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# NON-LINEAR LAGRANGIANS AND COSMOLOGICAL THEORY

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SUMMARY

In relativistic cosmology the theory of uniform model universes is based on Einstein's equations, which derive from a variational principle the field-Lagrangian $L$ of which is the scalar curvature $R$ to within an arbitrary additive constant. In this work the possibility of taking $L$ to be a more general invariant of the Riemann tensor is contemplated. The consequences of choosing $L$ to be a function $\phi$ of $R$ alone are tentatively examined under specialized circumstances, with particular attention to an open world-model oscillating between non-singular states. Difficulties revolving about the actual form which $\phi$ might take are discussed.

## 1. INTRODUCTION

Cosmological theory based on the general theory of relativity leads to the conclusion (1) that an evolving Robertson–Walker universe either developed from a singular state or a state of complete emptiness in the past or will reach such a state in the future, whilst every oscillatory universe goes through a succession of singular states; granted throughout that pressure $p$ and energy density $\rho$ are non-negative quantities, and that $d\rho/dp \nless 0$. As long as one adheres, on the one hand, to the Robertson–Walker metric $g_{kl}$ as the appropriate instrument for the description of the Universe as a whole and, on the other, to the general principle that the geometry of the world is determined covariantly by the distribution of matter and radiation within it, one will seek to adopt an equation of the form

$$P^{kl} = T^{kl}, \tag{1.1}$$

where $T^{kl}$ is a phenomenological tensor which represents the sources of the gravitational field other than itself, whereas $P^{kl}$ is a geometrical tensor which depends on the $g_{kl}$ and their derivatives alone. If one now enquires how one might construct modified theories which do not imply the necessary occurrence of states of complete emptiness or of singular states, in particular in the case of oscillating models, two generic alternatives offer themselves: (i) modifications of the first kind which contemplate the abandonment of the requirement that $T_4{}^4$ and $-T_a{}^a$ $(a = 1, 2, 3)$ must be non-negative (2); and (ii) modifications of the second kind, in which Einstein's form† $G_{kl} := \frac{1}{2}g_{kl}R - R_{kl} - \lambda g_{kl}$ of $P_{kl}$ is abandoned in favour of some more general tensor. The first class of modifications evidently leaves the basic principles of Einstein's theory untouched and merely concerns itself with the

† The notation $:=$ and $=:$ means that the equation operates as a definition of the quantity on the left and right respectively.

sources, whereas the second affects the structure of the theory. Of course, they are not mutually exclusive, nor is there any *a priori* reason why the generic form of the field equations must necessarily be that represented by equation (1.1).

Now, whereas the consequences of various modifications of the first kind have been considered in the literature at various times (2), the same does not appear to be true of those of the second kind. In this note I therefore undertake a very tentative examination of the outcome of a class of modifications of the second kind. The class as a whole† is characterized by the requirement that $P^{kl}$ be the functional derivative with respect to $g_{kl}$ of some invariant $L$ of the Riemann tensor other than $R$. Then, whatever the form of the Lagrangian $L$ may be, the divergence $T^{kl}{}_{;l}$ of the source tensor will vanish as a consequence of equation (1.1), so that the conservation laws governing the sources are the same as those in Einstein's theory. Incidentally, an important consequence of this is that the resulting theory cannot accommodate the steady-state universe for any choice of $L$ since the required constancy of the energy density implies that the sum of pressure and energy density must vanish [see equation (2.6)], a conclusion which is incompatible with the requirement that these quantities shall be non-negative.

Various objections may be raised against the introduction of Lagrangians more general than $R-2\lambda$. For instance, even when $L$ is required not to involve covariant derivatives of the Riemann tensor (a restriction which will be adhered to throughout), $P_{kl}$ will contain fourth derivatives of the components of the metric tensor. In any event, it is difficult to see how a definite choice of $L$ might be motivated. Nevertheless, if a quantum-theoretic description of the metrical field is possible one may expect that on a phenomenological level the effects of 'vacuum polarization' will be to induce non-linear terms in the Lagrangian (4). (In lowest order perturbation theory the additional term in $L$ may be expected to be a linear combination of the quadratic invariants $R_{kl}R^{kl}$ and $R^2$, bearing in mind that the functional derivatives of the quadratic invariants $R_{klmn}R^{klmn}-4R_{kl}R^{kl}+R^2$ and $e^{klmn}R_{klst}R_{mn}{}^{st}$ vanish identically.) One has, indeed, a situation corresponding closely to that which obtains in quantum electrodynamics. There vacuum polarization induces effects which are described on a phenomenological (quasi-classical) level by the theory of Born and Infeld. This is a non-linear generalization of Maxwellian electrodynamics, non-linearity here being understood in the sense that the Lagrangian is a non-linear function of the two quadratic invariants of the field (5).

With these remarks in mind, the introduction of a non-linear Lagrangian is perhaps after all not as far-fetched as might appear to be the case at first sight—see, however, Section 7. The following formal point may also be raised. Suppose one writes $L$ as $R-2\lambda+{}^*L$. If ${}^*P^{kl}$ is the functional derivative of ${}^*L$, equation (1.1) becomes

$$G_{kl} = T_{kl}-{}^*P_{kl}, \tag{1.2}$$

which are Einstein's equations, but with the 'normal' source supplemented by an 'apparent' source

$${}^*T_{kl} := -{}^*P_{kl}. \tag{1.3}$$

In the terminology used earlier, this looks like a modification of the first kind, but it is not, to the extent that ${}^*T_{kl}$ depends upon the field alone. The same remark might

† Probably this was first contemplated by Eddington (3).

therefore be made about the work of Pachner (6), although his source appears to have no readily identifiable normal part, and lacks a geometrical foundation. Moreover, its dependence upon the metric tensor bears an *ad hoc* character of a kind which precludes the applicability of the field equations to more general situations, e.g. when the sources are unsymmetrical.

Since, in the first instance, no particular *a priori* choice of $L$ suggests itself it is not unreasonable to adopt, for the purposes of orientation, the simple class of Lagrangian

$$L = \phi(R) \tag{1.4}$$

where the function $\phi$ remains as yet unspecified. The corresponding field equations are given in Section 2, and specialized to the case of the Robertson–Walker metric. A particular open model, oscillating between non-singular states, satisfies these equations if the source filling the model obeys the equations of state $p = \frac{1}{3}(\rho - \rho_1)$, where $\rho_1$ is a constant. After a digression on Palatini's device, the model is considered in a little more detail in Sections 5 and 6. Finally, in Section 7 certain apparently rather severe difficulties, relating to the actual form $\phi(R)$ might take, are discussed in some detail.

It is, of course, understood throughout that, whatever the actual form of the Lagrangian may be, the effects of the part $^*P^{kl}$ of $P^{kl}$ are completely negligible under 'ordinary' circumstances. In other words, on a local scale, and under ordinary astrophysical conditions, the actual Lagrangian $L$ is effectively indistinguishable from $R$, and the theory reduces to Einstein's theory (without cosmical term).

## 2. THE FIELD EQUATIONS

In accordance with the remarks made in the introduction we contemplate the field equations generated by the Lagrangian $\phi(R)$. They are (see the Appendix)

$$(-R_{;k}R_{;l} + g_{kl}R_{;m}R^{;m})\phi''' + (-R_{;kl} + g_{kl}\Box R)\phi'' - R_{kl}\phi' + \tfrac{1}{2}g_{kl}\phi = T_{kl}, \tag{2.1}$$

where primes denote differentiation with respect to $R$. In the cosmological context one would not solve these equations in the ordinary sense of the term since the metric is required to be of the general form

$$ds^2 = -(1 + \tfrac{1}{4}\epsilon r^2/r_0^2)^{-2} S^2(t)(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + c^2\, dt^2, \tag{2.2}$$

where $r_0$ is a constant, $\epsilon$ has one or other of the values $+1$, $0$, $-1$, and $S(t)$ is an unspecified function of $t$ alone. For the time being units will be adopted such that the numerical values of $c$, $r_0$ and $8\pi G$ ($G$ = Newton's constant) become unity. The use of (2.2) in (2.1) now shows that

$$T_k{}^l = \mathrm{diag}(-p, -p, -p, \rho)$$

where

$$\rho = 3S^{-1}\dot{S}\dot{R}\phi'' - 3S^{-1}\ddot{S}\phi' + \tfrac{1}{2}\phi, \tag{2.3}$$

$$p = -\dot{R}^2\phi''' - (\ddot{R} + 2S^{-1}\dot{S}\dot{R})\phi'' + S^{-2}(S\ddot{S} + 2\dot{S}^2 + 2\epsilon)\phi' - \tfrac{1}{2}\phi, \tag{2.4}$$

with

$$R = 6S^{-2}(S\ddot{S} + \dot{S}^2 + \epsilon), \tag{2.5}$$

the dots denoting differentiation with respect to $t$.

In place of (2.4) one may use the relation

$$\dot{\rho} + 3S^{-1}\dot{S}(\rho + p) = 0. \tag{2.6}$$

## 3. EXAMPLE OF AN OSCILLATING MODEL

To gain some insight into the possibilities the theory has to offer one may simply make some more or less arbitrary but convenient choice of the function $S(t)$. To this end, take

$$S(t) = a(1-\eta \cos \omega t)^{1/2} \quad (a = \sqrt{2}\, \omega^{-1}), \tag{3.1}$$

where $\omega$ and $\eta$ ($<1$) are positive constants. Then in the first place

$$R = -3\omega^2[1-(\epsilon+1)a^2 S^{-2}]. \tag{3.2}$$

In the interests of gaining the greatest possible simplicity let it now further be supposed that the model is open, $\epsilon = -1$. Then

$$R = -3\omega^2, \tag{3.3}$$

and this result shows the *raison d'être* for the form of (3.1) of $S$. $\rho$ and $p$ may now be computed from (2.3) and (2.4), and one so finds that

$$\begin{aligned} \rho &= \tfrac{3}{4}\omega^2[1-(1-\eta^2)(1-\eta \cos \omega t)^{-2}]\phi' + \tfrac{1}{2}\phi \\ p &= -\tfrac{1}{4}\omega^2[3+(1-\eta^2)(1-\eta \cos \omega t)^{-2}]\phi' - \tfrac{1}{2}\phi. \end{aligned} \tag{3.4}$$

It follows that

$$\rho - 3p = -R\phi' + 2\phi, \tag{3.5}$$

an equation which, bearing (3.3) in mind, follows at once by taking the trace of both members of (2.1). The right-hand member of (3.5) is a constant, $\rho_1$, say. Accordingly, the equation of state of the fluid filling the model is

$$p = \tfrac{1}{3}(\rho - \rho_1). \tag{3.6}$$

Evidently $\phi$ cannot be chosen entirely arbitrarily, for the algebraic equation

$$R\phi' - 2\phi + \rho_1 = 0 \tag{3.7}$$

must have at least one root on the negative real axis. Granted that $\rho_1 \geqq 0$, the Lagrangian $L = R + bR^2$ ($b$ = constant), for example, is excluded by this condition, whereas $L = -2\lambda + R + bR^2$ is not, provided $4\lambda < -\rho_1$. Granted now that (3.7) has a root of the required kind, (3.4) can be given the form

$$\rho = \tfrac{1}{4}\rho_1 + (\tfrac{1}{2}\phi - \tfrac{1}{4}\rho_1)X, \quad 3p = -\tfrac{3}{4}\rho_1 + (\tfrac{1}{2}\phi - \tfrac{1}{4}\rho_1)X, \tag{3.8}$$

where

$$X := (1-\eta^2)(1-\eta \cos \omega t)^{-2}.$$

## 4. DIGRESSION ON PALATINI'S DEVICE

The results of the preceding section were based on the ansatz (3.1) together with the assignment of the value $-1$ to $\epsilon$. Suppose now that instead the equation of state (3.6) were adjoined to the equations of Section 2. Then, because of (2.6),

$$\rho = bS^{-4} + \tfrac{1}{4}\rho_1, \tag{4.1}$$

where $b$ is a constant of integration. Inserting this form of $\rho$ on the left of (2.4) and bearing (2.5) in mind one then has a third order differential equation for $S$ once the form of $\phi$ is prescribed ($\phi' \neq$ constant). Granted that $\epsilon = -1$, (3.1) is clearly not the most general solution of this equation and the situation is rather complicated. One is tempted to draw upon Palatini's device which hitherto seems to have been employed only in the context of vacuum field equations (7). Thus one replaces the previous field equation

$$\frac{\delta L}{\delta g_{kl}} = T^{kl} \tag{4.2}$$

by the pair

$$\frac{\delta L}{\delta g_{kl}} = T^{kl}, \quad \frac{\delta L}{\delta \Gamma^m{}_{kl}} = 0, \tag{4.3}$$

the $g_{kl}$ and the symmetric components of linear connection $\Gamma^m{}_{kl}$ being now treated as independent field functions. With $L = \phi(R)$ equations (4.3) become

$$\tfrac{1}{2}g_{kl}\phi - \phi' R_{kl} = T_{kl}, \quad (\phi' g^{kl}\sqrt{-g})_{;m} = 0. \tag{4.4}$$

Now, in the particular case of the equation of state (3.6) the first of these immediately leads to (3.7) and therefore to the constancy of $R$, which had previously to be imposed as a subsidiary condition. The second then shows the $\Gamma^m{}_{kl}$ to be the usual Christoffel symbols. By these means one indeed arrives uniquely at the solution (3.1). Although this may at first sight seem to be an attractive conclusion, it must be realized that it is tied to a particularly simple equation of state. In any event, Palatini's method is so full of difficulties (**8**) in all but the simplest cases that it seems wisest not to pursue it.

## 5. THE RADIATION-LIKE CASE

The substance filling the model will be called radiation-like if it obeys the equation of state of electromagnetic blackbody radiation, $\rho = 3p$. Then $\rho_1$ is to be taken as zero in the equations of the preceding section, so that (3.8) in particular becomes

$$\rho = 3p = \tfrac{1}{2}\phi X. \tag{5.1}$$

Physically this situation is presumably not very realistic except about the epoch of greatest contraction. Since $X$ is positive the value of $\phi$, evaluated at the (negative) root of equation (3.7), must be positive which is a further condition upon the form of $\phi$.

If $\rho_+$ and $\rho_-$ stands for the greatest and least values attained by $\rho$ respectively, (5.1) becomes

$$\rho = 4\rho_+\zeta^2[(1+\zeta)-(1-\zeta)\cos\omega t]^{-2}, \tag{5.2}$$

with

$$\zeta^2 := \rho_-/\rho_+.$$

For a universe filled with radiation whose density is proportional to the fourth power of the temperature $T$, (5.2) can be written as

$$T = [\zeta+(1-\zeta)\sin^2 \tfrac{1}{2}\omega t]^{-1/2}T_-, \quad (\zeta = T_-^2/T_+^2). \tag{5.3}$$

Under conditions even remotely resembling those in the actual universe the value of $\zeta$ will presumably be very small compared with unity. Even from the point of view of the redshift alone, if a value $z$ is observed then one must certainly already have $\zeta \leqq (z+1)^{-2}$. If, on the other hand, one takes the Universe to oscillate between states such that $T_+ = 10^{12}\,°K$ and $T_- = 1\,°K$—merely for the sake of illustration—then $\zeta = 10^{-24}$. At any rate, assuming that $\zeta \ll 1$, the time dependence of $T$ will be effectively independent of $T_+$ almost all the time, i.e. except when $\sin\frac{1}{2}\omega t$ is comparable in magnitude with $\surd\zeta$. On the other hand the 'half-life' $t_1$, defined as the time required for $T$ to drop from $T_+$ to $\frac{1}{2}T_+$ is very nearly just the fraction $T_-/T_+$ of the period of oscillation $\tau\ (=2\pi/\omega)$. With the values of $T_+$ and $T_-$ adopted above $t_1$ will therefore be measured in days if $\tau$ is of the order of $10^{19}$ sec.

## 6. THE CASE $\rho_1 \neq 0$

The condition that $\rho$ and $p$ be non-negative now requires that

$$\phi \geqq \rho_1(2+\eta)/(1-\eta), \tag{6.1}$$

and this entails the inequalities

$$\rho_+ \geqq \rho_1(1+\eta+\eta^2)/(1-\eta)^2, \quad \rho_- \geqq \rho_1. \tag{6.2}$$

The density can be written in a form analogous to (5.2). With

$$\begin{aligned} \xi_\pm &:= (4\rho_\pm - \rho_1)^{-1/2}, \\ \rho &= \tfrac{1}{4}\rho_1 + [(\xi_+ + \xi_-) + (\xi_+ - \xi_-)\cos\omega t]^{-2}. \end{aligned} \tag{6.3}$$

From this it follows incidentally that if $t'$ is a time sufficiently close to a state of greatest expansion one has generically

$$\rho = \rho_- + 3\Gamma t'^2 + O(t'^4), \quad p = \tfrac{1}{3}(\rho_- - \rho_1) + \Gamma t'^2 + O(t'^4). \tag{6.4}$$

The equation of state reflects the fact that about the times of greatest contraction the universe is likely to be radiation-like. On the other hand, although the velocity of sound is, rather unnaturally, $1/\surd 3$ at all times, there exists now the possibility for the ratio $3p/\rho$ to have any value between 0 and 1 about the times of greatest expansion.

The Hubble parameter $H$ and the acceleration parameter $q$ are, as usual, defined as

$$H := \dot{S}/S, \quad q := -S\ddot{S}/\dot{S}^2. \tag{6.5}$$

With (3.1) these are known functions of $t$. When $\eta \ll 1$ one finds that $qH^2$ is independent of $t$:

$$qH^2 = \tfrac{1}{4}\omega^2, \tag{6.6}$$

except at times such that $\sin^2 \frac{1}{2}\omega t$ is of the order of $1-\eta$. At all times

$$(1-\eta^2)(q-1)^2(H/\omega)^4 + [1+(q-1)\eta^2](H/\omega)^2 - \tfrac{1}{4}\eta^2 = 0. \tag{6.7}$$

## 7. REMARKS ON THE FUNCTION $\phi(R)$

Perhaps the least attractive feature of the kind of modification contemplated above revolves about the problem of the actual form of $\phi(R)$. It is true that the following essentially qualitative considerations are tied to the severely restrictive assumptions made earlier, namely those embodied in equations (1.4), (3.1) and (3.3). Nevertheless I am inclined to the view that the situation is not likely to be substantially more favourable under more general circumstances.

Recall that $R$ must be a negative root of the algebraic equation (3.7), and that, further, $\phi(R) > 0$. These conditions alone are very weak and without appeal to other theories no plausible choice of $\phi$ suggests itself. Still, in the first instance it is not unreasonable to make the qualitative assumption that $\phi$ can be written as a Taylor series in $R$:

$$\phi = \phi_0 + \phi_1 R + \phi_2 R^2 + \ldots, \tag{7.1}$$

where the $\phi_n$ are constants, $\phi_n$ having the physical dimensions of $(\text{length})^{2n-2}$. One might indeed suppose that the terms depending non-linearly upon $R$ are induced by quantum effects, either in the sources or in the gravitational field as such or both. (The analogy with the non-linear electrodynamics of Born and Infeld springs to mind.) The effects of these terms should therefore presumably be small under 'ordinary' circumstances, in which case $\phi_1 = 1$, $\phi_0 = -2\lambda$. Moreover, one may expect that $\phi_n$ is of the order of $l^{2n-2}$, where $l$ is some fundamental length:

$$\phi_n = a_n l^{2n-2} \quad (n > 1), \tag{7.2}$$

the constants $a_n$ being dimensionless and of the order of unity. However, equation (3.7) then entails that $|l^2 R|$ is a number of the order of unity, and therefore

$$\omega l \sim 1, \tag{7.3}$$

unless one has some very artificial Lagrangian, artificial in the sense that it involves a large dimensionless constant. It appears therefore that $l$ will be very large compared with any length which might be expected to appear in the Lagrangian as a consequence of quantum effects. This conclusion in turn implies an obvious conflict with the expectation expressed just before equation (7.2). Under ordinary circumstances, e.g. in the interior of stars, one may expect, if on no other than dimensional grounds, that the values of $R$ and of $\rho$ are of comparable magnitude, so that $Rl^2$ will be a number very large compared with unity. Therefore, far from requiring that $\phi \sim -2\lambda + R$ as $R \to 0$ one has to demand that $\phi \sim R$ for sufficiently large (though possibly not too large) values of $R$; and if $\phi$ can be written as a power series at all, it will have to be in descending powers. A simple example of a function which such a series might represent is

$$\phi = [(R + 2l^{-2})^2 + \tfrac{1}{2} l^{-4}]^{1/2}. \tag{7.4}$$

When $\rho_1 = 0$ one finds in this case that $\omega l = 1$. However, one can but view a Lagrangian such as (7.4) with misgivings.

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REFERENCES

(1) Tolman, R. C., 1934. *Relativity, Thermodynamics and Cosmology*, Chapter 10, pp. 394–419. Clarendon Press, Oxford.
(2) Rosen, N., 1969. *Int. J. Theor. Phys.*, **2,** 189, and references given therein.
(3) Eddington, A. S., 1923. *The Mathematical Theory of Relativity*, Chapter IV, § 62. Cambridge.
(4) Pechlaner, E. & Sexl, R., 1966. *Comm. Math. Phys.*, **2,** 165.
(5) Ivanenko, D. & Sokolov, A., 1953. *Klassische Feldtheorie*, Chapter 4, § 32. Berlin.
(6) Pachner, J., 1965. *Mon. Not. R. astr. Soc.*, **131,** 173.
(7) Stephenson, G., 1959. *Proc. Camb. phil. Soc.*, **55,** 375.
(8) Buchdahl, H. A., 1960. *Proc. Camb. phil. Soc.*, **56,** 396.
(9) Buchdahl, H. A., 1948. *Q. Jl math. Oxford, Ser.*, **19,** 150.

# APPENDIX

## THE FUNCTIONAL DERIVATIVE OF $\phi(R)$

The explicit form of the functional derivative of $\phi(R)$ which appears on the left of equation (2.1) is contained in a more general result (9) relating to the functional derivative of any second order invariant of the Riemann tensor. However, it may be found directly as follows.

For convenience, write

$$w := \sqrt{(-g)}, \quad h_{kl} := \delta g_{kl}.$$

Then

$$\begin{aligned}\delta(w\phi) &= w(\phi'\delta R+\tfrac{1}{2}g^{kl}\phi h_{kl}) \\ &= w[\phi' g^{st}\delta R_{st}+(\tfrac{1}{2}\phi g^{kl}-\phi' R^{kl})h_{kl}]. \end{aligned} \tag{A1}$$

Since

$$\tfrac{1}{2}R_{st} = \Gamma^n{}_{s[n,\,t]}+\Gamma^m{}_{s[n}\Gamma^n{}_{t]m}$$

it follows at once that

$$\tfrac{1}{2}\delta R_{st} = \delta\Gamma^n{}_{s[n;\,t]}, \tag{A2}$$

bearing in mind the tensorial character of $\delta\Gamma_{st}{}^n$. By the same token

$$\delta\Gamma^k{}_{mn} = \tfrac{1}{2}(h^k{}_{m;\,n}+h^k{}_{n;m}-h^k{}_{mn;}),$$

so that (A2), after transvection with $g^{st}$, becomes

$$g^{st}\delta R_{st} = \Box h_s{}^s - h^{st}{}_{;\,st}. \tag{A3}$$

Using the sign $\approx$ to denote equality to within an additive divergence one therefore has

$$\begin{aligned}\phi' g^{st}\delta R_{st} &\approx h_s{}^s\Box\phi' - h^{st}\phi'{}_{;\,st} \\ &= h_{kl}(g^{kl}\Box\phi' - \phi'{}^{;\,kl}),\end{aligned}$$

so that (A1) becomes

$$\delta(w\phi)\approx w(-\phi'{}^{;\,kl}+g^{kl}\Box\phi'-\phi' R^{kl}+\tfrac{1}{2}\phi g^{kl})h_{kl},$$

and this means that

$$P_{kl} = -\phi'{}_{;\,kl}-R_{kl}\phi'+g_{kl}(\Box\phi'+\tfrac{1}{2}\phi). \tag{A4}$$

Bearing in mind that

$$\phi'{}_{;\,kl} = R_{;\,kl}\phi''+R_{;\,k}R_{;\,l}\phi'''$$

one thus has finally

$$P_{kl} = -\phi''' R_{;\,k}R_{;\,l}-\phi'' R_{;\,kl}-\phi' R_{kl}+g_{kl}(\phi''' R_{;\,m}R^{;\,m}+\phi''\Box R+\tfrac{1}{2}\phi).$$