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Non-linear density wave solutions for different models of galaxies

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ABSTRACT

We have studied different geometry of the galaxy and the influence of certain geometry on the possible derivation of non-linear equation. We discussed soliton solutions of the derived non-linear equations and the properties of the morphologies resulting from these solutions. For thick disc, perturbations of the equilibrium state cause the non-linear Korteveg de Vries equation, and the stable solution of that equation results in the ring shape, while for the thin disc, for the similar type of perturbations, non-linear Schrodinger equation is derived with stable solution of the spiral shape.

Key words: waves – galaxies: general – galaxies: spiral – galaxies: structure.

1 INTRODUCTION

Spiral structure in galaxies has been studied in both observational and theoretical field, mainly during the 20th century. First, B. Lindblad formulated the hypothesis that the large-scale spiral structure in galaxies is quasi-stationary, in spite of the presence of differential rotation in the disc (Lindblad 1963). Due to observed differential rotation, any material structure could not persist for a long time, but would be stretched on a very short time-scale. This is known as the winding dilemma. In contrast, as the conclusion of the Lindblad work, the idea was introduced in the early 1960s that the spiral arms, if associated with a wave phenomenon, could survive differential rotation as a quasi-stationary pattern (Lin & Shu 1964). There are a few levels at which validity of the density wave theory has been questioned. The main discussions are posed on generation and maintenance of density waves (Lin & Bertin 1995).

There are, in general, three different theoretical models that can be used to investigate stellar component of galaxies, orbital, kinematic and dynamical one. Here, we recall just two of them to underline difference and difficulties. A kinematic model specifies the spatial density of stars and their kinematics at each point without questioning whether a gravitational potential exists in which the given density distribution and kinematics constitute a steady state. To treat galaxy dynamically is more complex and plausible, since it is necessary to relate the galactic gravitational potential, in which there is substantial contribution to the local acceleration from a disc, a bulge and a dark halo, to the mass density. Recently, more dynamical models of our Galaxy have been explored by Binney (2012), using complex gravitational potential that is generated by three discs (gas and both thin and thick stellar discs), a bulge and a dark halo. There are a number of papers studying the same problem from different points of research interests, such as (El-Zanta & Haßlera 1998; Bratek, Jalocha & Kutschera 2008).

Apart from these studies, there is also number of non-linear approaches to density waves. In numerical studies, using simulations, complex system has been treated in non-linear regime (Sellwood 1985, 1986). Turbulent behaviour of the interstellar medium was studied by number of authors such as Wada, Meurer & Norman (2002), and this kind of research is very useful since the turbulence is transient phase that could lead, under certain conditions, to soliton formation. There is also non-linear study of the accretion disc (Heinemann & Papaloizou 2012), resulting in a non-linear Burgers' equation and sawtooth waves, that could be used to understand transient physical processes from linear waves to possible stable non-linear structure. Theoretical research concerning possible soliton solution was given by Norman (1978) for the first time.

In this paper, our aim is not to investigate complexity but rather to study theoretically, weakly non-linear dynamics of different simplified galaxy models, using reductive perturbation method (RPM), with the primary emphasis on possible soliton solutions. Solitons are able to overcome mentioned difficulties in density wave theory. In order to use proper coordinate transformation, it is necessary to analyse stability of the linearized system of equations, and to define proper parameter regime. Each model is useful in verifying the more complex models, especially in testing the simulations and numerics used to explain dynamics of galaxies.

2 GOVERNING EQUATIONS

The density wave model consists of transport equations for the mass density ρ and the momentum ρv, together with the Poisson's equation that relates the density to the gravitational potential ϕ. The equilibrium state of the system is described as a rotation with an angular velocity $\Omega(r)$ about z-axis under the balance of centrifugal and gravitational forces in a frame rotating with constant angular velocity Ω_0. Then, the equilibrium velocity is $v_{0\varphi} = (\Omega - \Omega_0)r$,

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where $\Omega^2 r = \partial\phi_0/\partial r$. The quantities ϕ_0 and ρ_0 are the equilibrium potential and the density, respectively.

The dispersive property originates from the coupled Poisson equation, which is a second-order elliptic partial differential equation.

Depending on the Poisson equation three different geometries could be considered.

Case (a): the infinitely long cylinder. The simplest solution of Poisson's equation is obtained concerning one-dimensional motion of the infinite fluid. The Poisson equation reads as

$$\frac{\partial^2\phi}{\partial x^2} = \rho - v^2 \tag{1}$$

if rotation is present in the system, or

$$\frac{\partial^2\phi}{\partial x^2} = \rho - 1 \tag{2}$$

if there is no rotation.

Hence, the geometry of the model is infinitely long cylinder, and the coordinate x corresponds to azimuthal one. In this model, the galaxy is considered as fluid with both rotation and pressure, assuming radial velocity component to be much less than azimuthal one, and $\Omega = \text{const}$.

We consider the density wave that propagates in the φ direction and approximate spatial derivative as

$$\frac{1}{r}\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial x}. \tag{3}$$

Then, the set of equations that describes this model has the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{4}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -K\gamma\rho^{\gamma-2}\frac{\partial\rho}{\partial x} - \frac{\partial\phi}{\partial x} \tag{5}$$

$$\frac{\partial^2\phi}{\partial x^2} = \rho - v^2, \tag{6}$$

where v is the x component of the velocity and all variables are normalized as: $\rho = \rho_0\bar{\rho}$, $p = 2\pi G\rho_0^2 R^2\bar{p}$, $v = (2\pi G\rho_0^2)^{1/2}R\bar{v}$, $\phi = 2\pi G\rho_0^2 R^2\bar{\phi}$, $x = R/\sqrt{2}\bar{x}$, $t = (2\pi G\rho_0^2)^{-1/2}\bar{t}$.

We have supposed polytropic fluid and that the variations of ρ and p take place adiabatically: $\frac{1}{\rho}\nabla p = \nabla(\frac{\gamma K}{\gamma-1}\rho^{\gamma-1})$ for $\gamma \neq 1$, or $\frac{1}{\rho}\nabla p = \nabla(K\log\rho)$ for $\gamma = 1$.

Case (b): infinitely thin disc. The model of Lin and Shu assumes delta function for the density in z-direction and approximates Poissons equation by

$$\frac{\partial\phi(r, z=0)}{\partial r} = \pm 2\pi \mathrm{i} G\sigma \tag{7}$$

in the vicinity of spiral arms, where σ represents surface mass density. Then, relation between surface density and two-dimensional potential is $\sigma = -\frac{k}{2\pi G}\phi(z=0)$, where $k = -\frac{\mathrm{i}}{\phi}\frac{\partial\phi}{\partial r}$ (Lin & Shu 1964). Here, the geometry of the model is infinitely thin disc.

Within this approximation it is necessary to examine more complicated two-dimensional motion of the fluid model of the galaxy, but we simplified it neglecting the pressure. This simplification will have no influence on the possibility of deriving integrable non-linear equation.

In the cylindrical coordinates, the governing equations for two-dimensional fluid model describing the galaxy, are written as

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial\varphi}(\rho v_\varphi) = 0 \tag{8}$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\varphi^2}{r} = -\frac{\partial\phi}{\partial r} \tag{9}$$

$$\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r v_\varphi}{r} = -\frac{1}{r}\frac{\partial\phi}{\partial\varphi} \tag{10}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 4\pi G\rho, \tag{11}$$

where v_r and v_φ are radial and azimuthal velocity components.

The last equation will be approximated using Lin and Shu asymptotic solution, and we use notation ρ for surface density. Coordinates r and φ are normalized by mean wavelength of the carrier wave in the radial direction $2\pi R/\lambda$, where R is the radial size of the galaxy and $\lambda \gg 1$ is a dimensionless constant from the Lin–Shu derivation, t by the period of the carrier wave $2\pi/\omega$, ρ by ρ_0, both components of velocity by the phase velocity $\omega R/\lambda$, ϕ by $\omega^2 R^2/\lambda^2$ and G by $\omega^2 R/(2\rho_0\lambda)$.

Case (c): thick disc. In this paper, we propose more realistic solution, introducing Gaussiansin the z-direction instead of delta function, $f(z)$ for potential and $g(z)$ for density. Then, we can approximately express Poissons equation in dimensionless form as follows:

$$A\nabla_\perp^2\bar{\phi} + B\bar{\phi} = \bar{\rho}, \tag{12}$$

where $\bar{\phi}$, $\bar{\rho}$ are two-dimensional potential and density, respectively, normalized in the same way as in case (b), $A = a/(4Gc)$, $B = b/(4Gc)$ are constants dependent on the thickness of the disc L by way of a, b and c given by

$$a = \frac{1}{2L}\int_{-L}^{L} f(z)\,\mathrm{d}z = f_1(L) \tag{13}$$

$$b = \frac{1}{2L}\int_{-L}^{L} f''(z)\,\mathrm{d}z = f_2(L) \tag{14}$$

$$c = \frac{1}{2L}\int_{-L}^{L} g(z)\,\mathrm{d}z = f_3(L) \tag{15}$$

and $\nabla_\perp^2$ denotes two-dimensional Laplacian in the plane perpendicular to z. In order to find analytical solution, we assume $\nabla_\perp^2 \ll \frac{\partial^2}{\partial^2 z}$, and $v_z \ll v_\perp$, which is correct as long as the disc is not too thick. Note that for $B = 0$ we can restore infinitesimally thin disc approximation, taking $g(z) = \delta(z)$. Governing equations in this case are the same as in case (b), only the Poisson equation (11) is replaced by equation (12).

3 LINEAR STABILITY ANALYSIS

Before making the choice of transformation of coordinates and expansion of variables in order to derive possible non-linear equation, it is necessary to discuss parameter regime. Linear dispersion relation is very useful since its form can suggest the type of the non-linear equation. We do it invoking linear stability analysis for each case. The wavenumber and density for each case is defined separately, although the same notation is used.

Case (a): concerning the gravitational force, there is known Jeans criterion for the stability of a finite spherical, non-rotating system with gravity and pressure (Jeans 1902) as

$$k^2c^2 > 4\pi G\rho, \tag{16}$$

where k is the total wavenumber in radial direction, due to spherical symmetry, and c is the sound velocity. Jeans criterion is valid only locally, as long as the inhomogeneity of the system needs to be recognized. For a uniformly rotating infinite-length cylindrical column, Chandrasekhar proved that Jeans criterion is unaffected by rotation, except for modes with wave numbers perpendicular to the axis of rotation (Chandrasekhar 1981). In the previous section, perturbing and linearizing the system equations (4)–(6), assuming that all quantities are proportional to $e^{i(\omega t - kx)}$, we obtain the dispersion relation

$$(\omega - k)^2k^2 + k^2 - \gamma k^4 - 2(\omega - k)k = 0. \tag{17}$$

Note that in this model gravitational instability is suppressed by the rotation, since we treat perturbations in azimuthal direction. Then, we consider only linearly stable waves in order to apply RPM (Jaffrey & Kawahara 1982) and to obtain the non-linear equation.

Case (b): For differentially rotating thin disc, linearizing equations (8)–(10), using equation (7) and assuming plane wave type variation as $f = f(r)e^{i(kr+m\varphi-\omega t)}$, we obtain the dispersion relation

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\rho_0|k|, \tag{18}$$

where $\omega - m\Omega$ is Doppler-shifted frequency and κ is epicyclic frequency due to differential rotation

$$\kappa^2 = 2\Omega\left(2\Omega + r\left(\frac{d\Omega}{dr}\right)\right). \tag{19}$$

Equation (18) is the same as obtained by Lin & Shu (1964), but for pressureless medium. For a pressureless medium, $(\omega - m\Omega)$ becomes negative if

$$\kappa^2 < 0, \tag{20}$$

so the disc is unstable. This is the rotational instability due to exponentially growing departure of particles from circular orbits, and the growing rate is given by κ.

Stability parameter is defined by $k_2 = \frac{\kappa^2}{2\pi G\rho_0}$, so all waves with $k < k_2$ are purely stable. For this parameter regime, dark soliton solution was obtained (Kondoh, Teramoto & Yoshida 2000). The problem is that such consideration results in dark soliton solution with diminishing density, and has no spiral pattern. It is due to improper coordinate transformation used in reductive perturbation expansion.

Taking initial limitation on the wavenumber into account, namely $k > k_1$, where $k_1 = \max\left\{\frac{1}{r}, \frac{\rho_0'(r)}{\rho_0(r)}\right\}$ (sign ′ denotes derivative with respect to r), one finds that observational data suggests having $k_1 \approx k_2$ in real galaxy (Bertin 2000). Marginal stability, as introduced above in terms of local dispersion relation, identifies a very important condition for the basic state. In fact, if the system is far from it on the side of instability, then it is expected to be subject to rapidly growing perturbations, which are bound to change the properties of the basic state in a short dynamical time-scale. It is often said for this point of astrophysical applications, that violently unstable models are just the wrong choice of basic state (Bertin 2000), (Toomre 1964). Observed systems are generally well beyond such a transient dynamical state, or such a rapidly evolving dynamical state would be very hard to catch by the observer. On the other side of marginal

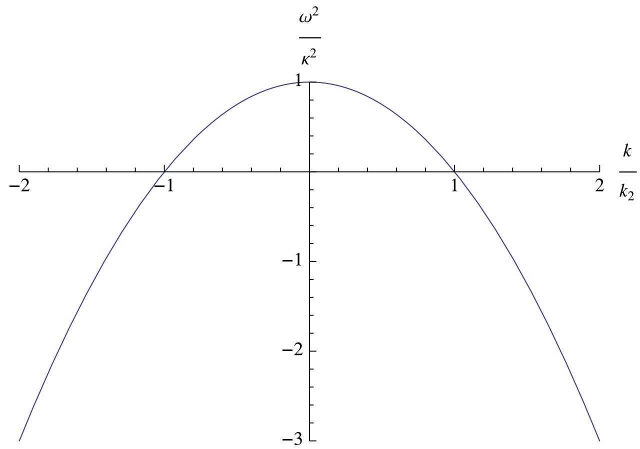

Figure 1. Marginal stability curve for the zero-thickness fluid model without pressure; ω^2 is Doppler-shifted frequency and is normalized by epicyclic frequency; wavenumber k is normalized by critical wavenumber k_2; part defined by $\frac{\omega^2}{\kappa^2} > 0$ is stable and for $\frac{\omega^2}{\kappa^2} < 0$ is unstable region.

stability, if the disc is well within a locally stable regime, not only would the local instabilities be absent, but wave propagation would also be inhibited altogether. Hence, the relevant regimes for the galaxy disc must be close to the threshold of instability (Fig. 1). In this case, new transformation of variables has to be introduced, different from the stable case (the reason is that in marginal stability frequency goes to zero, so group velocity becomes infinite).

Before taking thickness of the disc into account, it is necessary to underline two points. First, the sign of wavenumber can be positive or negative (equation 18) defining two possible branches of waves, leading and trailing. An important effect that might distinguish between these two branches is differential rotation. Density waves are propagated primarily by gravitational forces, but they would be modified by differential rotation, if the non-linear terms, omitted in linear description, were included. This effect is similar to that of fluid motion that leads to the distortion of acoustic waves. In that case, a density decrease in the direction of wave propagation tends to be accentuated into compression shock, while a density decrease would tend to be smoothed out by the motion of fluid. Thus, only the trailing waves are stable in the presence of non-linear effects. Next, inspecting the dispersion relation for gaseous disc when the pressure cannot be neglected, there is additional term c^2k^2. However, pressure and differential rotation work in the same way, balancing the non-linearity. That means, omitting the pressure, the general conclusion on the non-linear effects is not reduced; it is just modified through the parameters related with dispersive and non-linear terms.

Case (c): finite thickness of the disc is responsible for the different value of critical wavenumber due to appearance of parameters A and B in Poisson's equation (12). It will result in the more complex dispersion relation, comparing to one obtained in Lin–Shu model, for the zero-thickness fluid model. Dispersion relation in this case has a form

$$(\omega - m\Omega)^2 = \kappa^2 - \frac{4\pi G\rho_0 m\hat{k}^2}{1 + \hat{k}^2}, \tag{21}$$

where $\hat{k}^2 = \frac{k^2}{n}$, and $m = 1/A$, $n = 1/B$.

Marginal stability criterion holds for this case as well, so the coordinate transformation can be used in the same way as for thin disc. Non-linear equation will be of the same type as for thin disc, but with different coefficients of non-linear and dispersive terms. Using those coefficients, one can control the validity of applied finite thickness approximation, comparing with observed values for the galaxies.

4 NON-LINEAR EQUATIONS

In the previous section, we have summarized instabilities that are possible to occur in different galaxy models. Using WKJB analysis, Lin and Shu have proposed that gravitational instability is the basis for formation of the spiral pattern in the infinitesimally thin disc galaxies. Proposed theory resolved winding dilemma problem, assuming that the matter in the galaxy can maintain density waves through gravitational interaction in the presence of a differential rotation (even neglecting pressure). How density waves can persist quasi-stationary for a long time, remains unresolved. Several authors searched for different possible mechanisms that can replenish waves (Toomre 1969; Mark 1976; Bertin et al. 1989), but still there is no complete understanding. One possibility could be derivation of non-linear equation, that has stable soliton solution, mainly because that approach avoids involvement of some other objects which additionally complicate analysis.

We try to overcome that difficulty just keeping the higher order terms in perturbation expansions, which were omitted in linear approach, deriving the non-linear equation with localized solution. Such solutions exist whenever dispersive effects are counterbalanced by non-linear effects and coherent structure can be formed. The Korteveg de Vries (KdV) and the non-linear Schrodinger (NLS) equation are expressions of that balancing. Some of these coherent structures are stable and have been found experimentally (Mitchell & Driscoll 1996).

In order to obtain either of these two equations, we introduce asymptotic, RPM, which has been developed for non-linear dispersive wave problem (Jaffrey & Taniuti 1964). The scale transformation,

$$\xi = \epsilon^{\alpha}(x - \lambda t), \quad \tau = \epsilon^{\beta} t, \tag{22}$$

introduced by Gardner and Morikawa, may be derived from the linearized asymptotic behaviour of long waves (Jaffrey & Taniuti 1964). Using combination of this transformation of coordinates with a perturbation expansion of the dependent variables, one can obtain single non-linear equation (KdV or NLS). This type of perturbation has generally been developed and formulated by Taniuti and his collaborators (Jaffrey & Taniuti 1964).

Case (a): we transform coordinates and expand variables as

$$\xi = \epsilon^{1/2}(x - Vt), \quad \tau = \epsilon^{3/2} t \tag{23}$$

$$\rho = 1 + \sum_{\infty}^{n=1} \sum_{m=-\infty}^{\infty} \epsilon^n \rho^{(n,m)}(\xi, \tau) E, \tag{24}$$

$$v = 1 + \sum_{\infty}^{n=1} \sum_{m=-\infty}^{\infty} \epsilon^n v^{(n,m)}(\xi, \tau) E, \tag{25}$$

$$\phi = \sum_{\infty}^{n=1} \sum_{m=-\infty}^{\infty} \epsilon^n \phi^{(n,m)}(\xi, \tau) E, \tag{26}$$

where ϵ is a small parameter, $E = \mathrm{e}^{\mathrm{i}(\omega t - kx)}$ with k belonging to the linearly stable domain discussed in the previous section for this case and V is group velocity. Set of equations is obtained from the lowest order of $\epsilon^{3/2}$:

$$v^{(1,0)} = \frac{1}{2}\rho^{(1,0)}, \quad \phi^{(1,0)} = \frac{1 - 4K\gamma}{4}\rho^{(1,0)}, \quad V = \frac{3}{2}. \tag{27}$$

KdV-type equation is obtained from the order of $\epsilon^{5/2}$, as following:

$$\frac{\partial}{\partial \tau}\phi^{(1,0)} + \frac{3}{1 - 4K\gamma}\phi^{(1,0)} - \frac{1 - 4K\gamma}{8}\frac{\partial^3}{\partial \xi^3}\phi^{(1,0)} = 0. \tag{28}$$

This type of non-linear equation has a solution in the form:

$$\phi(\xi, \tau) = \phi_\infty + a\,\mathrm{sech}^2\left[(\xi - V\tau)\left(\frac{a}{12b}\right)^{1/2}\right], \tag{29}$$

where ϕ_∞ denotes the boundary value of $\phi^{(1,0)}$ at $(\xi - V\tau) \to \pm\infty$, a is amplitude of the wave relative to the constant solution ϕ_∞ at infinity, $b = \frac{27}{8(1-4K\gamma)^2}$ and V is the speed of the soliton. The solution of such non-linear equation represents the soliton, stable and localized solution, which, although derived under certain assumptions, can be used as a control parameter in numerical simulations. Also, it can be understood as a kind of equilibrium that could be perturbed and create some new structures. This soliton is travelling along azimuthal direction creating the ring structure. The width of the soliton b represents the width of the ring and could be used to compare properties of the obtained structure with the properties of observed rings.

Case (b): as we mentioned in the previous section, a new transformation of variables has to be introduced for this case, according to Watanabe (Watanabe 1969), contrary to the stable case (the reason is that frequency goes to zero in marginal stability, so group velocity becomes infinite). Stretched coordinates and expansion of variables in this case are given as

$$\xi = \epsilon(\tau - cr), \quad \eta = \epsilon^2 r, \tag{30}$$

where $\tau = t + \Omega\varphi$. Consequently, spatial and time derivatives will be

$$\frac{\partial}{\partial r} = -\epsilon c\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial \tau} = \epsilon\frac{\partial}{\partial \xi}, \tag{31}$$

together with $\frac{1}{r} = \epsilon^2\frac{1}{\eta}$.

Variable expansions have the form

$$\rho = \rho_0 + \sum_{\infty}^{n=1} \sum_{m=-\infty}^{\infty} \epsilon^n \rho^{(n,m)}(\xi, \eta) E, \tag{32}$$

$$v_r = \sum_{\infty}^{n=1} \sum_{m=-\infty}^{\infty} \epsilon^n v^{(n,m)_r}(\xi, \eta) E, \tag{33}$$

$$v_\varphi = r\Omega + \sum_{\infty}^{n=1} \sum_{m=-\infty}^{\infty} \epsilon^n v_\varphi^{(n,m)}(\xi, \eta) E, \tag{34}$$

where $E = \mathrm{e}^{\mathrm{i}(kr - \omega\tau)}$.

Substituting equations (30) and (31) into equations (8)–(10) with respect to equation (7), we derive non-linear equation following procedure of RPM. We separate terms with respect to the order of small parameter ϵ as follows:

$$\epsilon^1 : m = 0, \quad v_\varphi^{1,0} = a_1\rho^{1,0}, \quad a_1 = \frac{-\mathrm{i}\pi G}{\Omega}, \quad v_r^{1,0} = 0; \tag{35}$$

$$m = 1, \quad \omega^2 = \kappa^2 - 2\pi G\rho_0 k, \quad v_r^{1,1} = a_2\rho^{1,1}, \quad a_2 = \frac{-\omega}{k\rho_0}, \tag{36}$$

$$v_\varphi^{1,1} = a_3\rho^{1,1}, \quad a_3 = \frac{-\mathrm{i}\kappa^2}{2\Omega k\rho_0}. \tag{37}$$

$$\epsilon^2 : m = 0, \quad \rho^{1,0} = 0, \quad v_\varphi^{2,0} = a_4\rho^{2,0}, \quad a_4 = \frac{-\mathrm{i}\pi G}{\Omega}, \tag{38}$$

$$v_r^{2,0} = a_5\rho^{1,1}, \quad a_5 = \frac{2\omega}{k\rho_0^2}; \tag{39}$$

$$m = 1, \quad \frac{\partial\omega}{\partial k} = \frac{\pi G\rho_0}{\omega} = c, \quad \rho^{2,1} = 0, \tag{40}$$

$$v_r^{2,1} = b_1\frac{\partial}{\partial\xi}\rho^{1,1}, \quad b_1 = \frac{\rho_0 c a_2 - 1}{i\rho_0 k}, \tag{41}$$

$$v_\varphi^{2,1} = b_2\frac{\partial}{\partial\xi}\rho^{1,1}, \quad b_2 = \frac{a_3 - \frac{\kappa^2}{2\Omega}b_1}{i\omega}; \tag{42}$$

$$m = 2, \quad v_r^{2,2} = b_3(\rho^{1,1})^2, \quad b_3 = \frac{\frac{1}{2}\mathrm{i}ka_2^2 + \frac{1}{2}\frac{k\Omega}{\omega}a_2a_3 + \frac{\mathrm{i}\pi Gk}{\omega}a_2}{\mathrm{i}\omega + \frac{\kappa^2}{4i\omega} - \frac{\mathrm{i}\pi Gk\rho_0}{\omega}}, \tag{43}$$

$$\rho^{2,2} = b_4(\rho^{1,1})^2, \quad b_4 = \frac{k}{\omega}a_2 + \frac{\rho_0 k}{\omega}, \tag{44}$$

$$v_\varphi^{2,2} = b_5(\rho^{1,1})^2, \quad b_5 = \frac{1}{2}\frac{k}{\omega}a_2a_3 - \frac{\kappa^2}{4i\omega\Omega}b_3; \tag{45}$$

$$\epsilon^3 : m = 0, \rho^{2,0} = 0; \quad m = 1, \frac{\omega}{2\pi G}b_2 + \frac{2\Omega}{2i\pi G}b_3 = b_1; \tag{46}$$

From equation (46), after substituting all coefficients, we obtain the NLS equation:

$$\mathrm{i}\frac{\partial}{\partial\eta}\rho^{1,1} + P\frac{\partial^2}{\partial\xi^2}\rho^{1,1} + Q\left|\rho^{1,1}\right|^2\rho^{1,1} = 0, \tag{47}$$

where $P = -\frac{k_2}{\kappa^2} = \frac{1}{2}\frac{\partial^2 k}{\partial\omega^2} < 0$, and $Q = -\frac{3}{2}\frac{\kappa^2}{k_2\rho_0^2} < 0$, which implies $PQ > 0$ and consequently bright soliton solution. The solution of equation (47) has the form:

$$\rho^{1,1}(\xi,\eta) = \rho_a\frac{\mathrm{e}^{\mathrm{i}\psi}}{ch\left(\sqrt{\frac{Q}{2P}}\rho_a(\xi - 2P\eta)\right)}, \tag{48}$$

$$\psi = P\left(\frac{Q}{2P}\rho_a^2 - 1\right)\eta + \xi. \tag{49}$$

Here, ρ_a is relative amplitude of the soliton, its velocity of travel is the coefficient P, $\sqrt{Q/2P}\rho_a$ is the width of the soliton, all in dimensionless units, and ψ is the phase.

Going back to the original coordinates, we have obtained solitary structure with enhanced density along the spiral, which explains the observed pattern (Fig. 2). The solitary solution resolves the main difficulty from the linear theory is removed, e.g. the problem of searching for generators of spiral wave and mechanism that maintain waves for a long time-scale (quasi-stationarity assumption). Also, it is likely that the transport of the mass by solitons away from the considered region of the disc into outer regions will keep the disc in a state close to the threshold of stability for a long time. This fact might be responsible for the relative stability of the spiral structure as a whole. Finally, this solution provides a fine oscillatory structure inside the soliton, with a space period much smaller than the width of the soliton. This property could explain the appearance of large density gradients within spiral arms, responsible for understanding the star formation process.

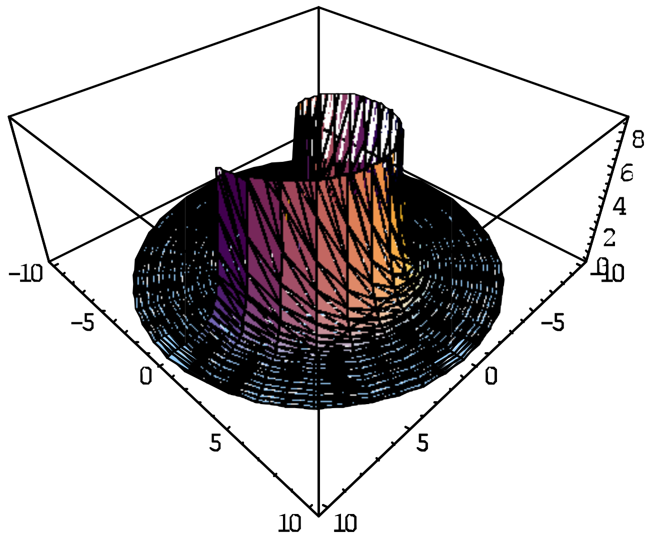

Figure 2. Bright soliton solution.

In order to make some rough estimates for the arms of the Galaxy in the neighbourhood of the Solar system, we take the following values from the observations: mass density in the disc (1–3)10^{-24} g cm^{-3}, thickness of the flat system (0.1–0.2) kpc, which implies surface density (3–5)10^{-4} g cm^{-2}, half-width of the arm 0.5 kpc, enhancement of the mean density in the arm ≈5/100, which implies $\rho_a \approx 0.3$, $\Omega \approx 10^{-15} 1/s$, and $\kappa \approx \sqrt{2}\Omega$. Then, equation (48) indicates that group velocity of the soliton is $P = 3 \times 10^5$ cm s^{-1}. A more detailed comparison of the observed structures with our proposed model would request solving the non-linear integro-differential equations involving the boundary conditions at the centre.

Case (c): we extend non-linear analysis in the more realistic case, taking finite thickness effect into account. It will result in the dispersion relation (21), in a contrast with the Lin–Shu model, where dispersion relation is linear with respect to k. Resulting non-linear equation will be of the same type as in case (b), but with different coefficients of non-linear and dispersive terms. Using those coefficients, one can control the validity of used finite thickness approximation, comparing it to the observed galactic parameters.

We transform coordinates as in case (b), invoking again physical restrictions for the galaxy (marginal stability case), and expanding variables in the same way, but we have approximated the potential using Poisson's equation given by equation (12), as follows:

$$\phi = -\frac{r^2\Omega^2}{2} + \sum_{\infty}^{n=1}\sum_{m=-\infty}^{\infty}\epsilon^n\phi^{(n,m)}(\xi,\eta)\mathrm{e}^{\mathrm{i}(kr-\omega\tau)}. \tag{50}$$

Here, spatial and time derivatives are

$$\frac{\partial}{\partial r} = -\epsilon c\frac{\partial}{\partial\xi} + \epsilon^2\frac{\partial}{\partial\eta}, \quad \frac{\partial}{\partial\tau} = \epsilon\frac{\partial}{\partial\xi}. \tag{51}$$

Following the same procedure as in case (b), one obtains the NLS equation

$$\mathrm{i}\frac{\partial}{\partial\eta}\rho^{1,1} + W\frac{\partial^2}{\partial\xi^2}\rho^{1,1} + Z\left|\rho^{1,1}\right|^2\rho^{1,1} = 0, \tag{52}$$

but in this case coefficients related to dispersive and non-linear terms, $W = -\frac{k_2}{n\kappa^2}$ and $Z = -\frac{3}{2}\frac{n\kappa^2}{k_2\rho_0^2}$ will be dependent on the thickness of the disc n. Since those coefficients give velocity and width of the soliton, respectively, comparing with observed structure, it is possible to evaluate when the finite thickness approximation is necessary to be involved for the given galaxy. Comparing these two soliton parameters for zero-thickness model and finite thickness model, one finds that spiral arm in the first case is wider than in the latter one. It is expectable because the same material amount would be redistributed taking vertical direction into account. Also, the fact that the same type of non-linear equation is obtained is in agreement with the linear stability analysis.

5 SUMMARY AND CONCLUSION

In this paper, we have studied weakly non-linear dynamics of different galaxy models, using RPM, with the primary emphasis on possible soliton solutions. Linear stability analysis of different galaxy models, which is necessary for defining parameter regime, has been conducted. We have studied the influence of finite-thickness of the galaxy disc on dispersive properties of the system. We underline that the purpose of this paper is not to describe full galactic dynamics but rather to derive possible non-linear equation which has stable localized solution for simplified models. The soliton existence gives physical answer for the permanent density wave phenomenon as a balance between the tendency of the dispersion to propagate the wave inwards, and non-linearity that tends to hold it up. This type of solution is useful for comparison with numerical simulations that treat more realistic models and with the observations, or for evaluation of dominance of each mechanism that occurs in real galaxy. We have not treated terms which we have already found that would not influence the type of the non-linear equation; for example, pressure, that undoubtedly exists in most galaxies, would not change the necessary and sufficient conditions for derivation of integrable non-linear equation. It will change only the parameters of non-linear and dispersive term, giving better agreement with observed patterns. In one-dimensional model, KdV type of non-linear equation has been derived. For a twodimensional model, using Lin–Shu approximation, the NLS equation has been derived. The solution is bright soliton propagating along the spiral. We have extended two-dimensional analysis for galaxies solving Poisson's equation in a different manner and obtain NLS equation. The last one is with different coefficients for non-linear and dispersive terms, which means different properties of soliton. The first type of non-linear equation, namely KdV, is applicable as long as the azimuthal velocity is dominant, together with the assumption of relatively thick disc. In that case, the ring shape could be formed, with restriction that certain structure cannot be formed at any distance from the galactic centre, but the radial distance is defined by restriction $\ll \frac{1}{r}\frac{\partial}{\partial\varphi} = \frac{m}{r}$, where m is positive integer and could be understood as number of rings. The second type of non-linear equation, namely NLS, is applicable when radial and azimuthal components of motions are coupled, as long as $\frac{1}{kr} \ll 1$. The last restriction is related with the condition of the finite amplitude perturbations. We have shown that derived velocity of the soliton is in good agreement with the observations. The advantage of the second type is fine structure inside the solitary envelope, that might be used to explain any process within spiral arms with scales shorter than the width of soliton. Comparing soliton properties with observational data, it is possible to control validity of the approximation that was made for each model. However, neither of these models is able to explain stretched structures, such as barred or elliptical galaxies. It would be very interesting to investigate the transient models, the dynamical scale on which the thickness of the galaxy changes and consequently, change the structure. If there is any correlation between the age of the galaxies and their thickness, how does it evolve? There is the remaining problem of the inner part of galaxy with the singularity in the centre, that would also be interesting to consider in non-linear regime. Both aspects will be treated in further research.

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