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Non-linear kinetics underlying generalized statistics

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Abstract

The purpose of the present effort is threefold. Firstly, it is shown that there exists a principle, that we call kinetical interaction principle (KIP), underlying the non-linear kinetics in particle systems, independently on the picture (Kramers, Boltzmann) used to describe their time evolution. Secondly, the KIP imposes the form of the generalized entropy associated to the system and permits to obtain the particle statistical distribution, both as stationary solution of the non-linear evolution equation and as the state which maximizes the generalized entropy. Thirdly, the KIP allows, on one hand, to treat all the classical or quantum statistical distributions already known in the literature in a unifying scheme and, on the other hand, suggests how we can introduce naturally new distributions. Finally, as a working example of the approach to the non-linear kinetics here presented, a new non-extensive statistics is constructed and studied starting from a one-parameter deformation of the exponential function holding the relation $f(-x)f(x)=1$.

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1. Introduction

In the last few decades, there has been an intensive discussion on non-conventional classical or quantum statistics. Up to now several entropies with the ensuing statistics have been considered. For instance, in classical statistics, beside the additive Boltzmann–Gibbs–Shannon entropy which leads to the standard Maxwell–Boltzman statistics, one can find in the literature, the entropies and/or related statistics introduced

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by Druyvenstein [1,2], Renyi [3], Sharma–Mittal [4], Tsallis [5], Abe [6], Papa [7], Borges–Roditi [8], Landsberg–Vedral [9], Anteneodo–Plastino [10], Frank–Daffertshofer [11], among others. On the other hand, in the literature we can find, besides the standard Bose–Einstein and Fermi–Dirac quantum statistics and/or entropies, the ones introduced by Gentile [12], Green [13], Greenberg–Mohapatra [14], Biedenharn [15], Haldane–Wu [16,17], Acharya–Narayana Swamy [18], Buyukkilic–Dimirhan [19] etc. This plethora of entropies poses naturally some questions.

A first question is if it is possible and how to treat the above entropies in the frame of a unifying context and from a more general prospective, in such a way to distinguish the common properties of the entropies, from the ones depending on the particular form of the single entropy.

A second question is if it is possible to obtain the stationary statistical distribution of the various non-linear systems in the frame of a time-dependent scheme. Fermion and boson kinetics were introduced in 1935 by Uehling and Uhlenbeck [20]. On the other hand, the non-linear kinetics associated with the anomalous diffusion has been considered in the last few decades in several papers by the mathematicians and by the physicists of the condensed matter. After 1995, in the frame of the Fokker–Planck picture, the anomalous diffusion has been linked with the time-dependent Tsallis statistical distribution [21–30] and the kinetics of the particles obeying the Haldane statistics [31] and the quon statistics [32] has been considered.

The problem of the non-linear kinetics from a more general point of view has been considered only in 1994. In Ref. [33] it has been proposed an evolution equation (Eq. (7) of the reference) describing a generic non-linear kinetics. Subsequently, some properties of this kinetics in the frame of the Fokker–Planck picture has been studied in Ref. [31] and in the frame of the Boltzmann picture in Ref. [34]. Finally, the kinetics described by non-linear Fokker–Planck equations has been reconsidered recently in Refs. [11,35].

It is well known that the formalism used to describe the time evolution of a statistical system, depends on the picture used to describe the system. For instance, for a particle system interacting with a bath, we can study its time evolution in the phase space in the frame of the Kramers picture (Fokker–Planck picture in the velocity space). Besides, for an isolated system, we can study its time evolution in the phase space adopting the Boltzmann picture.

A third question which arises at this point is if the entropy of a system, or its stationary statistical distribution, depends and how on the particular picture used to describe the system.

A fourth and last question is if it exists a principle underlying the time evolution of the system, in the two pictures. Obviously if this principle exists, it must define both the entropy and the stationary statistical distribution of the system.

The present paper is concerned with the above questions. Its principal goal is to show, that exists a principle in the following called kinetical interaction principle (KIP), which governs the particle kinetics and imposes the form of the entropy of the system independently on the particular picture used to describe the system. Within the two pictures and in a unifying context, the H-theorem is proved and the form of the generalized entropy together with the stationary statistical distribution for a generic non-linear system is obtained.

The paper is organized as it follows. In Section 2, we introduce the KIP underlying the kinetics of a particle system, without regard if it interacts with its environment or if it is an isolated system. In Section 3, we study the non-linear kinetics in the Kramers picture implied by KIP. In particular, after writing the evolution equation of the system we obtain its entropy and also its statistical distribution both as the stationary state of the evolution equation and by using the maximum entropy principle. The stability of this equilibrium distribution is also studied. In Section 4, we examine an isolated particle system and describe its non-linear kinetics governed by the KIP in the Boltzmann picture. In particular, the equilibrium and the stability of the system are studied. In Section 5, we consider in an unifying context some examples of already known classical and quantum statistical distributions, in order to test and highlight the utility of the formalism here developed. In Sections 6 and 7, we consider, just as a working example, a new statistical distribution, the κ-deformed distribution, which arises naturally in the frame of this formalism. This distribution is obtained according to the KIP both as stationary solution of a non-linear evolution equation and by using the maximum entropy principle. In Section 8, we consider some concrete physical systems where the κ-deformed distribution can be adopted. Finally, in Section 9, some concluding remarks are reported.

2. A principle underlying the kinetics

Isolated systems: Let us consider an isolated system composed by N identical particles. We make the hypothesis that the system is a low density gas so that we can describe it by a one particle distribution function. The interaction of a particle in the site $\boldsymbol{r}=(\boldsymbol{x},\boldsymbol{v})$ where the particle density is $f=f(t,\boldsymbol{r})$, with a second particle in the site $\boldsymbol{r}_1=(\boldsymbol{x}_1,\boldsymbol{v}_1)$ where the particle density is $f_1=f(t,\boldsymbol{r}_1)$, changes the states of both the particles. After the interaction we find the first particle to the site $\boldsymbol{r}'=(\boldsymbol{x}',\boldsymbol{v}')$, where the particle density is $f'=f(t,\boldsymbol{r}')$ and the second particle to the site $\boldsymbol{r}_1'=(\boldsymbol{x}_1',\boldsymbol{v}_1')$, where the particle density is $f_1'=f(t,\boldsymbol{r}_1')$. We postulate that the transition probability from the state where the particles occupy the sites $\boldsymbol{r}$ and $\boldsymbol{r}_1$, after interaction, to the state where the particles occupy the sites $\boldsymbol{r}'$ and $\boldsymbol{r}_1'$ is given by

$$\pi(t,\boldsymbol{r}\to\boldsymbol{r}',\boldsymbol{r}_1\to\boldsymbol{r}_1')=T(t,\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}_1,\boldsymbol{r}_1')\gamma(f,f')\gamma(f_1,f_1')\,. \tag{1}$$

The first factor in (1) is the transition rate which depends only on the nature of the two-body particle interaction. Then this factor is proportional to the cross section of the two body interaction and does not depend on the particle population of the four sites. Being the system composed by identical particles, the second and third factors are given by the same function and differ only on the arguments. The factor $\gamma(f,f')$ in (1) is an arbitrary function of the particle populations of the starting and of arrival sites. Eq. (1) takes into account two body interactions and for this reason the function $\gamma(f,f')$ must satisfy the condition $\gamma(0,f')=0$ because, if the starting site is empty, the transition probability is equal to zero. The dependence of the function $\gamma(f,f')$ on the particle population f' of the arrival site plays a very important role in the particle kinetics because can stimulate or inhibite the particle transition $\boldsymbol{r}\to\boldsymbol{r}'$ in such a way that interactions originated from collective effects can be taken into account. The condition $\gamma(f,0)\neq 0$ requires that in the case the arrival site is empty the transition probability must depend only on the population of the starting site. We note that for the standard linear kinetics the relation $\gamma(f,f')=f$ holds.

Systems interacting with a bath: We consider now the case of a particle system interacting with its environment which we consider as a bath. The particle which transits from the site $\boldsymbol{r}$ to the site $\boldsymbol{r}'$ is now the test particle while the one that transits from the site $\boldsymbol{r}_1$ to the site $\boldsymbol{r}_1'$ is a particle of the bath and the factor entering into the transition probability is indicated with $\tilde{\gamma}(f_1,f_1')$ and depends on the nature of the bath. If we address our attention to the test particle and after posing

$$W(t,\boldsymbol{r},\boldsymbol{r}')=\int \mathrm{d}^{2n}r_1\,\mathrm{d}^{2n}r_1'\,T(t,\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}_1,\boldsymbol{r}_1')\tilde{\gamma}(f_1,f_1')\,, \tag{2}$$

the transition probability (1) transforms immediately into

$$\pi(t,\boldsymbol{r}\to\boldsymbol{r}')=W(t,\boldsymbol{r},\boldsymbol{r}')\gamma(f,f')\,. \tag{3}$$

We remark that the transition rate $W(t,\boldsymbol{r},\boldsymbol{r}')$ depends only on the nature of the interaction between the test particle and the bath and does not depend on the population of the test particle in the starting and arrival sites.

Kinetical interaction principle: In this paper, we will study kinetics coming from the transition probabilities (3) and (1) when the function γ satisfies the condition

$$\frac{\gamma(f,f')}{\gamma(f',f)}=\frac{\kappa(f)}{\kappa(f')}\,, \tag{4}$$

where $\kappa(f)$ is a positive real function. This condition implies that $\gamma(f,f')/\kappa(f)$ is a symmetric function. Then we can pose $\gamma(f,f')=\kappa(f)b(f)b(f')c(f,f')$ where $b(f)$ and $c(f,f')=c(f',f)$ are two real arbitrary functions. It will be convenient later on to introduce the real arbitrary function $a(f)$ by means of

$$\kappa(f)=\frac{a(f)}{b(f)} \tag{5}$$

and write $\gamma(f,f')$ under the guise

$$\gamma(f,f')=a(f)b(f')c(f,f')\,. \tag{6}$$

We claim at this point that $\gamma(f,f')$ given by (6) with $a(f)$ and $b(f)$ linked through (5), is the most general function obeying condition (4). We wish to note that $\gamma(f,f')$ is given as a product of three factors. The first factor $a(f)$ is an arbitrary function of the particle population of the starting site and satisfies the condition $a(0)=0$ because if the starting site is empty the transition probability is equal to zero. The second factor $b(f')$ is an arbitrary function of the arrival site particle population. For this function we have the condition $b(0) = 1$ which requires that the transition probability does not depend on the arrival site if, in it, particles are absent. The expression of the function $b(f')$ plays a very important role in the particle kinetics, because stimulates or inhibites the transition $\boldsymbol{r} \to \boldsymbol{r}'$, allowing in such a way to consider interactions originated from collective effects. Finally, the third factor $c(f, f')$ takes into account that the populations of the two sites, namely f and f', can eventually affect the transition, collectively and symmetrically.

The function $\gamma(f, f')$ given by (6) defines a special interaction which involves, separately and/or together, the two particle bunches entertained in the starting and arrival sites. We observe that this interaction is different from the one depending on the coordinates of the sites involved in the transition which one takes into account by means of the functions T (cross section) in (1) or by W (transition rate) in (3). In order to explain the nature of the interaction introduced by the function $\gamma(f, f')$ we start by considering the case

$$\gamma(f, f') = f(1 - f') . \tag{7}$$

It is well known [20,36] that this particular expression for the $\gamma(f, f')$ given by (7) takes into account the Pauli exclusion principle and defines completely the fermion kinetics. Other expressions of the function $\gamma(f, f')$ take into account interactions introduced by the generalized exclusion–inclusion principle [33], the Haldane generalized exclusion principle [16,31], the Tsallis principle underlying the non-extensive statistics [5,21] etc. We observe that the above-mentioned principles impose the form of the collisional integral in the kinetic equations through the choice of $\gamma(f, f')$. It is worth noting that in the cases of Haldane statistics the particular expression of $\gamma(f, f')$ is originated from the fractal structure of the single particle Hilbert space, its dimension depending on the particle number in the considered state [16,31]. Also the Tsallis statistics is originated from the fractal structure of the relevant particle phase space [5].

Taking into account that particular choices of $\gamma(f, f')$ reproduce the already known principles above mentioned, we can see the function $\gamma(f, f')$ as describing a general principle which we call *kinetical interaction principle* (KIP). The KIP defines a special collective interaction which could be very useful to describe the dynamics of many-body systems. As we will see in the following sections, the KIP both governs the system evolving toward the equilibrium and imposes the stationary state of the system.

3. Kramers generalized kinetics

In the following we study the particle kinetics in a $2n$-dimensional phase space of a dilute system composed by N identical particles interacting with an equilibrated bath. The procedure which we use in the present section to derive the evolution equation of the system, is a generalization to the non-linear case of the standard procedure, involving the Kramers–Moyal expansion and the first neighbor approximation, which was introduced firstly to study the linear kinetics. We indicate with $\boldsymbol{x}$ and $\boldsymbol{v}$ the position and the velocity variables, respectively. The particles evolve under an external potential $V=V(\boldsymbol{x})$. The evolution equation for the distribution function $f=f(t,\boldsymbol{x},\boldsymbol{v})$ is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathcal{R}} [\pi(t,\boldsymbol{x},\boldsymbol{v}' \to \boldsymbol{v}) - \pi(t,\boldsymbol{x},\boldsymbol{v} \to \boldsymbol{v}')]\, \mathrm{d}^n v' \,, \tag{8}$$

where $\mathrm{d}f/\mathrm{d}t$ is the total time derivative while the transition probability according to the KIP is given by

$$\pi(t,\boldsymbol{x},\boldsymbol{v} \to \boldsymbol{v}') = W(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{v}')\gamma(f,f') \,. \tag{9}$$

Let us write the transition rate as $W(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{v}')=w(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{v}'-\boldsymbol{v})$, where the last argument in w represents the change of the velocity during the transition. In the following, for simplicity, we indicate explicitly only the dependence on the velocity variables of the functions $w(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{v}'-\boldsymbol{v})$ and $f(t,\boldsymbol{x},\boldsymbol{v})$. We start by rewriting Eq. (8) as follows:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathcal{R}} w(\boldsymbol{v}+\boldsymbol{y},\boldsymbol{y})\gamma[f(\boldsymbol{v}+\boldsymbol{y}),f(\boldsymbol{v})]\, \mathrm{d}^n y - \int_{\mathcal{R}} w(\boldsymbol{v},\boldsymbol{y})\gamma[f(\boldsymbol{v}),f(\boldsymbol{v}-\boldsymbol{y})]\, \mathrm{d}^n y \,. \tag{10}$$

For physical systems evolving very slowly $w(\boldsymbol{v},\boldsymbol{y})$ decreases very expeditiously as $\boldsymbol{y}$ increases and we can consider only the transitions for which $\boldsymbol{v} \pm \boldsymbol{y} \approx \boldsymbol{v}$. At this point we make use of the two following Taylor expansions:

$$w(\boldsymbol{v}+\boldsymbol{y},\boldsymbol{y})\gamma[f(\boldsymbol{v}+\boldsymbol{y}),f(\boldsymbol{v})] = \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\partial^m \{w(\boldsymbol{u},\boldsymbol{y})\gamma[f(\boldsymbol{u}),f(\boldsymbol{v})]\}}{\partial u_{\alpha_1} \partial u_{\alpha_2} \ldots \partial u_{\alpha_m}} \right]_{\boldsymbol{u}=\boldsymbol{v}} y_{\alpha_1} y_{\alpha_2} \cdots y_{\alpha_m} \,,$$

$$\gamma[f(\boldsymbol{v}),f(\boldsymbol{v}-\boldsymbol{y})] = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left[\frac{\partial^m \gamma[f(\boldsymbol{v}),f(\boldsymbol{u})]}{\partial u_{\alpha_1} \partial u_{\alpha_2} \ldots \partial u_{\alpha_m}} \right]_{\boldsymbol{u}=\boldsymbol{v}} y_{\alpha_1} y_{\alpha_2} \cdots y_{\alpha_m}$$

and after substitution in (10) we obtain the following Kramers–Moyal expansion:

$$\frac{\mathrm{d}f(t,\boldsymbol{x},\boldsymbol{v})}{\mathrm{d}t} = \sum_{m=1}^{\infty} \left[\frac{\partial^m \{\zeta_{\alpha_1\alpha_2\ldots\alpha_m}(t,\boldsymbol{x},\boldsymbol{u})\gamma[f(t,\boldsymbol{x},\boldsymbol{u}),f(t,\boldsymbol{x},\boldsymbol{v})]\}}{\partial u_{\alpha_1} \partial u_{\alpha_2} \ldots \partial u_{\alpha_m}} + (-1)^{m-1} \zeta_{\alpha_1\alpha_2\ldots\alpha_m}(t,\boldsymbol{x},\boldsymbol{v}) \frac{\partial^m \gamma[f(t,\boldsymbol{x},\boldsymbol{v}),f(t,\boldsymbol{x},\boldsymbol{u})]}{\partial u_{\alpha_1} \partial u_{\alpha_2} \ldots \partial u_{\alpha_m}} \right]_{\boldsymbol{u}=\boldsymbol{v}} \,, \tag{11}$$

where the mth-order momentum $\zeta_{\alpha_1\alpha_2\ldots\alpha_m}(t,\boldsymbol{x},\boldsymbol{v})$ of the transition rate is defined as

$$\zeta_{\alpha_1\alpha_2\ldots\alpha_m}(t,\boldsymbol{x},\boldsymbol{v}) = \frac{1}{m!} \int_{\mathcal{R}} y_{\alpha_1} y_{\alpha_2} \ldots y_{\alpha_m} w(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{y})\, \mathrm{d}^n y. \tag{12}$$

We remark that from Eq. (11), where the dependence on all the variables $t,\boldsymbol{x},\boldsymbol{v}$ is indicated explicitly, we can obtain as a particular case Eq. (7) of Ref. [33].

In the frame of the first neighbor approximation only the first-order (drift coefficient) ζ_i and the second-order (diffusion coefficient) ζ_{ij} momenta of the transition rate are considered. Indicating again explicitly only the dependence on the velocity variables Eq. (11) reduces to the following non-linear second-order partial differential equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial\{\zeta_i(\boldsymbol{u})\gamma[f(\boldsymbol{u}), f(\boldsymbol{v})]\}}{\partial u_i} + \zeta_i(\boldsymbol{v})\frac{\partial\gamma[f(\boldsymbol{v}), f(\boldsymbol{u})]}{\partial u_i}\right.$$
$$\left. + \frac{\partial^2\{\zeta_{ij}(\boldsymbol{u})\gamma[f(\boldsymbol{u}), f(\boldsymbol{v})]\}}{\partial u_i \partial u_j} - \zeta_{ij}(\boldsymbol{v})\frac{\partial^2\gamma[f(\boldsymbol{v}), f(\boldsymbol{u})]}{\partial u_i \partial u_j}\right]_{\boldsymbol{u}=\boldsymbol{v}}, \tag{13}$$

which after taking into account the two identities

$$\left[\frac{\partial\{\zeta_i(\boldsymbol{u})\gamma[f(\boldsymbol{u}), f(\boldsymbol{v})]\}}{\partial u_i} + \zeta_i(\boldsymbol{v})\frac{\partial\gamma[f(\boldsymbol{v}), f(\boldsymbol{u})]}{\partial u_i}\right]_{\boldsymbol{u}=\boldsymbol{v}}$$
$$= \frac{\partial}{\partial v_i}\{\zeta_i(\boldsymbol{v})\gamma[f(\boldsymbol{v}), f(\boldsymbol{v})]\}$$

and

$$\left[\frac{\partial^2\{\zeta_{ij}(\boldsymbol{u})\gamma[f(\boldsymbol{u}), f(\boldsymbol{v})]\}}{\partial u_i \partial u_j} - \zeta_{ij}(\boldsymbol{v})\frac{\partial^2\gamma[f(\boldsymbol{v}), f(\boldsymbol{u})]}{\partial u_i \partial u_j}\right]_{\boldsymbol{u}=\boldsymbol{v}}$$
$$= \frac{\partial}{\partial v_i}\left\{\frac{\partial\zeta_{ij}(\boldsymbol{v})}{\partial v_j}\gamma[f(\boldsymbol{v}), f(\boldsymbol{v})]\right.$$
$$\left. + \zeta_{ij}(\boldsymbol{v})\left[\frac{\partial\gamma[f(\boldsymbol{u}), f(\boldsymbol{v})]}{\partial u_j} - \frac{\partial\gamma[f(\boldsymbol{v}), f(\boldsymbol{u})]}{\partial u_j}\right]_{\boldsymbol{u}=\boldsymbol{v}}\right\},$$

assumes the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial v_i}\left\{\left[\zeta_i(\boldsymbol{v}) + \frac{\partial\zeta_{ij}(\boldsymbol{v})}{\partial v_j}\right]\gamma[f(\boldsymbol{v}), f(\boldsymbol{v})]\right.$$
$$\left. + \zeta_{ij}(\boldsymbol{v})\left[\frac{\partial\gamma[f(\boldsymbol{u}), f(\boldsymbol{v})]}{\partial u_j} - \frac{\partial\gamma[f(\boldsymbol{v}), f(\boldsymbol{u})]}{\partial u_j}\right]_{\boldsymbol{u}=\boldsymbol{v}}\right\}. \tag{14}$$

Finally Eq. (14) can be rewritten as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial v_i}\left[\left(\zeta_i + \frac{\partial\zeta_{ij}}{\partial v_j}\right)\gamma(f) + \zeta_{ij}\gamma(f)\lambda(f)\frac{\partial f}{\partial v_j}\right] \tag{15}$$

with $\gamma(f) = \gamma(f, f)$ and

$$\lambda(f) = \left[\frac{\partial}{\partial f}\ln\frac{\gamma(f, f')}{\gamma(f', f)}\right]_{f'=f}.$$

By taking into account condition (4), the function $\lambda(f)$ simplifies as

$$\lambda(f) = \frac{\partial \ln \kappa(f)}{\partial f} \tag{16}$$

and Eq. (15) becomes

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial v_i}\left[\left(\zeta_i + \frac{\partial\zeta_{ij}}{\partial v_j}\right)\gamma(f) + \gamma(f)\frac{\partial \ln\kappa(f)}{\partial f}\zeta_{ij}\frac{\partial f}{\partial v_j}\right]. \tag{17}$$

We assume the independence of motion among the n directions of the homogeneous and isotropic n-dimensional velocity space and pose $\zeta_i = J_i$, $\zeta_{ij} = D\delta_{ij}$, being $\boldsymbol{J}$ and D the drift and diffusion coefficients, respectively. Moreover, we introduce the function U by means of

$$\beta \frac{\partial U}{\partial \boldsymbol{v}} = \frac{1}{D}\left(\boldsymbol{J} + \frac{\partial D}{\partial \boldsymbol{v}}\right) \tag{18}$$

with β a constant. In the following, we will consider the case where $U = U(\boldsymbol{v})$ depends exclusively on the velocity. Taking into account that the potential $V = V(\boldsymbol{x})$ depends only on the spatial variable, Eq. (17) can be written as

$$\frac{\mathrm{d}f(t,\boldsymbol{x},\boldsymbol{v})}{\mathrm{d}t} = \frac{\partial}{\partial \boldsymbol{v}}\left(D(\boldsymbol{v})\gamma(f)\frac{\partial}{\partial \boldsymbol{v}}\left\{\beta\left[V(\boldsymbol{x}) + U(\boldsymbol{v}) - \mu\right] + \ln \kappa(f)\right\}\right) \tag{19}$$

with μ a constant and the expression of the total time derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial U(\boldsymbol{v})}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{x}} - \frac{1}{m}\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{v}}\,.$$

Eq. (19) represents the evolution equation of the particle system in the Kramers picture and describe a non-linear kinetics. This non-linear evolution equation can be written in the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\partial}{\partial \boldsymbol{v}}\left[D\gamma(f)\frac{\partial}{\partial \boldsymbol{v}}\frac{\delta \mathscr{K}}{\delta f}\right] = 0\,, \tag{20}$$

where $\delta\mathscr{K}/\delta f$ is the functional derivatives of the functional $\mathscr{K}$ defined through

$$\mathscr{K} = -\int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v \int \mathrm{d}f \ln \frac{\kappa(f)}{\kappa(f_s)}\,, \tag{21}$$

where the stationary distribution $f_s = f(\infty, \boldsymbol{x}, \boldsymbol{v})$ is defined through

$$\ln \kappa(f_s) = -\beta[V(\boldsymbol{x}) + U(\boldsymbol{v}) - \mu]\,. \tag{22}$$

We remark now that from Eq. (20) it follows immediately that the stationary distribution maximizes $\mathscr{K}$ and can be obtained from a variational principle

$$\frac{\delta\mathscr{K}}{\delta f} = 0 \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = 0; \quad f = f_s\,.$$

It is easy to verify that the functional $\mathscr{K}$ increases in time

$$\begin{aligned}\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} &= \int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v \frac{\delta\mathscr{K}}{\delta f}\frac{\mathrm{d}f}{\mathrm{d}t} \\ &= -\int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v \frac{\delta\mathscr{K}}{\delta f}\frac{\partial}{\partial \boldsymbol{v}}\left[D\gamma(f)\frac{\partial}{\partial \boldsymbol{v}}\frac{\delta\mathscr{K}}{\delta f}\right] \\ &= \int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v D\gamma(f)\left(\frac{\partial}{\partial \boldsymbol{v}}\frac{\delta\mathscr{K}}{\delta f}\right)^2 \geqslant 0\,.\end{aligned} \tag{23}$$

In order to study the behavior of the functional $\mathscr{K}(t)$ when $t \to \infty$ we introduce the function $\sigma(f) = -\int \mathrm{d}f \ln \kappa(f)$ so that $\kappa(f)$ can be written as $\kappa(f) = \exp[-\mathrm{d}\sigma/\mathrm{d}f]$. Now we are able to calculate, in the limit $t \to \infty$, the following difference:

$$\mathscr{K}(t) - \mathscr{K}(\infty) = \int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v[\sigma(f) - \sigma(f_s) + (f - f_s)\ln \kappa(f_s)]$$

$$= \int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v \left[\sigma(f) - \sigma(f_s) - (f - f_s)\frac{\mathrm{d}\sigma(f_s)}{\mathrm{d}f_s}\right]$$

$$\approx \int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v \left[\frac{1}{2}\frac{\mathrm{d}^2\sigma(f_s)}{\mathrm{d}f_s^2}(f - f_s)^2\right] \quad (24)$$

and to assume that $\mathrm{d}^2\sigma(f)/\mathrm{d}f^2 \leqslant 0$. This requirement is satisfied if the function $\kappa(f)$ obeys to the condition $\mathrm{d}\kappa(f)/\mathrm{d}f \geqslant 0$ and consequently we have $\mathscr{K}(t) \leqslant \mathscr{K}(\infty)$, then $\mathscr{K}$ assumes its maximum value for $t=\infty$. The inequalities $\mathrm{d}\mathscr{K}(t)/\mathrm{d}t \geqslant 0$ and $\mathscr{K}(t) \leqslant \mathscr{K}(\infty)$ imply that $-\mathscr{K}$ is a Lyapunov functional and demonstrate the H-theorem. The functional $\mathscr{K}$ is the constrained entropy of the system and results to be the sum of two terms: $\mathscr{K} = S + S_c$ where

$$S = -\int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v \int \mathrm{d}f \ln \kappa(f)\,, \quad (25)$$

is the entropy of the system and $S_c = -\beta(E - \mu N)$. The energy E of the system is given by

$$E = \int_{\mathscr{R}} \mathrm{d}^n x\, \mathrm{d}^n v [V(\boldsymbol{x}) + U(\boldsymbol{v})] f\,. \quad (26)$$

We remark that, being $\kappa(f)$ an arbitrary function, the H-theorem has been verified in a unified way, for a very large class of non-linear systems interacting with a bath.

4. Boltzmann generalized kinetics

In the diffusive approximation, adopted in the previous section to describe the changes of the particle states, the system is coupled with its environment. In this frame, meanwhile the particle system, interacting with the bath, evolves toward the equilibrium, both its entropy S and $\mathscr{K}$ increases monotonicaly. In the present section we will consider the particle system isolated from its environment and describe its time evolution in the frame of the more rigorous Boltzmann picture. In presence of external forces derived from a potential the total time derivative is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\frac{\partial}{\partial \boldsymbol{x}} - \frac{1}{m}\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{v}}$$

and the evolution equation assumes the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathscr{R}} \mathrm{d}^n v'\, \mathrm{d}^n v_1\, \mathrm{d}^n v_1' [\pi(t, \boldsymbol{x}, \boldsymbol{v}' \to \boldsymbol{v}, \boldsymbol{v}_1' \to \boldsymbol{v}_1) - \pi(t, \boldsymbol{x}, \boldsymbol{v} \to \boldsymbol{v}', \boldsymbol{v}_1 \to \boldsymbol{v}_1')]\,. \quad (27)$$

Eq. (27) describes a non-linear generalized kinetics, the transition probabilities being defined, taking into account the KIP, as

$$\pi(t, \boldsymbol{x}, \boldsymbol{v} \to \boldsymbol{v}', \boldsymbol{v}_1 \to \boldsymbol{v}_1') = T(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{v}', \boldsymbol{v}_1, \boldsymbol{v}_1')\, \gamma(f, f')\gamma(f_1, f_1')\,. \quad (28)$$

In (28) we have posed $f = f(t, \boldsymbol{x}, \boldsymbol{v})$, $f' = f(t, \boldsymbol{x}, \boldsymbol{v}')$ and analogously $f_1 = f(t, \boldsymbol{x}, \boldsymbol{v}_1)$, $f_1' = f(t, \boldsymbol{x}, \boldsymbol{v}_1')$ in order to consider point-like binary collisions. A symmetry is imposed to $T = T(t, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{v}', \boldsymbol{v}_1, \boldsymbol{v}_1')$ by the principle of detailed balance

$T(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}_1,\boldsymbol{v}_1') = T(t,\boldsymbol{x},\boldsymbol{v}',\boldsymbol{v},\boldsymbol{v}_1',\boldsymbol{v}_1)$, so that Eq. (27) assumes the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathscr{R}} \mathrm{d}^n v'\,\mathrm{d}^n v_1\,\mathrm{d}^n v_1'\,T(t,\boldsymbol{x},\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}_1,\boldsymbol{v}_1') \times[\gamma(f',f)\gamma(f_1',f_1) - \gamma(f,f')\gamma(f_1,f_1')] . \tag{29}$$

Alternatively, by taking into account (6), one can write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathscr{R}} \mathrm{d}^n v'\,\mathrm{d}^n v_1\,\mathrm{d}^n v_1'\,Tc(f,f')c(f_1,f_1') \times[a(f')b(f)a(f_1')b(f_1) - a(f)b(f')a(f_1)b(f_1')] . \tag{30}$$

We note that when $c(f,f') = c(f_1,f_1') = 1$, Eq. (30) reduces to the equation recently considered in Ref. [34]. In the following, we will describe the system using Eq. (29) which can be rewritten as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathscr{R}} \mathrm{d}^n v'\,\mathrm{d}^n v_1\,\mathrm{d}^n v_1'\,T\mathscr{Q}\gamma(f',f)\gamma(f_1',f_1) , \tag{31}$$

where the auxiliary function $\mathscr{Q} = \mathscr{Q}(f,f',f_1,f_1') \leqslant 1$ is defined as

$$\mathscr{Q} = 1 - \frac{\gamma(f,f')\gamma(f_1,f_1')}{\gamma(f',f)\gamma(f_1',f_1)} .$$

Taking into account the condition (4), the function $\mathscr{Q}$ becomes $\mathscr{Q} = 1 - [\kappa(f)\kappa(f_1)]/[\kappa(f')\kappa(f_1')]$ and can be written immediately in the following form:

$$\mathscr{Q} = 1 - \exp[\ln\kappa(f) + \ln\kappa(f_1) - \ln\kappa(f') - \ln\kappa(f_1')] .$$

We consider now the system at the equilibrium. From the evolution equation (31) we have $\mathrm{d}f/\mathrm{d}t = 0$, and $\mathscr{Q} = 0$. The first condition $\mathrm{d}f/\mathrm{d}t = 0$ taking into account the definition of the total time derivative, implies for the stationary distribution $f = f_s$ that $f_s = f_s[m\boldsymbol{v}^2/2 + V(\boldsymbol{x})]$. On the other hand, from the condition $\mathscr{Q} = 0$ we have

$$\ln\kappa(f_s) + \ln\kappa(f_{s1}) - \ln\kappa(f_s') - \ln\kappa(f_{s1}') = 0 .$$

This last equation allows us to conclude that the quantity $\ln\kappa(f_s)$ is a collisional invariant for the particle system. If we suppose that the binary interparticle collisions conserve the particle number and the kinetic energy

$$\tfrac{1}{2}m\boldsymbol{v}^2 + \tfrac{1}{2}m\boldsymbol{v}_1^2 = \tfrac{1}{2}m\boldsymbol{v}'^2 + \tfrac{1}{2}m\boldsymbol{v}_1'^2 ,$$

we have that the quantity $-\beta[\frac{1}{2}m\boldsymbol{v}^2 + V(\boldsymbol{x}) - \mu]$ is the more general collisional invariant. Then, we obtain the condition

$$\ln\kappa(f_s) = -\beta\left[\tfrac{1}{2}m\boldsymbol{v}^2 + V(\boldsymbol{x}) - \mu\right] , \tag{32}$$

which defines the stationary distribution, so that $\mathscr{Q}$ becomes

$$\mathscr{Q} = 1 - \exp\left[\ln\frac{\kappa(f)}{\kappa(f_s)} + \ln\frac{\kappa(f_1)}{\kappa(f_{s1})} - \ln\frac{\kappa(f')}{\kappa(f_s')} - \ln\frac{\kappa(f_1')}{\kappa(f_{s1}')}\right] .$$

After introducing the functional $\mathscr{K}$ by means of (21) with f_s is given by (32) the quantity $\mathscr{Q}$ can be written in the form

$$\mathscr{Q} = 1 - \exp\left(-\frac{\delta\mathscr{K}[f]}{\delta f} - \frac{\delta\mathscr{K}[f_1]}{\delta f_1} + \frac{\delta\mathscr{K}[f']}{\delta f'} + \frac{\delta\mathscr{K}[f_1']}{\delta f_1'}\right) .$$

Finally, the evolution equation (31) assumes the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \int_{\mathcal{R}} \mathrm{d}^n v' \, \mathrm{d}^n v_1 \, \mathrm{d}^n v_1' T \gamma(f', f)\gamma(f_1', f_1) \times \left\{ 1 - \exp\left(-\frac{\delta \mathcal{K}[f]}{\delta f} - \frac{\delta \mathcal{K}[f_1]}{\delta f_1} + \frac{\delta \mathcal{K}[f']}{\delta f'} + \frac{\delta \mathcal{K}[f_1']}{\delta f_1'} \right) \right\} . \quad (33)$$

From the structure of (33) we have that the stationary distribution $f_s = f(\infty, \boldsymbol{x}, \boldsymbol{v})$ can be obtained from a variational principle

$$\frac{\delta \mathcal{K}}{\delta f} = 0 \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = 0, \quad f = f_s .$$

We study now how the functional $\mathcal{K}$ evolves in time. Its time derivative is given by

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -\int_{\mathcal{R}} \mathrm{d}^n x \, \mathrm{d}^n v \ln \frac{\kappa(f)}{\kappa(f_s)} \frac{\mathrm{d}f}{\mathrm{d}t} = -\int_{\mathcal{R}} \mathrm{d}^n x \, \mathrm{d}^n v \, \mathrm{d}^n v' \, \mathrm{d}^n v_1 \, \mathrm{d}^n v_1' T \ln \frac{\kappa(f)}{\kappa(f_s)} \times [\gamma(f', f)\gamma(f_1', f_1) - \gamma(f, f')\gamma(f_1, f_1')] . \quad (34)$$

The symmetry of the integrand function permit us to write (34) as it follows

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -\int_{\mathcal{R}} \mathrm{d}^n x \, \mathrm{d}^n v \, \mathrm{d}^n v' \, \mathrm{d}^n v_1 \, \mathrm{d}^n v_1' \frac{1}{4} T \times \left[\ln \frac{\kappa(f)}{\kappa(f_s)} + \ln \frac{\kappa(f_1)}{\kappa(f_{s1})} - \ln \frac{\kappa(f')}{\kappa(f_s')} - \ln \frac{\kappa(f_1')}{\kappa(f_{s1}')} \right] \times [\gamma(f', f)\gamma(f_1', f_1) - \gamma(f, f')\gamma(f_1, f_1')] . \quad (35)$$

After taking into account the expressions of $\mathcal{Q}$ we have

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \int_{\mathcal{R}} \mathrm{d}^n x \, \mathrm{d}^n v \, \mathrm{d}^n v' \, \mathrm{d}^n v_1 \, \mathrm{d}^n v_1' \frac{1}{4} T \times [-\mathcal{Q} \ln(1 - \mathcal{Q})]\gamma(f', f) \, \gamma(f_1', f_1) . \quad (36)$$

We observe now that $\mathcal{Q} \leqslant 1$ and then we have $-\mathcal{Q}\ln(1 - \mathcal{Q}) \geqslant 0$. This implies that the integrand function in (36) is a non-negative function. Then we conclude that $\mathrm{d}\mathcal{K}/\mathrm{d}t \geqslant 0$. Starting from the definition of $\mathcal{K}$ and following the procedure adopted in Section 3 we obtain $\mathcal{K}(t) \leqslant \mathcal{K}(\infty)$. We can conclude at this point that $-\mathcal{K}$ is a Lyapunov functional. It is easy to verify that

$$\mathcal{K} = S - \beta(E - \mu N) , \quad (37)$$

where the entropy of the system S is defined through (25) while its energy E is given by

$$E = \int_{\mathcal{R}} \mathrm{d}^n x \, \mathrm{d}^n v \left[\frac{1}{2} m \boldsymbol{v}^2 + V(\boldsymbol{x}) \right] f , \quad (38)$$

which is a conserved quantity, $\mathrm{d}E/\mathrm{d}t = 0$, as the particle number N. Consequently (37) can be written as

$$\mathcal{K}(t) = S(t) + \text{constant} . \quad (39)$$

The H-theorem for the isolated non-linear system follows immediately:

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geqslant 0; \quad S(t) \leqslant S(\infty) . \quad (40)$$

5. Some known statistics

In this section, we will show that the formalism previously developed permit us to consider, in a unitary way, the already known statistical distributions. For simplicity, we will discuss the case of distributions depending exclusively on the velocity. Firstly, we observe that the stationary distribution f, defined through $\kappa(f) = \exp(-\varepsilon)$ with $\varepsilon = \beta(mv^2/2 - \mu)$, can be obtained as steady state of a Fokker–Planck (FP) equation describing the kinetics of brownian particles for which results $U = mv^2/2$ and $D = const$. The same distribution can be viewed as steady state of a Boltzmann equation, describing free particles interacting by means of binary collisions, conserving the particle number, momentum and energy. In this section, we will write the evolution equations (Fokker–Planck and/or Boltzmann) of some distributions available in literature to illustrate the relevance of the approach adopted in the previous sections describing the non-linear particle kinetics.

Maxwell–Boltzmann statistics: We start by considering the MB statistics given by $f = \exp(-\varepsilon)$. It is readily seen that the related kinetics is defined starting from $a(f) = f$, $b(f') = 1$, while the symmetric function $c(f, f')$ remains arbitrary. Then we have $\kappa(f) = f$ and $\gamma(f) = fc(f)$. In the Boltzmann picture the evolution equation becomes

$$\frac{\partial f}{\partial t} = \int_{\mathscr{R}} \mathrm{d}^n v' \, \mathrm{d}^n v_1 \, \mathrm{d}^n v_1' \, T c(f, f') c(f_1, f_1')(f' f_1' - f f_1) \,, \tag{41}$$

while, in the FP picture we have

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \left[D c(f) \left(\beta m \boldsymbol{v} f + \frac{\partial f}{\partial \boldsymbol{v}} \right) \right] \,. \tag{42}$$

In the simplest case $c(f, f') = 1$ we obtain the standard linear Boltzmann and Fokker–Planck equations. We observe that there is an infinity of ways (one for any choice of $c(f, f')$) to obtain the MB distribution.

Bosonic and fermionic statistics: We consider now the case of quantum statistics namely the Fermi–Dirac ($\eta = -1$) and Bose–Einstein ($\eta = 1$) statistics defined by means of $f = (\exp\varepsilon - \eta)^{-1}$. The kinetics now is defined through $a(f) = f$ and $b(f') = 1 + \eta f'$ while again the function $c(f, f')$ remains arbitrary. We have consequently $\kappa(f) = f/(1 + \eta f)$. In the Boltzmann picture the evolution equation becomes

$$\frac{\partial f}{\partial t} = \int_{\mathscr{R}} \mathrm{d}^n v' \, \mathrm{d}^n v_1 \, \mathrm{d}^n v_1' \, T c(f, f') c(f_1, f_1') \times [f'(1 + \eta f) f_1'(1 + \eta f_1) - f(1 + \eta f') f_1 (1 + \eta f_1')] \tag{43}$$

and reduces to the well-known Uehling–Uhlenbeck equation if we choose $c(f, f') = 1$. This choice for $c(f, f')$, in the frame of the FP picture, implies $\gamma(f) = f(1 + \eta f)$ and we obtain the following evolution equation [33]:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \left[D \beta m \boldsymbol{v} f (1 + \eta f) + D \frac{\partial f}{\partial \boldsymbol{v}} \right] \,. \tag{44}$$

In order to show that a kinetics, different from the one described by (44), also reproducing the bosonic and fermionic statistics, exists, we consider $c(f, f') = (1 + \eta \sqrt{f f'})^{-1}$

or alternatively $c(f,f') = [1 + \eta(f + f')/2]^{-1}$. It is easy to verify that in both the cases we have $c(f) = (1 + \eta f)^{-1}$ and $\gamma(f) = f$. Now the evolution equation in the FP picture becomes [11]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\left[D\beta m\boldsymbol{v} f + \frac{D}{\eta}\frac{\partial}{\partial \boldsymbol{v}}\ln(1+\eta f)\right] . \tag{45}$$

Intermediate statistics: The quantum statistics interpolating between the bosonic and fermionic statistics has captured the attention of many researchers in the last few years. A first example of intermediate statistics can be realized by considering in the distribution $f = Z^{-1}(\exp\varepsilon - \eta)^{-1}$, previously examined, the parameter η as being continuous: $0 \leqslant \eta \leqslant 1$. For $\eta \neq \pm 1$ we have a quantum statistics different from the Bose or Fermi statistics. A second intermediate statistics is the boson-like (+) or fermion-like (−) quon statistics [32], which can be obtained easily by posing $a(f) = [f]_q$ and $b(f') = [1 \pm f']_q$, where $[x]_q = (q^x - q^{-x})/2\ln q$ and $q \in \mathbf{R}$. If we choose for simplicity $c(f) = c_q = 2\ln q/(q - q^{-1})$, the evolution equation in the FP picture becomes [32]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\left(c_q D\beta m\boldsymbol{v}[f]_q[1 \pm f]_q + D\frac{\partial f}{\partial \boldsymbol{v}}\right) . \tag{46}$$

A third intermediate statistics is the Haldane–Wu exclusion statistics which can be obtained starting from the kinetics defined by setting $a(f) = f$ and $b(f') = (1 - gf')^g[1 + (1-g)f']^{1-g}$ with $0 \leqslant g \leqslant 1$ [31].

Tsallis statistics: We consider the non-extensive thermostatistics introduced by Tsallis [5]. The relevant distribution $f = Z^{-1}[1 - (1-q)\varepsilon]^{1/(1-q)}$ can be obtained naturally starting from the kinetics defined through $\ln\kappa(f) = (f^{1-q} - 1)/(1-q) \equiv \ln_q f$. The Boltzmann equation (30) becomes now

$$\frac{\partial f}{\partial t} = \int_{\mathscr{R}} d^n v' d^n v_1 d^n v_1'\, T\, c(f,f')c(f_1,f_1')b(f)b(f_1)b(f')b(f_1') \times \left[\exp(\ln_q f' + \ln_q f_1') - \exp(\ln_q f + \ln_q f_1)\right] . \tag{46}$$

In the FP picture the evolution equation is given by

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\left[D\gamma(f)\left(\beta m\boldsymbol{v} + f^{-q}\frac{\partial f}{\partial \boldsymbol{v}}\right)\right] , \tag{47}$$

which for $\gamma(f) = f$ reduces to the one proposed in Ref. [21].

6. The κ-deformed analysis

In the previous section, we have considered some statistical distributions, quantum or classical, already known in the literature, depending on one continuous parameter. We will now turn our attention to the distribution $f = Z^{-1}(\exp\varepsilon - \eta)^{-1}$. We note that this quantum distribution can be viewed as a deformation of the MB one, which can be recovered as the deformation parameter η approaches to zero. Another classical distribution which can be obtained by deforming the MB one, is the Tsallis distribution

$f = Z^{-1}[1-(1-q)\varepsilon]^{1/(1-q)}$. The MB distribution emerges again as the deformation parameter $q \to 1$.

In the present section we will study the main mathematical properties of a new, one parameter, deformed exponential function, while in the next section we will consider the induced deformed statistics. The deformed exponential is indicated by $\exp_{\{\kappa\}}(x)$, where κ denote the deformation parameter, and we postulate that it obeys the following condition:

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(-x) = 1 \ . \tag{48}$$

We start by observing that any function $A(x)$ can be written in the form $A(x) = A_e(x) + A_o(x)$ where $A_e(x) = A_e(-x)$ is an even function and $A_o(x) = -A_o(-x)$ an odd one. The condition $A(x)A(-x) = 1$ allows us to express $A_e(x)$ in terms of $A_o(x)$ by means of $A_e(x) = \sqrt{1+A_o(x)^2}$ and consequently write the function $A(x)$ in the form: $A(x) = \sqrt{1+A_o(x)^2} + A_o(x)$. At this point it is obvious that a deformed exponential $\exp_{\{\kappa\}}(x)$ obeying (48) and depending only on one deformation parameter κ, so that $\exp_{\{\kappa\}}(x) \underset{\kappa\to 0}{\sim} \exp x$, can be written as

$$\exp_{\{\kappa\}}(x) = \left[\sqrt{1+g_\kappa(x)^2} + g_\kappa(x)\right]^{1/\kappa} \tag{49}$$

or alternatively as

$$\exp_{\{\kappa\}}(x) = \exp\left(\frac{1}{\kappa}\,\text{arcsinh}\, g_\kappa(x)\right) \ . \tag{50}$$

In (49) the generator $g_\kappa(x)$ of the deformed exponential is an arbitrary function depending on the parameter κ and obeying the conditions

$$g_\kappa(-x) = -g_\kappa(x), \quad g_\kappa(x) \underset{\kappa\to 0}{\sim} \kappa x \ . \tag{51}$$

Since it will be useful later on, we introduce the inverse function of $\exp_{\{\kappa\}}(x)$ indicated by $\ln_{\{\kappa\}}(x)$ and defined through $\exp_{\{\kappa\}}[\ln_{\{\kappa\}}(x)] = x$. It is easy to verify that

$$\ln_{\{\kappa\}}(x) = g_\kappa^{-1}\left(\frac{x^\kappa - x^{-\kappa}}{2}\right) \ , \tag{52}$$

where $g_\kappa^{-1}(x)$ is the inverse function of $g_\kappa(x)$. We remark that the deformed exponential can be defined by fixing the expression of the generator $g_\kappa(x)$. We note that by choosing $g_\kappa(x) = \sinh \kappa x$ we can generate the standard undeformed exponential $\exp_{\{\kappa\}}(x) = \exp x$ and logarithm $\ln_{\{\kappa\}}(x) = \ln x$ functions.

The κ-exponential: In the following we consider the simplest deformed exponential (in following called κ-exponential), which is generated from $g_\kappa(x) = \kappa x$ and is given by

$$\exp_{\{\kappa\}}(x) = \left(\sqrt{1+\kappa^2 x^2} + \kappa x\right)^{1/\kappa} \tag{53}$$

or equivalently by

$$\exp_{\{\kappa\}}(x) = \exp\left(\frac{1}{\kappa}\text{arcsinh}\,\kappa x\right) \ . \tag{54}$$

Obviously, we have $\exp_{\{0\}}(x) = \exp x$ and for $\forall x \in \mathbf{R}$ it results $\exp_{\{\kappa\}}(x) \in \mathbf{R}^+$. Furthermore we have $\exp_{\{\kappa\}}(0) = 1$ and $\exp_{\{-\kappa\}}(x) = \exp_{\{\kappa\}}(x)$, so that we can consider simply that $\kappa \in \mathbf{R}^+$. A relevant property of $\exp_{\{\kappa\}}(x)$ is that for $\forall a \in \mathbf{R}$

$$\exp_{\{\kappa\}}(ax) = [\exp_{\{a\kappa\}}(x)]^a \ . \tag{55}$$

Concerning the asymptotic behavior of κ-exponential we easily obtain that $\exp_{\{\kappa\}}(x) \underset{x\to+\infty}{\sim} |2\kappa x|^{1/|\kappa|}$ and $\exp_{\{\kappa\}}(x) \underset{x\to-\infty}{\sim} |2\kappa x|^{-1/|\kappa|}$. We observe that the deformed exponential is an increasing function $\mathrm{d}\exp_{\{\kappa\}}(x)/\mathrm{d}x > 0$, $\forall \kappa \in \mathbf{R}$. Finally, its concavity is $\mathrm{d}^2\exp_{\{\kappa\}}(x)/\mathrm{d}x^2 > 0$ for $|\kappa| \leqslant 1$ while for $|\kappa| > 1$ we obtain $\mathrm{d}^2\exp_{\{\kappa\}}(x)/\mathrm{d}x^2 > 0$ for $x < x_c$ and $\mathrm{d}^2\exp_{\{\kappa\}}(x)/\mathrm{d}x^2 < 0$ for $x > x_c$ being $x_c = (\kappa^4 - \kappa^2)^{-1/2}$.

The κ-sum: Of course if we take into account that $\operatorname{arcsinh} x + \operatorname{arcsinh} y = \operatorname{arcsinh}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right)$ we arrive immediately at the following relationship:

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(y) = \exp_{\{\kappa\}}(x \overset{\kappa}{\oplus} y) \ , \tag{56}$$

where the κ-deformed sum or simply κ-sum $x \overset{\kappa}{\oplus} y$, is defined through

$$x \overset{\kappa}{\oplus} y = x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2} \ . \tag{57}$$

We note that the κ-sum obeys the associative law $(x \overset{\kappa}{\oplus} y) \overset{\kappa}{\oplus} z = x \overset{\kappa}{\oplus} (y \overset{\kappa}{\oplus} z)$, admits a neutral element $x \overset{\kappa}{\oplus} 0 = 0 \overset{\kappa}{\oplus} x = x$ and for any x exists its opposite $x \overset{\kappa}{\oplus} (-x) = 0$. Moreover the commutativity property $x \overset{\kappa}{\oplus} y = y \overset{\kappa}{\oplus} x$ holds, so that the real numbers constitute an abelian group with respect to the κ-sum. Since it will be useful later on, we define the κ-difference as: $x \overset{\kappa}{\ominus} y = x \overset{\kappa}{\oplus} (-y)$.

The κ-trigonometry: Firstly we define the κ-deformed hyperbolic sine and cosine, starting from $\exp_{\{\kappa\}}(\pm x) = \cosh_{\{\kappa\}}(x) \pm \sinh_{\{\kappa\}}(x)$ which is the κ-Euler formula and obtain

$$\sinh_{\{\kappa\}}(x) = \frac{1}{2}\left[\exp_{\{\kappa\}}(x) - \exp_{\{\kappa\}}(-x)\right] \ ,$$

$$\cosh_{\{\kappa\}}(x) = \frac{1}{2}\left[\exp_{\{\kappa\}}(x) + \exp_{\{\kappa\}}(-x)\right] \ .$$

It is straightforward to introduce the κ-hyperbolic trigonometry which reduces to the ordinary hyperbolic trigonometry as $\kappa \to 0$. For instance, the formulas

$$\tanh_{\{\kappa\}}(x) = \frac{\sinh_{\{\kappa\}}(x)}{\cosh_{\{\kappa\}}(x)} \ ,$$

$$\cosh^2_{\{\kappa\}}(x) - \sinh^2_{\{\kappa\}}(x) = 1 \ ,$$

$$\sinh_{\{\kappa\}}(x \overset{\kappa}{\oplus} y) + \sinh_{\{\kappa\}}(x \overset{\kappa}{\ominus} y) = 2\sinh_{\{\kappa\}}(x)\cosh_{\{\kappa\}}(x) \ ,$$

$$\tanh_{\{\kappa\}}(x) + \tanh_{\{\kappa\}}(y) = \frac{\sinh_{\{\kappa\}}(x \overset{\kappa}{\oplus} y)}{\cosh_{\{\kappa\}}(x)\cosh_{\{\kappa\}}(y)}$$

and so on, still hold true. All the formulas of the ordinary hyperbolic trigonometry still hold true after expediently deformed. The deformation of a given trigonometric formula can be obtained starting from the corresponding undeformed formula, making in the argument of the trigonometric functions the substitution $x+y \to x \overset{\kappa}{\oplus} y$ and obviously $nx \to x \overset{\kappa}{\oplus} x \ldots \overset{\kappa}{\oplus} x$ (n times).

The κ-De Moivre formula involving trigonometric functions having arguments of the type rx with $r \in \mathbf{R}$, assumes the form

$$[\cosh_{\{\kappa\}}(x) \pm \sinh_{\{\kappa\}}(x)]^r = \cosh_{\{\kappa/r\}}(rx) \pm \sinh_{\{\kappa/r\}}(rx)\,.$$

The κ-derivative: It is important to emphasize that we can naturally introduce a κ-calculus which reduces to the usual one as the deformation parameter $\kappa \to 0$. In fact we can define the κ-derivative of course through

$$D_{\{\kappa\}} f(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x_{\{\kappa\}}} = \lim_{y\to x} \frac{f(x)-f(y)}{x \overset{\kappa}{\ominus} y}\,, \tag{58}$$

where the κ-differential $\mathrm{d}x_{\{\kappa\}} = \lim_{y\to x} \overset{\kappa}{\ominus} y$, after taking into account the identity $(x \overset{\kappa}{\oplus} y)(x \overset{\kappa}{\ominus} y) = x^2 - y^2$, is given by

$$\mathrm{d}x_{\{\kappa\}} = \lim_{y\to x} x \overset{\kappa}{\ominus} y = \lim_{y\to x} \frac{x^2-y^2}{x \overset{\kappa}{\oplus} y} = \frac{\mathrm{d}x}{\sqrt{1+\kappa^2x^2}}\,.$$

The function $x_{\{\kappa\}}$ can be easily calculated after integration

$$x_{\{\kappa\}} = \frac{1}{\kappa}\ln(\sqrt{1+\kappa^2x^2}+\kappa x) = \frac{1}{\kappa}\,\mathrm{arcsinh}\,\kappa x \tag{59}$$

and satisfies the relationship $x_{\{\kappa\}} + y_{\{\kappa\}} = (x \overset{\kappa}{\oplus} y)_{\{\kappa\}}$. We observe that it is possible to write the κ-derivative in the form

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x_{\{\kappa\}}} = \sqrt{1+\kappa^2x^2}\,\frac{\mathrm{d}f(x)}{\mathrm{d}x}\,, \tag{60}$$

from which it appears clearly that the κ-calculus is governed by the same rules of the ordinary one. For instance, we can write

$$\frac{\mathrm{d}\exp_{\{\kappa\}}(x)}{\mathrm{d}x_{\{\kappa\}}} = \exp_{\{\kappa\}}(x), \qquad \frac{\mathrm{d}\sinh_{\{\kappa\}}(x)}{\mathrm{d}x_{\{\kappa\}}} = \cosh_{\{\kappa\}}(x)$$

and so on. We observe now that the introduction of the function $x_{\{\kappa\}}$ permit us to write $\exp_{\{\kappa\}}(x) = \exp(x_{\{\kappa\}})$ and analogously for the κ-deformed trigonometric functions. For instance, we have $\sinh_{\{\kappa\}}(x) = \sinh(x_{\{\kappa\}})$, $\cosh_{\{\kappa\}}(x) = \cosh(x_{\{\kappa\}})$, etc. This property of the κ-deformed exponential and trigonometric functions permits us to consider their Taylor expansion in terms of the function $x_{\{\kappa\}}$. For instance, it holds

$$\exp_{\{\kappa\}}(x) = \sum_{m=0}^{\infty} \frac{1}{m!}(x_{\{\kappa\}})^m\,. \tag{61}$$

The κ-integral: Starting from the definition of κ-derivative, we define the κ-integral through

$$\int f(x)\,\mathrm{d}x_{\{\kappa\}} = \int \frac{f(x)}{\sqrt{1+\kappa^2x^2}}\,\mathrm{d}x\,. \tag{62}$$

All the standard rules of the undeformed integral calculus still hold true.

The κ-cyclic functions: The κ-deformed sine and cosine can be defined through $\sin_{\{\kappa\}}(x) = -\mathrm{i}\sinh_{\{\kappa\}}(\mathrm{i}x)$ and $\cos_{\{\kappa\}}(x) = \cosh_{\{\kappa\}}(\mathrm{i}x)$, respectively. It results: $\sin_{\{\kappa\}}(x)=\sin(x_{\{i\kappa\}})$ and $\cos_{\{\kappa\}}(x)=\cos(x_{\{i\kappa\}})$ being $x_{\{i\kappa\}}=\kappa^{-1}\arcsin(\kappa x)$. Of course the κ-cyclic trigonometry, naturally can be introduced.

The κ-logarithm: We study now the inverse function of the κ-exponential namely the κ-logarithm which, after remembering its definition (52) and the expression of the generator $g_\kappa(x) = \kappa x$, can be written as

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} . \tag{63}$$

From (63) we can see that $\forall x \in \mathbf{R}^+ \Rightarrow \ln_{\{\kappa\}}(x) \in \mathbf{R}$ and for $\ln_{\{0\}}(x) = \ln x$. Furthermore, we have $\ln_{\{\kappa\}}(1) = 0$ and $\ln_{\{-\kappa\}}(x) = \ln_{\{\kappa\}}(x)$. Two relevant properties of the κ-logarithm are $\ln_{\{\kappa\}}(xy) = \ln_{\{\kappa\}}(x) \overset{\kappa}{\oplus} \ln_{\{\kappa\}}(y)$ and $\ln_{\{\kappa\}}(x^m) = m\ln_{\{m\kappa\}}(x)$ while its asymptotic behavior is $\ln_{\{\kappa\}}(x) \underset{x\to 0^+}{\sim} -|2\kappa|^{-1}x^{-|\kappa|} \to -\infty$ and $\ln_{\{\kappa\}}(x) \underset{x\to +\infty}{\sim} |2\kappa|^{-1}x^{|\kappa|} \to +\infty$.

In closing this section, we consider the relationship linking the functions $\exp_{\{\kappa\}}(x)$ and $\ln_{\{\kappa\}}(x)$, here introduced with the already known in the literature Tsallis exponential $\exp_q(x) = [1 + (1-q)x]^{1/(1-q)}$ and its inverse function, the Tsallis logarithm $\ln_q(x) = (x^{1-q} - 1)/(1-q)$ namely

$$\exp_{\{\kappa\}}(x) = \exp_{1+\kappa}\left(x + \frac{\sqrt{1+\kappa^2x^2} - 1}{\kappa}\right) ,$$

$$\ln_{\{\kappa\}}(x) = \tfrac{1}{2}\left[\ln_{1+\kappa}(x) + \ln_{1-\kappa}(x)\right] .$$

7. The κ-deformed statistics

In this section we will consider a new statistical distribution, just as a working example of the theory presented in the previous sections. We start by considering a particle system in the velocity space and postulating the following density of entropy:

$$\sigma_\kappa(f) = -\int \mathrm{d}f \ln_{\{\kappa\}}(\alpha f) , \tag{64}$$

where the real constant $\alpha > 0$ is not specified, for the moment. Eq. (64), for $\kappa \to 0$, gives the standard Boltzmann–Gibbs–Shannon density of entropy if we pose $\alpha = 1$. We note that the above definition of $\sigma_\kappa(f)$ implies that $\kappa(f)$ is related to the κ-logarithm, through $\ln\kappa(f) = \ln_{\{\kappa\}}(\alpha f)$. It is immediate to see that for $\forall\kappa \in \mathbf{R}$, $\mathrm{d}^2\sigma_\kappa(f)/\mathrm{d}f^2 \leqslant 0$, independently on the value of α. Then for the entropic functional $\mathscr{K}$, we have $\mathrm{d}\mathscr{K}(t)/\mathrm{d}t \geqslant 0$ and $\mathscr{K}(t) \leqslant \mathscr{K}(\infty)$. The entropy of the system given by $S_\kappa = \int_{\mathscr{R}} \mathrm{d}^n v\, \sigma_\kappa(f)$ assumes now the form

$$S_\kappa = -\frac{1}{2\kappa}\int_{\mathscr{R}} \mathrm{d}^n v \left(\frac{\alpha^\kappa}{1+\kappa} f^{1+\kappa} - \frac{\alpha^{-\kappa}}{1-\kappa} f^{1-\kappa}\right) \tag{65}$$

and reduces to the standard Boltzmann–Gibbs–Shannon entropy $S_0 = -\int_{\mathscr{R}} d^n v[\ln(\alpha f) - 1]f$ as the deformation parameter approaches zero. This κ-entropy is linked to the Tsallis entropy $S_q^{(T)}$ through the following relationship:

$$S_\kappa = \frac{1}{2}\frac{\alpha^\kappa}{1+\kappa} S_{1+\kappa}^{(T)} + \frac{1}{2}\frac{\alpha^{-\kappa}}{1-\kappa} S_{1-\kappa}^{(T)} + \text{const.} \tag{66}$$

First choice of α: We discuss now the case $\alpha = 1$. The stationary statistical distribution corresponding to the entropy S_κ can be obtained by maximizing the functional $\mathscr{K}$

$$\delta\left[S_\kappa + \int_{\mathscr{R}} d^n v(\beta\mu f - \beta U f)\right] = 0\,. \tag{67}$$

One arrives to the following distribution:

$$f = \exp_{\{\kappa\}}(-\varepsilon); \quad \varepsilon = \beta(U - \mu)\,, \tag{68}$$

which gives the standard classical statistical distribution as $\kappa \to 0$.

Second choice of α: Before introducing the second choice of the constant α we consider briefly the concept of entropy. It is to be understood that the entropy is an absolute measure of information only in the case of an isolated system, where the particle number and energy are conserved. In the case of a system interacting with a bath, only the particle number is conserved and the free entropy can't be used as an absolute measure of information. For this reason an entropy constrained by the particle number conservation and by the relevant energy mean value must be introduced. The constrain introduced by the particle number conservation is a special one, and because of its presence both in the case of free and interacting systems, can be taken into account directly in the definition of the entropy. For instance, for the linear kinetics, the stationary distribution $f = Z^{-1}\exp(-\beta U)$ with relevant partition function given by $Z = \int_{\mathscr{R}} d^n v \exp(-\beta U)$, can be obtained using the variational principle, namely $\delta[S_0 - \beta\int_{\mathscr{R}} d^n v\, Uf] = 0$, where the functional $S_0 = -\int_{\mathscr{R}} d^n v[\ln(Zf) - 1]f$ depending on the constant Z is the above mentioned constrained entropy, the particle number (here we pose $N = 1$) being conserved.

It is clear that, for analogy, also in the case of the non-linear kinetics with $\kappa \neq 0$, we can choose $\alpha = Z$ in the expression of S_κ given by (65), so that the stationary statistical distribution

$$f = \frac{1}{Z}\exp_{\{\kappa\}}(-\beta U) \tag{69}$$

with $Z = \int_{\mathscr{R}} d^n v \exp_{\{\kappa\}}(-\beta U)$ can be obtained by considering the following variational principle:

$$\delta\left[S_\kappa - \beta\int_{\mathscr{R}} d^n v U f\right] = 0\,. \tag{70}$$

Of course, the expression of f depends on the potential U and, in the particularly interesting case of Brownian particles, $U = mv^2/2$, after tedious but straightforward calculations we can write the distribution function as

$$f = \left[\frac{\beta m|\kappa|}{\pi}\right]^{n/2}\left[1 + \frac{1}{2}n|\kappa|\right]\frac{\Gamma(1/2|\kappa| + n/4)}{\Gamma(1/2|\kappa| - n/4)}\exp_{\{\kappa\}}\left(-\frac{\beta}{2}mv^2\right)\,, \tag{71}$$

where $n=1,2,3$ is the dimension of the velocity space and $|\kappa|<2/n$. The distribution given by (71) reduces to the standard Maxwell–Boltzmann one $f=(\beta m/2\pi)^{n/2}\exp(-\beta m v^2/2)$ as the deformation parameter κ approaches to zero. This limit can be performed easily by taking into account the Stirling's formula: $\Gamma(z)\sim\sqrt{2\pi}\,z^{z-1/2}\exp(-z)$, holding for $z\to+\infty$.

We write now the evolution equation, whose stationary state is described by (71). After indicating with $f(t,\boldsymbol{v})$ the time depending statistical distribution, which for $t\to\infty$ reduces to (71), we introduce the new function $p=Z_\infty f$, and remember that $\alpha=Z_\infty$. The evolution equation of the function p, in the Fokker–Planck picture, in the case of Brownian particles, and by choosing for simplicity $\gamma(f)=f$, assumes the form

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial \boldsymbol{v}}\left[k\boldsymbol{v}p+\frac{D}{2}(p^{\kappa}+p^{-\kappa})\frac{\partial p}{\partial \boldsymbol{v}}\right], \tag{72}$$

where $k=Dm\beta_\infty$. In the Boltzmann picture, the evolution equation is structurally similar with the one of Tsallis kinetics. The only difference is that the Tsallis logarithm is replaced now by the κ-logarithm.

We conclude this section by considering a new quantum distribution, describing particles with a behavior intermediate between bosons and fermions, which can be constructed starting from (68). We impose for the entropy density in the following expression:

$$\sigma_\kappa(f)=-\int \mathrm{d}f\,\ln_{\{\kappa\}}\left(\frac{f}{1+\eta f}\right), \tag{73}$$

being η a real parameter. After maximization of the constrained entropy or, equivalently, after obtaining the stationary solution of the proper evolution equation associated to (73), one arrives to the following distribution:

$$f=\frac{1}{\exp_{\{\kappa\}}(\varepsilon)-\eta}. \tag{74}$$

We note that (74) for $\eta=0$ reduces to (68) while for $\eta=1$ becomes a κ-deformed Bose–Einstein distribution and for $\eta=-1$ becomes a κ-deformed Fermi–Dirac distribution. Finally, for $\eta\neq 0,\pm1$ (74) becomes a quantum distribution which defines an intermediate statistics and can describe anyons likewise of the distribution considered in Refs. [18,33]. Obviously, other quantum intermediate statistical distributions (generalized Haldane statistics, generalized quon statistics, etc.), can be constructed starting from (68).

8. Applications of κ-deformed statistics

It is worth to note here that κ-deformed statistical distribution given by (68) and (69) is obtained by extremization, under constraints, of the entropy S_κ which is given by (66) in terms of Tsallis entropy. This interesting result permits us to use κ-deformed statistical distribution to study physical systems where we can find experimental evidences for the physical relevance of the Tsallis entropy like, for instance, in 2d turbulent pure-electron plasma [37], solar neutrinos [38–40], bremsstrahlung [41], anomalous diffusion of correlated and Levy type [27,42], self-gravitating systems [43], cosmology [44] among many others. In the following we consider two examples of application of κ-deformed distribution, just to see the values the parameter κ assumes.

In Refs. [38–40], firstly, the problem of the solar neutrinos is considered and a solution, in the frame of the non-extensive statistics is proposed. It is well known that the solar core, where the energy is produced, is a weakly non-ideal plasma. In fact, density and temperature condition suggest that the microscopic diffusion of ions is non-standard: diffusion and friction coefficients are energy dependent, collisions are not two-body processes and retain memory beyond the single scattering event. For this reason, the equilibrium velocity distribution of the ions is slightly different in the tail from the Maxwellian one, as argued also by Clayton [45,46]. Consequently, the reaction rates are sensibly modified and, at last, the neutrino fluxes which are experimentally detected, have calculated values different from the standard ones. With the hypothesis that the velocity distribution of the ions in the solar core is a κ-deformed distribution, we can reproduce the experimental data, analyzed in Refs. [38–40], when $\kappa = 0.15$.

In Ref. [44] it is shown that the observational data concerning the velocity distribution of clusters of galaxies can be fitted by a non-extensive statistical distribution. If we adopt the κ-deformed statistical distribution to analyze the data reported in Ref. [44] we obtain a remarkable good fitting when $\kappa = 0.51$.

9. Conclusions

In the present effort, we have studied from a general prospective the kinetics of non-linear systems in the frame of two pictures, namely Kramers and Boltzmann. The results, can be summarized as follows:

The KIP governs the time evolution of the non-linear system by means of the function $\gamma(f, f') = a(f)b(f')c(f, f')$ and imposes its steady state through the function $\kappa(f) = a(f)/b(f)$. The steady-state f_s can be obtained as stationary solution of its evolution equation.

The KIP imposes the entropy form of the non-linear system, which is given by (25) both in Kramers and in Boltzmann pictures. The constrained entropy $\mathscr{K}$, given by (21), satisfies the H-theorem when $\mathrm{d}\kappa(f)/\mathrm{d}f \geqslant 0$ and the f_s can be obtained also from the maximum entropy principle for $\mathscr{K}$.

In the case of Brownian particles where $U(\boldsymbol{v})=\frac{1}{2}m\boldsymbol{v}^2$, the stationary state f_s and then the constrained entropy $\mathscr{K}$ assume the same values both in Kramers and in Boltzmann pictures. $\kappa(f)$ being an arbitrary function, the H-theorem has been demonstrated in a unified way for a very large class of isolated or interacting with a bath non-linear systems.

Finally, we have considered within the formalism here developed, some statistical distributions already known in the literature. On the other hand, as a working example of the theory here presented, we have introduced the new κ-deformed statistics. After studying the main properties of the new statistics, we have discussed two applications to real physical systems.

References

[1] M. Druyvenstein, Physica (Eindhoven) 10 (1930) 6.
[2] M. Druyvenstein, Physica (Eindhoven) 1 (1934) 1003.
[3] A. Renyi, Probability Theory, North-Holland, Amsterdam, 1970.
[4] B.D. Sharma, D.P. Mittal, J. Math. Sci. 1 (1975) 28.
[5] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[6] S. Abe, Phys. Lett. A 224 (1997) 326.
[7] A.R.R. Papa, J. Phys. A 31 (1998) 5271.
[8] E.P. Borges, I. Roditi, Phys. Lett. A 246 (1998) 399.
[9] P.T. Landsberg, V. Vedral, Phys. Lett. A 247 (1998) 211.
[10] C. Anteneodo, A.R. Plastino, J. Phys. A 32 (1999) 1089.
[11] T.D. Frank, A. Daffertshofer, Physica A 272 (1999) 497.
[12] G. Gentile, Nuovo Cimento 17 (1940) 493.
[13] H.S. Green, Phys. Rev. 90 (1953) 270.
[14] O.W. Greenberg, R.N. Mohapatra, Phys. Rev. D 39 (1989) 2032.
[15] L.C. Biedenharn, J. Phys. A 22 (1989) L873.
[16] F.D.M. Haldane, Phys. Rev. Lett. 66 (1991) 1529.
[17] Y.S. Wu, Phys. Rev. Lett. 73 (1994) 922.
[18] R. Acharya, P. Narayana Swamy, J. Phys. A 27 (1994) 7247.
[19] F. Buyukkilic, D. Demirhan, Phys. Lett. A 181 (1993) 24.
[20] E.A. Uehling, G.E. Uhlenbeck, Phys. Rev. 43 (1933) 552.
[21] A.R. Plastino, A. Plastino, Physica A 222 (1995) 347.
[22] C. Tsallis, D.J. Buckman, Phys. Rev. E 54 (1996) R2197.
[23] A. Compte, D. Jou, J. Phys. A 29 (1996) 4321.
[24] A. Compte, D. Jou, Y. Katayama, J. Phys. A 30 (1997) 1023.
[25] D.A. Stariolo, Phys. Rev. E 55 (1997) 4806.
[26] S. Martinez, A.R. Plastino, A. Plastino, Physica A 259 (1998) 183.
[27] L. Borland, Phys. Rev. E 57 (1998) 6634.
[28] L. Borland, Phys. Lett. A 245 (1998) 67.
[29] L. Borland, F. Pennini, A.R. Plastino, A. Plastino, Eur. Phys. J. B 12 (1999) 285.
[30] G. Kaniadakis, G. Lapenta, Phys. Rev. E 62 (2000) 3246.
[31] G. Kaniadakis, A. Lavagno, P. Quarati, Nucl. Phys. B 466 (1996) 527.
[32] G. Kaniadakis, A. Lavagno, P. Quarati, Phys. Lett. A 227 (1997) 227.
[33] G. Kaniadakis, P. Quarati, Phys. Rev. E 49 (1994) 5103.
[34] A. Rossani, G. Kaniadakis, Physica A 277 (2000) 349.
[35] T.D. Frank, Phys. Lett. A 267 (2000) 298.
[36] G. Kaniadakis, P. Quarati, Phys. Rev. E 48 (1993) 4263.
[37] B.M. Boghosian, Phys. Rev. E 53 (1996) 4754.
[38] G. Kaniadakis, A. Lavagno, P. Quarati, Phys. Lett. B 369 (1996) 308.
[39] G. Kaniadakis, A. Lavagno, M. Lissia, P. Quarati, Physica A 261 (1998) 359.
[40] M. Coraddu, G. Kaniadakis, A. Lavagno, M. Lissia, G. Mezzorani, P. Quarati, Braz. J. Phys. 29 (1999) 153.
[41] C. Tsallis, A.M.C. de Souza, Phys. Lett. A 235 (1994) 444.
[42] P.A. Alemany, D.H. Zanette, Phys. Rev. E 49 (1994) R956.
[43] A.R. Plastino, A. Plastino, Physica A 174 (1993) 384.
[44] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati, C. Tsallis, Astro. Lett. Comm. 35 (1998) 449.
[45] D.D. Clayton, Nature 249 (1974) 131.
[46] D.D. Clayton, E. Eliahu, M.J. Newman, R.J. Talbor Jr., Astrophys. J. 199 (1975) 494.