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Non-linear modal interactions in the oscillations of a liquid drop in a gravitational field

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Abstract

Non-linear modal interactions in the dynamics of a vibrating drop are examined. The partial differential equations governing the drop vibrations are formulated assuming potential flow and incompressibility. The solution is expressed in terms of the eigenfunctions of the (linearized) Laplace operator in spherical coordinates. A small parameter ε is introduced to scale the (small) deformation of the drop surface from its position of equilibrium. A 2 : 1 internal resonance is then imposed between the second and third modes of the resulting discretized system, and the ensuing non-linear modal interactions are studied using the method of multiple scales. A bifurcation in the slow dynamics of the system is detected that leads to amplitude modulations of the drop oscillations. The method employed in this work is general and can be used to study other types of non-linear interactions involving two or more drop modes.

Keywords: Liquid drop; Modal interactions; Bifurcations

1. Introduction

The study of the shape dynamics of drops and bubbles has a long history. The review of previous work in this area can be found in [1–3]. Viscous and inviscid, linear and non-linear models of free and forced interface oscillations were investigated by various theoretical and numerical schemes for the drop surrounded by gas and bubbles surrounded by fluid [1–13]. The emphasis in these previous studies was on corrections to the eigenfrequencies of the drop (bubble) and to the drop (bubble) shape due to non-linearities and (or) viscous effects. The response to the different initial conditions as well as the dynamics and stability of the drop (bubble) in the presence of external perturbing fields have been also investigated in the previous works. Nevertheless, the behaviour of the drop (bubble) in cases of internal resonances, i.e., when non-linear transfer of energy occurs between pairs of modes due to non-linearities, has not received much attention. As exception is [13] where a 2 : 1 internal resonance in bubble oscillations was considered. In [6] numerical simulations are utilized to study modal interactions in drops. In [2]

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modal interactions in drops are studied experimentally. Modulations in the drop oscillations (similar to the ones predicted herein) were detected numerically in that work.

Usually, the theoretical consideration starts with assumptions on the properties of the fluid flow and on the (often axisymmetric spherical) geometry of the drop (bubble). The assumption of potential flow and incompressibility of the fluid within the drop or outside the bubble lead to a Laplace equation for velocity potential and, consequently, to an analytical solution in terms of spherical harmonics. Then, the analysis for bubbles and drops diverges. The fluid contained in drops is treated as incompressible and therefore, does not allow pulsation-type solutions (i.e., the zeroth-order spherical harmonic); this is not necessarily the case for bubbles where the outside fluid although incompressible can still lead to bubble pulsations. This difference between drop and bubble dynamics becomes important if we want to study non-linear modal interactions. For the bubble, the pulsating spherical mode (harmonic of zeroth order) leads to parametric excitation of higher harmonics beginning with that of order 2 (since the first-order harmonic corresponds to zero natural frequency, i.e., to pure translation of the bubble), since the mean radius of the bubble is a time-periodic function. Therefore, the bubble dynamics are characterized by stability–instability regions arising from Floquet analysis. On the contrary, in the absence of pulsations, the mean radius of the drop can be considered constant, and no parametric excitation of higher modes is possible. In this case internal resonances between drop modes can produce interesting non-linear energy exchanges and modulate the drop oscillations. It is the aim of this work to analyze in detail such a non-linear modal interaction and to predict changes in the drop dynamics due to modal bifurcations.

2. General formulation

We consider an incompressible and inviscid free liquid drop. The drop is surrounded by a vacuum or a tenuous gas. We assume potential flow within the drop, and preservation of volume during drop oscillations. Assuming that the initial equilibrium shape of the drop is spherical, we are interested in the dynamics of its surface under small perturbations of its shape.

First we formulate the equations governing the motion of the drop and the associated boundary conditions. The liquid satisfies the equation of continuity (Laplace equation for the velocity potential, ϕ). Employing a spherical coordinate system with the origin at the center of mass of the drop, and assuming axial symmetry, Laplace equation is given by,

$$\nabla^2\phi = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0. \quad (1)$$

The solution of (1) can be expressed as the linear combination of spherical functions, $r^n P_n(\theta)$, where $P_n(\theta)$ are Legendre polynomials of nth order $(n = 0, 1, 2\ldots)$.

In addition, Bernoulli's equation for the pressure within the drop is as follows:

$$\rho\frac{\partial\phi}{\partial t} + \rho\frac{v^2}{2} + \rho g r\cos\theta + p = f(t), \quad (2)$$

where ρ is the density of the fluid, and g is the gravity constant. In (2) we can set $f(t) = 0$, since $\mathbf{v} = \nabla\phi$, where p denotes the pressure. Expressing v in (2) in terms of ϕ, using spherical coordinates, we obtain

$$p = -\rho\frac{\partial\phi}{\partial t} - \frac{\rho}{2}\left(\frac{\partial\phi}{\partial r}\right)^2 - \frac{\rho}{2r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 - \rho g r\cos\theta. \quad (3)$$

At this point we introduce the following non-dimensional variables

$$p^* = \frac{p}{\rho g r_0}, \quad r^* = \frac{r}{r_0}, \quad t^* = \frac{t}{(r_0/g)^{1/2}}, \quad \phi^* = \frac{\phi}{(r_0^3 g)^{1/2}} \quad (4)$$

where, r_0 is the radius of the drop in equilibrium, and the superscript (*) will be omitted in the following analysis. Using (4), the pressure assumes the following non-dimensional form:

$$p = -\frac{\partial \phi}{\partial t} - \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 - \frac{1}{2r^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 - r\cos\theta \qquad (5)$$

The perturbed drop surface is described by the relation $F(r,\theta,t) = 0$, where

$$F(r,\theta,t) = r_s - 1 - \eta(\theta,t), \qquad (6)$$

and η is a small perturbation. Taking the total time derivative of the above relation describing the surface, we obtain a kinematic condition satisfied point-wise at the surface of the drop:

$$\left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla\right)F(r,\theta,t) = 0, \qquad (7)$$

where r_s denotes the perturbed radius of the drop, given by $r_s = 1 + \eta$. Taking into account (6) and employing direct differentiation, (7) yields:

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} = 0. \qquad (8)$$

Considering that the pressure difference on the two sides of the surface of the drop is solely due to surface tension, we obtain the following expression for the pressure within the drop:

$$p = p_e + \frac{1}{Bo}\nabla\cdot\hat{\mathbf{n}}, \qquad (9)$$

where p_e is the external pressure, $Bo = g\rho r_0^2/\sigma$ is the Bond number in dimensional variables, $\hat{\mathbf{n}}$ is the outward unit normal to the surface $r_s = 1 + \eta$, and

$$\nabla\cdot\hat{\mathbf{n}} = \frac{2}{r} - \frac{1}{r^2}\mathbf{L}_s(\eta), \qquad (10)$$

where $\mathbf{L}_s$ denotes the surface Laplacian. For the axisymmetric case this Laplacian is given by, $(\partial/\partial\theta)(\sin\theta(\partial/\partial\theta))$. Expanding (10) in Taylor series about the equilibrium $r = 1$ we get,

$$\nabla\cdot\hat{n} = 2 - (2\eta + \mathbf{L}_s(\eta)) + 2\eta(\eta + \mathbf{L}_s(\eta)). \qquad (11)$$

The Taylor series in (11) was truncated up to order η^2 (e.g. Longuett–Higgins [1]). Substituting (9) and (11) into (5) and combining the resulting equation with (8), we obtain a set of non-linear differential equations governing η and ϕ. Rescaling the variables as, $\eta \mapsto \varepsilon\eta, \phi \mapsto \eta\phi$, where ε is a small perturbation parameter, the governing equations take the form:

$$\begin{aligned} p_e^{(0)} + \frac{2}{Bo} + \cos\theta \\ = -\varepsilon\frac{\partial \phi}{\partial t} - \varepsilon\eta\cos\theta - \varepsilon p_e^{(1)} + \frac{\varepsilon}{Bo}(2\eta + \mathbf{L}_s(\eta)) \\ - \frac{\varepsilon^2}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 - \frac{\varepsilon^2}{2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 - \varepsilon^2 p_e^{(2)} \\ - \frac{2\varepsilon^2}{Bo}(2\eta^2 + \eta\mathbf{L}_s(\eta)) - \varepsilon^2\eta\frac{\partial^2 \phi}{\partial r\partial t} \\ \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial r} = -\varepsilon\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} + \varepsilon\eta\frac{\partial^2 \phi}{\partial r^2}. \end{aligned} \qquad (12)$$

In deriving (12) we expanded the external pressure in the regular perturbation expansion, $p_e \sim p_e^{(0)} + \varepsilon p_e^{(1)} + \cdots$ and retained terms up to order ε^2.

Before proceeding with the analysis of Eq. (12) we discuss certain restrictions imposed on the variables η and p_e:

(i) The assumption of uniform acceleration $\mathbf{g}$ of the center of mass of the drop in the vertical direction leads to, $\overline{p_e} = -\frac{2}{3}$, where "bar" denotes averaging with respect to θ in the vertical direction [3].

(ii) The liquid in the drop was assumed to be incompressible. Taking into account the following expression for the instantaneous volume of the drop, we get:

$$\begin{aligned} V &= \frac{1}{3}\int_0^{\pi}\int_0^{2\pi}(1 + \varepsilon\eta)^3 \sin\theta\, d\theta\, d\varphi \\ &= \frac{4\pi}{3}[1 + 3\varepsilon\bar{\eta} + 3\varepsilon^2\overline{\eta^2} + O(\varepsilon^3)] \end{aligned} \qquad (13)$$

where "bar" denotes averaging with respect to θ. Thus, the requirement of volume preservation to $O(\varepsilon^3)$ during drop oscillations leads to the requirement, $\bar{\eta} = \overline{\eta^2} = 0$.

(iii) The origin of the spherical coordinate system remains fixed at the center of mass of the drop, i.e., $\overline{\eta\cos\theta} = 0$

Combining (i), (ii) and (iii) we obtain [3],

$$p_e^{(0)} = \frac{2}{Bo} + \cos\theta, \tag{14}$$

$$p_e^{(1)} = C\cos\theta, \tag{15}$$

where

$$C = -\frac{3}{2}\int_0^{\pi} \eta^{(0)} P_2(\cos\theta)\sin\theta\,\mathrm{d}\theta$$

and $\eta^{(0)}$ is the first term of the regular expansion $\eta \sim \eta^{(0)} + \varepsilon\eta^{(1)} + \cdots$. Similarly, it can be shown, using (ii), that

$$p_e^{(2)} = \tilde{C}\cos\theta \tag{16}$$

where the coefficient $\tilde{C}$ depends on both $\eta^{(0)}$ and $\eta^{(1)}$ (note that correct to order ε^0, $\tilde{C}$ depends only on $\eta^{(0)}$).

Employing the above relations, (12) is rewritten as,

$$\frac{\partial\phi}{\partial t} + \eta\cos\theta + C\cos\theta - \frac{1}{Bo}(2\eta + \mathbf{L}_s(\eta))$$
$$= -\varepsilon\tilde{C}\cos\theta - \frac{\varepsilon}{2}\left(\frac{\partial\phi}{\partial r}\right) - \frac{\varepsilon}{2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 - \varepsilon p_e^{(2)}$$
$$-\frac{2\varepsilon}{Bo}(2\eta^2 + \eta\mathbf{L}_s(\eta)) - \varepsilon\eta\frac{\partial^2\phi}{\partial r\partial t}$$
$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial r} = -\varepsilon\frac{\partial\phi}{\partial\theta}\frac{\partial\eta}{\partial\theta} + \varepsilon\eta\frac{\partial^2\phi}{\partial r^2} \tag{17}$$

Note, that due to the previous perturbation expansions, both Eqs. (17) are applied at $r = 1$, i.e., at the non-dimensional equilibrium value of the radius; in addition, in (17) we retain only linear terms in ε. We see, that the Bond number Bo is the sole parameter for our non-dimensionalized system. Harper et al. [3] performed a linearized analysis of this problem, and studied the stability of the surface shape to small perturbations. In doing so they neglected the non-linear terms in (17), and determined, that the lowest critical Bond number for surface drop instability is

$$(Bo)_{\mathrm{cr}} = 11.22. \tag{18}$$

Below this value of Bo the surface is "absolutely stable" in the linear sense, i.e., it does not blow up [3]. In this work we include first-order non-linear effects in the analysis, and study the drop oscillations near a particular value of Bo, which turns out to be less than the linearized instability limit (18); thus, no instability phenomena are expected to occur in our analysis.

First we need to decouple the linear parts of Eq. (17). Since we are interested in studying the surface deformation of the drop, we eliminate the velocity potential from these equations. Retaining only O(1) terms in (17) we obtain the following linearized system (similar to the system studied by [3]):

$$\frac{\partial\phi^{(0)}}{\partial t} + \eta^{(0)}\cos\theta + C\cos\theta$$
$$-\frac{1}{Bo}(2\eta^{(0)} + \mathbf{L}_s(\eta^{(0)})) = 0,$$
$$\frac{\partial\eta^{(0)}}{\partial t} - \frac{\partial\phi^{(0)}}{\partial r} = 0. \tag{19}$$

Expressing the solution in terms of the eigenfunctions of the Laplace operator in spherical coordinates, we have that,

$$\phi(r,\theta,t) = \sum_{n=2}^{\infty} r^n b_n(t) P_n(\cos\theta). \tag{20}$$

Similarly, we expand the surface deformation in series,

$$\eta = \sum_{n=2}^{\infty} a_n(t) P_n(\cos\theta), \tag{21}$$

where $a_n(t)$ and $b_n(t)$ are the amplitudes of the spherical harmonics; note, that $a_0 = a_1 = b_0 = b_1 = 0$ due to the previous condition (ii). From the second of equations in (19) we obtain

$$\phi_n^{(0)} = \frac{1}{n}\eta_n^{(0)}, \tag{22}$$

where the following notations were employed:

$$\eta_n^{(0)} = a_n(t)P_n(\cos\theta),$$
$$\phi_n^{(0)} = r^n b_n(t) P_n(\cos\theta). \tag{23}$$

Thus,

$$\frac{\partial \phi_n^{(0)}}{\partial t} = \frac{1}{n}\frac{\partial \eta_n^{(0)}}{\partial t}. \tag{24}$$

Substituting (24) into the first of Eqs. (19) we eliminate the potential and obtain a single ordinary differential equation governing $\eta_n^{(0)}$:

$$\sum_2^\infty \frac{1}{n}\frac{\partial^2 \eta_n^{(0)}}{\partial t^2} + \sum_2^\infty \eta_n^{(0)}\cos\theta + C\cos\theta + \sum_2^\infty \frac{(n-1)(n+2)}{Bo}\eta_n^{(0)} = 0. \tag{25}$$

3. Solution by direct expansion

We now suppose that only two harmonics in (25) are excited. The justification for such truncation of the infinite series will be discussed later, and will be based on a careful study of internal resonances in the dynamical system under consideration. Setting all amplitudes other than the ones corresponding to the second and third harmonics equal to zero, we obtain the representation,

$$\eta^{(0)} = a_2^{(0)}P_2(\cos\theta) + a_3^{(0)}P_3(\cos\theta), \tag{26}$$

in terms of which (25) yields:

$$\frac{1}{2}\frac{\partial^2 a_2^{(0)}}{\partial t^2}P_2(\cos\theta) + \frac{1}{3}\frac{\partial^2 a_3^{(0)}}{\partial t^2}P_3(\cos\theta) + a_2^{(0)}P_2(\cos\theta)P_1(\cos\theta) + a_3^{(0)}P_3(\cos\theta)P_1(\cos\theta) - CP_1(\cos\theta) + \frac{4}{Bo}a_2^{(0)}P_2(\cos\theta) + \frac{10}{Bo}a_3^{(0)}P_3(\cos\theta) = 0. \tag{27}$$

Multiplying (27) by P_2, taking the average of the result with respect to θ, and repeating the same operations with P_3, we obtain a set of two coupled linear ordinary differential equations governing the dynamics of our system at O(1):

$$\frac{\partial^2 a_2^{(0)}}{\partial t^2} + \frac{8}{Bo}a_2^{(0)} + \frac{6}{7}a_3^{(0)} = 0,$$
$$\frac{\partial^2 a_3^{(0)}}{\partial t^2} + \frac{9}{5}a_2^{(0)} + \frac{30}{Bo}a_3^{(0)} = 0. \tag{28}$$

In deriving (28) we have used the following properties of Legendre polynomials:

$$\overline{P_n^2} = \frac{2}{2n+1},$$

$$\sum_2^\infty a_nP_1P_n = \sum_2^\infty \frac{n}{2n-1}a_{n-1}P_n + \sum_2^\infty \frac{n+1}{2n+3}a_{n+1}P_n + \tfrac{2}{5}a_2P_1.$$

Due to the previous truncation we have that $a_1 = a_4 = \cdots = a_n = 0$, and, in addition [3],

$$C = -\tfrac{2}{5}a_2P_1.$$

The system (28) gives us information about the normal modes of the linearized system; however, in this work we are interested in the next order of ε in order to obtain the modulation equations for time periodic solutions correct to O(ε). Hence, repeating the above manipulations for system (17) to order O(ε), and upon substitution of the perturbation expansion $a_2 = a_2^{(0)} + \varepsilon a_2^{(1)}$, $a_3 = a_3^{(0)} + \varepsilon a_3^{(1)}$, we obtain the following corrected version of Eqs. (28) that account for leading non-linear effects:

$$\frac{\partial^2 a_2}{\partial t^2} + \frac{8}{Bo}a_2 + \frac{6}{7}a_3 = -\varepsilon\left[\frac{9}{14}\left(\frac{\partial a_2^{(0)}}{\partial t}\right)^2 + \frac{16}{21}\left(\frac{\partial a_3^{(0)}}{\partial t}\right)^2 + \frac{5}{7}\frac{\partial^2 a_2^{(0)}}{\partial t^2}a_2^{(0)} + \frac{4}{7}\frac{\partial^2 a_3^{(0)}}{\partial t^2}a_3^{(0)} + \frac{40}{7Bo}a_2^{(0)^2} + \frac{176}{21Bo}a_3^{(0)^2}\right],$$

$$\frac{\partial^2 a_3}{\partial t^2} + \frac{9}{5}a_2 + \frac{30}{Bo}a_3 = -\varepsilon\left[\frac{16}{15}\frac{\partial a_2^{(0)}}{\partial t}\frac{\partial a_3^{(0)}}{\partial t} + \frac{14}{15}\frac{\partial^2 a_2^{(0)}}{\partial t^2}a_3^{(0)} + \frac{16}{15}\frac{\partial^2 a_3^{(0)}}{\partial t^2}a_3^{(0)} + \frac{384}{15Bo}a_2^{(0)}a_3^{(0)}\right]. \tag{29}$$

A special note regarding the notation used in (29) is in order. Due to the perturbation expansion used, the right-hand sides of these equations contain the O(1) approximations for the amplitudes, governed by the linearized system (28). The right-hand sides contain the non-linear estimates for the amplitudes, defined immediately above (29).

4. Multiple-scales formulation

The system (29) is in the form of a weakly non-linear dynamical system and, thus, is amenable to analysis techniques from the theory of non-linear dynamics. In what follows we will apply the method of multiple-scales to study the "slow" dynamics [14]. Before we proceed to the analysis a study of the linearized eigenfrequencies of (29) is undertaken in order to justify the truncation of the series (25). These eigenfrequencies are computed as,

$$\omega_1 = \sqrt{\frac{19}{Bo} + \sqrt{\frac{121}{Bo} + \frac{54}{35}}},$$

$$\omega_2 = \sqrt{\frac{19}{Bo} - \sqrt{\frac{121}{Bo} + \frac{54}{35}}}. \tag{30}$$

We now restrict our attention to the special case of 1 : 2 resonance between the second and third harmonics. This is achieved by selecting the parameter Bo so that ω_1 is nearly equal to $2\omega_2$. Under this condition, a 1 : 2 internal resonance is possible, that non-linearly couples the dynamics of the two modes giving rise to non-linear phenomena having no counterparts in linear theory. If no other low-order integral relationships (such as 1 : 4, 2 : 3, etc.) exist between the remaining harmonics and the second or third harmonic, no additional internal resonances exist and one expects that the remaining modes are uncoupled (to first-order) from the ones studied herein. In addition, the quadratic non-linearities of (29) are expected to excite only the 1 : 2 internal resonance and no other higher order resonances. Hence, our truncation to two harmonics is justified. From (30) the value of the parameter for exact 1 : 2 internal resonance is $Bo^* = 2.40986$. We introduce a tuning parameter σ_0 and choose the Bond number so that near resonance occurs:

$$Bo = Bo^* + \varepsilon\sigma_0. \tag{31}$$

Introducing (31) into (29) we obtain:

$$\frac{\partial^2 a_2}{\partial t^2} + \frac{8}{Bo^*}a_2 + \frac{6}{7}a_3 = \varepsilon\frac{8\sigma_0}{Bo^{*2}}a_2 - \varepsilon\left[\frac{9}{14}\left(\frac{\partial a_2}{\partial t}\right)^2 + \frac{16}{21}\left(\frac{\partial a_3}{\partial t}\right)^2 + \frac{5}{7}\frac{\partial^2 a_2}{\partial t^2}a_2 + \frac{4}{7}\frac{\partial^2 a_3}{\partial t^2}a_3 + \frac{40}{7Bo^*}a_2^2 + \frac{176}{21Bo^*}a_3^2\right]$$

$$\frac{\partial^2 a_3}{\partial t^2} + \frac{9}{5}a_2 + \frac{30}{Bo^*}a_3 = \varepsilon\frac{30\sigma_0}{Bo^{*2}}a_3 - \varepsilon\left[\frac{16}{15}\frac{\partial a_2}{\partial t}\frac{\partial a_3}{\partial t} + \frac{14}{15}\frac{\partial^2 a_2}{\partial t^2}a_3 + \frac{16}{15}\frac{\partial^2 a_3}{\partial t^2}a_3 + \frac{384}{15Bo^*}a_2 a_3\right] \tag{32}$$

and suppress the superscript star for Bo^* in further manipulations. We now introduce the following linear coordinate transformation that decouples the linear part of (32),

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = [\mathbf{C}]\begin{pmatrix} d_2 \\ d_3 \end{pmatrix}, \tag{33}$$

where

$[\mathbf{C}] = \begin{bmatrix} 1 & 1 \\ C_1 & C_2 \end{bmatrix}$ *is the matrix of eigenvectors, and*

$$C_1 = \frac{7}{6}\left(\omega_1^2 - \frac{8}{Bo}\right), \qquad C_2 = \frac{7}{6}\left(\omega_2^2 - \frac{8}{Bo}\right)$$

and analyze the resulting linearly decoupled system employing the method of multiple scales [14]. To this end, we introduce "slow" and "fast" scales, $T_0 = t$ and $T_1 = \varepsilon t$, and expand the amplitudes d_2 and d_3 in regular perturbation expansions using at the same time the new slow and fast independent variables. Then the following series of subproblems arises at different orders of approximation.

At order ε^0:

$$D_0^2 d_2^{(0)} + \omega_1^2 d_2^{(0)} = 0,$$

$$D_0^2 d_3^{(0)} + \omega_1^2 d_3^{(0)} = 0, \tag{34}$$

with solution,

$$d_2^{(0)} = A_2 e^{i\omega_1 T_0} + \text{c.c.}$$

$$d_3^{(0)} = A_3 e^{i\omega_2 T_0} + \text{c.c.}$$

where c.c. denotes the complex conjugate, $D_0 \equiv d/dT_0$ and $i = (-1)^{1/2}$. The complex amplitudes are in turn expressed as

$$A_2 = \tfrac{1}{2}\alpha_2 e^{i\beta_2}; \quad A_3 = \tfrac{1}{2}\alpha_3 e^{i\beta_3}$$

where the real amplitudes α_i and phases β_i, $i = 1, 2$, are determined at the next order of approximation.

At order ε^1:

$$D_0^2 d_2^{(1)} + \omega_1^2 d_2^{(1)} = U_4 \sigma_0 d_2^{(0)} - 2D_0 D_1 d_2^{(0)} + U_1[D_0 d_3^{(0)}]^2 + U_2[D_0^2 d_3^{(0)}] d_3^{(0)} + U_3[d_3^{(0)}]^2 + \text{c.c.} + \text{unimportant terms}$$

$$D_0^2 d_3^{(1)} + \omega_1^2 d_3^{(1)} = V_5 \sigma_0 d_3^{(0)} - 2D_0 D_1 d_3^{(0)} + V_1[D_0 d_2^{(0)}][D_0 d_3^{(0)}] + V_2[D_0^2 d_2^{(0)}] d_3^{(0)} + V_3[D_0^2 d_3^{(0)}] d_2^{(0)} + V_4 d_2^{(0)} d_3^{(0)} + \text{c.c.} + \text{unimportant terms}, \quad (35)$$

where $D_1 \equiv d/dT_1$, we retained only secular terms for 1 : 2 resonance [3] and the various coefficients are defined by,

$$U_1 = \frac{1}{C_2 - C_1}\left(\tfrac{16}{15}C_2 - \tfrac{9}{14}C_2 - \tfrac{16}{21}C_2^2\right),$$

$$U_2 = \frac{1}{C_2 - C_1}\left(\tfrac{16}{15}C_2 - \tfrac{5}{7}C_2 - \tfrac{4}{7}C_2^2\right),$$

$$U_3 = \frac{1}{C_2 - C_1}\left(\frac{384}{15Bo}C_2 - \frac{176}{21Bo}C_2^3 - \frac{40}{7Bo}C_2\right),$$

$$U_4 = \frac{8C_2 - 30C_1}{C_2 - C_1}\frac{\sigma_0}{Bo^2}, \quad (36)$$

$$V_1 = \frac{1}{C_2 - C_1}\left(\tfrac{9}{7}C_1 + \tfrac{32}{21}C_1^2 C_2 - \tfrac{16}{15}(C_1 + C_2)\right),$$

$$V_2 = \frac{1}{C_2 - C_1}\left(\tfrac{5}{7}C_1 + \tfrac{4}{7}C_1^2 C_2 - \tfrac{14}{15}C_2\right),$$

$$V_3 = \frac{1}{C_2 - C_1}\left(\tfrac{5}{7}C_1 + \tfrac{4}{7}C_1^2 C_2 - \tfrac{14}{15}C_1\right),$$

$$V_4 = \frac{1}{C_2 - C_1}\left(\frac{80}{7Bo}C_1 + \frac{352}{21Bo}C_1^2 C_2 - \frac{384}{15Bo}(C_1 + C_2)\right),$$

$$V_5 = \frac{30C_2 - 8C_1}{C_2 - C_1}\frac{\sigma_0}{Bo^2}.$$

Introducing the new tuning parameter σ so that the two frequencies are related by $\omega_1 = 2\omega_2 + \sigma\varepsilon$,

$$\sigma = k\sigma_0, \quad (37)$$

where

$$k = \frac{1}{2\omega_2^2 Bo^2}\left(19 - \frac{363}{Bo^2}\frac{1}{\left(\frac{121}{Bo^2} + \frac{54}{35}\right)^{1/2}}\right),$$

and eliminating secular terms from (35), we get the following modulation equations:

$$i\frac{\partial\alpha_2}{\partial T_1}e^{i\beta_2} - \alpha_2\frac{\partial\beta_2}{\partial T_1}e^{i\beta_2} + U_4\frac{\sigma\alpha_2}{2\omega_1 k}e^{i\beta_2} + \frac{\alpha_3^2}{4\omega_1}(\omega_2^2(U_1 + U_2) - U_3)e^{2i\beta_3 - i\sigma T_1} = 0, \quad (38)$$

$$i\frac{\partial\alpha_3}{\partial T_1}e^{i\beta_3} - \alpha_3\frac{\partial\beta_3}{\partial T_1}e^{i\beta_3} + V_5\frac{\sigma\alpha_3}{2\omega_2 k}e^{i\beta_3} + \frac{\alpha_2\alpha_3}{4\omega_2}(\omega_2\omega_1 V_1 + V_2\omega_1^2 + V_3\omega_2^2 - V_4)e^{i\beta_2 + i\sigma T_1 - i\beta_3} = 0,$$

or

$$i\frac{\partial\alpha_2}{\partial T_1} - \alpha_2\frac{\partial\beta_2}{\partial T_1} + \alpha_3^2 I_1 e^{2i\beta_3 - i\beta_2 - i\sigma T_1} + I_3\sigma\alpha_2 = 0,$$

$$i\frac{\partial\alpha_3}{\partial T_1} - \alpha_3\frac{\partial\beta_3}{\partial T_1} + \alpha_3\alpha_2 I_2 e^{i\beta_2 - 2i\beta_3 + i\sigma T_1} + I_4\sigma\alpha_3 = 0. \quad (39)$$

Setting $\phi = 2\beta_3 - \beta_2 - \sigma T_1$, and separating real and imaginary parts in (39) we obtain the following set of four first-order equations:

$$\frac{\partial \alpha_2}{\partial T_1} + \alpha_3^2 I_1 \sin\phi = 0,$$

$$\frac{\partial \alpha_3}{\partial T_1} - \alpha_3 \alpha_2 I_2 \sin\phi = 0, \tag{40}$$

$$\alpha_2 \frac{\partial \beta_2}{\partial T_1} - \alpha_3^2 I_1 \cos\phi - I_3 \sigma \alpha_2 = 0,$$

$$\alpha_3 \frac{\partial \beta_3}{\partial T_1} - \alpha_3 \alpha_2 I_2 \cos\phi - I_4 \sigma \alpha_2 = 0,$$

where

$$I_1 = \frac{\omega_2^2 (U_1 + U_2) - U_3}{4\omega_1},$$

$$I_2 = \frac{\omega_2 \omega_1 V_1 + V_2 \omega_1^2 + V_3 \omega_2^2 - V_4}{4\omega_2},$$

$$I_3 = \frac{U_4}{2\omega_1 k}, \quad I_4 = \frac{V_5}{2\omega_2 k}.$$

Introducing the polar transformation

$$\alpha_2 = \alpha \cos\gamma, \quad \alpha_3 = \alpha \left(\frac{I_2}{I_1}\right)^{1/2} \sin\gamma,$$

where $\gamma = \beta_2 - \beta_3$, (41)

we get the relations

$$\tfrac{1}{2}(\alpha^2)' = 0 \Rightarrow \alpha = \text{const},$$

$$\phi' = \sigma I_5 + \alpha I_2 \cos\phi \left(2\cos\gamma - \frac{\sin\gamma^2}{\cos\gamma}\right),$$

$$\gamma' = \alpha I_2 \sin\gamma \sin\phi, \tag{42}$$

where $I_5 = 2I_4 - I_3 - 1$. Eqs. (42) constitute a dynamical system of the two-Torus and $\gamma, \phi \in S \times S$. The first of the above equations is equivalent to a statement of energy conservation during the non-linear modal interactions between the second and third harmonics of the drop. Hence, the analysis is performed on an isoenergetic manifold. Moreover, it turns out that system (42) is integrable with a first integral of motion given by

$$K(\gamma, \phi) = s\cos\gamma^2 + 2\cos\gamma \sin\phi^2 \cos\phi, \tag{43}$$

where $s = \sigma I_5 / I_2 \alpha$. Equilibrium (singular) points of (41) correspond to points where $\mathrm{d}K/\mathrm{d}\gamma = 0$ and $\mathrm{d}K/\mathrm{d}\phi = 0$. The latter relation gives

$$\cos\gamma \sin\phi^2 \cos\phi = 0 \Rightarrow \phi = 0, \pi$$

and

$$\gamma = 0, \pi, \pi/2, 3\pi/2. \tag{44}$$

Substituting each of these possibilities into the relation $\mathrm{d}K/\mathrm{d}\gamma = 0$, gives

$$2\cos\phi - s = 0,$$

$$3\cos\gamma^2 - s\cos\gamma - 1 = 0. \tag{45}$$

Taking into account that $|\cos\gamma| \leqslant 1$ and $|\cos\phi| \leqslant 1$ we get the bifurcation value of s:

$$s_{\text{bif}} = \pm 2 \tag{46}$$

The lines $\gamma = 0, \pi$ are invariant lines in the phase plot of system (42), and $\gamma = \pi/2$ and $3\pi/2$ lead to $\cos\phi = 0$ or $\phi = \pi/2, 3\pi/2$. So the only possible bifurcations can occur at $s = \pm 2$ which corresponds to $\sigma = \pm 7.86$ for $\alpha = 1$.

In Fig. 1 we present the phase trajectories of (42) for two different values of the tuning parameter, $\sigma = 7.0$ (Fig. 1a, before the bifurcation) and $\sigma = 8.0$ (Fig. 1b, after the bifurcation). We note that the Saddle-node bifurcation as σ is decreased past 7.86 generates an stable-unstable pair of equilibrium points and a homoclinic loop connecting the unstable equilibrium with itself.

In order to determine what is the effect of the bifurcation on the surface shape of the drop we resort to the old amplitudes, computed by

$$\begin{aligned} a_2 = {} & \cos\gamma(\varepsilon t)\cos(2\omega_2 t + 2\gamma(\varepsilon t) + \phi(\varepsilon t) \\ & + \varepsilon\sigma t + \mathrm{O}(\varepsilon^2)) + \sin\gamma(\varepsilon t)\cos(\omega_2 t + \gamma(\varepsilon t) \\ & + \phi(\varepsilon t) + \varepsilon\sigma t + \mathrm{O}(\varepsilon^2)) + \mathrm{O}(\varepsilon). \end{aligned}$$

$$\begin{aligned} a_3 = {} & C_1 \cos\gamma(\varepsilon t)\cos(2\omega_2 t + 2\gamma(\varepsilon t) + \phi(\varepsilon t) \\ & + \varepsilon\sigma t + \mathrm{O}(\varepsilon^2)) + C_2 \sin\gamma(\varepsilon t)\cos(\omega_2 t + \gamma(\varepsilon t) \\ & + \phi(\varepsilon t) + \varepsilon\sigma t + \mathrm{O}(\varepsilon^2)) + \mathrm{O}(\varepsilon). \end{aligned} \tag{47}$$

Now, in order to observe the effect of the bifurcation on the drop oscillations we select the initial

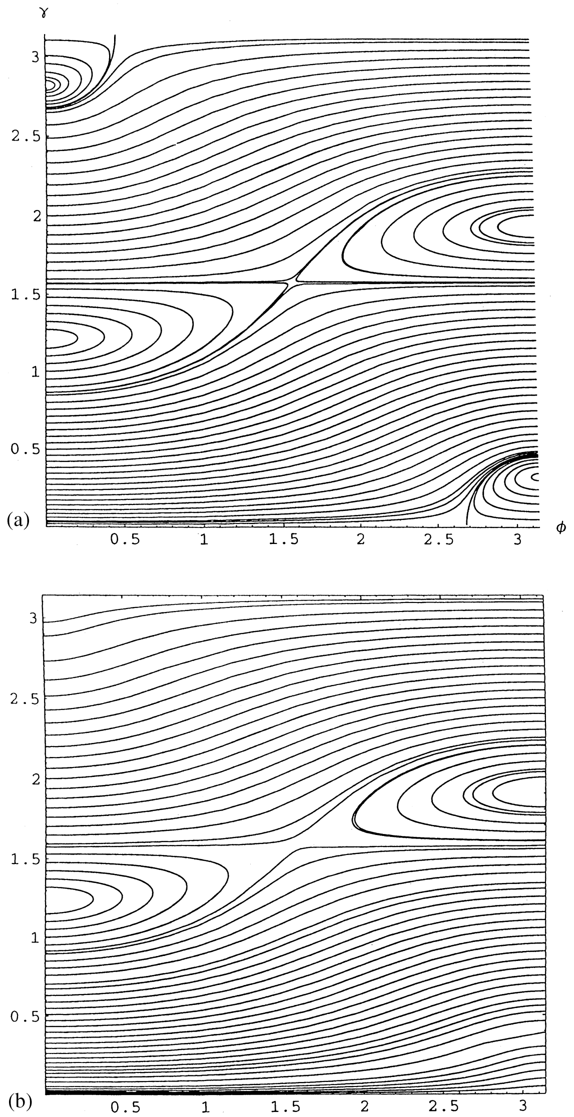

Fig. 1. (a) Phase portrait of the system before the bifurcation $\sigma = 7.0$. (b) Phase portrait of the system after bifurcation $\sigma = 8.0$.

conditions, corresponding to the homoclinic loop for "slow" variables, before ($\sigma = 7.0$) and after ($\sigma = 8.0$) the bifurcation. The resulting transient responses for variables $a_2(t)$ and $a_3(t)$ are depicted in Figs. 2 and 3. We note that on the homoclinic orbit the envelope of mode 2 decays with time, and, as a result, after sufficiently long time the drop oscillates predominantly in the third spherical mode. Close to the homoclinic orbit the second mode undergoes fast oscillations with slow envelope modulations of long periods. When no homoclinic orbit exists the second mode undergoes large-amplitude modulated oscillations, with shorter period envelope modulations.

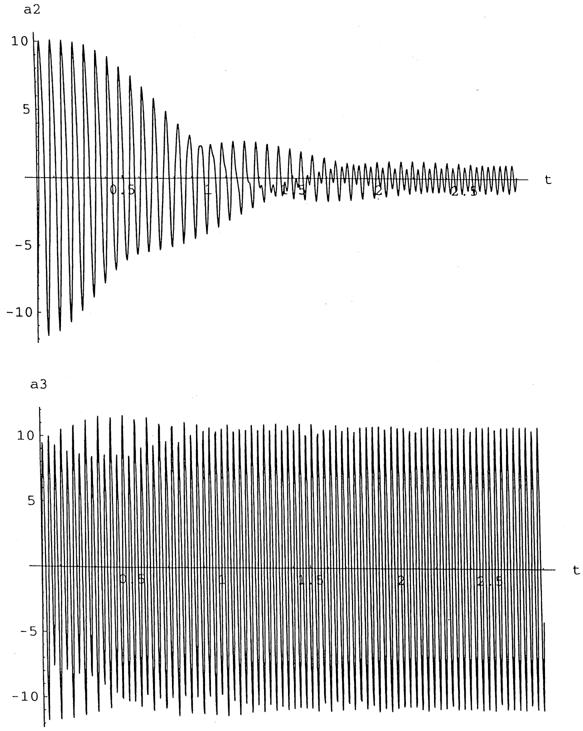

Fig. 2. Transient responses $\sigma = 7.0$.

5. Discussion

We considered a drop of inviscid, incompressible fluid in a gravitational field. In contrast to most previous linearized studies, we included leading-order non-linear effects in our model, and preceeded to an analytical study of the drop oscillations. Specifically, we studied a 2 : 1 internal resonance between the second and third spherical modes of the drop, and used the method of multiple-scales to study the slow-flow modal interactions (energy exchanges) between the two modes. An interesting feature of our analysis is that it leads

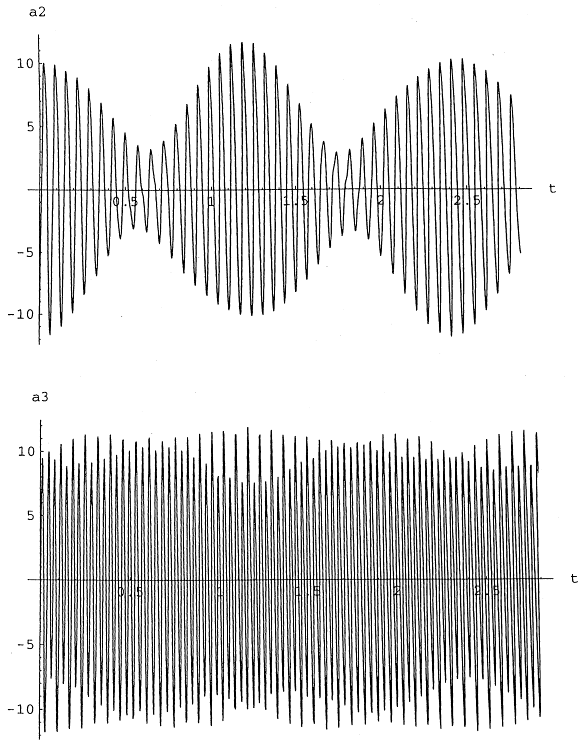

Fig. 3. Transient responses $\sigma = 8.0$.

to a completely integrable system on the 2-torus. The analysis reveals that a bifurcation occurs in the slow flow that leads to long-time amplitude modulations of the drop oscillations. After the bifurcation (when no homoclinic orbit in the slow dynamics exists), the amplitude modulations are of shorter duration and of larger amplitude and much more rigorous non-linear energy exchange occurs between modes. As a general result, the homoclinic orbit leads to energy localization in the third (higher) interacting mode. The analysis can be extended to bubbles (in the absence of pulsations), and to higher-order modal interactions in drops.

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