information.

We shall in the present discussion limit ourselves to states of plane stress. The simplest failure criterion is the so-called maximum stress criterion which states that failure occurs when either one of: stress in fiber direction, stress transverse to fibers, shear stress, reaches its critical value, these critical values being the same whether or not the stresses act simultaneously. In symbols the criterion is:

$$\sigma_{11} = \sigma_A$$

or

$$\sigma_{22} = \sigma_T \qquad (1)$$

or

$$\sigma_{12} = \tau_{AT}$$

where 1 is fiber direction and 2 is the transverse direction.

Generally, failure stresses σ_A and σ_T are different in tension and compression. This is known as Bauschinger effect. There is evidently no Bauschinger effect for the shear stress. The simplest generalization of (1) to account for Bauschinger effect would be to assume as failure criterion:

$$
\begin{aligned}
\sigma_{11} &= \sigma_A^{+} && \text{if} && \sigma_{11} > 0 \\
\sigma_{11} &= \sigma_A^{-} && \text{if} && \sigma_{11} < 0 \\
\sigma_{22} &= \sigma_T^{+} && \text{if} && \sigma_{22} > 0 \\
\sigma_{22} &= \sigma_{22}^{-} && \text{if} && \sigma_{22} < 0 \\
\sigma_{12} &= \tau_{AT} && \text{all} && \sigma_{12}
\end{aligned}
\tag{2}
$$

whichever occurs first, where (+) and (-) superscripts denote failure stresses in tension and compression respectively. The main drawback of these simple criteria is in that they take no account of interaction effects.

The most commonly used criterion which takes into account interaction is of quadratic form. For plane stress it has the form

$$A_{11}\sigma_{11}^2 + A_{22}\,\sigma_{22}^2 + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^2 = 1 \tag{3}$$

Here, products of shear stress with normal stress have been omitted since the material cannot distinguish between positive and negative shear stress. Therefore, odd powers (one, in this case) of shear stress cannot appear.

Applying (3) to failure for stress in fiber direction alone, stress transverse to fiber direction alone, shear stress alone, in turn, it is seen at once that

$$A_{11} = \frac{1}{\sigma_A^{\,2}}$$

$$A_{22} = \frac{1}{\sigma_T^{\,2}} \qquad (4)$$

$$A_{44} = \frac{1}{\tau_{AT}^{\,2}}$$

The coefficient A_{12} is troublesome since its determination requires a failure experiment under combined stress. Several authors have proposed to use failure experiments on off-axis specimens under uniaxial stress for the determination of A_{12}. See e.g. [18] for discussion.

The situation becomes more complicated if it is required to take into account Bauschinger effect, that is difference of failure stresses in tension and compression. One possibility to account for this effect is to assume that A_{11}, A_{22} assume different values for tension and compression. The situation regarding A_{12}, however, becomes very awkward as it would have to assume four different values to account for four different possibilities of sign combination in biaxial stressing and

It is also possible to add linear terms to (3) in which case it would assume the form:

$$A_{11}\sigma_{11}^{\,2}+A_{22}\sigma_{22}^{\,2} + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^{\,2} + \qquad (5)$$

$$B_1\sigma_{11} + B_2\sigma_{22} = 1$$

Such a device was suggested by Hoffman [19]. In this case it is possible to determine values of A_{11}, B_1, A_{22}, B_2 to account for different tensile and compressive uniaxial failure stresses in fiber direction and transverse to it. But the difficulty of assigning four different values to A_{12} remains, unfortunately.

In summary, the status of quadratic failure criteria has to date not been finalized. However, special versions of such criteria have been successfully fitted to experimental data.

It is of importance to realize that in the fiber reinforced materials used in practice failure predictions on the basis of maximum stress criterion or quadratic failure criterion are not very different. This is due to the large ratios between strength in fiber direction and transverse and shear strengths and is easiest realized by considering the failure criteria as surfaces in σ_{11}, σ_{22}, σ_{12} stress space. The maximum stress criterion is a very elongated rectangular parallelopiped while the quadratic failure criterion is an ellipsoid. For $A_{12}=0$, Fig. 25 shows this schematically on a cut in the σ_{11}, σ_{22} plane. Thus it is seen that stress points on the two failure surfaces are close together for most parts of the surfaces.

The situation would be entirely different for a material in which σ_A, and σ_T were of comparable magnitudes.

We shall now consider problem (b) i.e., the establishment of laminate failure criteria in terms of laminae failure criteria. The most conservative laminate failure criterion is to assume that once any lamina has failed the laminate has reached its ultimate load. There are cases of laminates in which all laminae would fail simultaneously and then this criterion would be justified. For example: a $\pm\theta$ laminate in which the external load direction bisects the angle between the fibers.

In most cases, however, a certain group of laminae will fail first and failure of remaining groups would require further increase of load. Therefore a more realistic alternative is to determine the load at which the first laminae group fails. At this state, the further carrying capacity of the laminate may be assumed to be given by the remaining undamaged laminae. The increase in load which fails another group of laminae is then determined. This process is continued until failure of all laminae has taken place.

Still another possibility is to assume that when a lamina has failed, certain of its stiffnesses reduce to zero. For example: suppose that a lamina or group of laminae has failed in shear. Such a failure implies a crack through the lamina in fiber direction. In that event, it is reasonable to assume that the shear and transverse stiffnesses of the lamina are zero, but it still retains its stiffness in fiber direction. If, however, a lamina fails because of the stress in fiber direction the damage is so widespread that all of its stiffnesses will be negligible. According to the type of failures encountered analysis is continued for the damaged laminate with the new stiffness rearrangement. This process is continued until failure of all laminae has taken place. This method of analysis seems to be the most realistic but is also the most complicated.

In almost all of the practical strength analyses of laminates in the literature, according to any of the methods outlined above, the stresses used for failure criteria have been determined on the basis of elastic laminate analysis. With the present inelastic laminate analysis, more realistic stresses are available in a better assessment of laminate failure loads.

APPENDIX E

MSC-NOLIN COMPUTER PROGRAM

1. General Description of the Program

This is a computer program developed for the inelastic analysis of a laminate subject to any constant, arbitrary combination of in-plane loading. Details of the method of analysis and of the numerical solution, using the Newton-Raphson method, have been described in the body of this report. The essential features of the program are summarized below.

The primary capability of MSC-NOLIN is to compute laminae properties when the laminate loads are defined. There is also a limited capability to work with constituent properties, rather than laminae properties, as the input. Details of the input options are discussed subsequently. Basically, the inputs required are the stress-strain characteristics of the individual laminae for each of the three in-plane stress components applied separately. The stress-strain curves for transverse stress and for stress and for axial shear stress are defined by Ramberg-Osgood stress-strain curves. The parameters for these curves along with the laminae elastic constants are the required material property inputs.

It has been observed that axial shear stresses in individual laminae are a major, perhaps the major, source of non-linearities in laminate response. Therefore, several additional options have been included in the MSC-NOLIN to accomodate more detailed characterization of shear response. First, the laminae shear stress-strain response may be input in tabular form and a least squares fit to the data is automatically obtained for the R-O yield stress (limited to the use of an exponent, n=3). Secondly, the matrix shear stress-strain curve can be input along with fiber elastic properties and the laminae shear stress-strain curve will be computed. In this latter case, the laminae elastic constants are also computed.

The input specifies one of two options for the determination of the initial set of stresses to be used in the iteration at each value of applied load on the laminate. In one case the stresses found at one load are increased to the load for which the stresses were evaluated. In the other, and generally used option, the increment between the initial stresses used at the nth laminate load value and the actual stresses found for the (n-1)st load value bears the same relation to the ratio of those two load values as the similar relation computed at the previous load cycle, that is,

$$\frac{\sigma_{ij}^{(n)} - \sigma_{ij}^{(n-1)}}{F_n/F_{n-1}} = \frac{\sigma_{ij}^{(n-1)} - \sigma_{ij}^{(n-2)}}{F_{n-1}/F_{n-2}}$$

The program contains a number of controls to define: the size and number of steps of loading at which computations are made; the maximum number of iterations to be permitted in the numerical solution; the desired accuracy to be obtained in convergence; the criteria for divergence of the solution in the iterative process to avoid the use of unnecessary execution time in the case of breakdown of the solution procedure. The program defines the failure of the laminate in a limited fashion, either on the basis of the maximum allowable stress in the fiber in tension or compression, or on the basis that the tangent modulus of the stress-strain curve of the laminate becomes less than a specified value. Failure due to shear or transverse stress are not included at this stage in the development of the program.

2. Input

The main features of input in this program are the following:

(a) Specify the number of laminates or problems to be solved;

(b) Define the geometrical properties of each layer;

(c) Define either the material properties of each layer or the properties of its constituents;

(d) Define either of the following for each layer:

   (i) yield stress in transverse direction and yield stress in shear;

   (ii) yield stress in transverse direction and a table of values defining shear stress-strain curve for the matrix plus a set of values of stresses to be used for the computation of yield stress in shear;

(e) Specify the type of Ramberg-Osgood relation to be used;

(f) Define the loadings; and

(g) Define the control parameters.

A guide to the preparation of input data for this program is given in section 4 below.

3. <u>Details of Output</u>

The output can be divided basically into two steps:

(a) Output of Input Data:

The first section of the output deals with the output of the input data. If the input is in the form of properties of constituents of the layer, it gives an output of the properties of the constituents first and then the computed value of the properties of the layer; otherwise, it gives output directly the properties of the layer.

(b) Output of Stresses and Strains:

For each set of loading, the computer prints the following:

   (1) value of the load applied;

   (2) number of iterations for convergence;

   (3) stresses for individual laminae with respect to principal elastic axes of the laminae; and

   (4) strains for individual laminae in terms of both laminae and laminate axes.

4. <u>Input Details for MSC-NOLIN</u>

(1) Read (I5) NSETS

NSETS: number of problems

(2) Read (I5) LAY

LAY: number of layers in this laminate analysis

(3) Read (I5) INP

INP: Option for reading in material properties

INP = 1; read in material properties of individual laminae;

INP = 2; compute properties of laminae from the properties of constituents.

(4) (a) If INP = 1

(i) Read (5D15.5) $E_{11}, E_{22}, \mu_{12}, \mu_{21}$

(ii) Read (5D15.5) G_{12}, SY, TY

(iii) Read (D15.5,I5) T, IANG

(b) If INP = 2

(i) Read (4D15.5) EF, MUF, GF, VF

(ii) Read (3D15.5) EM, MUM, GM

(iii) Read (I5) I2

If I 2 = 0; read in SY and TY
(i) Read (2D15.5) SY, TY

If I 2 = 1; TY is to be computed
(i) Read (5,1002) SYCE
(ii) Read (2 I 5) NUMT
NUMT = number of values in the table
(iii) Read (5D15.5) TAU (J), J=1, NUMT
(Table of shear stress values of matrix read in)
(iv) Read (5D15.5) GAM (J),J=1, NUMT
(Table of shear strain values of matrix read in)
(v) Read (5D15.5) SG12 (J), J=2,11
(Table of shear stress values of laminae read in)

(5) Read (5D15.5) XN, XM

XN: exponent in nonlinear transverse stress-strain law;

XM: exponent in nonlinear shear stress-strain law.

(6) Read (5D15.5) SO11, SO22, SO12

SO11: applied stress in X-direction
SO22: applied stress in Y-direction
SO12: shear stress in XY

(7) Read (I5, D15.5) KSGM, SMLT

KSGM: total number of loading increments

SMLT: ratio of load increment to the initial load.

(8) Read (D15.5) STIFF

STIFF: tangent modulus of stress-strain curve in terms of the laminate axes; specify a value of STIFF below which the program will not run.

Read (D15.5) SGR

SGR: maximum allowable stress in the fiber in tension or compression

(9) Read (I 5, 2D15.5) IT, EPS, UPBD

IT: maximum number of iteration permitted in Newton-Raphson analysis

EPS: convergence criteria; (ratio of values of two successive iterations should be less than EPS)

UPBD: divergence criteria (solution will stop if ratio of two successive iterations is greater than $10^{\pm 12}$)

(10) Read(I5) INMT

If INMT = 1, the program uses ratio of previous two solutions as the initial guess value iteration process;

If INMT = 2, the program uses extrapolated value of previous two solutions proportioned on the basis of stress ratio as the initial guess.

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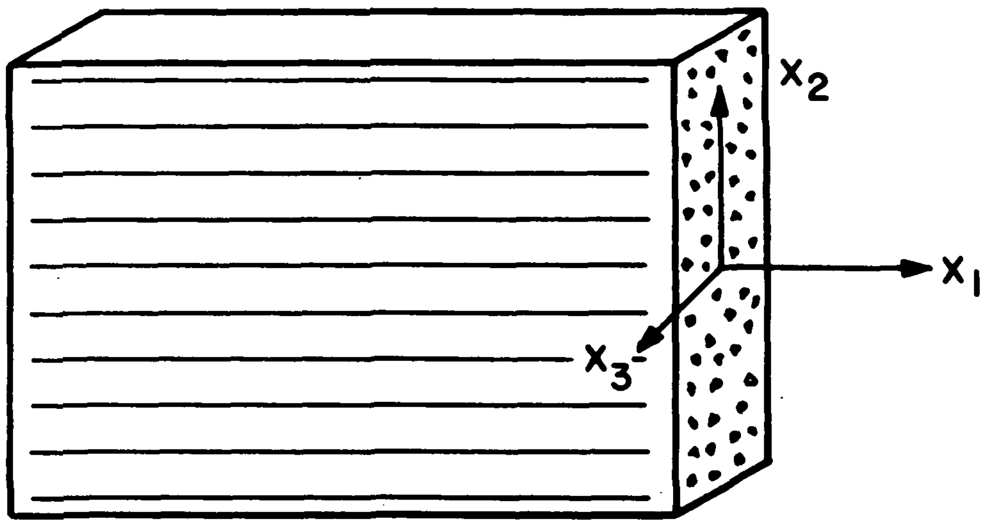

Fig. 1 - Coordinate system for unidirectional fiber composite material.

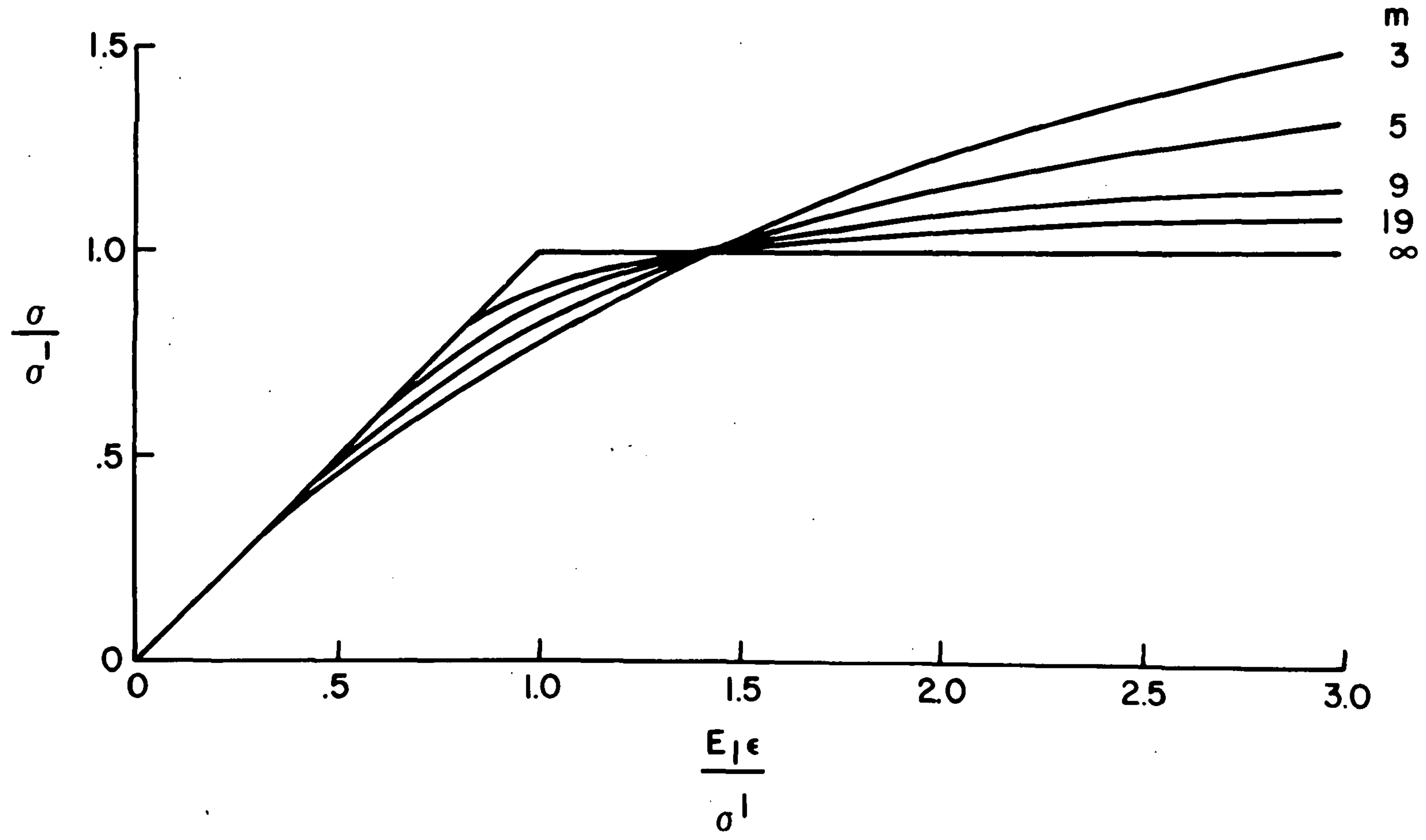

Fig. 2 - Nondimensional Ramberg-Osgood stress-strain curves. (e.g., 2.2.1 with k =0.4).

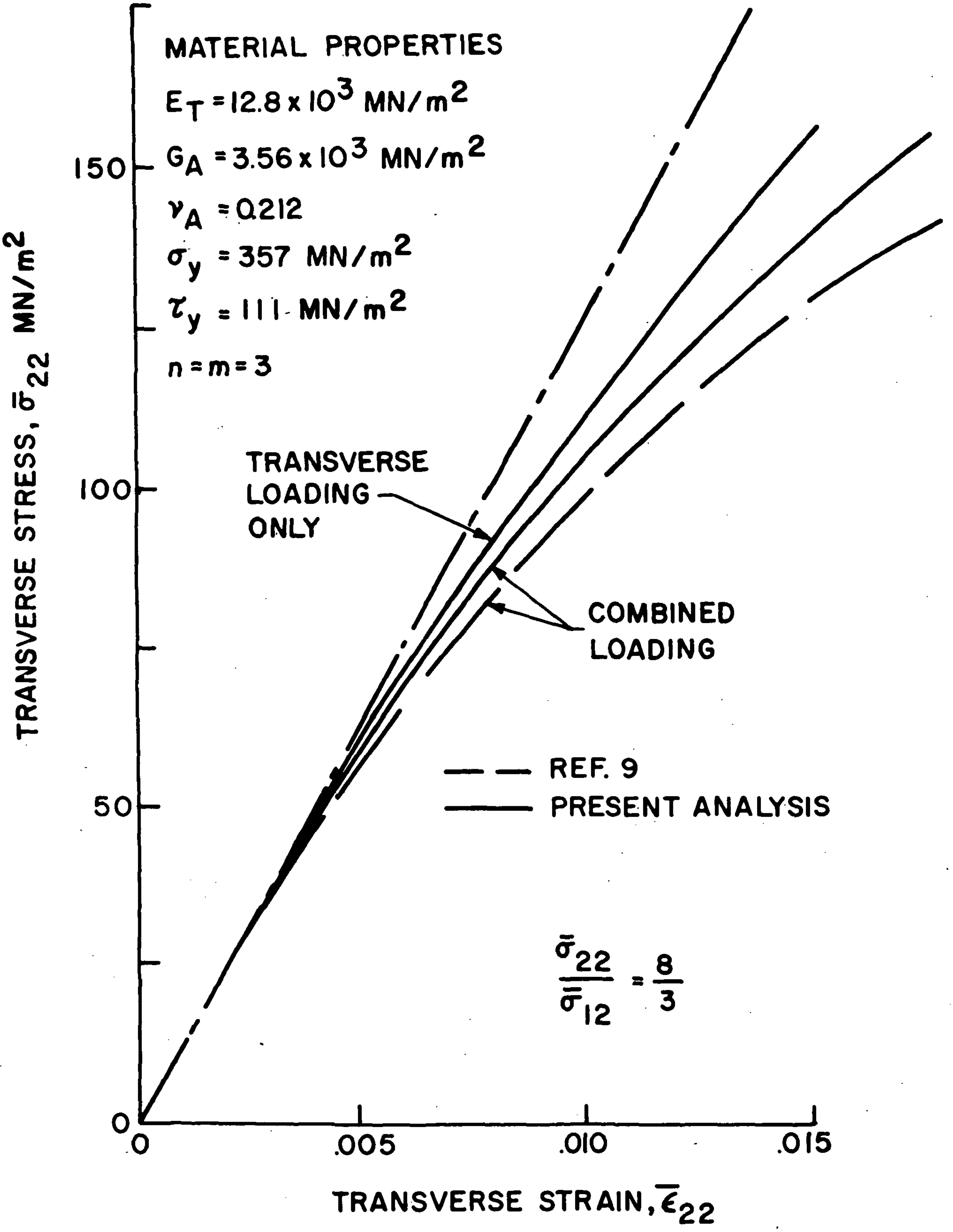

Fig. 3 - Transverse normal σ-ε curves for unidirectional Boron/ Epoxy (for normal loading and normal + shear loading).

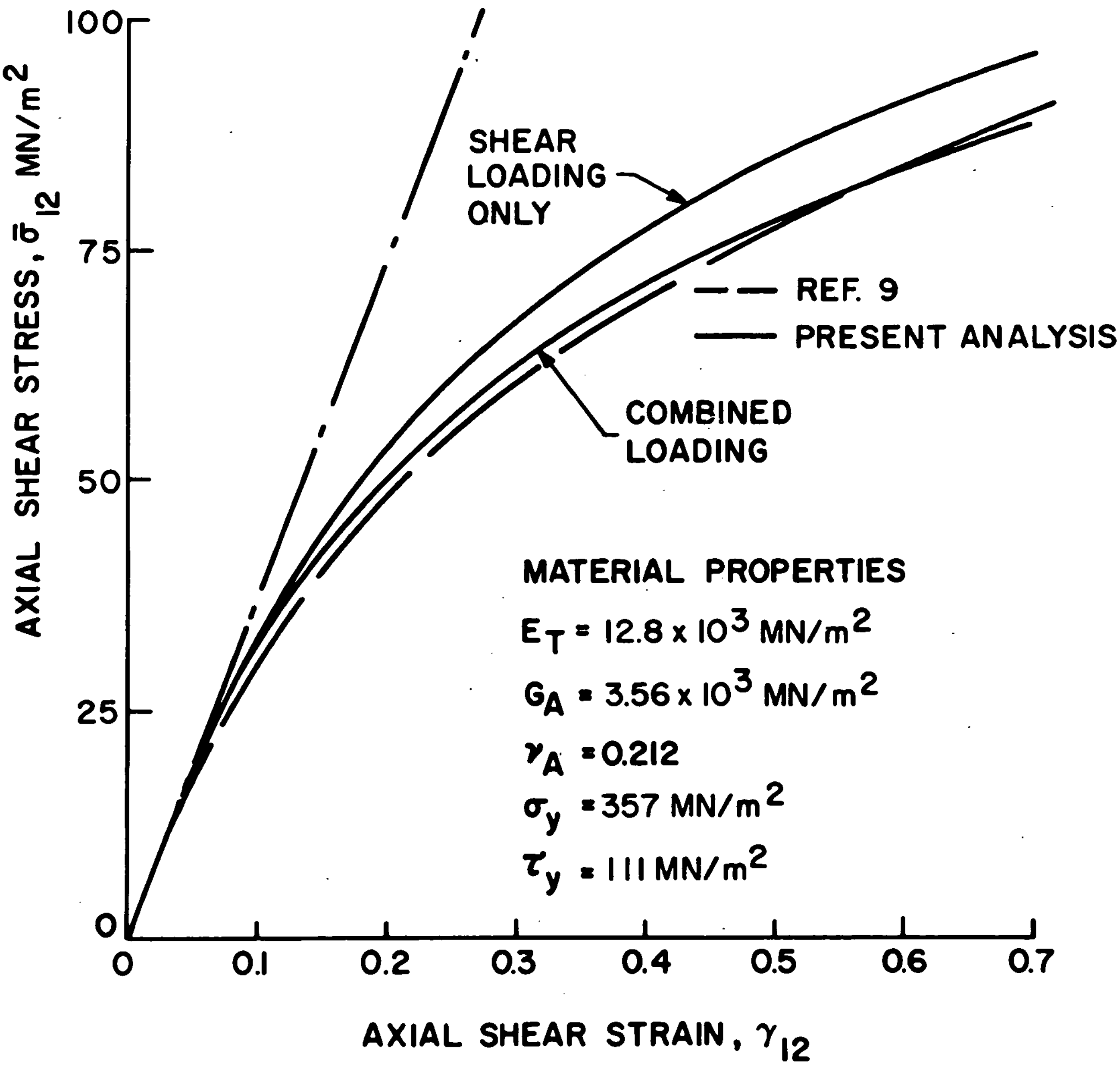

Fig. 4 - Shear σ-ε curves for unidirectional Boron/Epoxy (for shear loading and normal + shear loading).

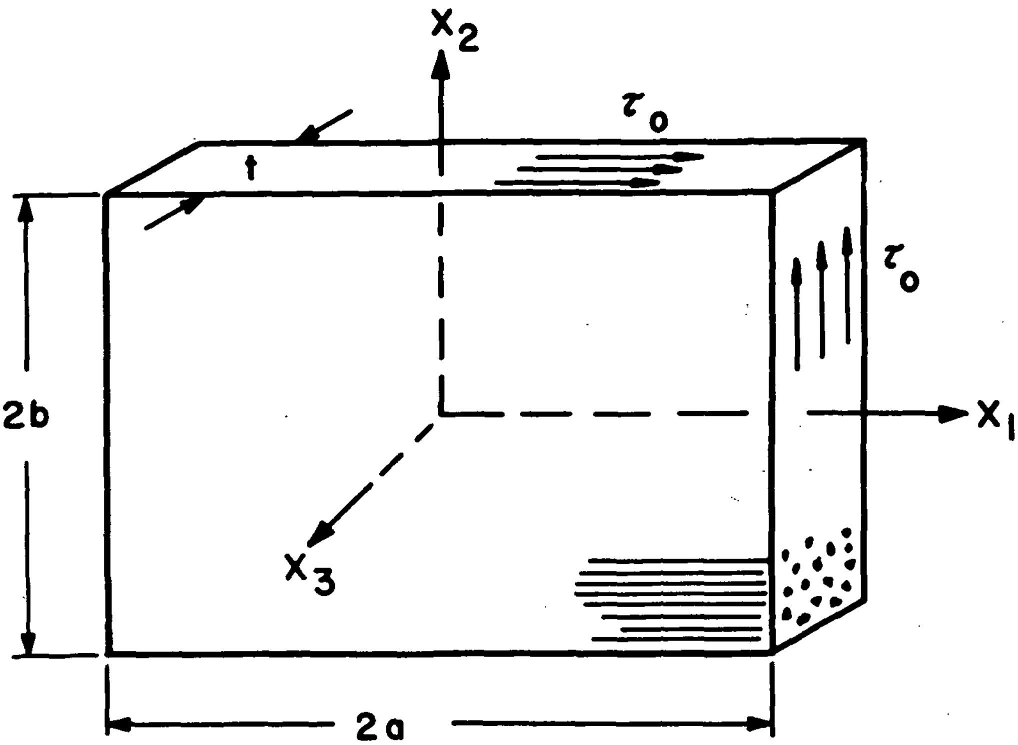

Fig. 5 - Unidirectional fiber composite material under axial shear stress.

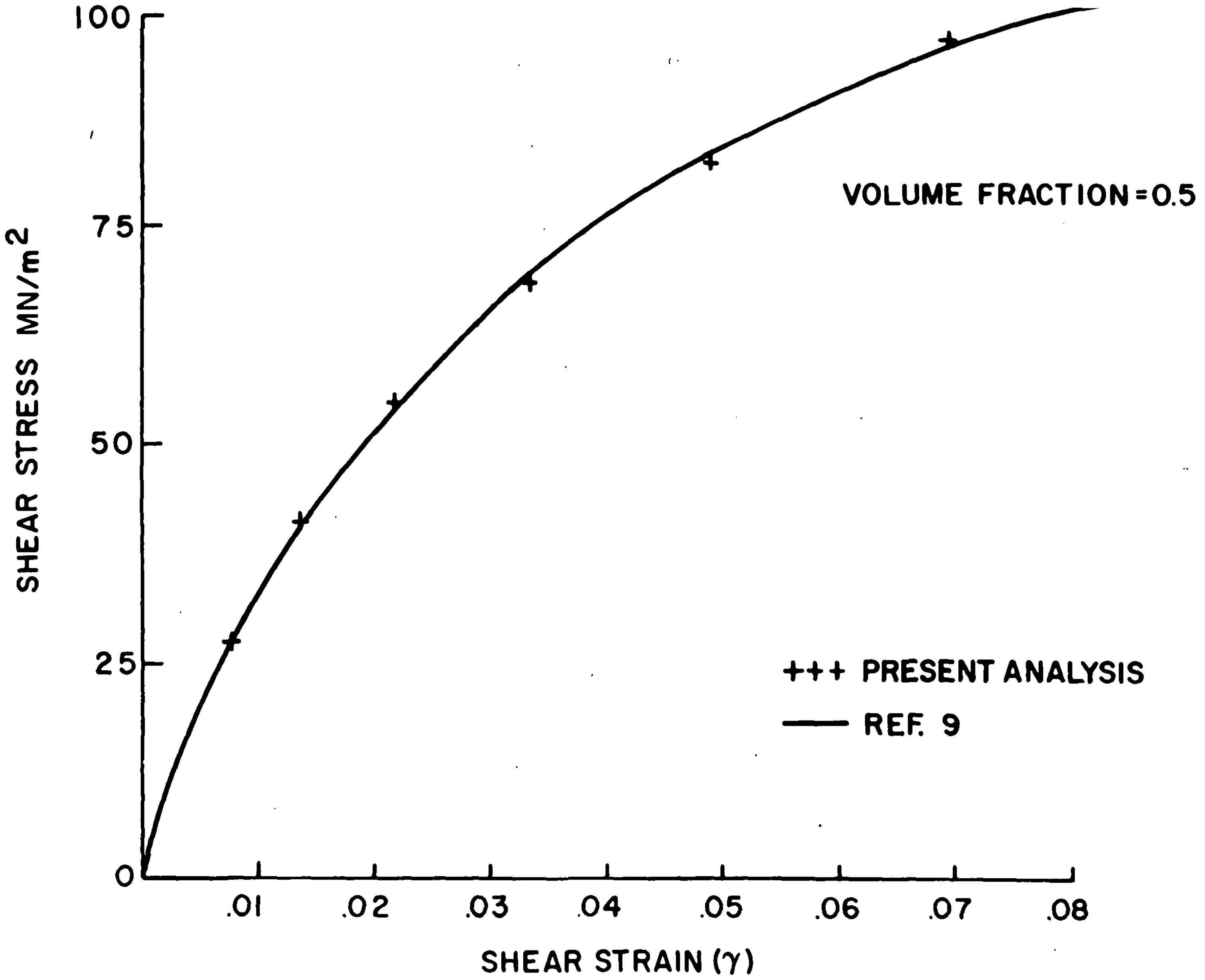

Fig. 6 - Axial shear stress-strain curve for composite computed from matrix properties.

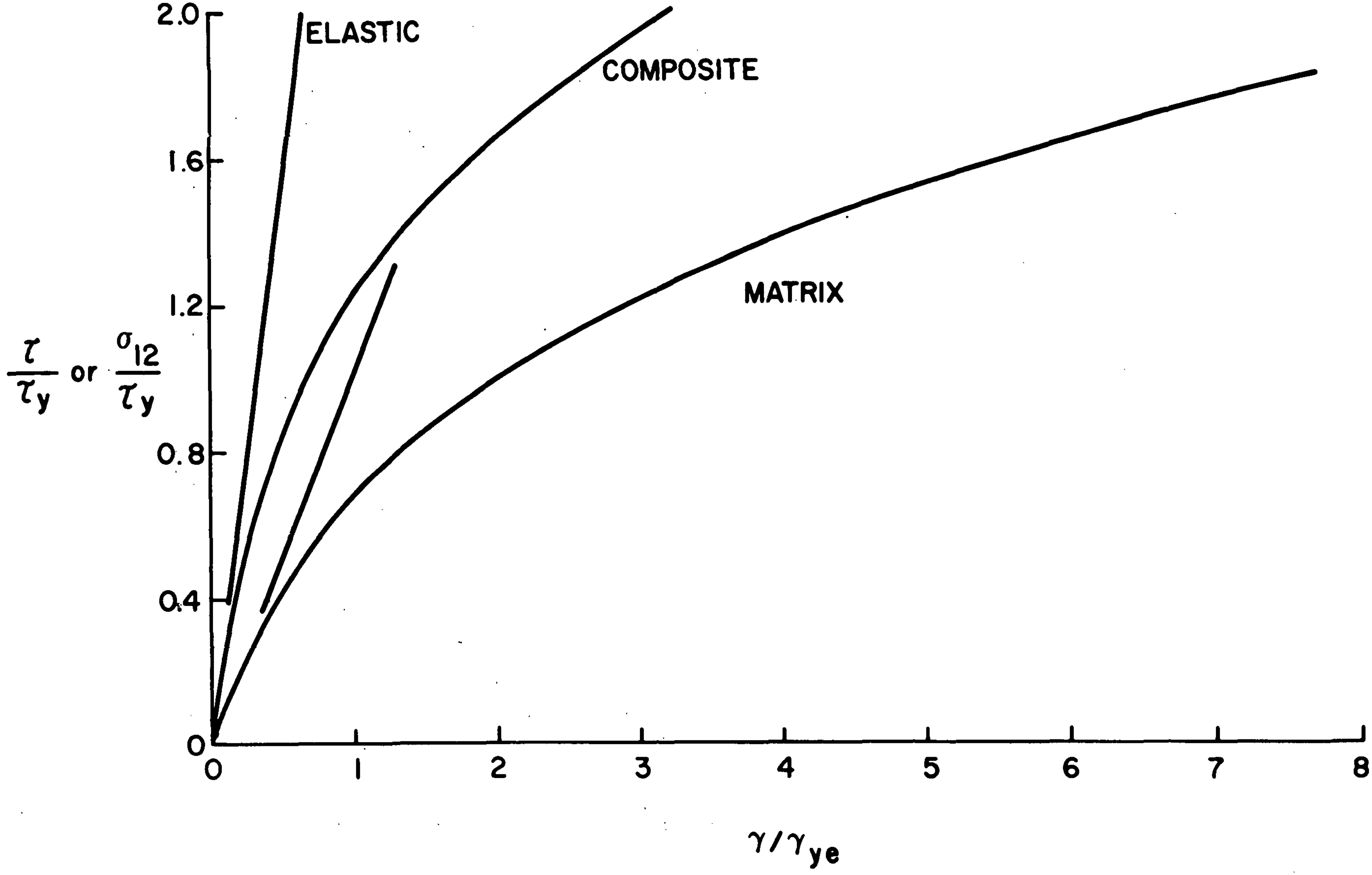

Fig. 7 - Generalized Ramberg-Osgood stress-strain curve in axial shear for composite and matrix (n=3).

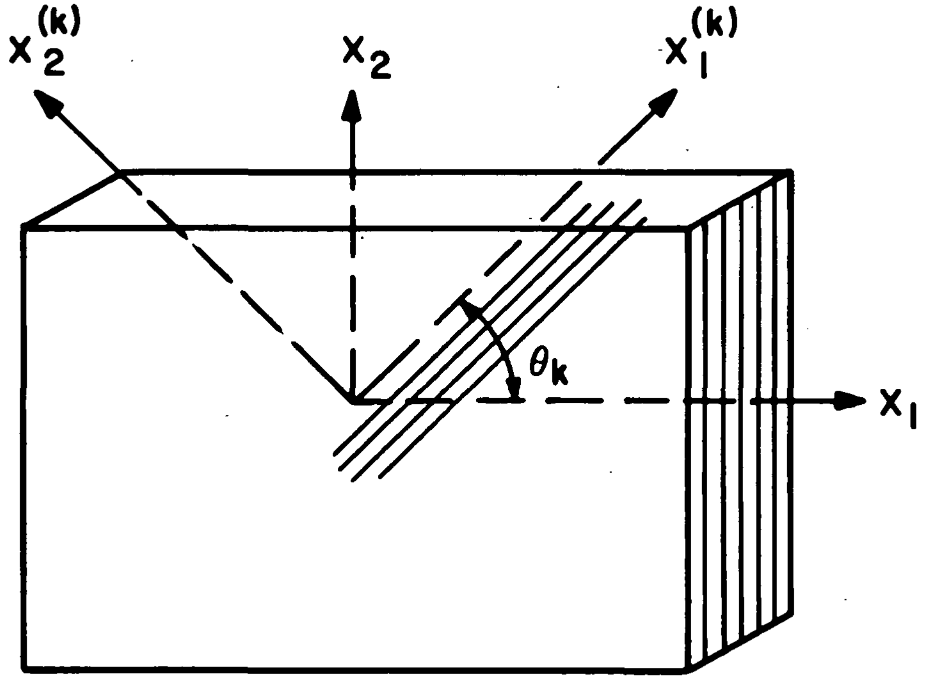

Fig. 8 - Laminate coordinate system.

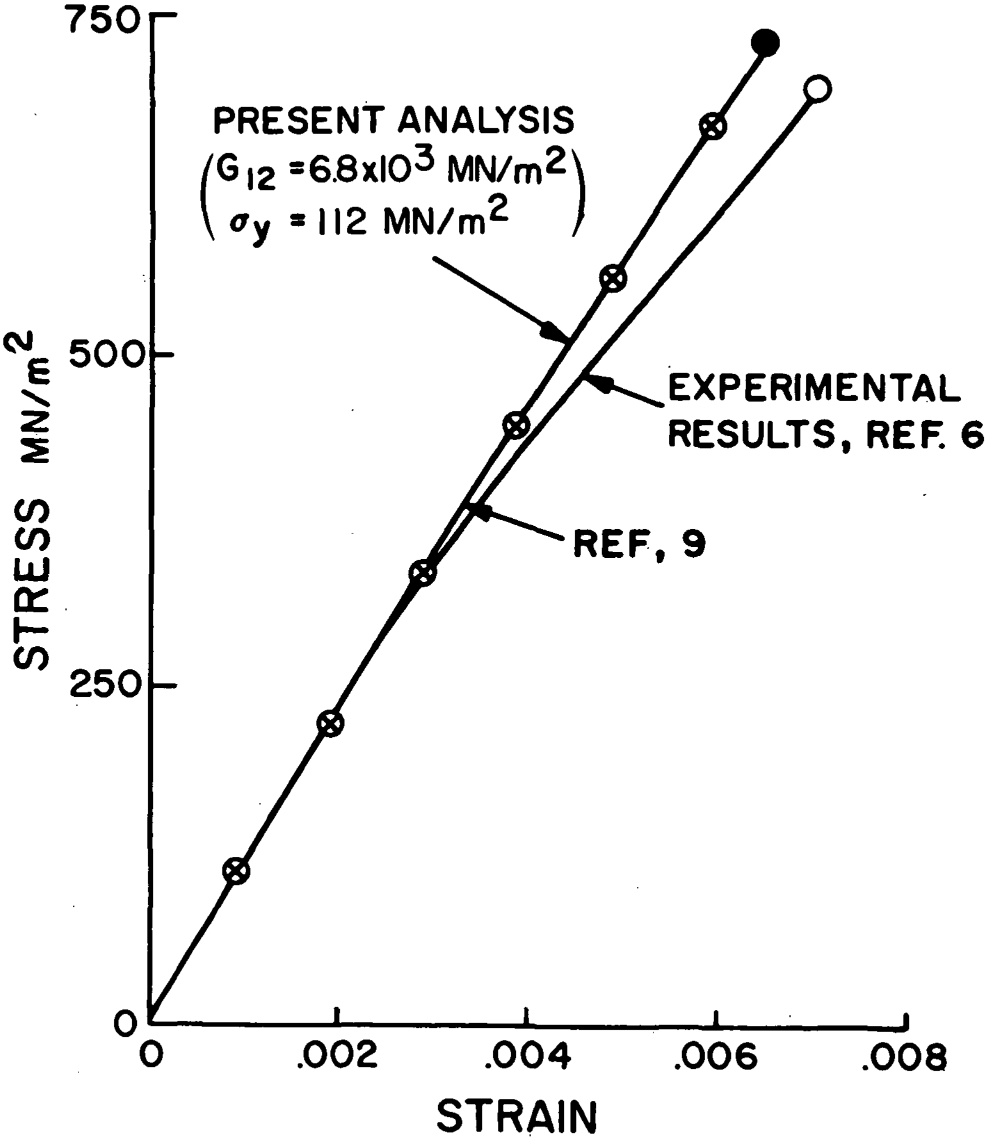

Fig. 9 - Tensile stress-strain curve with [0/90] B/Ep.

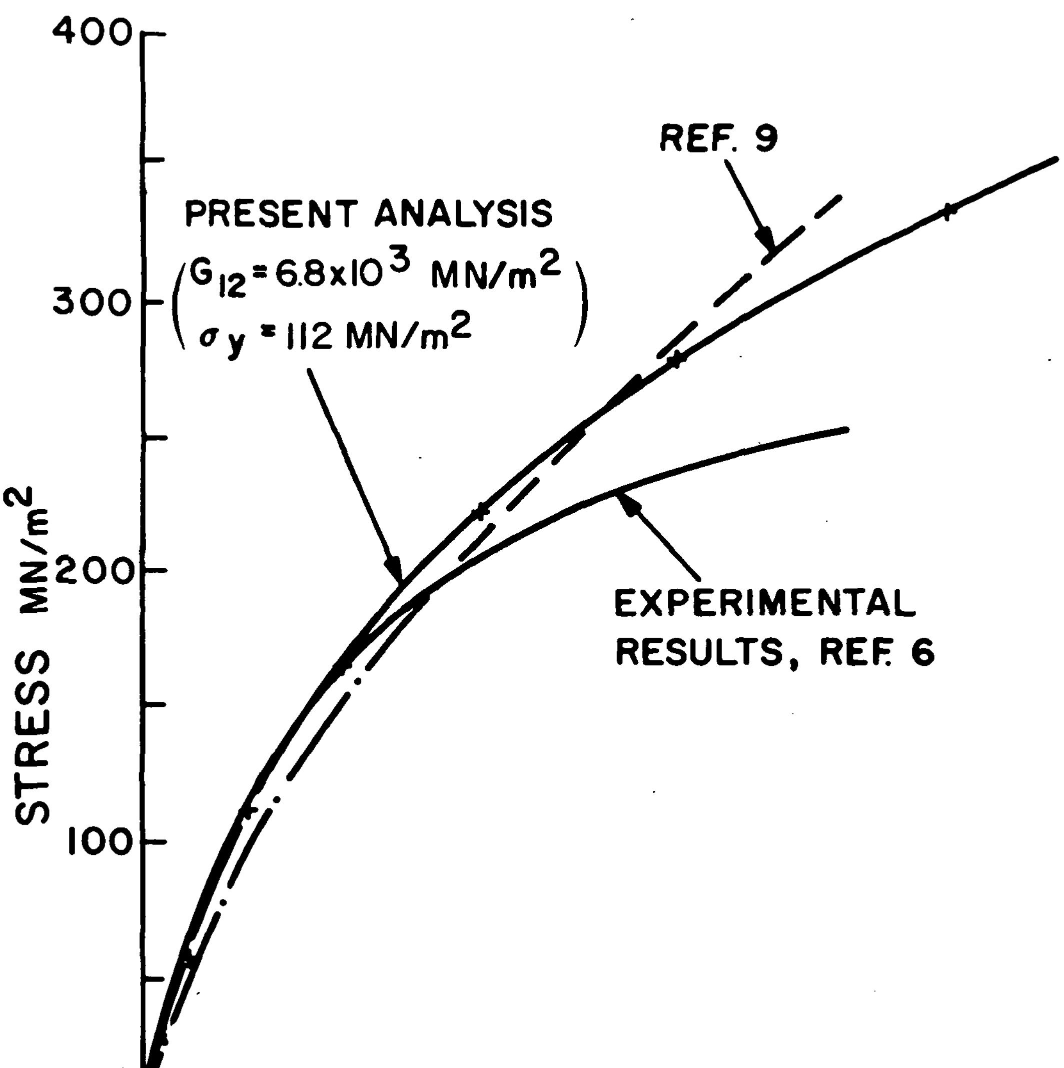

Fig. 10 - Tensile stress-strain curves for [$\pm$45] B/Ep.

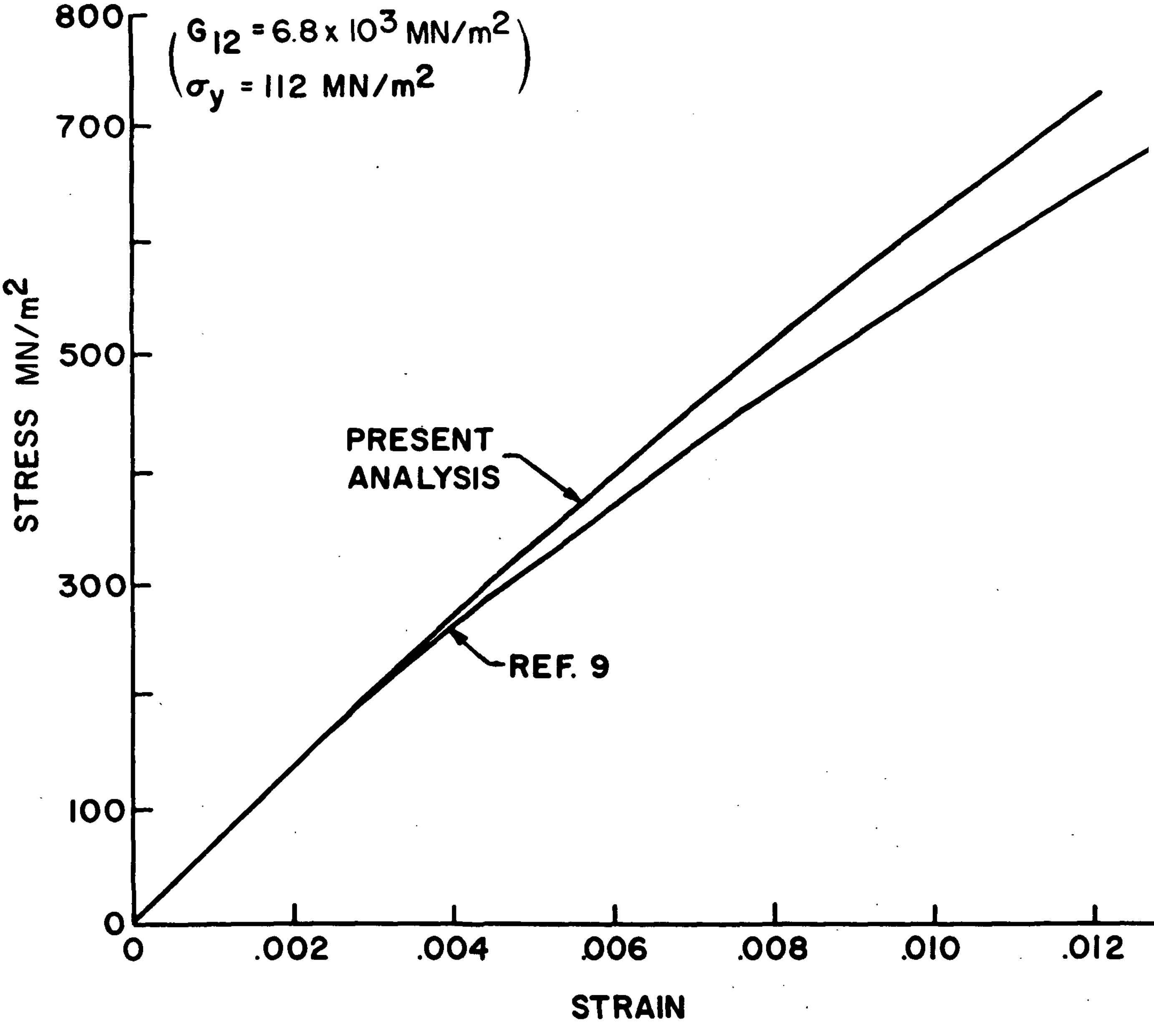

Fig. 11 - Tensile stress-strain curve with [$\pm$30] B/Ep.

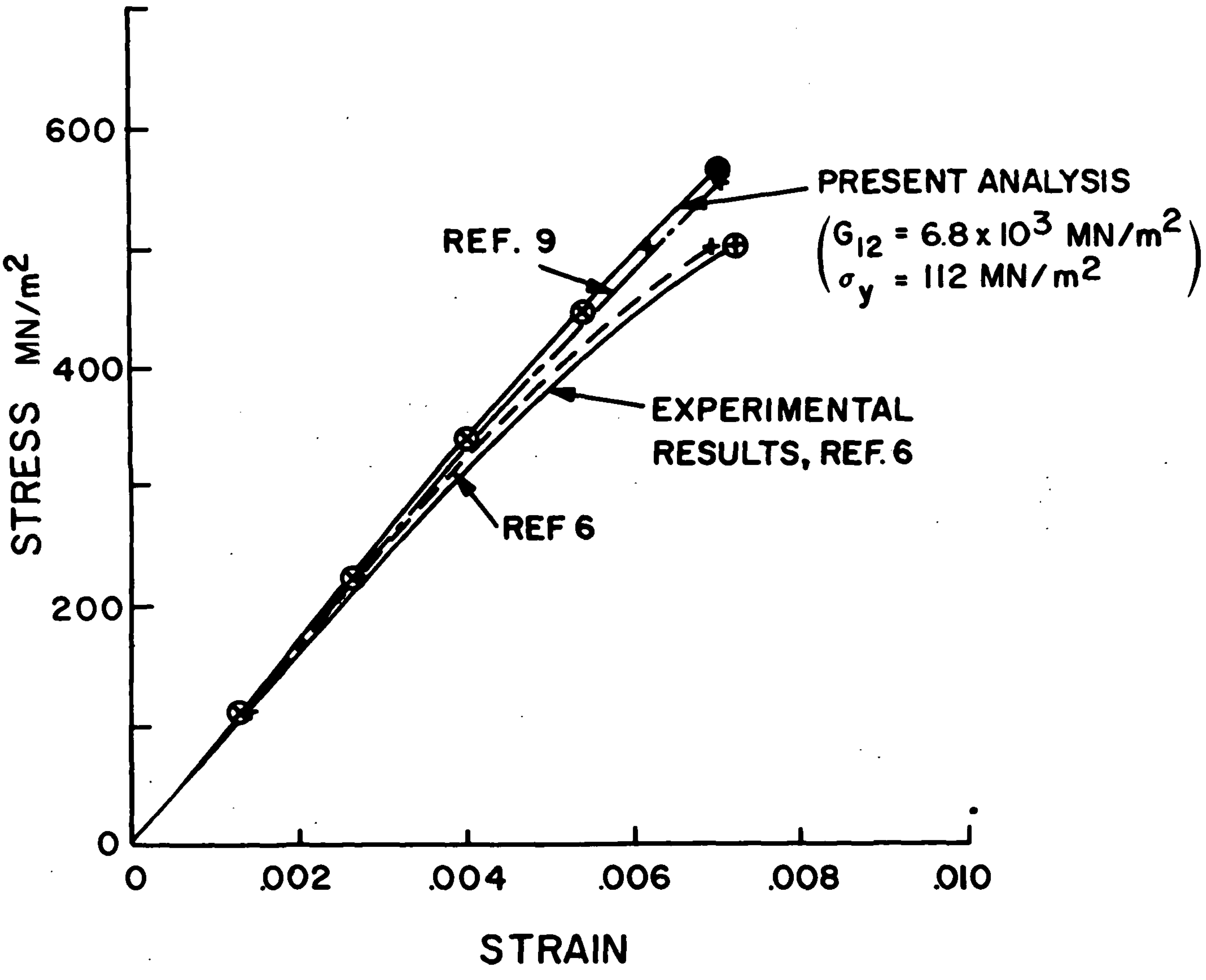

Fig. 12 - 0° Tensile stress-strain curves for [0/±45/90] B/Ep.

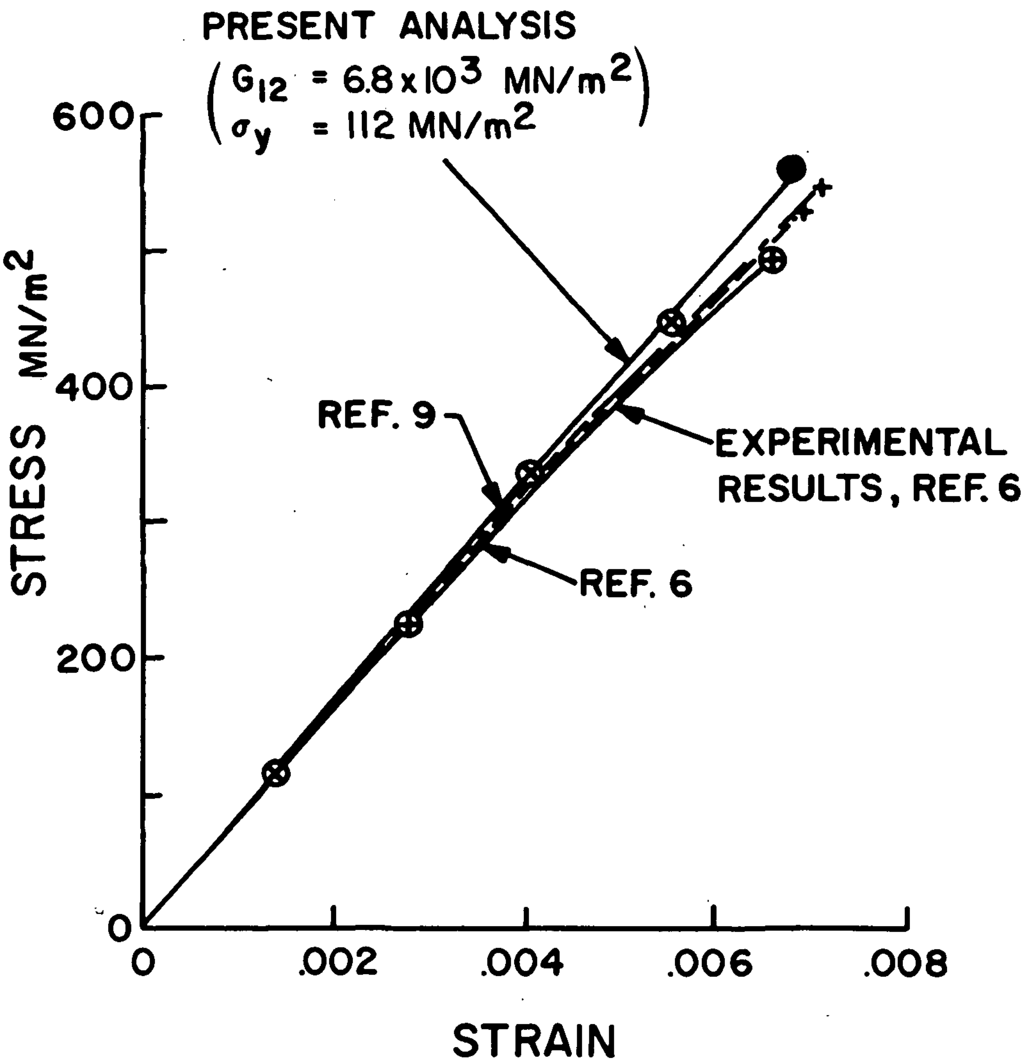

Fig. 13 - 0° Tensile stress-strain curves for [0/$\pm$60] B/Ep.

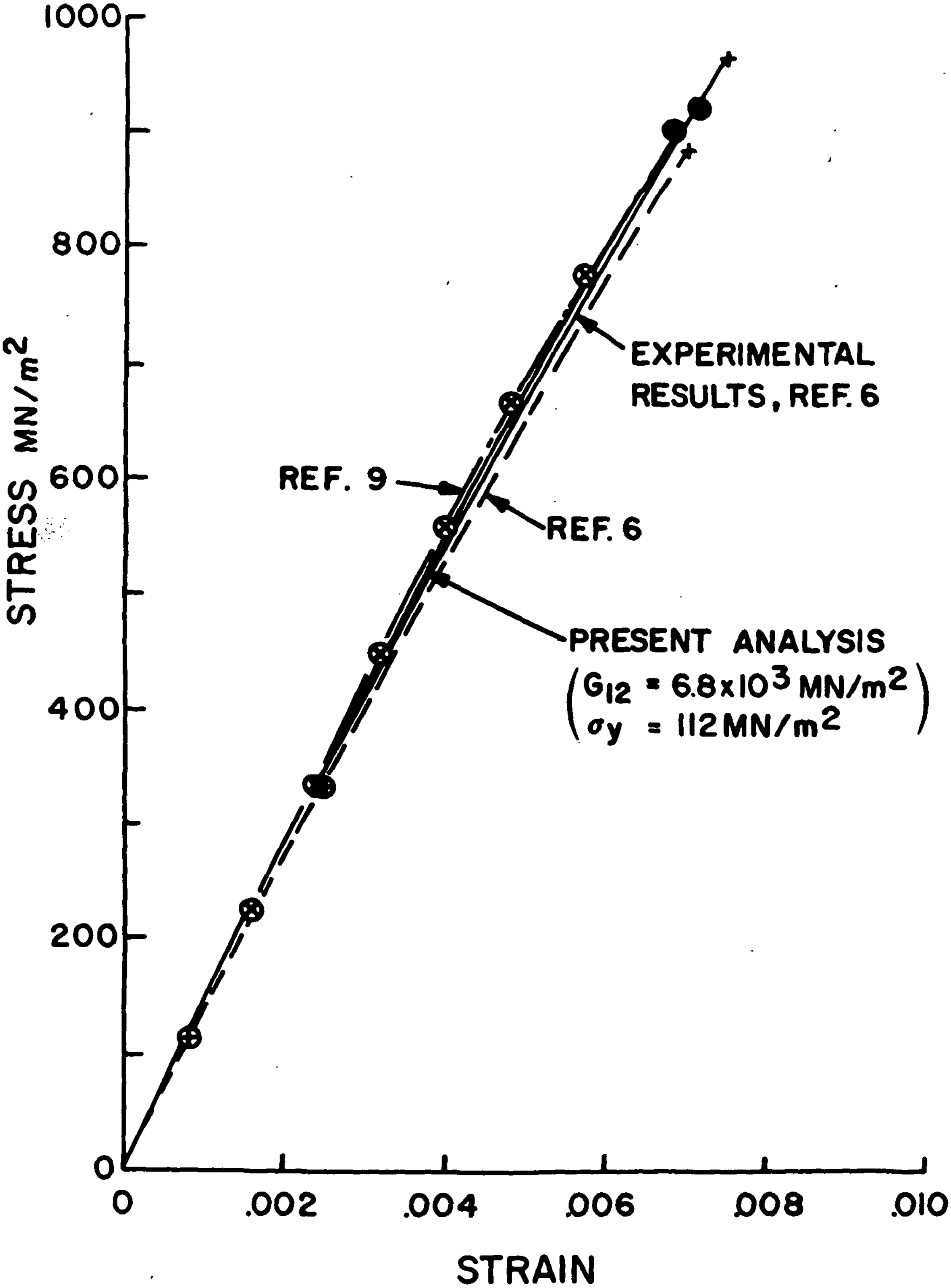

Fig. 14 - 0° Tensile stress-strain curves for $[0_3/\pm 45]$ B/Ep.

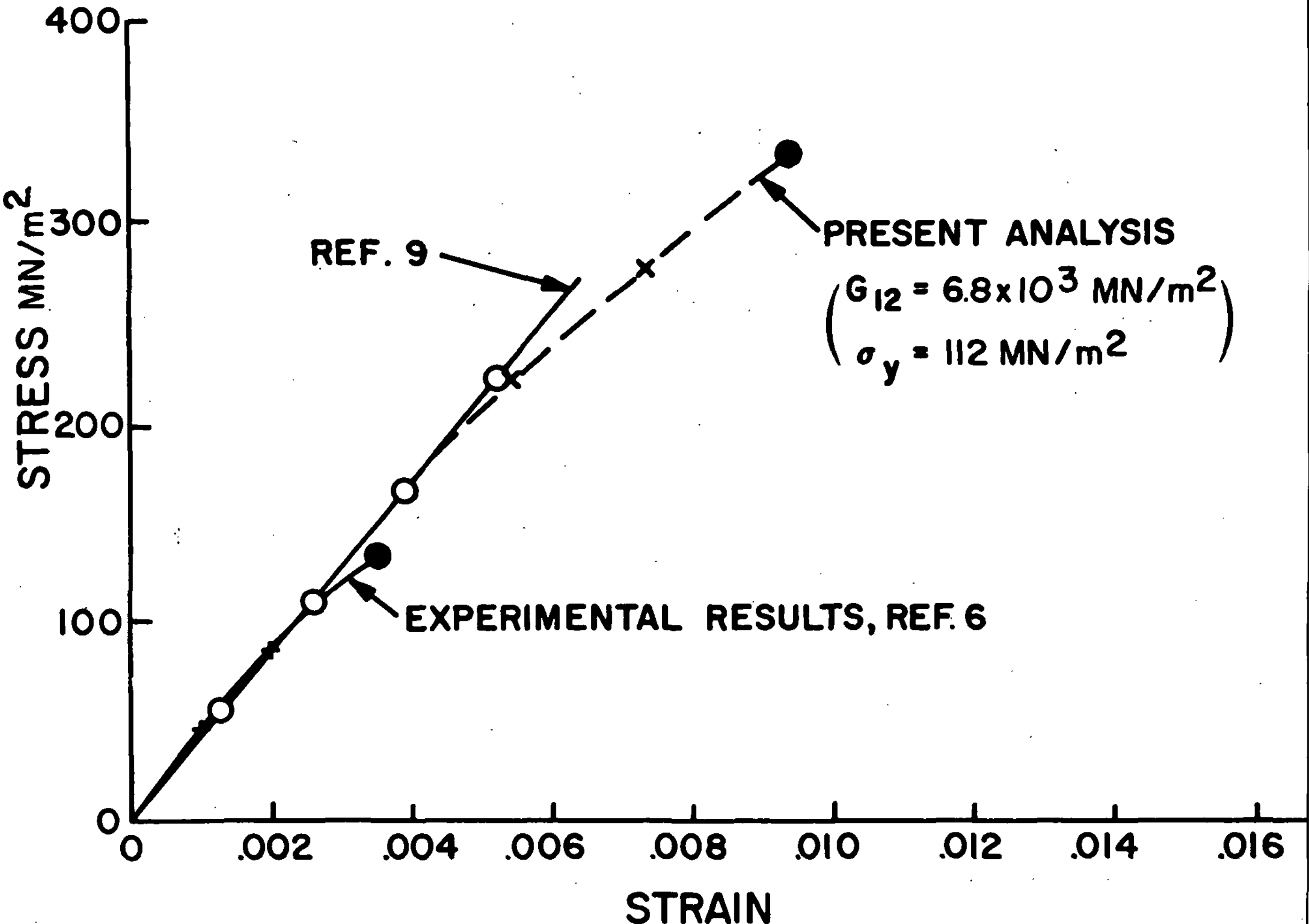

Fig. 15 - 0°Tensile stress-strain behavior of $[65_3/20/-70]$ B/Ep

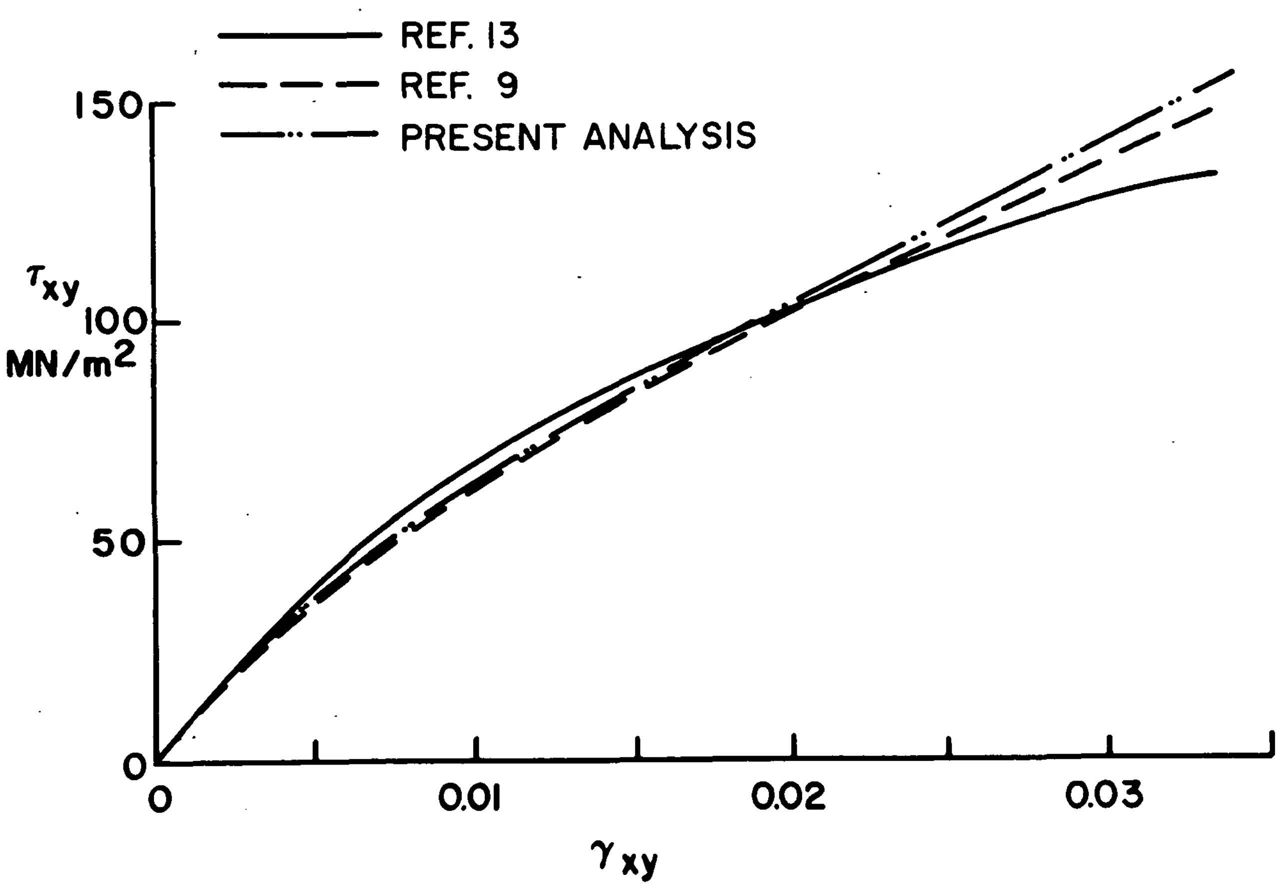

Fig. 16 - Comparison of present results with experimental data for Glass/Epoxy 90/$\pm$18 tubes in torsion, and with the results of Ref. 9.

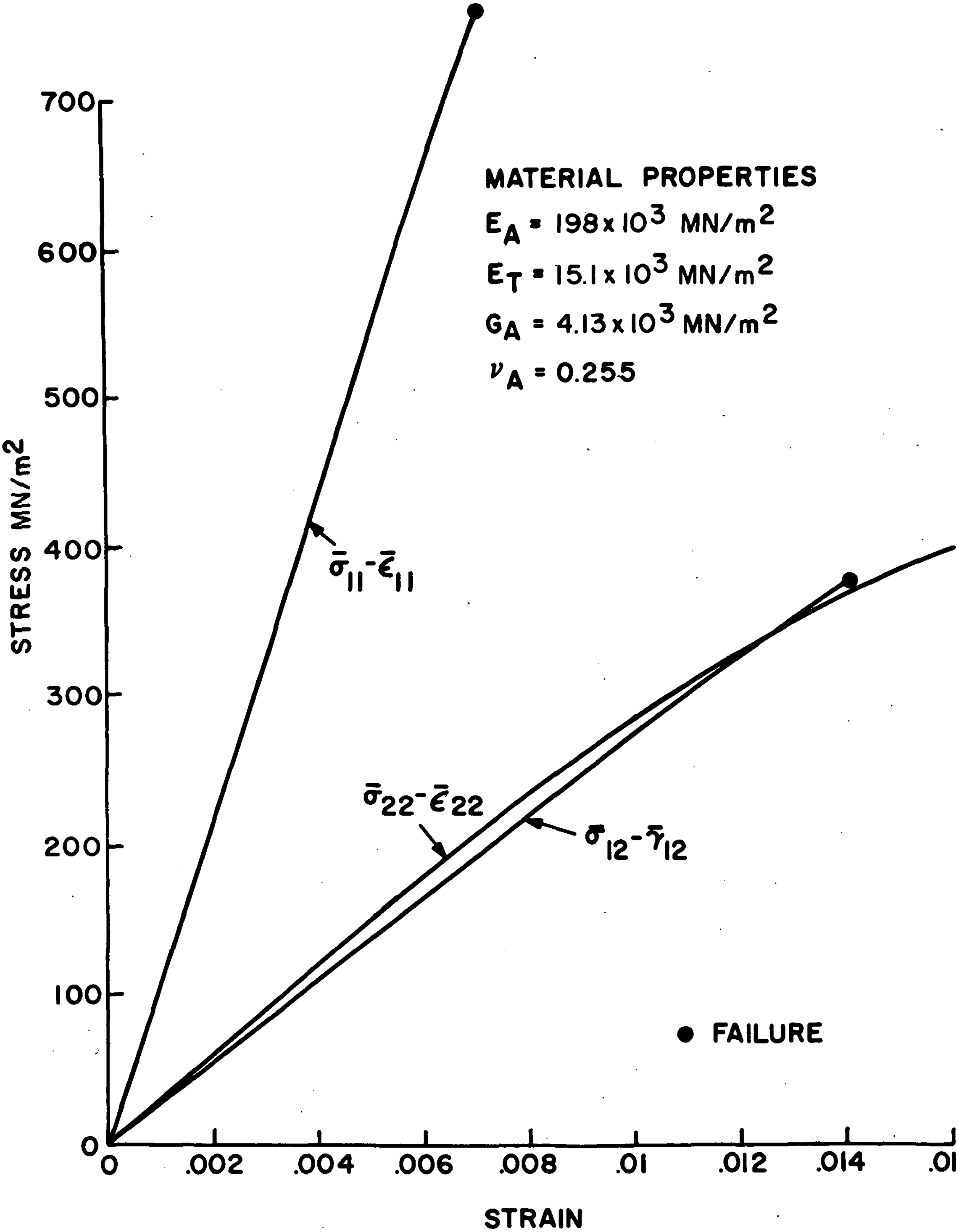

Fig. 17 - Stress-Strain curves for Axial, transverse and shear loading for [0°/±45°] laminate.

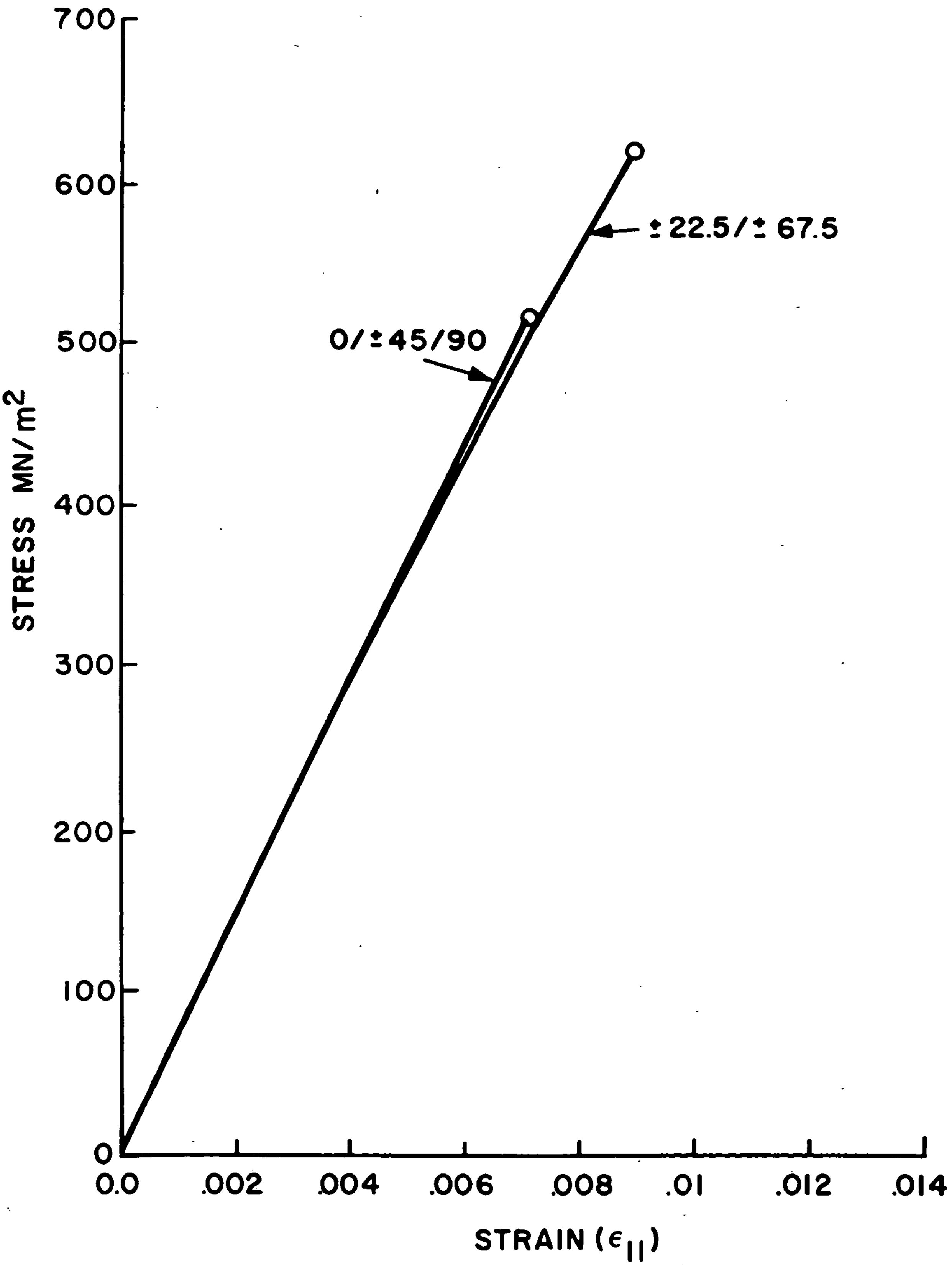

Fig. 18 - Four directional quasi-isotropic Boron/Epoxy plate under unidirectional tension in fiber direction and between fiber directions.

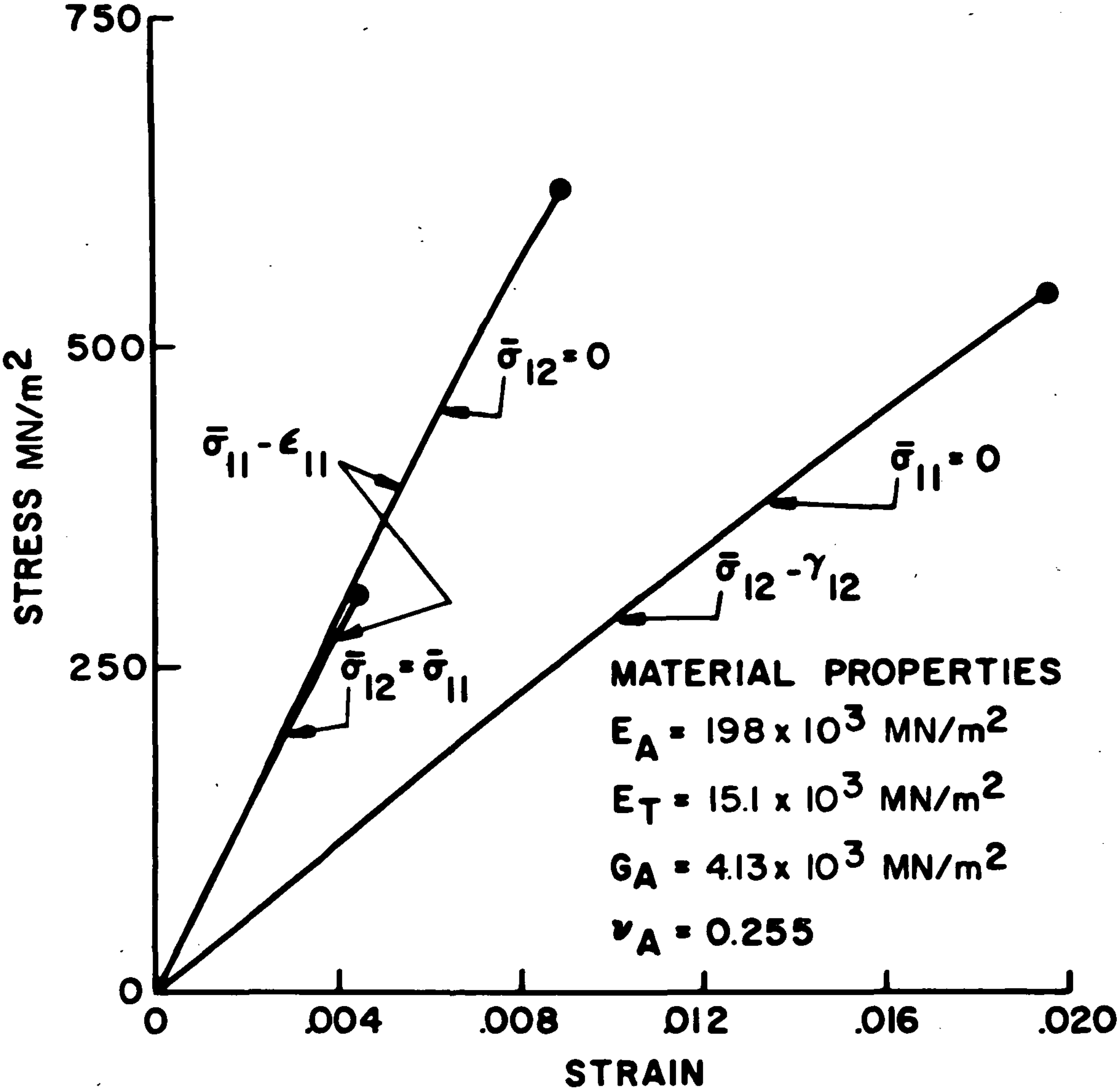

Fig. 19 - Combined stress effects on quasi-isotropic [$\pm$22.5/$\pm$67.5] laminate.

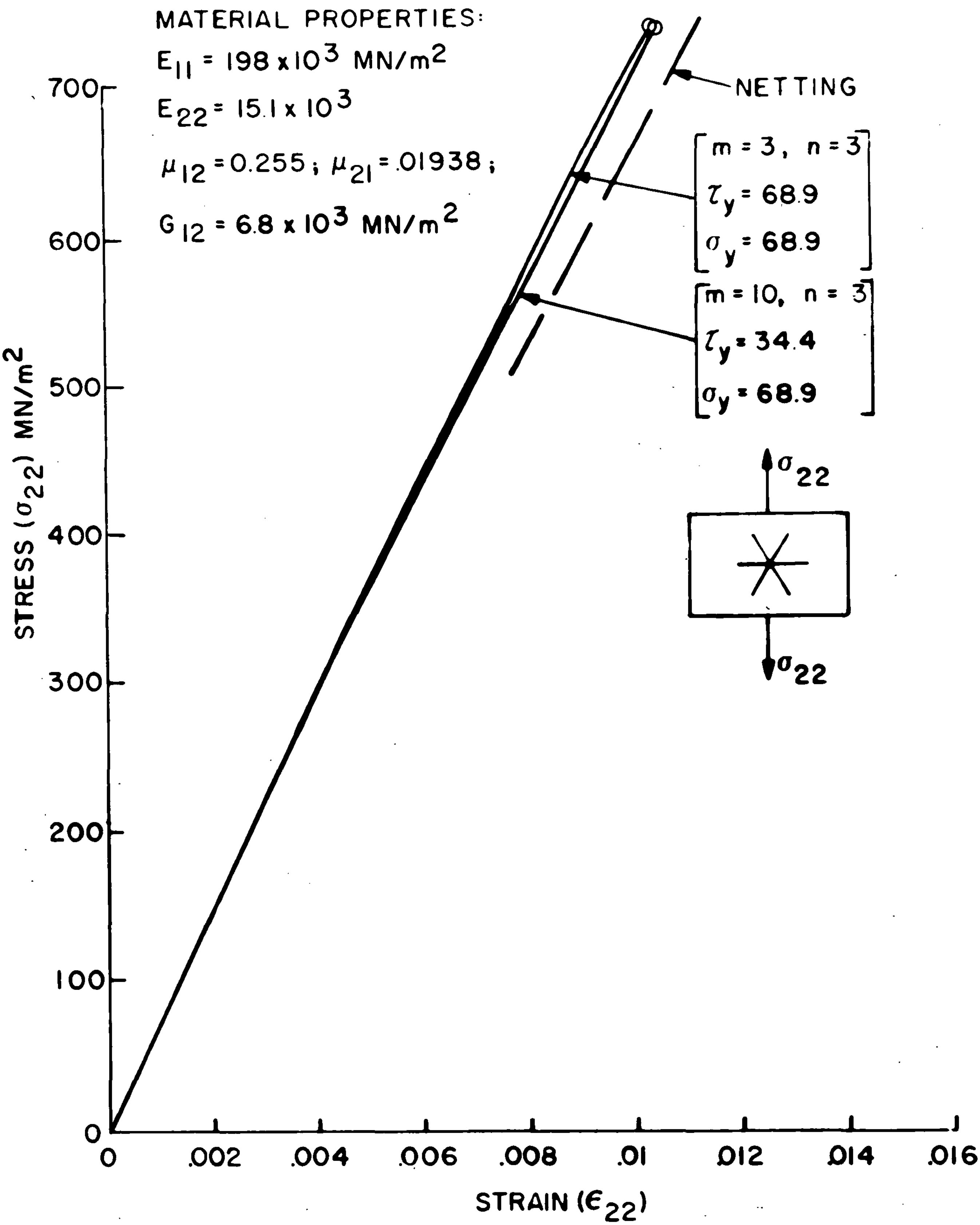

Fig. 20 - Effect of laminae inelasticity on transverse stress-strain curve for 0/$\pm$60 quasi-isotropic laminate.

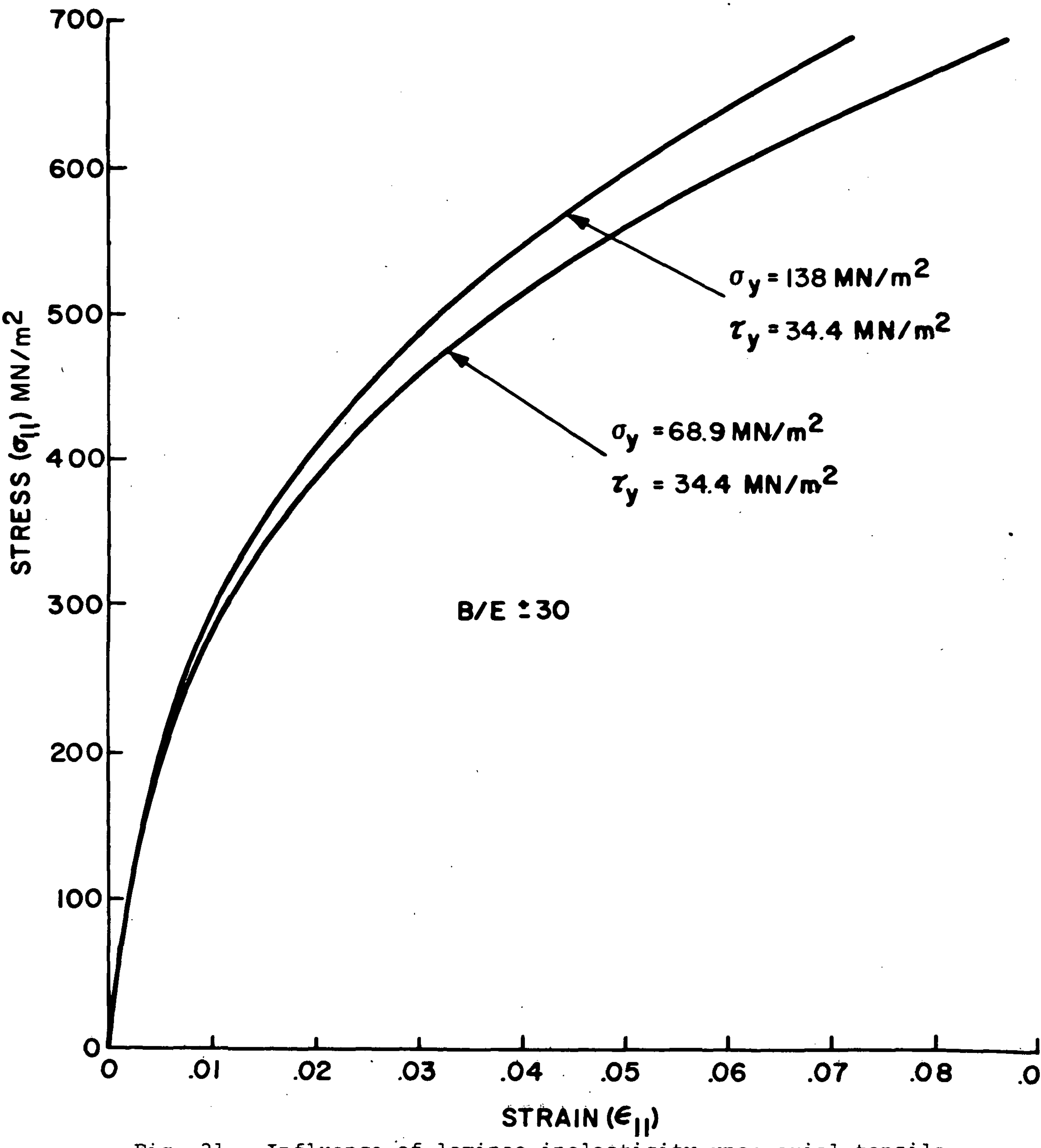

Fig. 21 - Influence of laminae inelasticity upon axial tensile stress-strain curve of ±30° Boron/Epoxy laminate.

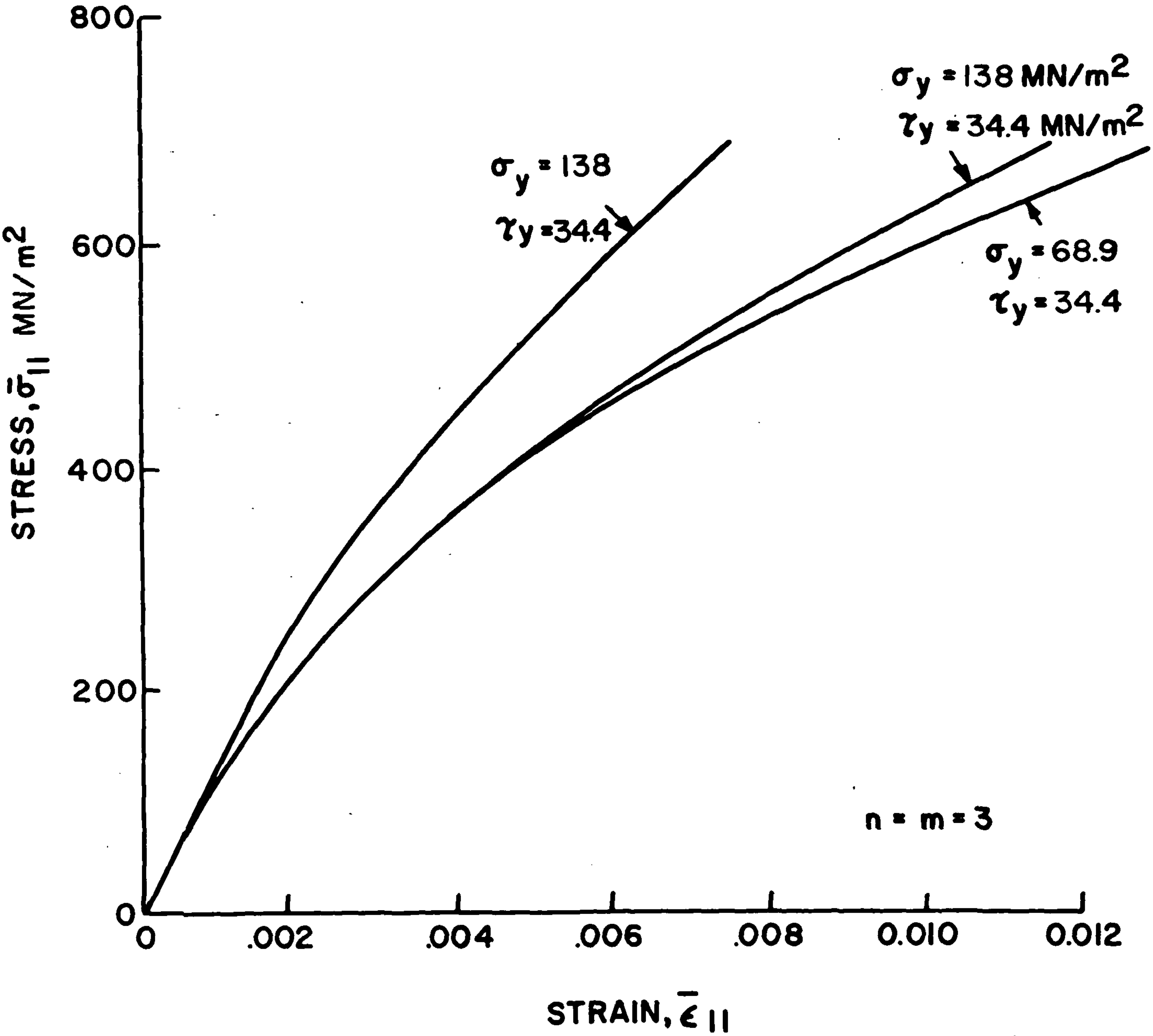

Fig. 22 - Influence of laminae inelasticity upon axial tensile stress-strain curve of ±30° Boron/Aluminum laminate.

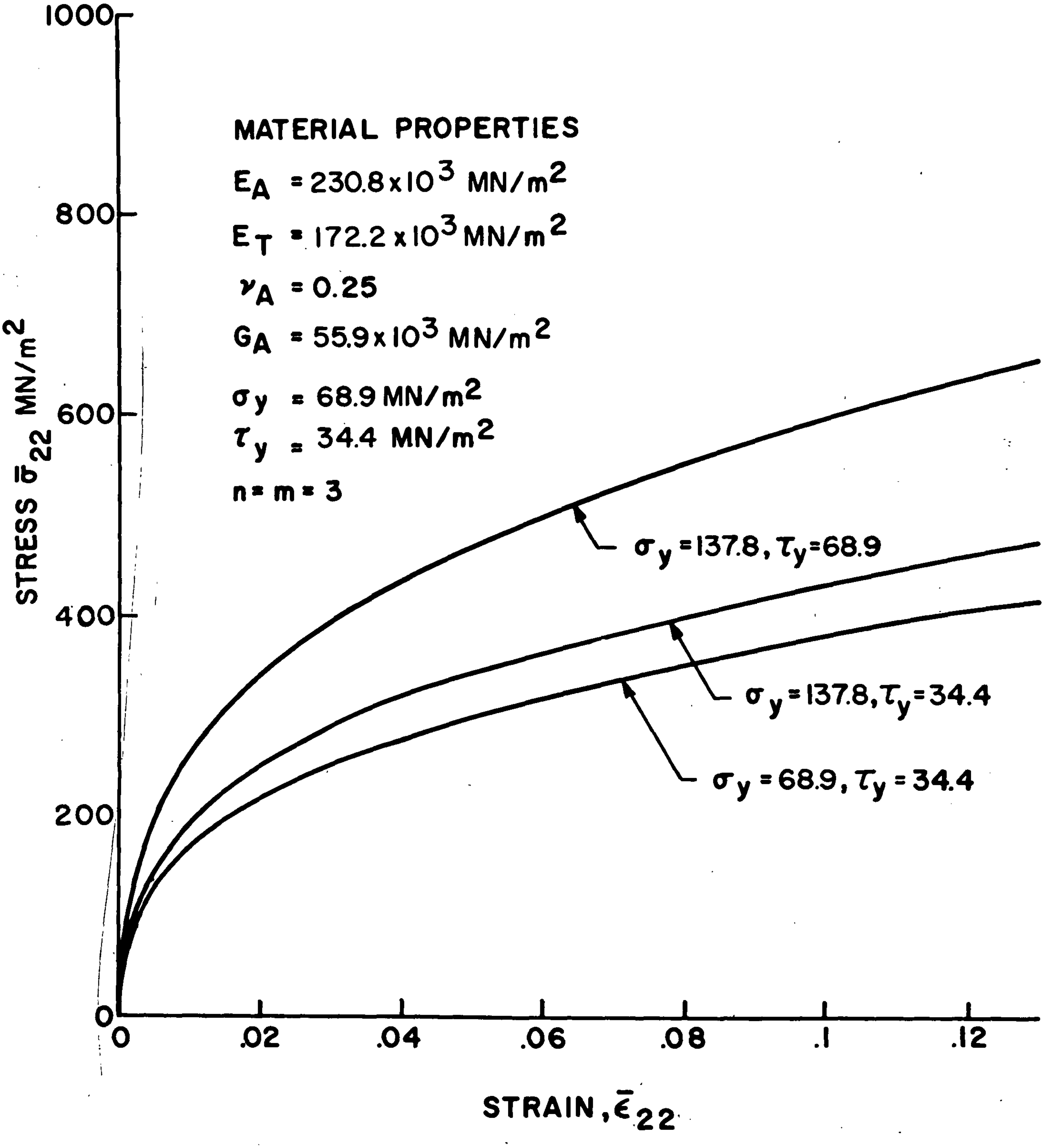

Fig. 23 - 0° Tensile stress-strain curves for $[0_3/\pm45]$ B/Ep.

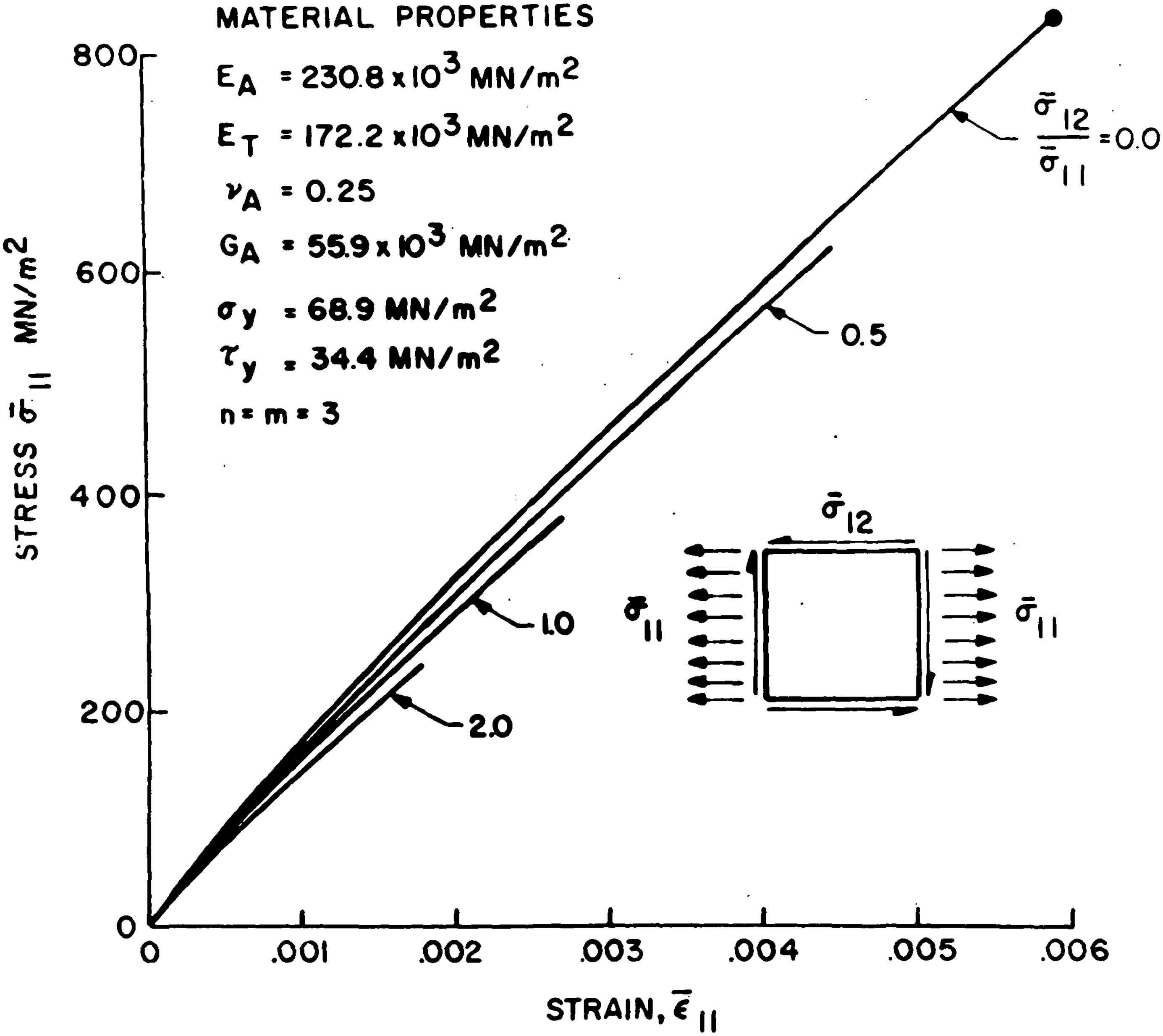

Fig. 24 - Stress-strain curves of Boron/Aluminum [0/$\pm$30] laminate under combined loading.

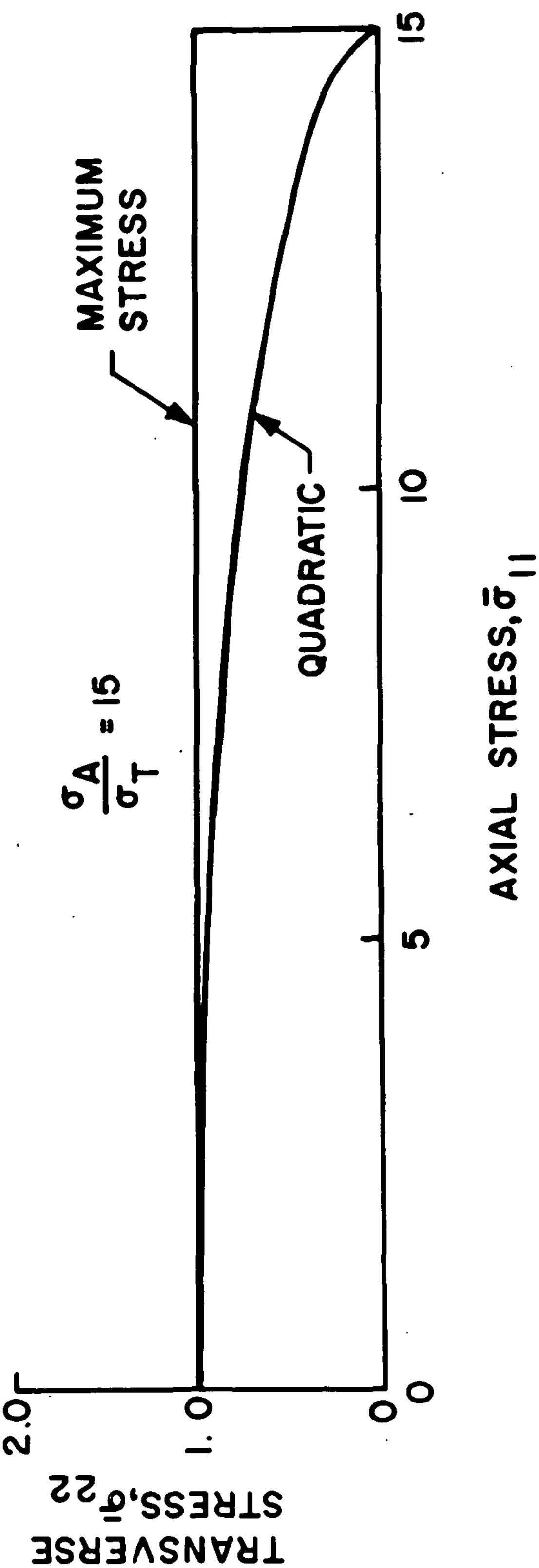

Fig. 25 - Comparison of failure criteria.

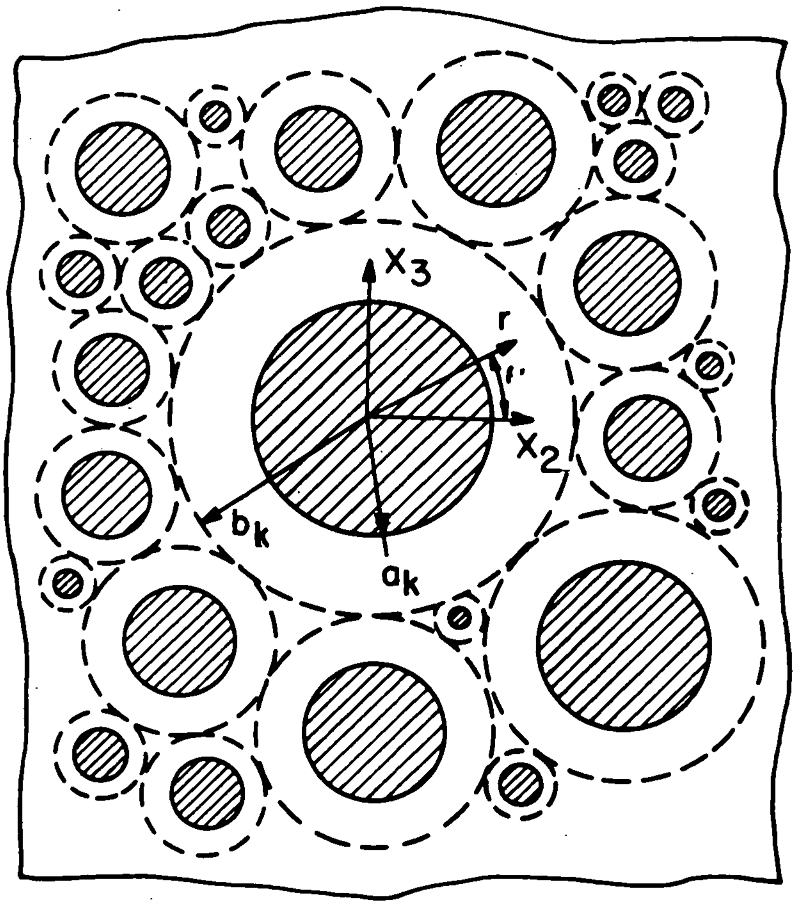

Fig. 26 - Composite Cylinder Assemblage

```
                C INELASTIC  LAMINATE  ANALYSIS
                C
                C
                C
                C
                C
0001                IMPLICIT          REAL*8(A-H,C-Z)
0002                DIMENSICN         E11(20),E22(20),V12(20),V21(20),G12(20),SY(20),
                   1                  S11(20),S12(20),S21(20),S22(20),S44(20),T(20),
                   2                  A(20,20),SGC(20,1),SG(20,1),SF(20),IANG(20),
                   3                  SINS(20),COSS(20),SIN2(20),COS2(20)
0003                DIMENSICN         P11(20),P22(20),P12(20),PS11(20),PS22(20),
                   1                  FS12(20),EP11(20),EP22(20),EP12(20),EPS11(20),
                   2                  EPS22(20),EPS12(20),SGS(20),DIF(20),SG2(20),
                   3                  SG1(20,1),DB(20,20),BT(20),DC(20),TY(20),
                   4                  F(3,20),G(3,2C),H(3,20)
0004                CGMMCN   /ANGRED/SIN2,COS2,SINS,COSS
0005                COMMON   /CCPARM/EPS,UPBD
                   1         /NCPARM/IT,NIT
0006                COMMON   /TLPARM/SGR,STIFF
                   1         /EQNTRM/EP11,EPS11,SM11,SO11
0007                CCMMON   /RAFNWT/S11,S12,S21,E22
0008                LOGICAL           MSING,MSINGC
                C
                C   INPUT
                C
                C   NOTE:   CONVENTION  FOR  V12  AND  V21  ESTABLISHED  BY  FCLLOWING
                C           RELATIONSHIPS-
                C                            S12 = -V12/E11
                C                            S21 = -V21/E22
                C
0009                WRITE(6,1500)
0010                READ(5,101C) NSETS
0011                DO 999 J = 1,NSETS
0012                READ(5,1010) LAY
0013                READ(5,1010) INP
                C
                C   LAMINATE  OUTPUT  HEADING
0014                WRITE(6,1509) J
0015                WRITE(6,1981)
0016                WRITE(6,1510) LAY
                C
0017                GO TO (20,25), INP
                C
                C   INPUT  1:
0018             20 CONTINUE
0019                DO 24 I = 1,LAY
0020                READ(5,1002) E11(I),E22(I),V12(I),V21(I)
0021                READ(5,1002) G12(I),SY(I),TY(I)
0022                READ(5,1022) T(I),IANG(I)
0023             24 CONTINUE
0024                GO TO 50
                C
                C   INPUT  2:
0025             25 CONTINUE
0026                WRITE(6,1561)
0027                WRITE(6,1563)
0028                WRITE(6,1565)
0029                DO 29 I = 1,LAY
0030                CALL INPLT2(E11,E22,V12,V21,G12,SY,TY,I)
0031                READ(5,1022) T(I),IANG(I)
```

```
0032          29 CONTINUE
           C
0033          50 CONTINUE
           C     EQUATICN  PARAMETERS
0034             READ(5,1002) XN,XM
           C     INITIAL  LCADING
0035             READ(5,1002) SO11,SO22,SO12
           C     INCRIMENTATICN  PARAMETERS
0036             READ(5,1024) KSGM,SMLT
           C     LAMINATE  TEST  PARAMETERS
0037             READ(5,1002) STIFF
0038             READ(5,1002) SGR
           C     CONTROL  PARAMETERS
0039             READ(5,1024) IT,EPS,UPBD
0040             READ(5,1010) INMT
           C
           C     OUTPUT  I.
0041             WRITE(6,1513)
0042             DO 60 I=1,LAY
0043             WRITE(6,1515)  I,IANG(I),T(I),E11(I),E22(I),V12(I),V21(I),
                1               G12(I),SY(I),TY(I)
0044          60 CONTINUE
0045             WRITE(6,1516)
0046             WRITE(6,1517) XM
0047             WRITE(6,1518) XN
0048             WRITE(6,1520)
0049             WRITE(6,1983)
0050             WRITE(6,1995)
           C
           C     ANGLE  REDUCTION  ROUTINE
0051             CALL ANGLE(LAY,IANG)
           C
           C     INITIAL  ASSIGNMENTS  AND  COMPUTATIONS
0052             TT     =  0.0D0
0053             DO 100 I = 1,LAY
0054             TT     =  TT + T(I)
0055         100 CONTINUE
0056             DO 105 I=1,20
0057             SG(I,1) =  0.0000 00
0058             SGS(I) =  0.0D00
0059             SF(I) =  1.0D0
0060         105 CONTINUE
0061             SM11 =  SMLT*SO11
0062             SM22 =  SMLT*SO22
0063             SM12 =  SMLT*SO12
0064             KSG  =  1
           C
0065             LT1  =  LAY
0066             N    =  LT1
0067             LP1  =  LAY  +1
0068             LT2  =  LAY*2
0069             LT21 =  LAY*2+1
0070             LT3  =  LAY*3
0071             LM1  =  LAY  -1
           C
0072             DO 107 I = 1,LAY
0073             S11(I) =  1.0D0/E11(I)
0074             S12(I) = -V12(I)/E11(I)
0075             S21(I) = -V21(I)/E22(I)
0076         107 CONTINUE
           C
```

```
0077        110 CONTINUE
          C
          C     RETURN  TO  110  AFTER  INCREMENTING  APPLIED  LOAD  OR
          C     FROM  LAMINATE  TESTS  3  OR  4
          C
0078            MSING = .FALSF.
0079            MSINGD = .FALSE.
0080            NIT  =  0
          C
0081            DO 111 K = 1,20
0082            DC(K)  =  0.0D0
0083            SGC(K,1) =  0.0D0
0084            SG1(K,1) =  0.0D0
0085            SG2(K) =  0.0D0
0086            DO 111 L=1,20
0087            DB(K,L)=  0.0D0
0088            A(K,L)   =  0.0D0
0089        111 CONTINUE
0090            DO 1115 I=1,N
0091            S22(I) =  1.0D0/E22(I)
0092            S44(I) =  1.0D0/(4.0D0*G12(I))
0093       1115 CONTINUE
          C
          C     AFTER  2ND  INCRIMENTATION  USE  MULTIPLICATIVE  FACTOR
          C     AS  INITIAL  STRESS  SOLUTION  ESTIMATE
0094        112 CONTINUE
0095            IF(KSG.GE.3)  GO TO 120
0096            DO 115 I=1,LM1
          C
0097            SNS  =  SINS(I)
0098            CSS  =  CCSS(I)
0099            SN2  =  SIN2(I)
0100            CS2  =  CCS2(I)
          C
0101            IF(N.EQ.1) GO TO 113
0102            SNSP =  SINS(I+1)
0103            CSSP =  CCSS(I+1)
0104            SN2P =  SIN2(I+1)
0105            CS2P =  COS2(I+1)
          C
0106        113 CONTINUE
0107            A(1,I     ) =  CSS*T(I)
0108            A(1,I+N   ) =  SNS*T(I)
0109            A(1,I+2*N) = -SN2*T(I)
          C
0110            A(2,I     ) =  SNS*T(I)
0111            A(2,I+N   ) =  CSS*T(I)
0112            A(2,I+2*N) =  SN2*T(I)
          C
0113            A(3,I     ) =  SN2*T(I)/2.0D0
0114            A(3,I+N   ) = -SN2*T(I)/2.0D0
0115            A(3,I+2*N) =  CS2*T(I)
0116            IF(N.EQ.1)  GO TO 116
          C
0117            A(3*I+1,I        ) = -S11(I  )*CSS -S21(I  )*SNS
0118            A(3*I+1,I+1      ) =  S11(I+1)*CSSP+S21(I+1)*SNSP
0119            A(3*I+1,I+N      ) = -S12(I  )*CSS -S22(I  )*SNS
0120            A(3*I+1,I+N+1    ) =  S12(I+1)*CSSP+S22(I+1)*SNSP
0121            A(3*I+1,I+2*N    ) =  2.0DC*S44(I  )*SN2
0122            A(3*I+1,I+2*N+1) = -2.0D0*S44(I+1)*SN2P
          C
```

```
0123          A(3*I+2,I      ) = -S11(I  )*SNS -S21(I  )*CSS
0124          A(3*I+2,I+1    ) =  S11(I+1)*SNSP+S21(I+1)*CSSP
0125          A(3*I+2,I+N    ) = -S12(I  )*SNS -S22(I  )*CSS
0126          A(3*I+2,I+N+1  ) =  S12(I+1)*SNSP+S22(I+1)*CSSP
0127          A(3*I+2,I+2*N  ) = -2.0D0*S44(I  )*SN2
0128          A(3*I+2,I+2*N+1) =  2.0D0*S44(I+1)*SN2P
      C
0129          A(3*I+3,I      ) = -(S11(I  )-S21(I  ))*SN2 /2.0D0
0130          A(3*I+3,I+1    ) =  (S11(I+1)-S21(I+1))*SN2P/2.0D0
0131          A(3*I+3,I+N    ) = -(S12(I  )-S22(I  ))*SN2 /2.0D0
0132          A(3*I+3,I+N+1  ) =  (S12(I+1)-S22(I+1))*SN2P/2.0D0
0133          A(3*I+3,I+2*N  ) = -2.0D0*S44(I  )*CS2
0134          A(3*I+3,I+2*N+1) =  2.0D0*S44(I+1)*CS2P
      C
0135      115 CONTINUE
      C
0136          A(1,N  ) =  CSSP*T(N)
0137          A(1,2*N) =  SNSP*T(N)
0138          A(1,3*N) = -SN2P*T(N)
0139          A(2,N  ) =  SNSP*T(N)
0140          A(2,2*N) =  CSSP*T(N)
0141          A(2,3*N) =  SN2P*T(N)
0142          A(3,N  ) =  SN2P*T(N)/2.0D0
0143          A(3,2*N) = -SN2P*T(N)/2.0D0
0144          A(3,3*N) =  CS2P*T(N)
0145          IF(MSING)  GO TO 117
      C
      C
0146      116 CONTINUE
      C
      C       INVERT  MATRIX  A
0147          CALL INVRTD(A,20,LT3,DET,1.0D-15,IRANK)
      C
      C       CHECK  FOR  SINGULAR  MATRIX
0148          IF(IRANK.EQ.LT3)  GO TO 118
0149          MSING = .TRUE.
0150          WRITE(6,1420) IRANK,DET
0151          GO TO 112
0152      117 CONTINUE
0153          WRITE(6,1425) ((A(K1,L1),L1=1,LT3),K1=1,LT3)
0154          MSING = .FALSE.
0155          GO TO 999
0156      118 CONTINUE
      C
      C       B.C.  VECTOR
0157          SGO(1,1) =  SO11
0158          SGO(2,1) =  SO22
0159          SGO(3,1) =  SO12
0160          CALL MXMULD(A,SGO,SG,20,20, 1,LT3,LT3,1)
      C
      C       RESET  STRESS = 0,  IF  RELATIVE  STRESS < 1.0D-06
0161          CALL RESET(LT3,SG,1.0D-06)
0162          GO TO 126
      C
      C       (MULTIPLICATIVE FACTOR) X (SOLUTION FROM PREVIOUS INCREMENT)
0163      120 CONTINUE
0164          DO 122 I=1,LT3
0165          SG(I,1) =  SF(I)*SGS(I)
0166      122 CONTINUE
0167          GO TO 126
      C
```

```
              C      RETURN  TO  125  FOR  NEXT  ITERATION  STEP
      0168       125 CONTINUE
      0169           NIT =  NIT + 1
      0170       126 CONTINUE
      0171           DO 127 I=1,LT3
      0172           SG1(I,1) =  SG(I,1)
      0173           BT(I) =  0.0D 00
      0174       127 CONTINUE
              C
              C
      0175       130 CONTINUE
      0176           DO 1305 K=1,LT3
      0177           DO 1305 L=1,LT3
      0178           DB(K,L) =  0.0D 00
      0179      1305 CONTINUE
      0180           IF(N.EQ.1) LM1=1
      0181           DO 151 I=1,LM1
      0182           CALL NRTRM (LAY,SG,F,G,H,G12,SY,TY,XN,XM,I)
      0183           CALL NRTRM (LAY,SG,F,G,H,G12,SY,TY,XN,XM,I+1)
              C
      0184           SNS  =  SINS(I)
      0185           CSS  =  COSS(I)
      0186           SN2  =  SIN2(I)
      0187           CS2  =  COS2(I)
              C
      0188           IF(N.EQ.1)  GO TO 131
      0189           SNSP =  SINS(I+1)
      0190           CSSP =  COSS(I+1)
      0191           SN2P =  SIN2(I+1)
      0192           CS2P =  COS2(I+1)
              C
      0193       131 CONTINUE
      0194           DB(1,I)       =  CSS*T(I)
      0195           DB(1,I+N)     =  SNS*T(I)
      0196           DB(1,I+2*N)   = -SN2*T(I)
      0197           DB(2,I)       =  SNS*T(I)
      0198           DB(2,I+N)     =  CSS*T(I)
      0199           DB(2,I+2*N)   =  SN2*T(I)
      0200           DB(3,I)       =  SN2*T(I)/2.
      0201           DB(3,I+N)     = -SN2*T(I)/2.
      0202           DB(3,I+2*N)   =  CS2*T(I)
      0203           IF(N.EQ.1)  GO TO 161
              C
      0204           DB(3*I+1,I)            = -F(1,I)
      0205           DB(3*I+1,I+1)          =  F(1,I+1)
      0206           DB(3*I+1,I+N)          = -F(2,I)
      0207           DB(3*I+1,I+N+1)        =  F(2,I+1)
      0208           DB(3*I+1,I+2*N)        = -F(3,I)
      0209           DB(3*I+1,I+2*N+1)      =  F(3,I+1)
              C
      0210           DB(3*I+2,I)            = -G(1,I)
      0211           DB(3*I+2,I+1)          =  G(1,I+1)
      0212           DB(3*I+2,I+N)          = -G(2,I)
      0213           DB(3*I+2,I+N+1)        =  G(2,I+1)
      0214           DB(3*I+2,I+2*N)        = -G(3,I)
      0215           DB(3*I+2,I+2*N+1)      =  G(3,I+1)
              C
      0216           DB(3*I+3,I)            = -H(1,I)
      0217           DB(3*I+3,I+1)          =  H(1,I+1)
      0218           DB(3*I+3,I+N)          = -H(2,I)
      0219           DB(3*I+3,I+N+1)        =  H(2,I+1)
```

```
0220            DB(3*I+3,I+2*N)     = -H(3,I)
0221            DB(3*I+3,I+2*N+1)   =  H(3,I+1)
          C
0222        151 CONTINUE
0223            DB(1,N)        =  CSSP*T(N)
0224            DB(1,2*N)      =  SNSP*T(N)
0225            DB(1,3*N)      = -SN2P*T(N)
0226            DB(2,N)        =  SNSP*T(N)
0227            DB(2,2*N)      =  CSSP*T(N)
0228            DB(2,3*N)      =  SN2P*T(N)
0229            DB(3,N)        =  SN2P*T(N)/2.
0230            DB(3,2*N)      = -SN2P*T(N)/2.
0231            DB(3,3*N)      =  CS2P*T(N)
0232            IF(MSINGD)  GO TO 167
0233        161 CONTINUE
          C
          C     UNDERFLOW  CHECK  ON  MATRIX
0234            DO 163 K=1,LT3
0235            DO 163 L=1,LT3
0236            IF(DABS(DB(K,L)).LT.1.0D-40)  DB(K,L) = 0.0D 00
0237        163 CONTINUE
          C
          C     INVERT  MATRIX  DB
0238            CALL INVFTD(DB,20,LT3,DET,1.0D-12,IRANK)
          C
          C     CHECK  FOR  SINGULAR  MATRIX
0239            IF(IRANK.EQ.LT3)  GO TO 168
0240            MSINGD = .TRUE.
0241            WRITE(6,1420) IRANK,DET
0242            GO TO 130
0243        167 CONTINUE
0244            WRITE(6,1425) ((DB(K1,L1),L1=1,LT3),K1=1,LT3)
0245            MSINGD = .FALSE.
0246            GO TO 999
0247        168 CONTINUE
          C
          C
          C
0248            DC(1)          = -SO11*TT
0249            DC(2)          = -SO22*TT
0250            DC(3)          = -SO12*TT
          C
0251            DO 6 I=1,LAY
0252            SNS  =  SINS(I)
0253            CSS  =  COSS(I)
0254            SN2  =  SIN2(I)
0255            CS2  =  COS2(I)
0256            DC(1)          =  DC(1) + SG(I,1)*CSS*T(I) + SG(LAY+I,1)*SNS*T(I)
               1                  -SG(2*LAY+I,1)*SN2*T(I)
0257            DC(2)          =  DC(2) + SG(I,1)*SNS*T(I) + SG(LAY+I,1)*CSS*T(I)
               1                  +SG(2*LAY+I,1)*SN2*T(I)
0258            DC(3)          =  DC(3) + SG(I,1)*T(I)*SN2/2. - SG(LAY+I,1)*T(I)*
               1                  SN2/2. + SG(2*LAY+I,1)*CS2*T(I)
0259          6 CONTINUE
          C
          C     REDEFINE  COEFFICIENTS  S22  AND  S44
0260            DO 7 K=1,N
0261            T12S =  (SG(K+2*N,1)/TY(K))**2
0262            TS22S =  (SG(K+N,1)/SY(K))**2
0263            S12S =  T12S + TS22S
0264            S22(K) =  (1.0D0+S12S**((XN-1.)/2.))/E22(K)
```

```
 0265            S44(K) =  (1.0D0+S12S**((XM-1.)/2.))/(4.0D0*G12(K))
 0266          7 CONTINUE
           C
 0267            DO 8 K=1,LM1
 0268            SNS  =  SINS(K)
 0269            CSS  =  CCSS(K)
 0270            SN2  =  SIN2(K)
 0271            CS2  =  CCS2(K)
 0272            SNSP =  SINS(K+1)
 0273            CSSP =  CCSS(K+1)
 0274            SN2P =  SIN2(K+1)
 0275            CS2P =  COS2(K+1)
           C
 0276            DC(3+3*K-2)       =  -(S11(K)*CSS+S21(K)*SNS)*SG(K,1) -
                1         SG(LAY+K,1)*(S12(K)*CSS+S22(K)*SNS)  + 2.*S44(K)*SN2*
                2                     SG(2*LAY+K,1) + (S11(K+1)*CSSP+S21(K+1)*SNSP)
                3                    *SG(K+1,1) + (S12(K+1)*CSSP+S22(K+1)*SNSP)*
                4                     SG(LAY+K+1,1) - 2.*S44(K+1)*SG(2*LAY+K+1,1)*
                5                     SN2P
 0277            DC(3+3*K-1)  = -(S11(K)*SNS+S21(K)*CSS)*SG(K,1) - (S12(K)*SNS+
                1              S22(K)*CSS)*SG(LAY+K,1) - 2.*S44(K)*SN2*
                2              SG(2*LAY+K,1) +
                3             (S11(K+1)*SNSP+S21(K+1)*CSSP)*SG(K+1,1) + (S12(K+1)*
                4              SNSP+S22(K+1)*CSSP)*SG(LAY+K+1,1)+ 2.*S44(K+1)*SN2P
                5             *SG(2*LAY+K+1,1)
 0278            DC(3+3*K)    =  -(S11(K)-S21(K))*SN2*SG(K,1)/2. -(S12(K)-S22(K))*
                1              SN2*SG(LAY+K,1)/2. - 2.*S44(K)*CS2*SG(2*LAY+K,1) +
                2              (S11(K+1)-S21(K+1))*SN2P*SG(K+1,1)/2. + (S12(K+1)-
                3              S22(K+1))*SN2P*SG(LAY+K+1,1)/2. + 2.*S44(K+1)*CS2P*
                4              SG(2*LAY+K+1,1)
 0279          8 CONTINUE
           C
 0280            DO 9 I=1,LT3
 0281            DC(I) = -DC(I)
 0282          9 CONTINUE
           C
 0283            DO 1 I=1,LT3
 0284            DO 1 K=1,LT3
 0285          1 BT(I) =  BT(I) + CB(I,K)*DC(K)
           C
 0286            DO 15 I=1,LT3
 0287            SG(I,1) =  SG1(I,1) + BT(I)
 0288         15 CONTINUE
           C
           C     RESET  STRESS = 0,  IF  RELATIVE  STRESS < 1.0D-06
 0289            CALL RESET(LT3,SG,1.0D-06)
 0290            IF(NIT.EQ.0)  GO TO 125
           C
           C     CONVERGENCE  CHECK
 0291            CALL CONVR(LAY,SG,SG1,KSG,&125,&900)
 0292        500 CONTINUE
           C
           C     STRAIN  COMPUTATIONS
 0293            DO 540 I = 1,LAY
 0294            SNS  =  SINS(I)
 0295            CSS  =  COSS(I)
 0296            SN2  =  SIN2(I)
 0297            CS2  =  COS2(I)
           C
 0298            P11(I) =  S11(I)*SG(I,1) + S12(I)*SG(I+N,1)
 0299            T12S =  (SG(I+2*N,1)/TY(I))**2
```

```
0300              TS22S =  (SG(I+N,1)/SY(I))**2
0301              S12S =  T12S + TS22S
0302              P22(I) =  S21(I)*SG(I,1) + SG(I+N,1)/E22(I)*(1.0D0+S12S**((XN
               1       -1.)/2.))
0303              P12(I) =  SG(I+2*N,1)/(2.*G12(I))*(1.0D0+S12S**((XM-1.)/2.))
0304         520 CONTINUE
           C
0305              EP11(I) =  P11(I)*CSS + P22(I)*SNS - P12(I)*SN2
0306              EP22(I) =  P11(I)*SNS + P22(I)*CSS + P12(I)*SN2
0307              EP12(I) =  (P11(I)-P22(I))*SN2/2.D0 + P12(I)*CS2
           C
0308         540 CONTINUE
           C
           C      LAMINATE  TESTS
0309              CALL LAMTST(LAY,SG,SGS,KSG,KSGM)
           C
           C      OUTPUT II.
0310         730 CONTINUE
0311              WRITE(6,1525)
0312              WRITE(6,1527) S011
0313              WRITE(6,1528) S022
0314              WRITE(6,1529) S012
0315              WRITE(6,1735) NIT
0316              WRITE(6,1536)
0317              WRITE(6,1537)
0318              WRITE(6,1538)
0319              DO 750 I = 1,LAY
0320              WRITE(6,1550) I,SG(I,1),SG(I+N,1),SG(I+2*N,1),
               1               EP11(I),EP22(I),EP12(I),P11(I),P22(I),P12(I)
0321         750 CONTINUE
0322              WRITE(6,1988)
           C
0323              IF(KSGM-KSG) 790,790,760
           C
           C      MULTIPLICATIVE  FACTOR  FOR  INITIAL  ESTIMATE  OF  SUCEEDING
           C      INCRIMENTATION
           C
0324         760 CONTINUE
0325              GO TO (762,766), INMT
           C
           C      RATIO  OF  PREVIOUS  SOLUTIONS
0326         762 CONTINUE
0327              DO 765 I=1,LT3
0328              IF(KSG.EQ.1)  GO TO 770
0329              IF(SGS(I).EQ.0.0D0)  GO TO 763
0330              SF(I) =  SG(I,1)/SGS(I)
0331              GO TO 765
0332         763 CONTINUE
0333              SF(I) =  1.0D0
0334         765 CONTINUE
0335              GO TO 770
           C
0336         766 CONTINUE
0337              DO 768 I=1,LT3
0338              IF(KSG.LT.2)  GO TO 767
0339              IF(SG(I,1).EQ.0.0D0)  GO TO 767
0340              VKSG =  KSG + 1
0341              CONS  =  VKSG*(VKSG-2)/(VKSG-1)**2
0342              SF(I) =  1.0D0 + CONS*(SG(I,1)-SGS(I))/SG(I,1)
0343              GO TO 768
0344         767 CONTINUE
```

```
0345              SF(I) =  1.0D0
0346          768 CONTINUE
              C
              C       STORE  STRESS  AND  STRAIN  VALUES
0347          770 CONTINUE
0348              DO 775 I = 1,LAY
0349              SGS(I     ) =  SG(I     ,1)
0350              SGS(I+  N) =  SG(I+  N,1)
0351              SGS(I+2*N) =  SG(I+2*N,1)
0352              PS11(I) =  P11(I)
0353              PS22(I) =  P22(I)
0354              PS12(I) =  P12(I)
0355              EPS11(I) =  EP11(I)
0356              EPS22(I) =  EP22(I)
0357              EPS12(I) =  EP12(I)
0358          775 CONTINUE
              C
              C       INCREMENT  APPLIED  LOADING
0359              KSG  =  KSG + 1
0360              SO11 =  SO11 + SM11
0361              SO22 =  SO22 + SM22
0362              SO12 =  SO12 + SM12
0363              GO TO 110
0364          790 CONTINUE
              C
0365          900 CONTINUE
              C
0366          999 CONTINUE
0367              WRITE(6,1599)
              C
              C
              C
0368         1002 FORMAT (5D15.5)
0369         1010 FORMAT (5I5)
0370         1022 FORMAT (D15.5,I5)
0371         1024 FORMAT (I5,5D15.5)
              C
0372         1420 FORMAT (//' MATRIX  IS  SINGULAR  OF  RANK  =  ',I2,
                 1          ';   DETERMINANT  =  ',D15.5/)
0373         1425 FORMAT (6D15.5)
              C
0374         1500 FORMAT (' INELASTIC  LAMINATE  ANALYSIS'////)
0375         1505 FORMAT (1H1,'LAMINATE ',I2)
0376         1510 FORMAT (////' NUMBER  OF  LAYERS = ',I2)
0377         1513 FORMAT ( /' LAYER  THETA',7X,'T',15X,'E11',11X,'E22',11X,
                 1          'V12',11X,'V21',11X,'G12',10X,'SIG Y',9X,'TAU Y'/)
0378         1515 FORMAT (I4,5X,I3,2X,D12.5,4X,7(2X,D12.5))
0379         1516 FORMAT (//' EQUATION  PARAMETERS')
0380         1517 FORMAT (/'     EXPONENT M = ',D12.5)
0381         1518 FORMAT ( '     EXPONENT N = ',D12.5)
0382         1520 FORMAT (//' LOADING  INCREMENTATION')
0383         1525 FORMAT (' EXTERNAL  APPLIED  LOADING'/)
0384         1527 FORMAT (' F X  = ',D13.5)
0385         1528 FORMAT (' F Y  = ',D13.5)
0386         1529 FORMAT (' F XY = ',D13.5)
0387         1536 FORMAT (/22X,'STRESS',38X,'STRAIN',37X,'STRAIN')
0388         1537 FORMAT (62X,'(LAMINATE AXES)',29X,'(LAYER AXES)')
0389         1538 FORMAT ( /' LAYER',4X,'SGM X',8X,'SGM Y',8X,'SGM XY',11X,'EPS X',
                 1          8X,'EPS Y',8X,'EPS XY',11X,'EPS X',8X,'EPS Y',8X,
                 2          'EPS XY'/)
0390         1550 FORMAT (I4,1X,2(3D13.5,4X),3D13.5)
```

```
0391      1561 FORMAT (//57X,'MATERIAL  PROPERTIES')
0392      1563 FORMAT ( /34X,'FIBER',54X,'MATRIX')
0393      1565 FORMAT (/' LAYER',3X,2(7X,'E',11X,'MU',12X,'G',12X,'V',12X)/)
          C
0394      1735 FORMAT (/' SOLUTION  FOR  STRESS  CONVERGES  WITHIN',I4,
             1         '  ITERATIONS'/)
          C
0395      1581 FORMAT (' -----------')
0396      1583 FORMAT (' -----------------------')
0397      1586 FORMAT (/62X,'------------'//)
0398      1595 FORMAT (1H0)
0399      1599 FORMAT (1H1)
0400           STOP
0401           END
```

```
0001            SUBROUTINE ANGLE(LAY,IANG)
          C
          C
          C     REDUCE  ANGLES  TO  VALUES  BETWEEN  0  AND  AND  PI/4  FOR
          C     COMPUTING  SIN  AND  COS
          C
          C
0002            IMPLICIT         REAL*8 (A-H,O-Z)
0003            DIMENSION        SINS(20),COSS(20),SIN2(20),COS2(20),IANG(20)
0004            COMMON    /ANGRED/SIN2,COS2,SINS,COSS
0005            DO 72 I = 1,LAY
0006            IANG2 =  2*IANG(I)
0007            ANG   =  IANG(I)
0008            ANG2  =  IANG2
0009            RAD   =  ANG /57.29577951300
0010            RAD2  =  ANG2/57.29577951300
0011            IAVAL =  IABS(IANG(I))
0012            IF(IAVAL.EQ. 0) GO TO 66
0013            IF(IAVAL.NE.90) GO TO 62
0014            SINS(I) =  DCOS(0.0D0)**2
0015            COSS(I) =  DSIN(0.0D0)**2
0016            SIN2(I) =  DSIN(0.0D0)
0017            COS2(I) = -DCOS(0.0D0)
0018            GO TO 72
0019         62 CONTINUE
0020            ISGN  =  IANG(I)/IAVAL
0021            SGN   =  ISGN
0022            IF(IAVAL.NE.45) GO TO 64
0023            SIN2(I) =  DCOS(0.0D0)*SGN
0024            COS2(I) =  DSIN(0.0D0)
0025            GO TO 68
0026         64 CONTINUE
0027            IF(IAVAL.LT.45) GO TO 66
0028            IRDA    =  2*IAVAL-90
0029            RDA     =  IRDA
0030            RDA     =  RDA/57.29577951300
0031            SIN2(I) =  SGN*DCOS(SGN*RDA)
0032            COS2(I) = -SGN*DSIN(SGN*RDA)
0033            GO TO 68
0034         66 CONTINUE
0035            SIN2(I) =  DSIN(RAD2)
0036            COS2(I) =  DCOS(RAD2)
0037         68 CONTINUE
0038            SINS(I) =  DSIN(RAD)**2
0039            COSS(I) =  DCOS(RAD)**2
0040         72 CONTINUE
0041            RETURN
0042            END
```

```
0001          SUBROUTINE CONVP(LAY,SG,SG1,KSG,*,*)
       C
       C
0002          IMPLICIT        REAL*8 (A-H,O-Z)
0003          DIMENSION       SG(20,1),SG1(20,1),DIF(20)
0004          COMMON    /CCPARM/EPS,UPBD
             1          /NCPARM/IT,NIT
       C
0005          ICON =  1
0006          N =  LAY
0007          LT3 =  LAY*3
       C
       C      CONVERGENCE  CHECK
0008          DO 375 J3=1,LT3
0009          SUB =  DABS(SG(J3,1))-DABS(SG1(J3,1))
0010          IF(SG1(J3,1).EQ.0.0D0) GO TO 330
0011          DIF(J3) =  DABS(SUB/SG1(J3,1))
0012          GO TO 335
0013      330 CONTINUE
0014          DIF(J3) =  SUB
0015      335 CONTINUE
0016          IF(DIF(J3).GT.EPS) GO TO 340
0017          GO TO 375
       C
       C      ITERATION  CHECK
0018      340 CONTINUE
0019          IF((NIT-IT).NE.0)  GO TO 350
0020          ICON =  3
0021          IF(DIF(J3).LE.UPBD)  GO TO 375
0022          ICON =  4
0023          GO TO 375
       C
       C      DIVERGENCE  CHECK
0024      350 IF(DIF(J3).LE.UPBD) GO TO 370
0025          ICON =  4
0026          GO TO 375
       C
       C
0027      370 CONTINUE
0028          ICON =  2
0029      375 CONTINUE
       C
0030          GO TO (500,400,382,386),ICON
       C
       C      NON-CONVERGENCE  DUMP
0031      382 CONTINUE
0032          WRITE(6,1720)
0033          WRITE(6,1722) EPS
0034          GO TO 395
0035      386 CONTINUE
0036          WRITE(6,1730)
0037          WRITE(6,1722) UPBD
0038      395 CONTINUE
0039          NITP =  NIT - 1
0040          WRITE(6,1741) NIT,NITP
0041          WRITE(6,1742)
0042          DO 397 I=1,LAY
0043          WRITE(6,1550) I,SG(I,1),SG(I+N,1),SG(I+2*N,1),SG1(I,1),SG1(I+N,1),
             1              SG1(I+2*N,1),DIF(I),DIF(I+N),DIF(I+2*N)
0044      397 CONTINUE
0045          RETURN 2
```

```
0046          400 RETURN 1
0047          500 RETURN
              C
0048          1550 FORMAT (I4,1X,2(3D13.5,4X),3D13.5)
0049          1720 FORMAT (' SOLUTION  FOR  STRESS  DOES  NOT  CONVERGE;')
0050          1722 FORMAT (' RELATIVE  ERROR  .GT.',D15.5)
0051          1730 FORMAT (' SOLUTION  FOR  STRESS  DIVERGES;')
0052          1741 FORMAT (/18X,'(ITERATION ',I3,')',28X,'(ITERATION ',I3,')')
0053          1742 FORMAT (  ' LAYER',4X,'SGM X',8X,'SGM Y',8X,'SGM XY',11X,'SGM X',
                 1         8X,'SGM Y',6X,'SGM XY',11X,'REL X',8X,'REL Y',8X,
                 2         'REL XY'/)
              C
0054                END
```

```
0001          SUBROUTINE LAMIST(LAY,SG,SGS,KSG,KSGM)
        C
        C
0002          IMPLICIT       REAL*8 (A-H,O-Z)
0003          DIMENSION      EP11(20),EPS11(20),SG(20,1),SGS(20)
0004          COMMON   /TLPARM/SGR,STIFF
             1         /EQNTRM/EP11,EPS11,SM11,SC11
        C
        C     TEST  1:  STIFFNESS  TEST
0005      550 CONTINUE
0006          DO 675 I = 1,LAY
0007          IF(KSG.EQ.1) GO TO 560
0008          IF(SO11.EQ.0.000)  GO TO 560
0009          RATIO = DABS(SM11/(EP11(I)-EPS11(I)))
0010          IF(RATIO.LT.STIFF) GO TO 677
        C
        C     TEST  2:  TEST  FOR  BOUND  ON  SGM X
0011      560 CONTINUE
0012          IF(DABS(SG(I,1)).GT.SGS) GO TO 679
        C
        C
        C
0013      675 CONTINUE
        C
        C
0014          WRITE(6,1995)
0015          RETURN
0016      677 LFAIL = 1
0017          KSG =  KSGM
0018          GO TO 700
0019      679 LFAIL = 2
0020          KSG =  KSGM
0021          GO TO 700
        C
        C
0022      700 CONTINUE
0023          GO TO (701,703), LFAIL
0024      701 WRITE(6,1450)
0025          GO TO 725
0026      703 WRITE(6,1452)
0027          GO TO 725
0028      725 WRITE(6,1495)
        C
0029     1450 FORMAT (//' LAMINATE  HAS  FAILED;   STIFFNESS  TEST  FAILURE')
0030     1452 FORMAT (//' LAMINATE  HAS  FAILED;   SGM X  EXCEEDS  MAXIMUM')
0031     1495 FORMAT (//' AT  FAILURE')
0032     1995 FORMAT (1HO)
        C
0033          RETURN
0034          END
```

```
FORTRAN IV G LEVEL  21                NETRM              DATE = 73027          16/26/39              PAGE 0001
0001               SUBROUTINE NBTRM(LAY,SG,E,G,H,G12,SY,TY,XN,XM,I)
             C
             C
0002               IMPLICIT          REAL*8(A-H,O-Z)
0003               DOUBLE PRECISION  DABS
0004               DIMENSION         S11(20),S12(20),S21(20),E22(20),G12(20),
                  1                  SINS(20),COSS(20),SIN2(20),COS2(20),SG(20,1),
                  2                  F(3,20),G(3,20),H(3,20),TY(20),SY(20)
0005               COMMON    /RAENWT/S11,S12,S21,E22
                  1          /ANGPED/SIN2,COS2,SINS,COSS
             C
0006               N =  LAY
0007               T12S =  (SG(I+2*N,1)/TY(I))**2
0008               T12D =   SG(I+2*N,1)/TY(I) **2
0009               T22S =  (SG(I+N,1)/TY(I))**2
0010               T22D =   SG(I+N,1)/TY(I) **2
0011               VAL =  SY(I)*1.00D-60
0012               RAT =  SG(I+2*N,1)*SG(I+N,1)/SY(I)
0013               VAL =  DABS(VAL)
0014               RAT =  DABS(RAT)
0015               IF(RAT.LE.VAL)  RAT =  0.0D 00
0016               C12S =  RAT/SY(I)
0017               C12T =  SG(I+2*N,1)*SG(I+N,1)/TY(I)**2
0018               TS22S =  (SG(I+N,1)/SY(I))**2
0019               S12S =  T12S + TS22S
             C
0020               SNS  =  SINS(I)
0021               CSS  =  COSS(I)
0022               SN2  =  SIN2(I)
0023               CS2  =  COS2(I)
             C
0024               F(1,I) =  S11(I)*CSS + S21(I)*SNS
0025               F(2,I) =  S12(I)*CSS + SNS/E22(I) + SNS/E22(I)*S12S**((XN-1.)/2.)
                  1        + SNS*TS22S*(XN-1.)/E22(I)*S12S**((XN-3.)/2.)
                  2        - SN2*(XM-1.)/(2.*G12(I))*S12S**((XM-3.)/2.)*C12S
0026               F(3,I) =  -SN2/(2.*G12(I)) - SN2/(2.*G12(I))*S12S**((XM-1.)/2.)
                  1         - SN2*T12S*(XM-1.)/(2.*G12(I))*S12S** ((XM-3.)/2.)
                  2                    + C12T*(XN-1.)/E22(I)*S12S**((XN-3.)/2.)
0027               G(1,I) =  S11(I)*SNS + S21(I)*CSS
0028               G(2,I) =  SNS*S12(I) + CSS/E22(I) + S12S**((XN-1.)/2.)*CSS/E22(I)
                  1         + CSS      *(XN-1.)*S12S**((XN-3.)/2.)/E22(I)*TS22S
                  2         + (XM-1.)*C12S*SN2*S12S**((XM-3.)/2.)/(2.*G12(I))
0029               G(3,I) =  SN2/(2.*G12(I)) + SN2*S12S**((XM-1.)/2.)/(2.*G12(I))
                  1        + SN2*(XM-1.)/(2.*G12(I))*S12S**((XM-3.)/2.)*T12S
                  2       + (XN-1.)*C12T*S12S**((XN-3.)/2.)*CSS/E22(I)
0030               H(1,I) =  (S11(I)-S21(I))*SN2/2.
0031               H(2,I) =  S12(I)*SN2/2. - SN2/(2.*E22(I)) - S12S**((XN-1.)/2.)*SN2
                  1       /(2.*E22(I)) - (XN-1.)*S12S**((XN-3.)/2.)*SN2/(2.*E22(I))
                  2            *TS22S                  +(XM-1.)*C12S*S12S**((XM-3.)/2.)
                  3            *CS2/(2.*G12(I))
0032               H(3,I)=  CS2/(2.*G12(I))+ S12S**((XM-1.)/2.)*CS2/(2.*G12(I))
                  1         + (XM-1.)*S12S**((XM-3.)/2.)*T12S*CS2/(2.*G12(I))
                  2         - (XN-1.)*C12T*S12S**((XN-3.)/2.)*SN2/(2.*E22(I))
             C
0033               RETURN
0034               END
```

```
0001          SUBROUTINE RESET(LT3,SG,RVAL)
          C
0002          IMPLICIT       REAL*8 (A-H,O-Z)
0003          DIMENSION    SG(20,1)
          C
0004          SGMX =  DABS(SG(1,1))
0005          DO 318 K=2,LT3
0006          IF(DABS(SG(K,1)).GT.SGMX)  SGMX =  DABS(SG(K,1))
0007      318 CONTINUE
0008          DO 319 K=1,20
0009          RAT =  DABS(SG(K,1)/SGMX)
0010          IF(RAT.LT.RVAL)     SG(K,1)  =  0.0D 00
0011      319 CONTINUE
0012          RETURN
0013          END
```

```
0001            SUBROUTINE INPUT2(E11,E22,V12,V21,G12,SY,TY,I)
          C
          C
          C
          C
0002            IMPLICIT         REAL*8(A-Z)
0003            DOUBLE PRECISION DSQRT
0004            DIMENSION        E11(20),E22(20),V12(20),V21(20),G12(20),SY(20),
               1                 GSA(11),SG12(11),GAM(20),TAU(20),TY(20)
0005            INTEGER          I,J,NUMT,NUMS,I2
          C
0006            READ(5,1002) EF,MUF,GF,VF
0007            READ(5,1002) EM,MUM,GM
0008            READ(5,1005) I2
0009            VM   =  1.0D 0 - VF
0010            TRM1  =  (1.0D0+VF)/(1.0D0-VF) * GM
0011            WRITE(6,1567) I,EF,MUF,GF,VF,EM,MUM,GM,VM
0012            IF(I2.EQ.1) GO TO 30
0013            READ(5,1002) SY(I),TY(I)
0014            G12(I) =  TRM1
0015            GO TO 60
0016         30 CONTINUE
0017            READ(5,1002) TY(I)
0018            READ(5,1005) NUMT
0019            READ(5,1002) (TAU(J),J=1,NUMT)
0020            READ(5,1002) (GAM(J),J=1,NUMT)
0021            READ(5,1002) (SG12(J),J=2,11)
          C
          C     COMPUTE  TAUY  FROM  TABLE
0022            SUMS6 =  0.0D0
0023            SUMS4 =  0.0D0
0024            SUMS3 =  0.0D0
0025            DO 40 J=1,NUMT
0026            SUMS6 =  SUMS6 + TAU(J)**6
0027            SUMS4 =  SUMS4 + TAU(J)**4
0028            SUMS3 =  SUMS3 + GAM(J)*TAU(J)**3
0029         40 CONTINUE
0030            TAUY =  DSQRT(SUMS6/(GM*SUMS3-SUMS4))
          C
          C     COMPUTE  SY  AND  G12
0031            SG12(1) =  0.0D0
0032            SUMS6 =  0.0D0
0033            SUMS4 =  0.0D0
0034            TRM2  =  (3.0D0+13.0D0*VF)/(3.0D0*(1.0D0+VF)**3) + VF**2/
               1         (3.0D0*(1.0D0+VF)**3) + (VF/(1.0D0+VF))**3/3.0D0
0035            DO 50 J=1,11
0036            GSA(J) =  TRM1/((SG12(J)/TAUY)**2*TRM2+1.0D0)
0037            SUMS6  =  SUMS6 + SG12(J)**6
0038            SUMS4  =  SUMS4 + SG12(J)**4*(GSA(1)/GSA(J)-1.0D0)
0039         50 CONTINUE
0040            SY(I)  =  DSQRT(SUMS6/SUMS4)
0041            G12(I) =  GSA(1)
          C
          C
0042         60 CONTINUE
0043            PI    =  3.14159D0
0044            KF    =  EF/(2.0D0*(1.0D0+MUF)*(1.0D0-2.0D0*MUF))
0045            KM    =  EM/(2.0D0*(1.0D0+MUM)*(1.0D0-2.0D0*MUM))
0046            M     =  (KF*(PI+8.0D0*VF*MUM) + 2.0D0*KM*(PI-4.0D0*VF)*MUM)/
               1         (KF*(PI-4.0D0*VF      ) + 2.0D0*KM*(PI*MUM+2.0D0*VF))
0047            KK    =  KM/2.0D0*((PI*M+4.0D0-PI)/4.0D0 +
```

```
                  1       4.0D0*M/(PI+(4.0D0-PI)*M))
0048                  E11(I)  =  EF*VF + EM*(1.0D0-VF)
0049                  V12(I)  =  MUF*VF + MUM*(1.0D0-VF)
0050                  MU23 =  MUF*VF + MUM*VM*(1.-MUM-V12(I)*(EM/E11(I)))
0051                  E22(I)  =  2.0D 0*KK*E11(I)*(1.0D0-MU23)/(E11(I)+4.0D0*KK*
                  1               V12(I)**2)
0052                  V21(I)  =  V12(I)*E22(I)/E11(I)
0053             1002 FORMAT (5D15.5)
0054             1005 FORMAT (5I5)
0055             1567 FORMAT (I4,2(7X, 4(D11.4,2X)))
               C
0056                  RETURN
0057                  END
```

```
      SUBROUTINE MXMULD(A,B,C,NROWA,NCOLA,NCOLB,MA,NA,NB)               R06D0001
C
C
      DIMENSION A(NROWA,NCOLA),B(NCOLA,NCOLB),C(NROWA,NCOLB)            R06D0002
      DOUBLE PRECISION A,B,C,X                                          R06D0003
      DO 20 I=1,MA                                                      R06D0004
      DO 20 J=1,NB                                                      R06D0005
      X=0.                                                              R06D0006
      DO 10 K=1,NA                                                      R06D0007
   10 X=X+A(I,K)*B(K,J)                                                 R06D0008
   20 C(I,J)=X                                                          R06D0009
      RETURN                                                            R06D0010
      END                                                               R06D0011
```

```
      SUBROUTINE INVRTD(A,NDIM,N,DETA,EPS,IRANK)                        R12D0001
C
C
      DIMENSION A(NDIM,NDIM)                                            R12D0002
      DOUBLE PRECISION PIV2,A,DETA,TEST,X,PIV,PIV1,TOL,TEMP,SUM,RMS,    R12D0003
     1 DMA,DSQRT,DABS,DBLE,EPS                                          R12D0004
      INTEGER IR(50),IC(50),P,S                                         R12D0005
      DETA=1.                                                           R12D0006
      SUM=0.                                                            R12D0007
      DO 10 I = 1,N                                                     R12D0008
      DO 10 J = 1,N                                                     R12D0009
   10 SUM=SUM+A(I,J)**2                                                 R12D0010
      SUM=DSQRT(SUM)                                                    R12D0011
      DMA = N**2                                                        R12D0012
      RMS=SUM/DMA                                                       R12D0013
      TOL=EPS*RMS                                                       R12D0014
      DO 20 I = 1,N                                                     R12D0015
      IR(I)=0                                                           R12D0016
   20 IC(I)=0                                                           R12D0017
      S=0                                                               R12D0018
      R = N                                                             R12D0019
   30 I=0                                                               R12D0020
      J=0                                                               R12D0021
      TEST=0.0                                                          R12D0022
      DO 50 K = 1,N                                                     R12D0023
      IF(IR(K).NE.0)GO TO50                                             R12D0024
      DO 40 L = 1,N                                                     R12D0025
      IF(IC(L).NE.0)GO TO40                                             R12D0026
      X=DABS(A(K,L))                                                    R12D0027
      IF(X.LT.TEST)GO TO40                                              R12D0028
      I=K                                                               R12D0029
      J=L                                                               R12D0030
      TEST=X                                                            R12D0031
   40 CONTINUE                                                          R12D0032
   50 CONTINUE                                                          R12D0033
      PIV=A(I,J)                                                        R12D0034
      IF(DETA.LT.1.0D-73)  DETA = 0.0D 00
      DETA=PIV*DETA                                                     R12D0035
      IF (DABS(PIV) .LE. TOL) GO TO 150                                 R12D0036
      IR(I)=J                                                           R12D0037
      IC(J)=I                                                           R12D0038
      PIV = 1.0D0/PIV                                                   R12D0039
      A(I,J)=PIV                                                        R12D0040
      DO 60 K = 1,N                                                     R12D0041
   60 IF(K.NE.J)A(I,K)=A(I,K)*PIV                                       R12D0042
      DO 90 K = 1,N                                                     R12D0043
      IF (K.EQ.I) GO TO 90                                              R12D0044
      PIV1 = A(K,J)                                                     R12D0045
   70 DO 80 L = 1,N                                                     R12D0046
   80 IF(L.NE.J)A(K,L)=A(K,L)-PIV1*A(I,L)                               R12D0047
   90 CONTINUE                                                          R12D0048
      DO 100 K = 1,N                                                    R12D0049
  100 IF(K.NE.I)A(K,J)=-PIV*A(K,J)                                      R12D0050
      S=S+1                                                             R12D0051
      IF(S.LT.R)GO TO 30                                                R12D0052
  110 DO 140 I = 1,N                                                    R12D0053
      K=IC(I)                                                           R12D0054
      M=IR(I)                                                           R12D0055
      IF(K.EQ.I)GO TO 140                                               R12D0056
      DETA=-DETA                                                        R12D0057
      DO 120 L = 1,N                                                    R12D0058
```

```
              TEMP=A(K,L)                       R12D0059
              A(K,L)=A(I,L)                     R12D0060
         120  A(I,L)=TEMP                       R12D0061
              DO 130 L = 1,N                    R12D0062
              TEMP=A(L,M)                       R12D0063
              A(L,M)=A(L,I)                     R12D0064
         130  A(L,I)=TEMP                       R12D0065
              IC(M)=K                           R12D0066
              IR(K)=M                           R12D0067
         140  CONTINUE                          R12D0068
         150  IRANK=S                           R12D0069
              RETURN                            R12D0070
              END
```

LAMINATE 1

SAMPLE PROBLEM

NUMBER OF LAYERS = 3

LAYER	THETA	T	E11	E22	V12	V21	G12	SIG Y	TAU Y
1	-45	0.33330D 00	0.28755D 08	0.21855D 07	0.25500D 00	0.19381D-01	0.98700D 06	0.10000D 05	0.50000D 04
2	0	0.33330D 00	0.28755D 08	0.21855D 07	0.25500D 00	0.19381D-01	0.98700D 06	0.10000D 05	0.50000D 04
3	45	0.33330D 00	0.28755D 08	0.21855D 07	0.25500D 00	0.19381D-01	0.98700D 06	0.10000D 05	0.50000D 04

EQUATION PARAMETERS

EXPONENT M = 0.30000D 01
EXPONENT N = 0.30000D 01

LOADING INCREMENTATION

EXTERNAL APPLIED LOADING

F X = 0.50000D 04
F Y = 0.00000D 00
F XY = 0.50000D 04

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.11659D 05	0.90057D 03	0.68930D 03	0.41473D-03	-0.30259D-03	0.47092D-03	-0.41485D-03	0.52699D-03	0.35866D-03
2	0.11819D 05	-0.41800D 03	0.89896D 03	0.41473D-03	-0.30259D-03	0.47092D-03	0.41473D-03	-0.30259D-03	0.47092D-03
3	-0.15000D 05	-0.60221D 03	-0.69222D 03	0.41473D-03	-0.30259D-03	0.47092D-03	0.52699D-03	-0.41485D-03	-0.35866D-03

EXTERNAL APPLIED LOADING

F X = 0.10000D 05
F Y = 0.00000D 00
F XY = 0.10000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY

1	-0.23726D 05	0.16851D 04	0.13138D 04	0.83867D-03	-0.62215D-03	0.94830D-03	-0.84004D-03	0.10566D-02	0.73041D-03
2	0.23911D 05	-0.80135D 03	0.16736D 04	0.83865D-03	-0.62215D-03	0.94829D-03	0.83865D-03	-0.62215D-03	0.94829D-03
3	0.30086D 05	-0.11556D 04	-0.13299D 04	0.83864D-03	-0.62215D-03	0.94830D-03	0.10565D-02	-0.84006D-03	-0.73039D-03

EXTERNAL APPLIED LOADING

F X = 0.15000D 05
F Y = 0.00000D 00
F XY = 0.15000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.36164D 05	0.23249D 04	0.18549D 04	0.12748D-02	-0.96472D-03	0.14333D-02	-0.12783D-02	0.15884D-02	0.11198D-02
2	0.36366D 05	-0.11447D 04	0.23077D 04	0.12748D-02	-0.96472D-03	0.14334D-02	0.12748D-02	-0.96472D-03	0.14334D-02
3	0.45257D 05	-0.16385D 04	-0.18898D 04	0.12748D-02	-0.96472D-03	0.14333D-02	0.15884D-02	-0.12783D-02	-0.11198D-02

EXTERNAL APPLIED LOADING

F X = 0.20000D 05
F Y = 0.00000D 00
F XY = 0.20000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.48963D 05	0.28439D 04	0.23224D 04	0.17222D-02	-0.13288D-02	0.19247D-02	-0.17280D-02	0.21214D-02	0.15255D-02
2	0.49151D 05	-0.14543D 04	0.28314D 04	0.17222D-02	-0.13288D-02	0.19247D-02	0.17222D-02	-0.13288D-02	0.19247D-02
3	0.60477D 05	-0.20542D 04	-0.23751D 04	0.17222D-02	-0.13288D-02	0.19247D-02	0.21214D-02	-0.17280D-02	-0.15255D-02

EXTERNAL APPLIED LOADING

F X = 0.25000D 05
F Y = 0.00000D 00
F XY = 0.25000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS SGM X	SGM Y	SGM XY	STRAIN (LAMINATE AXES) EPS X	EPS Y	EPS XY	STRAIN (LAYER AXES) EPS X	EPS Y	EPS XY
1	-0.62045D 05	0.32725D 04	0.27313D 04	0.21787D-02	-0.17107D-02	0.24207D-02	-0.21867D-02	0.26547D-02	0.19447D-02
2	0.62204D 05	-0.17361D 04	0.32748D 04	0.21786D-02	-0.17107D-02	0.24206D-02	0.21786D-02	-0.17107D-02	0.24206D-02
3	0.75719D 05	-0.24145D 04	-0.27987D 04	0.21785D-02	-0.17107D-02	0.24207D-02	0.26547D-02	-0.21868D-02	-0.19446D-02

EXTERNAL APPLIED LOADING

F X = 0.30000D 05
F Y = 0.00000D 00
F XY = 0.30000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS SGM X	SGM Y	SGM XY	STRAIN (LAMINATE AXES) EPS X	EPS Y	EPS XY	STRAIN (LAYER AXES) EPS X	EPS Y	EPS XY
1	-0.75348D 05	0.36347D 04	0.30941D 04	0.26424D-02	-0.21071D-02	0.29202D-02	-0.26526D-02	0.31879D-02	0.23747D-02
2	0.75471D 05	-0.19947D 04	0.36590D 04	0.26423D-02	-0.21071D-02	0.29201D-02	0.26423D-02	-0.21071D-02	0.29201D-02
3	0.90968D 05	-0.27309D 04	-0.31730D 04	0.26422D-02	-0.21071D-02	0.29202D-02	0.31878D-02	-0.26527D-02	-0.23746D-02

EXTERNAL APPLIED LOADING

F X = 0.35000D 05
F Y = 0.00000D 00
F XY = 0.35000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS SGM X	SGM Y	SGM XY	STRAIN (LAMINATE AXES) EPS X	EPS Y	EPS XY	STRAIN (LAYER AXES) EPS X	EPS Y	EPS XY
1	-0.88822D 05	0.39474D 04	0.34203D 04	0.31118D-02	-0.25152D-02	0.34224D-02	-0.31241D-02	0.37207D-02	0.28135D-02
2	0.88909D 05	-0.22338D 04	0.39960D 04	0.31118D-02	-0.25152D-02	0.34222D-02	0.31118D-02	-0.25152D-02	0.34222D-02
3	0.10622D 06	-0.30128D 04	-0.35082D 04	0.31116D-02	-0.25152D-02	0.34224D-02	0.37206D-02	-0.31242D-02	-0.28134D-02

EXTERNAL APPLIED LOADING

F X = 0.400000 05
F Y = 0.000000 00
F XY = 0.400000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.10244D 06	0.42228D 04	0.37171D 04	0.35859D-02	-0.29330D-02	0.39266D-02	-0.36001D-02	0.42530D-02	0.32595D-02
2	0.10249D 06	-0.24562D 04	0.43033D 04	0.35859D-02	-0.29330D-02	0.39263D-02	0.35859D-02	-0.29330D-02	0.39263D-02
3	0.12146D 06	-0.32670D 04	-0.38118D 04	0.35657D-02	-0.29330D-02	0.39266D-02	0.42529D-02	-0.36003D-02	-0.32593D-02

EXTERNAL APPLIED LOADING

F X = 0.450000 05
F Y = 0.000000 00
F XY = 0.450000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.11618D 06	0.44686D 04	0.39896D 04	0.40639D-02	-0.33589D-02	0.44324D-02	-0.40799D-02	0.47849D-02	0.37114D-02
2	0.11618D 06	-0.28643D 04	0.45603D 04	0.40639D-02	-0.33589D-02	0.44321D-02	0.40639D-02	-0.33589D-02	0.44321D-02
3	0.13669D 06	-0.34967D 04	-0.40896D 04	0.40636D-02	-0.33589D-02	0.44324D-02	0.47846D-02	-0.40800D-02	-0.37112D-02

EXTERNAL APPLIED LOADING

F X = 0.500000 05
F Y = 0.000000 00
F XY = 0.500000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.13001D 06	0.46909D 04	0.42416D 04	0.45451D-02	-0.37915D-02	0.49396D-02	-0.45628D-02	0.53164D-02	0.41683D-02
2	0.12996D 06	-0.26598D 04	0.48346D 04	0.45451D-02	-0.37915D-02	0.49393D-02	0.45451D-02	-0.37915D-02	0.49393D-02
3	0.15192D 06	-0.37120D 04	-0.43459D 04	0.45448D-02	-0.37915D-02	0.49396D-02	0.53162D-02	-0.45629D-02	-0.41681D-02

EXTERNAL APPLIED LOADING

F X = 0.55000D 05
F Y = 0.00000D 00
F XY = 0.55000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.14391D 06	0.43942D 04	0.44769D 04	0.50250D-02	-0.42299D-02	0.54478D-02	-0.50483D-02	0.58474D-02	0.46295D-02
2	0.14385D 06	-0.30443D 04	0.50701D 04	0.50290D-02	-0.42299D-02	0.54475D-02	0.50290D-02	-0.42299D-02	0.54475D-02
3	0.16714D 06	-0.39098D 04	-0.45842D 04	0.50287D-02	-0.42299D-02	0.54478D-02	0.58473D-02	-0.50484D-02	-0.46293D-02

EXTERNAL APPLIED LOADING

F X = 0.60000D 05
F Y = 0.00000D 00
F XY = 0.60000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.15789D 06	0.50817D 04	0.46973D 04	0.55153D-02	-0.46733D-02	0.59570D-02	-0.55360D-02	0.63760D-02	0.50943D-02
2	0.15777D 06	-0.32191D 04	0.52898D 04	0.55153D-02	-0.46733D-02	0.59567D-02	0.55153D-02	-0.46733D-02	0.59567D-02
3	0.18235D 06	-0.40944D 04	-0.48070D 04	0.55150D-02	-0.46733D-02	0.59570D-02	0.63779D-02	-0.55362D-02	-0.50942D-02

EXTERNAL APPLIED LOADING

F X = 0.65000D 05
F Y = 0.00000D 00
F XY = 0.65000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS

STRAIN (LAMINATE AXES)

STRAIN (LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.17193D 06	0.52559D 04	0.49049D 04	0.60036D-02	-0.51211D-02	0.64670D-02	-0.60257D-02	0.69083D-02	0.55624D-02
2	0.17177D 06	-0.33853D 04	0.54958D 04	0.60037D-02	-0.51211D-02	0.64667D-02	0.60037D-02	-0.51211D-02	0.64667D-02
3	0.19756D 06	-0.42676D 04	-0.50165D 04	0.60034D-02	-0.51211D-02	0.64670D-02	0.69081D-02	-0.60259D-02	-0.55622D-02

LAMINATE HAS FAILED; SGM X EXCEEDS MAXIMUM

AT FAILURE
EXTERNAL APPLIED LOADING

F X = 0.70000D 05
F Y = 0.00000D 00
F XY = 0.70000D 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.18602D 06	0.54189D 04	0.51014D 04	0.64937D-02	-0.55728D-02	0.69777D-02	-0.65172D-02	0.74382D-02	0.60332D-02
2	0.18562D 06	-0.35456D 04	0.56900D 04	0.64938D-02	-0.55728D-02	0.69773D-02	0.64936D-02	-0.55728D-02	0.69773D-02
3	0.21275D 06	-0.44311D 04	-0.52144D 04	0.64935D-02	-0.55728D-02	0.69777D-02	0.74350D-02	-0.65173D-02	-0.60331D-02

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