

Non-linear multipole interactions and gravitational-wave octupole modes for inspiralling compact binaries to third-and-a-half post-Newtonian order

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Abstract

This paper is motivated by the need to improve the post-Newtonian (PN) amplitude accuracy of waveforms for gravitational waves generated by inspiralling compact binaries, both for use in data analysis and in the comparison between post-Newtonian approximations and numerical relativity computations. It presents (i) the non-linear couplings between multipole moments of general post-Newtonian matter sources up to order 3.5PN, including all contributions from tails, tails-of-tails and the non-linear memory effect; and (ii) the source mass-type octupole moment of (non-spinning) compact binaries up to order 3PN, which permits completion of the expressions of the octupole modes (3, 3) and (3, 1) of the gravitational waveform to order 3.5PN. On this occasion we reconfirm by means of independent calculations our earlier results concerning the source mass-type quadrupole moment to order 3PN. Related discussions on factorized resummed waveforms and the occurrence of logarithmic contributions to high order are also included.

Keywords: post-Newtonian approximation, perturbation theory, related approximations, gravitational waves, neutron stars, black holes

1. Introduction

Coalescing compact binaries—two neutron stars or black holes in their late stage of evolution prior to final coalescence—should be the *workhorse* source driving the network of advanced

interferometric gravitational-wave detectors on ground. The post-Newtonian (PN) approximation is the appropriate technique to extract accurate and reliable predictions from general relativity theory for the inspiral phase of these systems. This constitutes the starting point, in data analysis, to construct templates for double neutron-star binaries and is a crucial input to validate the early inspiral phase of the numerical relativity waveforms for black-hole binaries.

The non-linear evolution of the orbital phase due to gravitational radiation reaction is the crucial ingredient in constructing these templates. It has been completed for non-spinning compact binaries up to order 3.5PN [1–7]³. The amplitude of the signal, including all signal harmonics besides the dominant one at twice the orbital frequency, has been computed over the years with increasing precision and is now complete to order 3PN [8–12]. Furthermore, the dominant quadrupole mode (2, 2) is also known to order 3.5PN [13]. Our current program consists in extending this computation and obtaining the full waveform up to order 3.5PN for all the modes (ℓ, m) in a spin-weighted spherical-harmonic decomposition. In this paper we will, as key milestones for this program:

1. Control all the non-linear couplings between multipole moments up to order 3.5PN for general matter sources. Those couplings involve the important contributions of tails, tails-of-tails, and the non-linear memory effect, as well as some extra contributions due to our specific definitions for the source multipole moments.
2. Obtain the source mass-type *octupole* moment of (non-spinning) compact binaries up to order 3PN, which allows us to obtain the expressions of the octupole modes (3, 3) and (3, 1) of the waveform to order 3.5PN. We will take this opportunity to recompute, using our new programs, the mass-type quadrupole moment to order 3PN and confirm the earlier results [5–7].

The full completion of our program will have to wait for the more difficult computation of the source current-type quadrupole moment to order 3PN, which is left for future work.

The plan of this paper is the following. Section 2 is a recapitulation of the basic definitions we use for source and canonical multipole moments within the multipolar-post-Minkowskian (MPM) formalism. In section 3 we present (without proof) the expressions of the radiative moments seen at infinity in terms of canonical ones up to order 3.5PN for general matter sources (including the various tail and memory effects), and the explicit links between canonical and source moments. Next section 4 deals with the waveform of non-spinning compact binary sources. Notably, in section 4.2, we compute the mass-type octupole moment for general orbits to the 3PN order, reduce it to the center-of-mass frame and then to circular orbits, and, in section 4.3, we obtain the gravitational-wave modes (3, 3) and (3, 1) up to order 3.5PN for circular orbits. Section 5 is devoted to the occurrence of logarithmic contributions in the MPM waveform to arbitrary high non-linear orders, with an application to factorized resummed waveforms in the effective-one-body (EOB) approach. The paper ends with three more technical appendices.

2. Multipolar post-Minkowskian expansion

We look for the solution of the Einstein field equations in the vacuum region outside the compact support of a general isolated matter source. With $h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$ denoting the ‘gothic’ metric deviation, where g and $g^{\alpha\beta}$ are respectively the determinant and the inverse of

³ As usual n PN order refers to the terms of order $1/c^{2n}$ in the waveform and energy flux, beyond the Einstein quadrupole formula, which is referred to as the Newtonian approximation.

the ‘covariant’ metric $g_{\alpha\beta}$ and where $\eta^{\alpha\beta} \equiv \text{diag}(-1, 1, 1, 1)$ stands for the Minkowski metric in Cartesian coordinates, the vacuum field equations relaxed by the harmonic-gauge condition read

$$\square h^{\alpha\beta} = \Lambda^{\alpha\beta} [h, \partial h, \partial^2 h], \quad (2.1a)$$

$$\partial_\beta h^{\alpha\beta} = 0. \quad (2.1b)$$

Here $\square \equiv \eta^{\alpha\beta} \partial_\alpha \partial_\beta$ denotes the flat d’Alembertian operator, whereas the non-linear gravitational source term $\Lambda^{\alpha\beta}$ is an expression of second order (at least) in the space-time components $h^{\gamma\delta}$, which is quadratic in the first space-time derivatives symbolized by ∂h and linear in the second space-time derivatives $\partial^2 h$.

The multipolar-post-Minkowskian (MPM) expansion [14] is an algorithmic procedure for generating iteratively the most general solution to the field equations (2.1) in the form of a post-Minkowskian (or non-linearity) expansion whose coefficients are themselves given by a multipole expansion that is physically valid outside the compact support of the source. The multipole expansion is parameterized by certain multipole moments characterizing the matter source but left, in a first stage, as some unspecified functions of time. However, among these moments, the mass monopole M as well as the mass and current dipoles, M_i and S_i respectively, are constrained to be constant or to vary linearly with time; they represent the ADM conserved mass, linear momentum, and total angular momentum of the source. In this paper we will work in a mass-centered frame such that $M_i = 0$. Furthermore, an important assumption of the MPM formalism is the stationarity in the past, namely the fact that the matter source has been stationary in the remote past, before some given date $-\mathcal{T}$. Thus, all multipole moments we will consider are assumed to be constant when $t \leq -\mathcal{T}$.

The starting point of the MPM algorithm is Thorne’s [15] linearized vacuum solution parameterized by two types of multipole moments, called the *source* moments: the mass-type moments $I_L(t)$ and the current-type moments $J_L(t)$; they are such that $I = M$, $I_i = M_i = 0$, and $J_i = S_i$ are constant. Such a general linearized solution, referred to as ‘canonical’, reads⁴

$$h_{\text{can}(1)}^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L [r^{-1} I_L(t_r)], \quad (2.2a)$$

$$h_{\text{can}(1)}^{0i} = \frac{4}{c^3} \sum_{\ell=1}^{+\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} [r^{-1} I_{iL-1}^{(1)}(t_r)] + \frac{\ell}{\ell+1} \varepsilon_{iab} \partial_{aL-1} [r^{-1} J_{bL-1}(t_r)] \right\}, \quad (2.2b)$$

⁴ Our notation is as follows. The retarded time is denoted as $t_r \equiv t - r/c$. The n th time derivatives of multipole moments are indicated by superscripts (n) . $L = i_1 \dots i_\ell$ denotes a multi-index composed of ℓ spatial indices (ranging from 1 to 3); $aL-1 = a i_1 \dots i_{\ell-1}$ and so on; $\partial_L = \partial_{i_1} \dots \partial_{i_\ell}$ is the ‘product’ of ℓ partial derivatives $\partial_i \equiv \partial/\partial x^i$; similarly $x_L = x_{i_1} \dots x_{i_\ell}$ with x_i being the spatial position, and $n_L = n_{i_1} \dots n_{i_\ell}$ with $n_i = x_i/r$. Symmetrization over indices is denoted by $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$. The symmetric-trace-free (STF) projection is indicated with a hat, i.e., $\hat{n}_L \equiv \text{STF}[n_L]$, or by angular brackets $\langle \rangle$ surrounding the relevant indices, e.g., $\hat{n}_{ijk} = n_{(ijk)} = n_i n_j n_k - \frac{1}{5}[\delta_{ij} n_k + \delta_{jk} n_i + \delta_{ki} n_j]$. Underlined indices mean that they should be excluded from the STF projection, e.g., $T_{(i\hat{a}j)} = \frac{1}{2}(T_{i\hat{a}j} + T_{\hat{a}ji}) - \frac{1}{3}\delta_{ij} T_{k\hat{a}k}$. The multipole moments we use, $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$, $\{M_L, S_L\}$, and $\{U_L, V_L\}$, are all STF; hence, e.g., $I_L = \hat{I}_L = I_{(L)}$. In the case of summed-up multi-indices L , we do not write the ℓ summations from 1 to 3 over the dummy indices. The Levi-Civita antisymmetric symbol is denoted ε_{iab} (with $\varepsilon_{123} = 1$).

$$h_{\text{can}(1)}^{ij} = -\frac{4}{c^4} \sum_{\ell=2}^{+\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-2} \left[r^{-1} I_{ijL-2}^{(2)}(t_r) \right] + \frac{2\ell}{\ell+1} \partial_{aL-2} \left[r^{-1} \varepsilon_{ab(i} J_{j)bL-2}^{(1)}(t_r) \right] \right\}. \quad (2.2c)$$

It satisfies the relaxed linearized vacuum field equations $\square h_{\text{can}(1)}^{\alpha\beta} = 0$ and the harmonic gauge condition $\partial_\beta h_{\text{can}(1)}^{\alpha\beta} = 0$ formally at any point except $r = 0$. However, this solution is not the most general one, as we can always perform an arbitrary linearized gauge transformation maintaining the harmonic-gauge condition. Introducing an arbitrary gauge vector $\varphi_{(1)}^\alpha$ satisfying $\square \varphi_{(1)}^\alpha = 0$ (except at $r = 0$), which will be parameterized by four supplementary types of (unconstrained) multipole moments $W_L(t)$, $X_L(t)$, $Y_L(t)$, and $Z_L(t)$, called the *gauge* moments, we can write

$$\varphi_{(1)}^0 = \frac{4}{c^3} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[r^{-1} W_L(t_r) \right], \quad (2.3a)$$

$$\varphi_{(1)}^i = -\frac{4}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_{iL} \left[r^{-1} X_L(t_r) \right] - \frac{4}{c^4} \sum_{\ell=1}^{+\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left[r^{-1} Y_{iL-1}(t_r) \right] + \frac{\ell}{\ell+1} \varepsilon_{iab} \partial_{aL-1} \left[r^{-1} Z_{bL-1}(t_r) \right] \right\}. \quad (2.3b)$$

The linear gauge terms take the form $\partial\varphi_{(1)}^{\alpha\beta} \equiv \partial^\alpha \varphi_{(1)}^\beta + \partial^\beta \varphi_{(1)}^\alpha - \eta^{\alpha\beta} \partial_\gamma \varphi_{(1)}^\gamma$ so that the most general linearized vacuum solution in harmonic coordinates reads

$$h_{\text{gen}(1)}^{\alpha\beta} = h_{\text{can}(1)}^{\alpha\beta} [I_L, J_L] + \partial\varphi_{(1)}^{\alpha\beta} [W_L, X_L, Y_L, Z_L]. \quad (2.4)$$

Starting from $h_{\text{gen}(1)}$ the MPM algorithm will generate a full post-Minkowskian solution of the field equations (2.1), i.e., a solution given as a formal non-linear expansion series in powers of Newton's constant G , as shown later in equation (2.10). Suppose one has succeeded in generating all the post-Minkowskian coefficients up to some order $n-1$, say, $h_{\text{gen}(2)}, \dots, h_{\text{gen}(n-1)}$. Then the precise procedure by which the next post-Minkowskian coefficient, i.e., $h_{\text{gen}(n)}$, is generated is as follows [14]. One decomposes this coefficient into two terms,

$$h_{\text{gen}(n)}^{\alpha\beta} = u_{\text{gen}(n)}^{\alpha\beta} + v_{\text{gen}(n)}^{\alpha\beta}. \quad (2.5)$$

The first one is defined as the standard (flat) retarded integral, denoted $\square_{\text{ret}}^{-1}$, of the iterated source term coming from the relaxed Einstein field equation (2.1a). Namely, after obtaining from the previous iterations the n th post-Minkowskian order source term as some $\Lambda_{(n)} = \Lambda_{(n)} [h_{\text{gen}(1)}, \dots, h_{\text{gen}(n-1)}]$, we pose

$$u_{\text{gen}(n)}^{\alpha\beta} = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\tilde{r}^B \Lambda_{(n)}^{\alpha\beta} \right]. \quad (2.6)$$

Crucial to the MPM algorithm is the regularization process based on analytic continuation in a complex parameter B which enters a regulator factor,

$$\tilde{r}^B \equiv \left(\frac{r}{r_0} \right)^B, \quad (2.7)$$

multiplying the source term. Here r_0 is an arbitrary constant length scale. The regulator (2.7) makes it possible, thanks to analytic continuation, to cure the divergency of the multipole expansion when $r \rightarrow 0$ that follows from the fact that the vacuum solution is physically valid only outside the matter source and is yet to be matched to the actual solution inside it⁵. Finally, an operation of taking the finite part (FP), i.e., picking up the term with zeroth power of B in the Laurent expansion of the expression when $B \rightarrow 0$, is applied. This fully defines the expression (2.6) as a particular solution of $\square u_{\text{gen}(n)} = \Lambda_{\text{gen}(n)}$ everywhere except at $r = 0$.

The second term in equation (2.5) ensures that the harmonic gauge condition $\partial_\beta h_{\text{gen}(n)}^{\alpha\beta} = 0$ is satisfied. It is algorithmically computed from the divergence of the first term, namely $w_{\text{gen}(n)}^\alpha \equiv \partial_\beta u_{\text{gen}(n)}^{\alpha\beta}$, which is necessarily a retarded solution of the source-free d'Alembertian equation, $\square w_{\text{gen}(n)}^\alpha = 0$. That solution can thus always be written as

$$w_{\text{gen}(n)}^0 = \sum_{\ell=0}^{+\infty} \partial_L \left[r^{-1} N_L(t_r) \right], \quad (2.8a)$$

$$w_{\text{gen}(n)}^i = \sum_{\ell=0}^{+\infty} \partial_{iL} \left[r^{-1} P_L(t_r) \right] + \sum_{\ell=1}^{+\infty} \left\{ \partial_{L-1} \left[r^{-1} Q_{iL-1}(t_r) \right] + \varepsilon_{iab} \partial_{aL-1} \left[r^{-1} R_{bL-1}(t_r) \right] \right\}, \quad (2.8b)$$

where the STF multipole moments $\{N_L, P_L, Q_L, R_L\}$ are given by some (very complicated at high post-Minkowskian orders n) functionals of the initial source and gauge moments $\{L, J_L, W_L, X_L, Y_L, Z_L\}$. We then pose [14, 17]

$$v_{\text{gen}(n)}^{00} = -c r^{-1} N^{(-1)} + \partial_a \left[r^{-1} \left(-c N_a^{(-1)} + c^2 Q_a^{(-2)} - 3P_a \right) \right], \quad (2.9a)$$

$$v_{\text{gen}(n)}^{0i} = r^{-1} \left(-c Q_i^{(-1)} + 3c^{-1} P_i^{(1)} \right) - \varepsilon_{iab} \partial_a \left[r^{-1} c R_b^{(-1)} \right] - \sum_{\ell=2}^{+\infty} \partial_{L-1} \left[r^{-1} N_{iL-1} \right], \quad (2.9b)$$

$$v_{\text{gen}(n)}^{ij} = -\delta_{ij} r^{-1} P + \sum_{\ell=2}^{+\infty} \left\{ 2\delta_{ij} \partial_{L-1} \left[r^{-1} P_{L-1} \right] - 6\partial_{L-2(i} \left[r^{-1} P_{j) L-2} \right] + \partial_{L-2} \left[r^{-1} \left(c^{-1} N_{ijL-2}^{(1)} + 3c^{-2} P_{ijL-2}^{(2)} - Q_{ijL-2} \right) \right] - 2\partial_{aL-2} \left[r^{-1} \varepsilon_{ab(i} R_{j)bL-2} \right] \right\}. \quad (2.9c)$$

It can readily be verified that $\partial_\beta v_{\text{gen}(n)}^{\alpha\beta} = -w_{\text{gen}(n)}^\alpha$; hence $\partial_\beta h_{\text{gen}(n)}^{\alpha\beta} = 0$. Since we also have $\square v_{\text{gen}(n)}^{\alpha\beta} = 0$, we see that the n th post-Minkowskian order piece of the gravitational field (2.5) satisfies the relaxed field equations in harmonic coordinates at order n . Note the presence in equations (2.9) of anti-derivatives, denoted e.g., $N^{(-1)}$, which are associated with the secular losses of energy, linear momentum, and angular momentum of the source through gravitational radiation.

⁵ The matching to a general isolated post-Newtonian matter source in the external near zone of this source has been elucidated within this formalism in [16–19].

Finally, we get a full solution to the vacuum Einstein field equations (2.1), parameterized by two sets of source moments I_L, J_L and four sets of gauge moments W_L, X_L, Y_L, Z_L , in the form of the post-Minkowskian expansion series

$$h_{\text{gen}}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_{\text{gen}(n)}^{\alpha\beta} [I_L, J_L, W_L, X_L, Y_L, Z_L]. \quad (2.10)$$

It was proved [14] that this represents physically the most general solution to the vacuum field equations outside an isolated matter system. Thanks to the matching, all the multipole moments therein have been given explicit closed-form expressions as integrals over the matter and gravitational fields of a general post-Newtonian source [17, 18].

The explicit MPM construction leading to equation (2.10) is quite complicated in practice but now entirely performed on a computer⁶. It is often convenient to simplify it by considering, instead of the six sets of source and gauge moments, only two, called the *canonical* mass-type and current-type multipole moments, $M_L(t)$ and $S_L(t)$ respectively. Indeed it has been proved [14, 15] that the most general solution is actually parameterized by two and only two sets of moments—by definition these canonical M_L and S_L moments. The simplest MPM construction, here referred to as ‘canonical’, is obtained by annulling all the gauge moments in equation (2.10) and starting with M_L, S_L in place of I_L, J_L , i.e.,

$$h_{\text{can}}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_{\text{gen}(n)}^{\alpha\beta} [M_L, S_L, 0, 0, 0, 0]. \quad (2.11)$$

This means that the iteration now begins at linearized order with the solution $h_{\text{can}(1)} [M_L, S_L]$. However, even if we proceed with the simpler construction (2.11), we still have to relate the canonical moments $\{M_L, S_L\}$ to the source and gauge moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ because only the latter are known as explicit integrals over the matter and gravitational fields of the source. To relate these two sets, we posit that the two constructions (2.10) and (2.11) are to be *isometric*, i.e., to differ by a coordinate transformation. It can be shown—see notably [12] for an explicit derivation at quadratic order—that this yields unique relations of the type

$$M_L = I_L + \mathcal{M}_L[I, J, W, X, Y, Z], \quad (2.12a)$$

$$S_L = J_L + \mathcal{S}_L[I, J, W, X, Y, Z], \quad (2.12b)$$

where \mathcal{M}_L and \mathcal{S}_L denote some non-linear functionals of the source and gauge moments that are at least quadratic and start only at the high order 2.5PN. When the relations (2.12) are satisfied, the two sets of moments $\{M_L, S_L\}$ and $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ describe the same physical matter source. We give in section 3.2 their most up-to-date explicit forms.

3. The radiative multipole moments

In the preceding section we reviewed the MPM solutions (2.10)–(2.11), which are valid all over the exterior of the source, in particular at future null infinity. However, these solutions exhibit a logarithmic far-zone structure $\sim(\ln r)^p/r^k$ when expanded as $r \rightarrow +\infty$ with $t_r \equiv t - r/c = \text{const}$, where t and r refer to the harmonic coordinates (see also section 5.1). This is due to the well-known logarithmic deviation of the null cones with respect to the retarded cones $t - r/c$ in this coordinate grid. It is thus convenient to introduce so-called *radiative* coordinates (T, R) such that $T_R \equiv T - R/c$ is a null coordinate, or becomes asymptotically null in the limit $R \rightarrow +\infty$. We then have (with the angular coordinates being untouched)

⁶ The MPM algorithm is implemented by using the algebraic computing software Mathematica together with the tensor package *xAct* [20]. Intermediate results along with the source codes can be provided by the authors on request.

$$T_R = t_r - \frac{2GM}{c^3} \ln\left(\frac{r}{cb}\right) + \mathcal{O}\left(\frac{1}{r}\right), \quad (3.1)$$

where M is the total mass of the source and b is an arbitrary constant time scale, *a priori* unrelated to the constant r_0 introduced in the MPM regulator (2.7). In radiative coordinates the structure of the expansion when $R \rightarrow +\infty$ with $T_R = \text{const}$ is merely $\sim 1/R^k$ [21].

The STF radiative moments $\{U_L, V_L\}$ are then defined from the leading $1/R$ term of the asymptotic waveform by [15]

$$g_{ij}^{\text{TT}} = \delta_{ij} + \frac{4G}{c^2 R} \left[\sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{ijL-2}(T_R) - \frac{2\ell}{c(\ell+1)} N_{aL-2} \varepsilon_{ab(i} V_{j)bL-2}(T_R) \right\} \right]^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (3.2)$$

where the superscript TT refers to the usual algebraic transverse-traceless projection. In what follows we present (without the full derivations) the expressions of the radiative multipole moments needed to control the waveform up to order 3.5PN. These results are obtained by implementing the MPM algorithm reviewed in the preceding section.

Our goal being to obtain the radiative moments $\{U_L, V_L\}$ as functionals of the source and gauge moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$, it is useful to know beforehand which types of interactions between any moments A_J and B_K (with j and k indices respectively), among the set of source and gauge moments, are allowed in a given radiative mass moment U_L or current moment V_L up to the 3.5PN order. To answer that question in the case of quadratic interactions, say, $A_J \times B_K$ where $A_J, B_K \in \{I_L, J_L, W_L, X_L, Y_L, Z_L\}$, we have developed some ‘selection rules’ following [12, 13]. The interactions allowed by those rules at order 3.5PN are given in table 1.

The two panels of table 1 show the maximal number of indices $\ell_{\text{max}}[A_J, B_K]$ on the mass moment U_L and the current moment V_L (respectively), beyond which U_L or V_L cannot contain products of the two multipole moments A_J and B_K or their time derivatives (or possibly time anti-derivatives) at the 3.5PN order in the waveform. Thus, when $\ell > \ell_{\text{max}}[A_J, B_K]$ we can safely ignore the multipole interaction $A_J \times B_K$ in U_L or V_L since it will be of higher PN order. When $\ell \leq \ell_{\text{max}}[A_J, B_K]$ we can deduce all the possible relevant interactions $A_J \times B_K$ by noticing that $j + k = \ell_{\text{max}}[A_J, B_K]$ if the product $A_J B_K$ has the same parity as the radiative moment containing the interaction (i.e., both A_J and B_K are mass moments or both are current moments in U_L ; one is a mass moment and the other is a current moment in V_L) and $j + k = \ell_{\text{max}}[A_J, B_K] + 1$ in all other cases⁷. Finally, the case $\ell_{\text{max}}[A_J, B_K] < 2$ is obviously impossible since radiative moments have at least $\ell = 2$. This case is indicated by dashes in the two panels of table 1. We emphasize that the latter rules apply to quadratic interactions, which are the most tricky to control thoroughly. In this paper we will also need to include some cubic interactions. These are simpler to look for and will be dealt with separately.

3.1. Radiative moments in terms of canonical moments

As in our previous papers [12, 13], to simplify the presentation, we will first present the radiative moments $\{U_L, V_L\}$ in terms of the canonical moments $\{M_L, S_L\}$, and only in a second stage will we give the canonical moments in terms of the set of source and gauge moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$. Clearly, the selection rules provided in table 1 apply to the

⁷ From equations (2.3) we see that the gauge moments $W_L, X_L,$ and Y_L have the same parity as the mass moment I_L , whereas the gauge moment Z_L has the same parity as the current moment J_L .

Table 1. Left panel: values of $\ell_{\max}[A_J, B_K]$ for the mass multipole moment U_L at the 3.5PN order for the various possible choices of multipole interactions between A_J and $B_K \in \{I, J, W, X, Y, Z\}$. Right panel: likewise, but for the current multipole moment V_L . We must have $j + k = \ell_{\max}[A_J, B_K]$ if the product $A_J B_K$ has the same parity as the radiative moment containing the interaction $A_J \times B_K$; i.e., both A_J and B_K belong to $\{I, W, X, Y\}$ or both belong to $\{J, Z\}$ in U_L , A_J belongs to $\{I, W, X, Y\}$ and B_K belongs to $\{J, Z\}$ or vice versa in V_L ; in all other cases $j + k = \ell_{\max}[A_J, B_K] + 1$. Impossible (because too low) values of $\ell_{\max}[A_J, B_K]$ are indicated by dashes.

| A_J | B_K | | | | | |
|-------|-------|-------|-------|-------|-------|-------|
| | I_K | J_K | W_K | X_K | Y_K | Z_K |
| I_J | 6 | 4 | 4 | 2 | 4 | 2 |
| J_J | 4 | 4 | 2 | – | 2 | 2 |
| W_J | 4 | 2 | 2 | – | 2 | – |
| X_J | 2 | – | – | – | – | – |
| Y_J | 4 | 2 | 2 | – | 2 | – |
| Z_J | 2 | 2 | – | – | – | – |

| A_J | B_K | | | | | |
|-------|-------|-------|-------|-------|-------|-------|
| | I_K | J_K | W_K | X_K | Y_K | Z_K |
| I_J | 5 | 5 | 3 | – | 3 | 3 |
| J_J | 5 | 3 | 3 | – | 3 | – |
| W_J | 3 | 3 | – | – | – | – |
| X_J | – | – | – | – | – | – |
| Y_J | 3 | 3 | – | – | – | – |
| Z_J | 3 | – | – | – | – | – |

full set of quadratic interactions with $A_J, B_K \in \{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ as well as to the restricted set with $A_J, B_K \in \{M_L, S_L\}$.

To display the results, it is also convenient to group the terms in the radiative moments U_L and V_L into different types, namely on the one hand all the instantaneous terms, and on the other hand the hereditary terms [22] which are made of the tails, the tails-of-tails, and the non-linear memory integrals:

$$U_L = U_L^{\text{inst}} + U_L^{\text{tail}} + U_L^{\text{tail} - \text{tail}} + U_L^{\text{mem}} + \delta U_L, \quad (3.3a)$$

$$V_L = V_L^{\text{inst}} + V_L^{\text{tail}} + V_L^{\text{tail} - \text{tail}} + V_L^{\text{mem}} + \delta V_L. \quad (3.3b)$$

The terms δU_L and δV_L in the foregoing decomposition represent the contributions occurring at 4PN or higher orders in the waveform (generally with a more complex non-linear structure), and they will be neglected here. In what follows we show the formulas needed to complete the waveform, including all its relevant harmonics up to order 3.5PN. Some of those formulas have already been partially published in [12, 13], but we reproduce them all in order to be self-contained for the convenience of the user.

3.1.1. Instantaneous terms. These terms, in which by definition all the canonical moments are evaluated at the current (retarded) time $T_R = T - R/c$, are the most intricate terms to obtain. For the mass-type moments they are given by:

$$\begin{aligned}
U_{ij}^{\text{inst}} = & M_{ij}^{(2)} \\
& + \frac{G}{c^5} \left[\frac{1}{7} M_{a\langle i}^{(5)} M_{j\rangle a} - \frac{5}{7} M_{a\langle i}^{(4)} M_{j\rangle a}^{(1)} - \frac{2}{7} M_{a\langle i}^{(3)} M_{j\rangle a}^{(2)} + \frac{1}{3} \varepsilon_{ab\langle i} M_{j\rangle a}^{(4)} S_b \right] \\
& + \frac{G}{c^7} \left[-\frac{1}{432} M_{ab} M_{ijab}^{(7)} + \frac{1}{432} M_{ab}^{(1)} M_{ijab}^{(6)} - \frac{5}{756} M_{ab}^{(2)} M_{ijab}^{(5)} + \frac{19}{648} M_{ab}^{(3)} M_{ijab}^{(4)} \right. \\
& + \frac{1957}{3024} M_{ab}^{(4)} M_{ijab}^{(3)} + \frac{1685}{1008} M_{ab}^{(5)} M_{ijab}^{(2)} + \frac{41}{28} M_{ab}^{(6)} M_{ijab}^{(1)} + \frac{91}{216} M_{ab}^{(7)} M_{ijab} \\
& - \frac{5}{252} M_{ab\langle i} M_{j\rangle ab}^{(7)} + \frac{5}{189} M_{ab\langle i} M_{j\rangle ab}^{(6)} + \frac{5}{126} M_{ab\langle i} M_{j\rangle ab}^{(5)} + \frac{5}{2268} M_{ab\langle i} M_{j\rangle ab}^{(4)} \\
& + \frac{5}{42} S_a S_{ija}^{(5)} + \frac{80}{63} S_a \langle i S_{j\rangle a}^{(5)} + \frac{16}{63} S_a \langle i S_{j\rangle a}^{(4)} - \frac{64}{63} S_a \langle i S_{j\rangle a}^{(3)} \\
& + \varepsilon_{ac\langle i} \left(\frac{1}{168} S_{j\rangle bc}^{(6)} M_{ab} + \frac{1}{24} S_{j\rangle bc}^{(5)} M_{ab}^{(1)} + \frac{1}{28} S_{j\rangle bc}^{(4)} M_{ab}^{(2)} + \frac{3}{56} S_{j\rangle bc}^{(2)} M_{ab}^{(4)} \right. \\
& + \frac{187}{168} S_{j\rangle bc}^{(1)} M_{ab}^{(5)} + \frac{65}{84} S_{j\rangle bc} M_{ab}^{(6)} + \frac{1}{189} M_{j\rangle bc}^{(6)} S_{ab} - \frac{1}{189} M_{j\rangle bc}^{(5)} S_{ab}^{(1)} \\
& + \frac{10}{189} M_{j\rangle bc}^{(4)} S_{ab}^{(2)} + \frac{32}{189} M_{j\rangle bc}^{(3)} S_{ab}^{(3)} + \frac{65}{189} M_{j\rangle bc}^{(2)} S_{ab}^{(4)} - \frac{5}{189} M_{j\rangle bc}^{(1)} S_{ab}^{(5)} \\
& \left. - \frac{10}{63} M_{j\rangle bc} S_{ab}^{(6)} - \frac{1}{6} S_{j\rangle bc} M_{ab}^{(3)} \right) \Big], \tag{3.4a}
\end{aligned}$$

$$\begin{aligned}
U_{ijk}^{\text{inst}} = & M_{ijk}^{(3)} \\
& + \frac{G}{c^5} \left[-\frac{4}{3} M_{a\langle i}^{(3)} M_{jk\rangle a}^{(3)} - \frac{9}{4} M_{a\langle i}^{(4)} M_{jk\rangle a}^{(2)} + \frac{1}{4} M_{a\langle i}^{(2)} M_{jk\rangle a}^{(4)} - \frac{3}{4} M_{a\langle i}^{(5)} M_{jk\rangle a}^{(1)} \right. \\
& + \frac{1}{4} M_{a\langle i}^{(1)} M_{jk\rangle a}^{(5)} + \frac{1}{12} M_{a\langle i}^{(6)} M_{jk\rangle a} + \frac{1}{4} M_{a\langle i} M_{jk\rangle a}^{(6)} \\
& + \frac{1}{5} \varepsilon_{ab\langle i} \left(-12 S_{j\dot{a}}^{(2)} M_{k\rangle b}^{(3)} - 8 M_{j\dot{a}}^{(2)} S_{k\rangle b}^{(3)} - 3 S_{j\dot{a}}^{(1)} M_{k\rangle b}^{(4)} \right. \\
& \quad \left. - 27 M_{j\dot{a}}^{(1)} S_{k\rangle b}^{(4)} - S_{j\dot{a}} M_{k\rangle b}^{(5)} - 9 M_{j\dot{a}} S_{k\rangle b}^{(5)} - \frac{9}{4} S_{\dot{a}} M_{jk\rangle b}^{(5)} \right) \\
& \left. + \frac{12}{5} S_{\langle i} S_{jk\rangle}^{(4)} \right], \tag{3.4b}
\end{aligned}$$

$$\begin{aligned}
U_{ijkl}^{\text{inst}} &= M_{ijkl}^{(4)} \\
&+ \frac{G}{c^3} \left[-\frac{21}{5} M_{\langle ij}^{(5)} M_{kl} \rangle - \frac{63}{5} M_{\langle ij}^{(4)} M_{kl}^{(1)} \rangle - \frac{102}{5} M_{\langle ij}^{(3)} M_{kl}^{(2)} \rangle \right] \\
&+ \frac{G}{c^5} \left[\frac{7}{55} M_{a\langle i} M_{jkl \rangle a}^{(7)} + \frac{7}{55} M_{a\langle i} M_{jkl \rangle a}^{(6)} + \frac{1}{25} M_{a\langle i} M_{jkl \rangle a}^{(2)} M_{jkl}^{(5)} - \frac{28}{11} M_{a\langle i} M_{jkl \rangle a}^{(3)} M_{jkl}^{(4)} \right. \\
&- \frac{273}{55} M_{a\langle i} M_{jkl \rangle a}^{(4)} M_{jkl}^{(3)} - \frac{203}{55} M_{a\langle i} M_{jkl \rangle a}^{(5)} M_{jkl}^{(2)} - \frac{49}{55} M_{a\langle i} M_{jkl \rangle a}^{(6)} M_{jkl}^{(1)} + \frac{14}{275} M_{a\langle i} M_{jkl \rangle a}^{(7)} \\
&+ \frac{14}{33} M_{a\langle ij} M_{kl \rangle a}^{(7)} + \frac{37}{33} M_{a\langle ij} M_{kl \rangle a}^{(6)} + \frac{9}{11} M_{a\langle ij} M_{kl \rangle a}^{(2)} M_{kl}^{(5)} + \frac{8}{33} M_{a\langle ij} M_{kl \rangle a}^{(3)} M_{kl}^{(4)} + \frac{9}{5} S_{\langle i} S_{jkl \rangle}^{(5)} \\
&+ \frac{16}{5} S_{\langle ij} S_{kl \rangle}^{(5)} + \frac{48}{5} S_{\langle ij}^{(1)} S_{kl}^{(4)} \rangle + \frac{32}{5} S_{\langle ij}^{(2)} S_{kl}^{(3)} \rangle + \varepsilon_{ab\langle i} \left(-\frac{3}{5} M_{ja} S_{kl \rangle b}^{(6)} - \frac{63}{25} M_{ja}^{(1)} S_{kl \rangle b}^{(5)} \right. \\
&+ \frac{3}{5} M_{ja}^{(2)} S_{kl \rangle b}^{(4)} + \frac{18}{5} M_{ja}^{(3)} S_{kl \rangle b}^{(3)} + \frac{9}{5} M_{ja}^{(4)} S_{kl \rangle b}^{(2)} + \frac{3}{5} M_{ja}^{(5)} S_{kl \rangle b}^{(1)} \\
&+ \frac{3}{25} M_{ja}^{(6)} S_{kl \rangle b} - \frac{8}{15} S_{ja} M_{kl \rangle b}^{(6)} - \frac{24}{25} S_{ja}^{(1)} M_{kl \rangle b}^{(5)} - \frac{8}{5} S_{ja}^{(2)} M_{kl \rangle b}^{(4)} + \frac{16}{3} S_{ja}^{(3)} M_{kl \rangle b}^{(3)} \\
&\left. + \frac{72}{5} S_{ja}^{(4)} M_{kl \rangle b} + \frac{56}{5} S_{ja}^{(5)} M_{kl \rangle b} + \frac{232}{75} S_{ja}^{(6)} M_{kl \rangle b} + \frac{29}{75} M_{jkl \rangle a}^{(6)} S_b \right] \Big], \quad (3.4c)
\end{aligned}$$

$$\begin{aligned}
U_{ijklm}^{\text{inst}} &= M_{ijklm}^{(5)} \\
&+ \frac{G}{c^3} \left[-\frac{710}{21} M_{\langle ij}^{(3)} M_{klm}^{(3)} \rangle - \frac{265}{7} M_{\langle ijk}^{(2)} M_{lm}^{(4)} \rangle - \frac{120}{7} M_{\langle ij}^{(2)} M_{klm}^{(4)} \rangle \right. \\
&\left. - \frac{155}{7} M_{\langle ijk}^{(1)} M_{lm}^{(5)} \rangle - \frac{41}{7} M_{\langle ij}^{(1)} M_{klm}^{(5)} \rangle - \frac{34}{7} M_{\langle ijk} M_{lm}^{(6)} \rangle - \frac{15}{7} M_{\langle ij} M_{klm}^{(6)} \rangle \right], \quad (3.4d)
\end{aligned}$$

$$\begin{aligned}
U_{ijklmn}^{\text{inst}} &= M_{ijklmn}^{(6)} \\
&+ \frac{G}{c^3} \left[-\frac{45}{28} M_{\langle ij} M_{klmn}^{(7)} \rangle - \frac{111}{28} M_{\langle ij}^{(1)} M_{klmn}^{(6)} \rangle - \frac{561}{28} M_{\langle ij}^{(2)} M_{klmn}^{(5)} \rangle \right. \\
&- \frac{1595}{28} M_{\langle ij}^{(3)} M_{klmn}^{(4)} \rangle - \frac{2505}{28} M_{\langle ij}^{(4)} M_{klmn}^{(3)} \rangle - \frac{2115}{28} M_{\langle ij}^{(5)} M_{klmn}^{(2)} \rangle \\
&- \frac{909}{28} M_{\langle ij}^{(6)} M_{klmn}^{(1)} \rangle - \frac{159}{28} M_{\langle ij}^{(7)} M_{klmn} \rangle - \frac{15}{7} M_{\langle ijk} M_{lmn}^{(7)} \rangle - \frac{75}{7} M_{\langle ijk}^{(1)} M_{lmn}^{(6)} \rangle \\
&\left. - \frac{135}{7} M_{\langle ijk}^{(2)} M_{lmn}^{(5)} \rangle - \frac{505}{21} M_{\langle ijk}^{(3)} M_{lmn}^{(4)} \rangle \right]. \quad (3.4e)
\end{aligned}$$

In the preceding expressions, the $1/c^5$ terms in U_{ijkl}^{inst} and $1/c^3$ terms in U_{ijklmn}^{inst} are new with this paper; the other terms were obtained in [12, 13]. For the current-type moments we have:

$$\begin{aligned}
V_{ij}^{\text{inst}} = S_{ij}^{(2)} &+ \frac{G}{7c^5} \left[4S_{a\langle i}^{(2)} M_{j\rangle a}^{(3)} + 8M_{a\langle i}^{(2)} S_{j\rangle a}^{(3)} + 17S_{a\langle i}^{(1)} M_{j\rangle a}^{(4)} - 3M_{a\langle i}^{(1)} S_{j\rangle a}^{(4)} + 9S_{a\langle i} M_{j\rangle a}^{(5)} \right. \\
&- 3M_{a\langle i} S_{j\rangle a}^{(5)} - \frac{1}{4} S_a M_{ija}^{(5)} - 7\varepsilon_{ab\langle i} S_a S_{j\rangle b}^{(4)} + \frac{1}{2} \varepsilon_{ac\langle i} \left(3M_{ab}^{(3)} M_{j\rangle bc}^{(3)} + \frac{353}{24} M_{j\rangle bc}^{(2)} M_{ab}^{(4)} \right. \\
&\left. \left. - \frac{5}{12} M_{ab}^{(2)} M_{j\rangle bc}^{(4)} + \frac{113}{8} M_{j\rangle bc}^{(1)} M_{ab}^{(5)} - \frac{3}{8} M_{ab}^{(1)} M_{j\rangle bc}^{(5)} + \frac{15}{4} M_{j\rangle bc} M_{ab}^{(6)} + \frac{3}{8} M_{ab} M_{j\rangle bc}^{(6)} \right) \right], \quad (3.5a)
\end{aligned}$$

$$\begin{aligned}
V_{ijk}^{\text{inst}} = S_{ijk}^{(3)} &+ \frac{G}{c^3} \left[\frac{1}{10} \varepsilon_{ab\langle i} M_{j\bar{a}}^{(5)} M_{k\rangle b} - \frac{1}{2} \varepsilon_{ab\langle i} M_{j\bar{a}}^{(4)} M_{k\rangle b}^{(1)} - 2S_{\langle i} M_{jk\rangle}^{(4)} \right] \\
&+ \frac{G}{c^5} \left[\frac{1}{12} M_{a\langle i} S_{jk\rangle a}^{(6)} + \frac{1}{12} M_{a\langle i}^{(1)} S_{jk\rangle a}^{(5)} + \frac{5}{12} M_{a\langle i}^{(2)} S_{jk\rangle a}^{(4)} \right. \\
&+ \frac{35}{12} M_{a\langle i}^{(4)} S_{jk\rangle a}^{(2)} + \frac{49}{12} M_{a\langle i}^{(5)} S_{jk\rangle a}^{(1)} + \frac{19}{12} M_{a\langle i}^{(6)} S_{jk\rangle a} \\
&+ \frac{2}{27} S_a \langle i M_{jk\rangle a}^{(6)} + \frac{10}{27} S_a \langle i M_{jk\rangle a}^{(5)} \\
&+ \frac{2}{27} S_a \langle i M_{jk\rangle a}^{(4)} + \frac{8}{9} S_a \langle i M_{jk\rangle a}^{(3)} \\
&\left. - \frac{10}{27} S_a \langle i M_{jk\rangle a}^{(2)} - \frac{38}{27} S_a \langle i M_{jk\rangle a}^{(1)} - \frac{2}{3} S_a \langle i M_{jk\rangle a}^{(6)} - \frac{1}{60} S_a M_{ijka}^{(6)} \right. \\
&+ \varepsilon_{ab\langle i} \left(-\frac{1}{180} M_{jk\rangle ac}^{(7)} M_{bc} + \frac{11}{900} M_{jk\rangle ac}^{(6)} M_{bc}^{(1)} + \frac{7}{300} M_{jk\rangle ac}^{(5)} M_{bc}^{(2)} \right. \\
&- \frac{37}{270} M_{jk\rangle ac}^{(4)} M_{bc}^{(3)} - \frac{191}{180} M_{jk\rangle ac}^{(3)} M_{bc}^{(4)} - \frac{65}{36} M_{jk\rangle ac}^{(2)} M_{bc}^{(5)} - \frac{367}{300} M_{jk\rangle ac}^{(1)} M_{bc}^{(6)} \\
&- \frac{133}{450} M_{jk\rangle ac} M_{bc}^{(7)} + \frac{1}{27} M_{j\bar{a}c} M_{k\rangle bc}^{(7)} + \frac{5}{162} M_{j\bar{a}c}^{(1)} M_{k\rangle bc}^{(6)} - \frac{5}{162} M_{j\bar{a}c}^{(2)} M_{k\rangle bc}^{(5)} \\
&\left. \left. - \frac{1}{81} M_{j\bar{a}c}^{(3)} M_{k\rangle bc}^{(4)} - \frac{11}{20} S_{jk\rangle b} S_a - \frac{88}{45} S_{j\bar{a}} S_{k\rangle b}^{(5)} - \frac{40}{9} S_{j\bar{a}}^{(1)} S_{k\rangle b}^{(4)} - \frac{32}{9} S_{j\bar{a}}^{(2)} S_{k\rangle b}^{(3)} \right) \right], \quad (3.5b)
\end{aligned}$$

$$\begin{aligned}
V_{ijkl}^{\text{inst}} = S_{ijkl}^{(4)} &+ \frac{G}{c^3} \left[-\frac{35}{3} S_{\langle ij}^{(2)} M_{kl\rangle}^{(3)} - \frac{25}{3} M_{\langle ij}^{(2)} S_{kl\rangle}^{(3)} \right. \\
&- \frac{65}{6} S_{\langle ij}^{(1)} M_{kl\rangle}^{(4)} - \frac{25}{6} M_{\langle ij}^{(1)} S_{kl\rangle}^{(4)} - \frac{19}{6} S_{\langle ij} M_{kl\rangle}^{(5)} \\
&- \frac{11}{6} M_{\langle ij} S_{kl\rangle}^{(5)} - \frac{11}{12} S_{\langle i} M_{jkl\rangle}^{(5)} \\
&+ \frac{1}{6} \varepsilon_{ab\langle i} \left(-5M_{j\bar{a}}^{(3)} M_{kl\rangle b}^{(3)} - \frac{11}{2} M_{j\bar{a}}^{(4)} M_{kl\rangle b}^{(2)} - \frac{5}{2} M_{j\bar{a}}^{(2)} M_{kl\rangle b}^{(4)} \right. \\
&\left. \left. - \frac{1}{2} M_{j\bar{a}}^{(5)} M_{kl\rangle b}^{(1)} + \frac{37}{10} M_{j\bar{a}}^{(1)} M_{kl\rangle b}^{(5)} + \frac{3}{10} M_{j\bar{a}}^{(6)} M_{kl\rangle b} + \frac{1}{2} M_{j\bar{a}} M_{kl\rangle b}^{(6)} \right) \right], \quad (3.5c)
\end{aligned}$$

$$\begin{aligned}
V_{ijklm}^{\text{inst}} = & S_{ijklm}^{(5)} \\
& + \frac{G}{c^3} \left[-\frac{3}{2} M_{\langle ij} S_{klm}^{(6)} - \frac{33}{10} M_{\langle ij}^{(1)} S_{klm}^{(5)} - 12 M_{\langle ij}^{(2)} S_{klm}^{(4)} - 27 M_{\langle ij}^{(3)} S_{klm}^{(3)} \right. \\
& \quad - \frac{69}{2} M_{\langle ij}^{(4)} S_{klm}^{(2)} - \frac{39}{2} M_{\langle ij}^{(5)} S_{klm}^{(1)} - \frac{21}{5} M_{\langle ij}^{(6)} S_{klm} - \frac{4}{3} S_{\langle ij} M_{klm}^{(6)} \\
& \quad - \frac{76}{15} S_{\langle ij}^{(1)} M_{klm}^{(5)} - \frac{16}{3} S_{\langle ij}^{(2)} M_{klm}^{(4)} - 8 S_{\langle ij}^{(3)} M_{klm}^{(3)} - \frac{28}{3} S_{\langle ij}^{(4)} M_{klm}^{(2)} \\
& \quad - \frac{20}{3} S_{\langle ij}^{(5)} M_{klm}^{(1)} - \frac{8}{5} S_{\langle ij}^{(6)} M_{klm} - \frac{3}{5} S_{\langle i} M_{jklm}^{(6)} \\
& \quad + \varepsilon_{ab} \langle i \left(\frac{1}{14} M_{j\bar{a}} M_{klm}^{(7)} \rangle_b + \frac{1}{2} M_{j\bar{a}}^{(1)} M_{klm}^{(6)} \rangle_b - \frac{3}{5} M_{j\bar{a}}^{(2)} M_{klm}^{(5)} \rangle_b - \frac{4}{3} M_{j\bar{a}}^{(3)} M_{klm}^{(4)} \rangle_b \right. \\
& \quad - \frac{3}{2} M_{j\bar{a}}^{(4)} M_{klm}^{(3)} \rangle_b - \frac{1}{2} M_{j\bar{a}}^{(5)} M_{klm}^{(2)} \rangle_b + \frac{1}{35} M_{j\bar{a}}^{(7)} M_{klm} \rangle_b + \frac{1}{7} M_{jk\bar{a}} M_{lm}^{(7)} \rangle_b \\
& \quad \left. + \frac{2}{3} M_{jk\bar{a}}^{(1)} M_{lm}^{(6)} \rangle_b + \frac{4}{3} M_{jk\bar{a}}^{(2)} M_{lm}^{(5)} \rangle_b + \frac{1}{3} M_{jk\bar{a}}^{(3)} M_{lm}^{(4)} \rangle_b \right) \Big]. \tag{3.5d}
\end{aligned}$$

In the preceding expressions for the current moments, the $1/c^5$ terms in V_{ijk}^{inst} and the $1/c^3$ terms in V_{ijklm}^{inst} are new with this paper. For all higher multipole moments it suffices, at this approximation level, to replace U_L^{inst} and V_L^{inst} with the corresponding $M_L^{(\ell)}$ and $S_L^{(\ell)}$.

3.1.2. Tail terms. Next we give the contributions due to tails which correspond to quadratic interactions and arise at (relative) order 1.5PN. They come from the interaction between the mass monopole M and a non-static multipole M_L or S_L (with $\ell \geq 2$). For these we dispose of a general formula valid for any ℓ [16, 22]. The following contributions have to be added to any mass and current multipole moments:

$$U_L^{\text{tail}}(T_R) = \frac{2GM}{c^3} \int_{-\infty}^{T_R} M_L^{(\ell+2)}(\tau) \left[\ln\left(\frac{T_R - \tau}{2b}\right) + \kappa_\ell \right] d\tau, \tag{3.6a}$$

$$V_L^{\text{tail}}(T_R) = \frac{2GM}{c^3} \int_{-\infty}^{T_R} S_L^{(\ell+2)}(\tau) \left[\ln\left(\frac{T_R - \tau}{2b}\right) + \pi_\ell \right] d\tau. \tag{3.6b}$$

Here the constant b entering the logarithmic kernel is the constant time scale that has been introduced into the relation (3.1) between the retarded time T_R in radiative coordinates and the retarded time t_r in harmonic coordinates. The numerical constants κ_ℓ and π_ℓ are given for general ℓ , in *harmonic coordinates*, by

$$\kappa_\ell = \frac{2\ell^2 + 5\ell + 4}{\ell(\ell+1)(\ell+2)} + H_{\ell-2}, \quad \pi_\ell = \frac{\ell-1}{\ell(\ell+1)} + H_{\ell-1}, \tag{3.7}$$

where $H_k \equiv \sum_{j=1}^k \frac{1}{j}$ denotes the k th harmonic number.

3.1.3. Tail-of-tail terms. The tails-of-tails arise at relative order 3PN and are due to the cubic interaction between two monopoles M and one non-static multipole, i.e., $M \times M \times M_L$ or

$M \times M \times S_L$. The tail-of-tail entering the mass quadrupole moment has been already computed in [2]. Those in the mass octupole and current quadrupole moments, new with this paper, have been obtained by the same method⁸. The relevant formulas to integrate the elementary cubic source terms are presented in appendix A. We find

$$U_{ij}^{\text{tail-tail}}(T_R) = \frac{G^2 M^2}{c^6} \int_{-\infty}^{T_R} M_{ij}^{(5)}(\tau) \left[2 \ln^2 \left(\frac{T_R - \tau}{2b} \right) + \frac{11}{3} \ln \left(\frac{T_R - \tau}{2b} \right) - \frac{214}{105} \ln \left(\frac{T_R - \tau}{2\tau_0} \right) + \frac{124627}{22050} \right] d\tau, \quad (3.8a)$$

$$U_{ijk}^{\text{tail-tail}}(T_R) = \frac{G^2 M^2}{c^6} \int_{-\infty}^{T_R} M_{ijk}^{(6)}(\tau) \left[2 \ln^2 \left(\frac{T_R - \tau}{2b} \right) + \frac{97}{15} \ln \left(\frac{T_R - \tau}{2b} \right) - \frac{26}{21} \ln \left(\frac{T_R - \tau}{2\tau_0} \right) + \frac{13283}{4410} \right] d\tau, \quad (3.8b)$$

$$V_{ij}^{\text{tail-tail}}(T_R) = \frac{G^2 M^2}{c^6} \int_{-\infty}^{T_R} J_{ij}^{(5)}(\tau) \left[2 \ln^2 \left(\frac{T_R - \tau}{2b} \right) + \frac{14}{3} \ln \left(\frac{T_R - \tau}{2b} \right) - \frac{214}{105} \ln \left(\frac{T_R - \tau}{2\tau_0} \right) - \frac{26254}{11025} \right] d\tau. \quad (3.8c)$$

Note the appearance of two constant time scales: (i) The time scale b , which is a pure gauge constant entering the definition of the particular radiative coordinates used in equation (3.1). Changing b simply means shifting the origin of time of the radiative coordinate system ($T, X^i = RN^i$) with respect to the harmonic coordinate grid ($t, x^i = m^i$), which clearly has no physical implication. (ii) The time scale $\tau_0 = r_0/c$, where r_0 is the regularization constant introduced in the regulator (2.7) of the MPM algorithm⁹; the constant τ_0 cannot be removed by a coordinate transformation, but it must disappear from the radiative moments (and hence from the physical waveform) once the *source* moments are explicitly related to the parameters of the matter source, e.g., the masses, trajectories, and velocities of the particles in a binary system. This has been verified explicitly in the case of the 3PN mass quadrupole moment in [3, 5]. In section (4) we verify the cancellation of τ_0 in the case of the 3PN mass octupole moment as well.

Accordingly, there are two types of logarithms in the kernels of equation (3.8): those containing b and those containing τ_0 . The coefficient of the leading logarithm square (which contains b) is always 2. We also know the coefficient of the logarithm containing τ_0 in the case of mass-type moments for general ℓ . Indeed, the coefficient of $\ln \tau_0$ in the tails-of-tails associated with mass multipole moments, resulting from the long computation of the multipole interactions $M \times M \times M_L$, is given by equation (A.6) in [25] as

$$\alpha_\ell = 2 \frac{15\ell^4 + 30\ell^3 + 28\ell^2 + 13\ell + 24}{\ell(\ell+1)(2\ell+3)(2\ell+1)(2\ell-1)}. \quad (3.9)$$

⁸ Tail-of-tail contributions to the gravitational field have recently been extensively used in a comparison of gravitational self-force results with high post-Newtonian orders [23, 24]. The corresponding sources of the relaxed Einstein equations can be found in appendix B of [24].

⁹ When studying the case of the mass quadrupole tail-of-tail in [2], the choice $b = \tau_0$ was adopted.

Thus, we have $\alpha_2 = 214/105$ and $\alpha_3 = 26/21$ in agreement with the coefficients displayed in equations (3.8a) and (3.8b). We will come back to the significance of this result in section (4) when we confirm that the value $\alpha_3 = 26/21$ is fully consistent with our 3PN computation of the source mass octupole moment I_{ijk} for compact binaries (see equation (4.12)–(4.13)). In section 5.1 we investigate the occurrence of the dominant powers of logarithms for more general tail interactions of the type $M \times \dots \times M \times (M_L \text{ or } S_L)$.

3.1.4. Memory terms. The contributions coming from the non-linear memory effect arise from quadratic interactions between two radiative moments. They have been computed in [22, 26–30] in the mass quadrupole moment at the lowest order, which is 2.5PN. More recently, the 3.5PN corrections beyond the leading order have been obtained for both circular [31] and eccentric orbits [32]. Note that the non-linear memory effect does not enter the current-type multipole moments V_L but only the mass-type multipole moments U_L ; hence $V_L^{\text{mem}} = 0$. Its contribution to the mass radiative moments needed for the 3.5PN waveform read

$$U_{ij}^{\text{mem}}(T_R) = \frac{G}{c^5} \int_{-\infty}^{T_R} \left[-\frac{2}{7} M_{a\langle i}^{(3)}(\tau) M_{j\rangle a}^{(3)}(\tau) \right] d\tau \\ + \frac{G}{c^7} \int_{-\infty}^{T_R} \left[-\frac{5}{756} M_{ab}^{(4)}(\tau) M_{ijab}^{(4)}(\tau) - \frac{32}{63} S_{a\langle i}^{(3)}(\tau) S_{j\rangle a}^{(3)}(\tau) \right. \\ \left. + \varepsilon_{ab\langle i} \left(\frac{5}{42} S_{j\rangle bc}^{(4)}(\tau) M_{ac}^{(3)}(\tau) - \frac{20}{189} M_{j\rangle bc}^{(4)}(\tau) S_{ac}^{(3)}(\tau) \right) \right] d\tau, \quad (3.10a)$$

$$U_{ijk}^{\text{mem}}(T_R) = \frac{G}{c^5} \int_{-\infty}^{T_R} \left[-\frac{1}{3} M_{a\langle i}^{(3)}(\tau) M_{jk\rangle a}^{(4)}(\tau) - \frac{4}{5} \varepsilon_{ab\langle i} M_{j\bar{a}}^{(3)}(\tau) S_{k\rangle b}^{(3)}(\tau) \right] d\tau, \quad (3.10b)$$

$$U_{ijkl}^{\text{mem}}(T_R) = \frac{G}{c^3} \int_{-\infty}^{T_R} \left[\frac{2}{5} M_{ij}^{(3)}(\tau) M_{kl}^{(3)}(\tau) \right] d\tau \\ + \frac{G}{c^5} \int_{-\infty}^{T_R} \left[\frac{12}{55} M_{a\langle i}^{(4)}(\tau) M_{jkl\rangle a}^{(4)}(\tau) \right. \\ - \frac{14}{99} M_{a\langle ij}^{(4)}(\tau) M_{kl\rangle a}^{(4)}(\tau) + \frac{32}{45} S_{ij}^{(3)}(\tau) S_{kl}^{(3)}(\tau) \\ \left. + \varepsilon_{ab\langle i} \left(-\frac{4}{5} M_{j\bar{a}}^{(3)}(\tau) S_{kl\rangle b}^{(4)}(\tau) + \frac{32}{45} S_{j\bar{a}}^{(3)}(\tau) M_{kl\rangle b}^{(4)}(\tau) \right) \right] d\tau, \quad (3.10c)$$

$$U_{ijklm}^{\text{mem}}(T_R) = \frac{G}{c^3} \int_{-\infty}^{T_R} \left[\frac{20}{21} M_{ij}^{(3)}(\tau) M_{klm}^{(4)}(\tau) \right] d\tau, \quad (3.10d)$$

$$U_{ijklmn}^{\text{mem}}(T_R) = \frac{G}{c^3} \int_{-\infty}^{T_R} \left[\frac{5}{7} M_{ijk}^{(4)}(\tau) M_{lmn}^{(4)}(\tau) - \frac{15}{14} M_{ij}^{(4)}(\tau) M_{klmn}^{(4)}(\tau) \right] d\tau. \quad (3.10e)$$

In fact, the non-linear memory terms are known for any multipolar order ℓ [31]; the foregoing expressions are a particular case of the general formula given for completeness in appendix C, in which we also present the corresponding modal decomposition of the non-linear memory.

3.2. Canonical moments in terms of source moments

Adding up all the previous contributions in equations (3.3) we obtain the radiative mass and current moments $\{U_L, V_L\}$ as full functionals of the canonical moments $\{M_L, S_L\}$, consistent

with our 3.5PN goal. However, there still remains the matter of relating with that same precision the canonical moments to the actual sets of source moments $\{I_L, J_L\}$ and gauge moments $\{W_L, X_L, Y_L, Z_L\}$, as shown schematically in equations (2.12). Here we present the most complete up-to-date results, with some repetition (for the sake of exhaustiveness) with respect to [12, 13]:

$$\begin{aligned}
M_{ij} = & I_{ij} + \frac{4G}{c^5} \left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right] \\
& + \frac{4G}{c^7} \left[-2X I_{ij}^{(3)} + \frac{4}{7} I_{a\langle i}^{(3)} W_{j\rangle a}^{(1)} + \frac{6}{7} I_{a\langle i}^{(4)} W_{j\rangle a} \right. \\
& - \frac{1}{7} I_{a\langle i} Y_{j\rangle a}^{(3)} - I_{a\langle i}^{(3)} Y_{j\rangle a} + \frac{1}{63} W_a^{(3)} I_{ija}^{(1)} \\
& - \frac{5}{21} W_a^{(4)} I_{ija} + \frac{5}{63} Y_a^{(1)} I_{ija}^{(2)} - \frac{22}{63} Y_a^{(2)} I_{ija}^{(1)} - \frac{25}{21} Y_a^{(3)} I_{ija} + 2W^{(2)} W_{ij}^{(1)} \\
& + 2W^{(3)} W_{ij} + 2W^{(2)} Y_{ij} - \frac{4}{3} W_{\langle i} W_{j\rangle}^{(3)} - 4W_{\langle i} Y_{j\rangle}^{(2)} \\
& + \varepsilon_{ab\langle i} \left(-I_{j\rangle a}^{(3)} Z_b + \frac{1}{3} I_{j\rangle a} Z_b^{(3)} + \frac{4}{9} J_{j\rangle a} W_b^{(3)} \right. \\
& \left. + \frac{8}{9} J_{j\rangle a}^{(1)} Y_b^{(1)} - \frac{4}{9} J_{j\rangle a} Y_b^{(2)} \right) \left. \right] + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{3.11a}
\end{aligned}$$

$$M_{ijk} = I_{ijk} + \frac{4G}{c^5} \left[W^{(2)} I_{ijk} - W^{(1)} I_{ijk}^{(1)} + 3 I_{\langle ij} Y_{k\rangle}^{(1)} \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{3.11b}$$

$$M_{ijkl} = I_{ijkl} + \frac{4G}{c^5} \left[-W^{(1)} I_{ijkl}^{(1)} + W^{(2)} I_{ijkl} + 4Y_{\langle i}^{(1)} I_{jkl\rangle} \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{3.11c}$$

$$\begin{aligned}
S_{ij} = & J_{ij} + \frac{2G}{c^5} \left[\varepsilon_{ab\langle i} \left(-I_{j\rangle b}^{(3)} W_a - 2I_{j\rangle b} Y_a^{(2)} + I_{j\rangle b}^{(1)} Y_a^{(1)} \right) + 3J_{\langle i} Y_{j\rangle}^{(1)} - 2J_{ij}^{(1)} W^{(1)} \right] \\
& + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{3.11d}
\end{aligned}$$

$$\begin{aligned}
S_{ijk} = & J_{ijk} + \frac{4G}{c^5} \left[-W^{(1)} J_{ijk}^{(1)} + \frac{8}{3} Y_{\langle i}^{(1)} J_{jk\rangle} \right. \\
& \left. + \varepsilon_{ab\langle i} \left(-\frac{1}{3} I_{jk\rangle a}^{(1)} Y_b^{(1)} + I_{jk\rangle a} Y_b^{(2)} + I_{ja}^{(3)} W_{kb} \right) \right] + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{3.11e}
\end{aligned}$$

The term $1/c^7$ in M_{ij} was already computed in [13], but the terms $1/c^5$ in M_{ijkl} and S_{ijk} are new with this paper. Finally, combining the previous formulas (3.11) with all the results of section 3.1 we control the full waveform in terms of the basic source and gauge moments up to order 3.5PN.

4. Gravitational-wave octupole modes of compact binaries

In this section we present the mass octupole source moment I_{ijk} of non-spinning compact (point-particle) binaries at the 3PN order for general orbits in the center-of-mass frame, as well as the associated octupole gravitational-wave modes. These results are part of our current program to obtain the waveform of compact binaries complete up to order 3.5PN. The mass

octupole moment I_{ijk} is not conceptually more difficult than the mass quadrupole moment I_{ij} , which has been obtained to order 3PN in [5–7] and extended to 3.5PN in [13]. Therefore, we will simply present the result of the long calculation, after a short recapitulation of the method, which is based, exactly as in [5–7], on a preliminary Hadamard-type self-field regularization, followed by dimensional regularization and renormalization. The more difficult computation of the current quadrupole moment J_{ij} at the 3PN order will be left for future work.

4.1. Dimensional regularization of the mass octupole moment

In the first stage of the calculation we obtain the mass octupole moment by means of the so-called pure-Hadamard–Schwartz (pHS) regularization to deal with the infinite self-field of the point particles. The pHS regularization is a specific minimal Hadamard-type regularization of integrals, used together with a minimal treatment of contact ambiguities and Schwartz distributional derivatives [33]. It is free of ambiguities but depends on the usual arbitrary UV regularization length scales s_A ($A = 1, 2$) associated with the Hadamard *partie finie* regularization of integrals that diverge at the locations of the two point particles [34]. In addition, the constant r_0 introduced into the regulator (2.7) is also involved and plays the role of an IR regularization scale when computing the multipole moments. The result of this initial calculation thus reads

$$I_{ijk}^{\text{pHS}} = I_{ijk}^{\text{pHS}}[\bar{\mathbf{y}}_A, s_A, r_0], \quad (4.1)$$

where we emphasize the dependence on both the UV and IR scales, s_A and r_0 respectively, and the functional dependence on the two trajectories, denoted $\bar{\mathbf{y}}_A$, with implicit dependence on the associated coordinate velocities $\bar{\mathbf{v}}_A = d\bar{\mathbf{y}}_A/dt$. The trajectories $\bar{\mathbf{y}}_A$ will later be understood as being ‘bare’ trajectories to be ‘dressed’ by renormalization (see equation (4.3)). Notice that the initial result (4.1) does not constitute by itself a *physical* solution to the problem. To make it physical within Hadamard’s regularization, it must be supplemented by certain ‘ambiguity terms’ [3, 5]. In dimensional regularization [35, 36], which is free of ambiguities and will thus be adopted here, the physical solution is obtained by augmenting the pHS result with some specific ‘pole part’ $\propto 1/(d-3)$ in the spatial dimension d considered as a complex number [6, 7, 33].

Therefore, in the second stage of the calculation, we add to the pHS result (4.1) the so-called ‘difference’, which is by definition what we precisely have to add in order to obtain the physical result produced by dimensional regularization (DR). The important point is that the latter difference can be computed purely *locally*, i.e., at the location of the two particles, in the limit where the dimension tends to three, or equivalently, $\varepsilon \rightarrow 0$ with $\varepsilon \equiv d - 3$, because it is determined only by the singular behavior of integrals in the neighborhood of the two singular source points. Moreover, this difference depends on the Hadamard UV regularization scales s_A as well as on the DR characteristic parameters, namely ε and an arbitrary length scale ℓ_0 entering Newton’s constant in d dimensions, $G^{(d)} = G\ell_0^\varepsilon$. To summarize, the mass octupole source moment I_{ijk}^{DR} in $d = 3 + \varepsilon$ dimensions is given, in the limit $\varepsilon \rightarrow 0$, by

$$I_{ijk}^{\text{DR}}[\bar{\mathbf{y}}_A, r_0, \varepsilon, \ell_0] = I_{ijk}^{\text{pHS}}[\bar{\mathbf{y}}_A, s_A, r_0] + \mathcal{D}I_{ijk}[\bar{\mathbf{y}}_A, s_A, \varepsilon, \ell_0], \quad (4.2)$$

where the second term is made of a polar part $\propto 1/\varepsilon$ plus a finite part contribution $\propto \varepsilon^0$, i.e., is of the type $\mathcal{D}I_{ijk} = \frac{1}{\varepsilon}A_{ijk} + B_{ijk} + \mathcal{O}(\varepsilon)$, all terms $\mathcal{O}(\varepsilon)$ being systematically neglected¹⁰. At

¹⁰ One can show that there is only a simple pole at order 3PN.

this stage we verify that the two parameters s_A are canceled out between the two terms on the right-hand side of (4.2), so I_{ijk}^{DR} is free of such arbitrary UV regularization scales.

Finally, in the third stage of the computation, we *renormalize* the pole part $\propto 1/\varepsilon$ of the DR result (4.2) by shifting the particle positions that can be thought of as the bare trajectories $\bar{y}_A \equiv y_A^{\text{bare}}$ into some physical positions, corresponding to renormalized trajectories $y_A \equiv y_A^{\text{renorm}}$ that will entirely absorb the pole. The precise *shifts* η_A of the trajectories, such that

$$\bar{y}_A = y_A + \eta_A[y_A, r'_A, \varepsilon, \ell_0], \quad (4.3)$$

will consist of a pole part followed by a finite part, neglecting a remainder $\mathcal{O}(\varepsilon)$. These shifts arise at the 3PN order and have been uniquely determined at the same approximation level in equations (1.13) and (6.41)–(6.43) of [33], or equation (6.8) of [7]. They are precisely those that ensure the complete equivalence between the Hadamard regularized equations of motion—the end result of [37]—and the DR equations of motion obtained in [33]. Note that the two UV regularization scales r'_A entering equation (4.3) are *a priori* different from the parameters s_A ; they have been chosen instead to match exactly their counterparts entering the Hadamard regularized 3PN equations of motion [37], in which they play the role of UV regularization scales in the context of Hadamard’s regularization. Finally our physical, renormalized (‘dressed’) result, which is numerically equal to the original, bare, result, modulo $\mathcal{O}(\varepsilon)$ corrections, i.e.,

$$I_{ijk}[y_A, r'_A, r_0] \equiv I_{ijk}^{\text{DR}}[\bar{y}_A, r_0, \varepsilon, \ell_0] + \mathcal{O}(\varepsilon), \quad (4.4)$$

is finite in the limit where $\varepsilon \rightarrow 0$ while the dressed trajectories y_A are kept constant. Using the link (4.3) we can rewrite:

$$I_{ijk}[y_A, r'_A, r_0] = \lim_{\varepsilon \rightarrow 0} \left\{ I_{ijk}^{\text{DR}}[y_A, r_0, \varepsilon, \ell_0] + \delta_{\eta[y_A, r'_A, \varepsilon, \ell_0]} I_{ijk} \right\}, \quad (4.5)$$

where the modification $\delta_{\eta} I_{ijk} = 3 \sum_{A=1,2} m_A y_A^{(i} y_A^j \eta_A^{k)}$ due to the latter shifts follows from the variation of the Newtonian mass octupole moment $I_{ijk}^{\text{N}} = \sum_{A=1,2} m_A y_A^{(i} y_A^j y_A^{k)}$ (valid in any dimension d) induced by η_A . By construction, the poles $\sim 1/\varepsilon$ are canceled out between the two terms on the right-hand side of equation (4.5) so that the result is indeed finite (and does not depend on ℓ_0) in the limit $\varepsilon \rightarrow 0$. The final scales it depends on are the two UV scales r'_A and the IR scale r_0 .

Now, the scales r'_A have been shown to be gauge constants, i.e., removable by a suitable gauge transformation, both in the equations of motion [37] and in the radiation field [3, 5]. We will indeed verify that these constants disappear when we compute the time derivatives of the octupole moment (4.5) by means of the 3PN equations of motion for insertion into the waveform. Our final invariant results, namely the gravitational modes (3, 3) and (3, 1) obtained in section 4.3, are thus independent of r'_A .

At the same time, the dependence on the IR constant r_0 offers the possibility of an interesting consistency check with the expression of the tails-of-tails for the mass octupole derived previously in equation (3.8b). Indeed, as we have already mentioned, the tail-of-tail integrals depend on the constant $\tau_0 = r_0/c$, where r_0 is defined by equation (2.7), in such a way that it will exactly cancel the constant r_0 coming from the expressions of the source multipole moments written in terms of the source parameters (i.e., the positions and velocities of the particles). That such a cancellation between tails-of-tails and source moments actually occurs has been proved very generally for any isolated matter system [2, 17]. This has also been explicitly verified in the case of point-particle binaries for the mass quadrupole moment

at the 3PN order in [3, 5]. In section 4.2 we extend this check to the case of the mass octupole moment at the 3PN order for general orbits in the center-of-mass frame.

4.2. The 3PN mass octupole in the center-of-mass frame

We have computed the 3PN mass octupole moment (4.5) of two point masses y_A for general orbits in an arbitrary frame. We then reduced that result to the frame of the center of mass defined by the nullity of the center-of-mass integral associated with the 3PN equations of motion, given by equation (2.13) in [38]. An interesting point about this calculation is that it requires the full 3PN relations between the variables in the center-of-mass frame and the relative variables, in contrast with what happens for the 3PN mass quadrupole moment where only the 2PN center-of-mass relations are needed [5].

The center-of-mass relations for point-particle binaries take the form¹¹

$$y_1 = [X_2 + \nu \Delta P]x + \nu \Delta Q v, \quad (4.6a)$$

$$y_2 = [-X_1 + \nu \Delta P]x + \nu \Delta Q v, \quad (4.6b)$$

where all the PN corrections are proportional to the symmetric mass ratio ν and the mass difference $\Delta = X_1 - X_2$. The two dimensionless coefficients P and Q are given with the full 3PN precision in equations (3.13)–(3.14) of [38].

As an interesting feature the gauge-constants r'_A appear at the 3PN order, in the coefficient P only, in the form of the particular combination r'_0 defined by

$$\Delta \ln r'_0 = X_1^2 \ln r'_1 - X_2^2 \ln r'_2 \quad (4.7)$$

(see equation (3.15) in [38]), due to the use of the full 3PN center-of-mass relations for this calculation. However, our final 3PN mass octupole source moment in the center-of-mass frame (equations (4.9)–(4.11) hereafter) will depend on r'_A through a combination that differs from that of equation (4.7), namely

$$\ln r'_0 = X_1 \ln r'_1 + X_2 \ln r'_2. \quad (4.8)$$

Indeed we have found that the r'_A in the 3PN center-of-mass relations combine nicely with another combination of these constants in the 3PN mass octupole moment (4.5) for general orbits, i.e., after applying the required shift of the world lines (4.3) but before the center-of-mass reduction, so that the final center-of-mass expression of the mass octupole moment contains the classic combination of these constants as given by equation (4.8). This is perfectly consistent with the fact that the constants r'_A should *in fine* disappear from gauge-invariant results such as our final polarization (3, 3) and (3, 1) modes (4.18).

The mass octupole moment defined by equation (4.5) at the 3PN order, i.e., after the processes of dimensional regularization and renormalization by shifts of the world lines as reviewed in section 4.1, and reduced to the center-of-mass frame using equations (4.6) with full 3PN precision, is finally of the form

$$I_{ijk} = -\nu m \Delta \left\{ A x_{\langle i} x_j x_k \rangle + B \frac{r}{c} v_{\langle i} x_j x_k \rangle + C \frac{r^2}{c^2} v_{\langle i} v_j x_k \rangle + D \frac{r^3}{c^3} v_{\langle i} v_j v_k \rangle \right\}. \quad (4.9)$$

¹¹ Our notation for point-particle binaries is as follows: m_A stands for the two masses ($A=1,2$); $m = m_1 + m_2$ for the total mass; $X_A = m_A/m$ for the two mass fractions; $\nu = X_1 X_2$ for the symmetric mass ratio; $\mu = m\nu$ for the reduced mass; $\Delta = X_1 - X_2$ for the relative mass difference; $\mathbf{x} = (x^i) = y_1 - y_2$ and $\mathbf{v} = (v^i) = d\mathbf{x}/dt = v_1 - v_2$ for the relative separation and velocity; $v^2 = v^i v^i$ and $\dot{r} = \mathbf{n} \cdot \mathbf{v}$, where $\mathbf{n} = \mathbf{x}/r$ and $r = |\mathbf{x}|$.

The coefficients for general orbits in the center-of-mass frame are found to be

$$\begin{aligned}
A = & 1 + \frac{1}{c^2} \left\{ v^2 \left(\frac{5}{6} - \frac{19}{6} \nu \right) + \frac{Gm}{r} \left[-\frac{5}{6} + \frac{13}{6} \nu \right] \right\} \\
& + \frac{1}{c^4} \left\{ v^4 \left(\frac{257}{440} - \frac{7319}{1320} \nu + \frac{5501}{440} \nu^2 \right) \right. \\
& + \frac{Gm}{r} \left[v^2 \left(\frac{3853}{1320} - \frac{14257}{1320} \nu - \frac{17371}{1320} \nu^2 \right) \right. \\
& + \dot{r}^2 \left(-\frac{247}{1320} + \frac{531}{440} \nu - \frac{1347}{440} \nu^2 \right) \left. \right] \\
& + \frac{G^2 m^2}{r^2} \left[-\frac{47}{33} - \frac{1591}{132} \nu + \frac{235}{66} \nu^2 \right] \left. \right\} \\
& + \frac{1}{c^5} \left\{ -\frac{56 G^2 m^2 \nu \dot{r}}{9 r^2} \right\} \\
& + \frac{1}{c^6} \left\{ v^6 \left(\frac{3235}{6864} - \frac{7667}{1040} \nu + \frac{10319}{286} \nu^2 - \frac{129707}{2288} \nu^3 \right) \right. \\
& + \frac{Gm}{r} \left[v^4 \left(\frac{11633}{2640} - \frac{93167}{2640} \nu + \frac{5289}{110} \nu^2 + \frac{203299}{2640} \nu^3 \right) \right. \\
& + v^2 \dot{r}^2 \left(-\frac{841}{11440} + \frac{2039}{3432} \nu - \frac{121393}{11440} \nu^2 + \frac{589981}{17160} \nu^3 \right) \\
& + \dot{r}^4 \left(\frac{1}{208} - \frac{1379}{2288} \nu + \frac{44841}{11440} \nu^2 - \frac{63699}{11440} \nu^3 \right) \left. \right] \\
& + \frac{G^2 m^2}{r^2} \left[v^2 \left(\frac{40497}{20020} - \frac{397027}{8580} \nu + \frac{120069}{2860} \nu^2 - \frac{29429}{858} \nu^3 \right) \right. \\
& + \dot{r}^2 \left(-\frac{60413}{120120} + \frac{20801}{1560} \nu - \frac{623351}{17160} \nu^2 - \frac{70735}{3432} \nu^3 \right) \left. \right] \\
& + \frac{G^3 m^3}{r^3} \left[\frac{4553429}{229320} + \frac{902873}{15015} \nu - \frac{31673}{1716} \nu^2 \right. \\
& + \frac{27085}{5148} \nu^3 - \frac{26}{7} \ln \left(\frac{r}{r_0} \right) - 22 \nu \ln \left(\frac{r}{r_0} \right) \left. \right] \left. \right\} \\
& + \mathcal{O} \left(\frac{1}{c^7} \right), \tag{4.10a}
\end{aligned}$$

$$\begin{aligned}
B = & \frac{\dot{r}}{c} \left\{ -1 + 2\nu \right\} \\
& + \frac{\dot{r}}{c^3} \left\{ v^2 \left(-\frac{13}{22} + \frac{107}{22}\nu - \frac{102}{11}\nu^2 \right) + \frac{Gm}{r} \left[-\frac{2461}{660} + \frac{8689}{660}\nu + \frac{1389}{220}\nu^2 \right] \right\} \\
& + \frac{1}{c^4} \left\{ -\frac{12 Gm\nu v^2}{5r} + \frac{232G^2m^2\nu}{15r^2} \right\} \\
& + \frac{\dot{r}}{c^5} \left\{ v^4 \left(-\frac{2461}{5720} + \frac{37321}{5720}\nu - \frac{34627}{1144}\nu^2 + \frac{127447}{2860}\nu^3 \right) \right. \\
& + \frac{Gm}{r} \left[v^2 \left(-\frac{80629}{17160} + \frac{47979}{1144}\nu - \frac{167122}{2145}\nu^2 - \frac{267081}{5720}\nu^3 \right) \right. \\
& \left. \left. + \dot{r}^2 \left(\frac{5}{572} + \frac{1851}{1144}\nu + \frac{3059}{1560}\nu^2 - \frac{299171}{17160}\nu^3 \right) \right] \right\} \\
& + \frac{G^2m^2}{r^2} \left[\frac{229}{9240} + \frac{33459}{440}\nu - \frac{1283}{330}\nu^2 + \frac{2287}{264}\nu^3 \right] \left. \right\} \\
& + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.10b}
\end{aligned}$$

$$\begin{aligned}
C = & 1 - 2\nu \\
& + \frac{1}{c^2} \left\{ v^2 \left(\frac{61}{110} - \frac{519}{110}\nu + \frac{504}{55}\nu^2 \right) + \dot{r}^2 \left(-\frac{1}{11} + \frac{4}{11}\nu - \frac{3}{11}\nu^2 \right) \right. \\
& \left. + \frac{Gm}{r} \left[\frac{1949}{330} + \frac{62}{165}\nu - \frac{483}{55}\nu^2 \right] \right\} \\
& + \frac{1}{c^4} \left\{ v^4 \left(\frac{465}{1144} - \frac{35777}{5720}\nu + \frac{3057}{104}\nu^2 - \frac{25071}{572}\nu^3 \right) \right. \\
& + v^2 \dot{r}^2 \left(-\frac{197}{1430} + \frac{1637}{1430}\nu - \frac{849}{286}\nu^2 + \frac{3063}{1430}\nu^3 \right) \\
& + \frac{Gm}{r} \left[v^2 \left(\frac{91379}{17160} - \frac{169537}{5720}\nu + \frac{83211}{5720}\nu^2 + \frac{504721}{8580}\nu^3 \right) \right. \\
& \left. + \dot{r}^2 \left(-\frac{3037}{3432} - \frac{125}{104}\nu + \frac{62143}{17160}\nu^2 + \frac{113089}{8580}\nu^3 \right) \right] \\
& + \frac{G^2m^2}{r^2} \left[-\frac{34515967}{1261260} - \frac{75373}{8580}\nu \right. \\
& \left. + \frac{428669}{8580}\nu^2 - \frac{62665}{2574}\nu^3 + \frac{52}{7} \ln\left(\frac{r}{r_0}\right) \right] \left. \right\} \\
& + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.10c}
\end{aligned}$$

$$\begin{aligned}
D = & \frac{\dot{r}}{c} \left\{ \frac{13}{55} - \frac{52}{55}\nu + \frac{39}{55}\nu^2 \right\} \\
& + \frac{\dot{r}}{c^3} \left\{ v^2 \left(\frac{333}{1430} - \frac{3181}{1430}\nu + \frac{1849}{286}\nu^2 - \frac{7247}{1430}\nu^3 \right) \right. \\
& + \dot{r}^2 \left(\frac{112}{2145} - \frac{224}{715}\nu + \frac{224}{429}\nu^2 - \frac{448}{2145}\nu^3 \right) \\
& \left. + \frac{Gm}{r} \left[\frac{26641}{8580} - \frac{8341}{2860}\nu - \frac{1655}{156}\nu^2 + \frac{24367}{4290}\nu^3 \right] \right\} \\
& + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{4.10d}
\end{aligned}$$

Let us also give the result of the reduction to quasi-circular orbits. Introducing the post-Newtonian parameter $\gamma = Gm/(rc^2)$, we have

$$\begin{aligned}
A^{\text{circ}} = & 1 - \gamma\nu + \gamma^2 \left(-\frac{139}{330} - \frac{11923}{660}\nu - \frac{29}{110}\nu^2 \right) \\
& + \gamma^3 \left(\frac{1229440}{63063} + \frac{610499}{20020}\nu + \frac{319823}{17160}\nu^2 \right. \\
& \left. - \frac{101}{2340}\nu^3 - \frac{26}{7} \ln\left(\frac{r}{r_0}\right) - 22\nu \ln\left(\frac{r}{r'_0}\right) \right) \\
& + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{4.11a}
\end{aligned}$$

$$B^{\text{circ}} = \frac{196}{15}\gamma^2\nu + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.11b}$$

$$\begin{aligned}
C^{\text{circ}} = & 1 - 2\nu + \gamma \left(\frac{1066}{165} - \frac{1433}{330}\nu + \frac{21}{55}\nu^2 \right) \\
& + \gamma^2 \left(-\frac{1130201}{48510} - \frac{989}{33}\nu + \frac{20359}{330}\nu^2 - \frac{37}{198}\nu^3 + \frac{52}{7} \ln\left(\frac{r}{r_0}\right) \right) \\
& + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.11c}
\end{aligned}$$

$$D^{\text{circ}} = \mathcal{O}\left(\frac{1}{c^4}\right). \tag{4.11d}$$

We observe that the coefficients A^{circ} and C^{circ} for quasi-circular orbits are purely conservative, while the other ones, B^{circ} and D^{circ} , are purely dissipative, i.e., due to radiation reaction. The results (4.11) extend equation (5.15a) in [12] to the 3PN order.

As already mentioned, the IR constant r_0 coming from the MPM regulator (2.7) in the 3PN octupole moment is exactly compensated by the same constant $r_0 = cr_0$ coming from the kernel of the mass octupole tail-of-tail integral displayed in equation (3.8b). Indeed we can verify from equations (4.9)–(4.10) that the octupole moment for general orbits depends on this constant through the combination

$$I_{ijk} = \dots - \frac{26}{21} \frac{G^2 m^2}{c^6} I_{ijk}^{(2)} \ln r_0 + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (4.12)$$

where we have re-expressed the 3PN coefficient by means of the Newtonian octupole $I_{ijk}^N = \sum_A m_A y_A^{(ijk)}$. The ellipsis denotes all the other terms in equations (4.9)–(4.10), which are independent of r_0 . At the same time, equation (3.8b) immediately gives

$$U_{ijk}^{\text{tail-tail}} = \dots + \frac{26}{21} \frac{G^2 M^2}{c^6} M_{ijk}^{(5)} \ln r_0, \quad (4.13)$$

which is nicely consistent with the source moment (4.12) and shows that the octupole tail-of-tail indeed compensates the r_0 present in the source octupole, at leading order. Recall that the coefficient $\alpha_3 = 26/21$ is a particular case of the general formula (3.9).

4.3. The gravitational-wave octupole modes (3, 3) and (3, 1)

With the 3PN mass octupole source moment in hand, and using all the non-linear multipole interactions computed in section 3, we obtain the complete 3PN mass octupole radiative moment U_{ijk} . Now recall that for non-spinning compact binaries, there is a clean separation of the modes (ℓ, m) into those with $\ell + m$ even, which depend only on the mass-type moments U_L , and those with $\ell + m$ odd, which depend only on the current-type moments V_L ¹². From the radiative octupole U_{ijk} we can thus compute the associated modes (3, 3) and (3, 1) in the usual spin-weighted spherical-harmonic decomposition of the waveform (3.2) for quasi-circular orbits. For $\ell + m$ even we have, adopting the conventions of [12, 13],

$$h_{\ell m} = -\frac{2G}{Rc^{\ell+2}\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{\ell(\ell-1)}} \alpha_{\ell m}^L U_L, \quad (4.14)$$

where the STF tensorial factor $\alpha_{\ell m}^L$ connects the usual basis of spherical harmonics $Y^{\ell m}$ to the set of STF products of unit direction vectors \hat{N}_L (see equation (C.10)). As in [12, 13], we pose

$$h_{\ell m} = \frac{2G}{R} \frac{m \nu x}{c^2} \sqrt{\frac{16\pi}{5}} H_{\ell m} e^{-im\psi}, \quad (4.15)$$

where the post-Newtonian parameter $x = (\frac{Gm\omega}{c^3})^{2/3}$ is defined from the orbital frequency of circular motion ω , and where ψ denotes a particular phase variable related to the actual orbital phase of the binary, namely $\varphi = \int \omega dt$, by

$$\psi = \varphi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right), \quad (4.16)$$

the constant frequency ω_0 being directly linked to the time scale b entering the relation (3.1) between harmonic and radiative coordinates:

$$\omega_0 = \frac{e^{\frac{11}{12} - \gamma_E}}{4b}, \quad (4.17)$$

with γ_E denoting the Euler constant. The 1.5PN logarithmic phase modulation in equation (4.16) originates physically from tails that propagate in the far zone [8, 9]. The mass M therein is the ADM mass; it *must* include the relevant post-Newtonian corrections, up to the 1PN order in the current case (see equation (5.23) in [12]).

¹² This fact is more generally true for ‘planar’ binaries, whose motion takes place in a fixed orbital plane, which is the case of spinning binaries with spins aligned or anti-aligned with the orbital angular momentum; see [13] for a proof.

Our final results for the gravitational-wave modes (3, 3) and (3, 1) at order 3.5PN in the waveform for quasi-circular orbits read

$$\begin{aligned}
H_{33} = & -\frac{3}{4}i\sqrt{\frac{15}{14}}\Delta\left[x^{1/2} + x^{3/2}(-4 + 2\nu) + x^2\left(3\pi + i\left[-\frac{21}{5} + 6\ln(3/2)\right]\right)\right] \\
& + x^{5/2}\left(\frac{123}{110} - \frac{1838\nu}{165} + \frac{887\nu^2}{330}\right) + x^3\left(-12\pi + \frac{9\pi\nu}{2}\right. \\
& \left. + i\left[\frac{84}{5} - 24\ln(3/2) + \nu\left(-\frac{48103}{1215} + 9\ln(3/2)\right)\right]\right) \\
& + x^{7/2}\left(\frac{19388147}{280280} + \frac{492}{35}\ln(3/2) - 18\ln^2(3/2)\right. \\
& \left. - \frac{78}{7}\gamma_E + \frac{3}{2}\pi^2 + 6i\pi\left[-\frac{41}{35} + 3\ln(3/2)\right]\right) \\
& + \frac{\nu}{8}\left[-\frac{7055}{429} + \frac{41}{8}\pi^2\right] - \frac{318841}{17160}\nu^2 + \frac{8237}{2860}\nu^3 - \frac{39}{7}\ln(16x) \\
& + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{4.18a}
\end{aligned}$$

$$\begin{aligned}
H_{31} = & \frac{i\Delta}{12\sqrt{14}}\left[x^{1/2} + x^{3/2}\left(-\frac{8}{3} - \frac{2\nu}{3}\right) + x^2\left(\pi + i\left[-\frac{7}{5} - 2\ln 2\right]\right)\right] \\
& + x^{5/2}\left(\frac{607}{198} - \frac{136\nu}{99} - \frac{247\nu^2}{198}\right) + x^3\left(-\frac{8\pi}{3} - \frac{7\pi\nu}{6}\right. \\
& \left. + i\left[\frac{56}{15} + \frac{16\ln 2}{3} + \nu\left(-\frac{1}{15} + \frac{7\ln 2}{3}\right)\right]\right) \\
& + x^{7/2}\left(\frac{10753397}{1513512} - 2\ln 2\left[\frac{212}{105} + \ln 2\right] - \frac{26}{21}\gamma_E + \frac{\pi^2}{6} - 2i\pi\left[\frac{41}{105} + \ln 2\right]\right) \\
& + \frac{\nu}{8}\left(-\frac{1738843}{19305} + \frac{41}{8}\pi^2\right) + \frac{327059}{30888}\nu^2 - \frac{17525}{15444}\nu^3 - \frac{13}{21}\ln x \\
& + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{4.18b}
\end{aligned}$$

This extends equation (9.4d) and (9.4f) in [12] by one-half PN order. We have verified the complete agreement in the test-mass limit $\nu \rightarrow 0$ with the corresponding modes computed by black-hole perturbation techniques and reported in equations (4.9) of [39]. Notice that the latter work uses a phase variable which differs from our definition ψ given by equation (4.16); in particular it happens to be different for each mode (ℓ, m) ¹³.

¹³ It is related to ours in the test-mass limit ($m_1 \rightarrow 0$) by

$$\psi_{\ell m}^{\text{FI}} = \psi + 2x^{3/2}\left(\gamma_E + \frac{3}{2}\ln(4x) - \frac{17}{12}\right) + \psi_{\ell m}^{(3\text{PN})},$$

with $\psi_{\ell m}^{(3\text{PN})}$ being defined by equation (4.5) of [39].

The other modes (3, 2) and (3, 0) are known at order 3PN but cannot be computed at order 3.5PN for now; this computation will have to wait for the completion of the current quadrupole moment J_{ij} (currently known at order 2.5PN [12]) up to order 3PN. The derivation of the 3PN current quadrupole presents new difficulties with respect to the 3PN mass quadrupole or octupole moments and will be left for future work.

5. Tail-induced resummed waveform

5.1. Resummation of IR logarithms

By implementing the MPM algorithm, it has been shown by induction that the n th post-Minkowskian coefficient in harmonic coordinates in an expansion when $r \rightarrow +\infty$ with $t_r = \text{const}$ involves powers of $1/r$ and powers of the logarithm $\ln r$ up to $n - 1$, so that its general structure at future null infinity reads [14]¹⁴

$$h_{(n)}^{\alpha\beta}(\mathbf{x}, t) = \sum_{\ell=0}^{+\infty} \hat{n}_L \left\{ \sum_{\substack{1 \leq k \leq N \\ 0 \leq p \leq n-1}} \frac{(\ln r/b)^p}{r^k} F_{L(n)kp}^{\alpha\beta}(t_r) + R_{L(n)N}^{\alpha\beta}(r, t_r) \right\}. \quad (5.1)$$

Here $h_{(n)}$ stands for the n th order post-Minkowskian piece of the gravitational field either in the canonical or in the general MPM algorithms reviewed in section 2. The functions $F_{L(n)kp}$ are complicated functionals of the canonical moments $\{M_L, S_L\}$ or the source and gauge moments, $\{I_L, J_L\}$ and $\{W_L, X_L, Y_L, Z_L\}$ respectively, depending on the chosen algorithm. The angular part is expressed with the STF products \hat{n}_L of unit vectors $n^i = x^i/r$. The remainder $R_{L(n)N}(r, t_r)$ is $\mathcal{O}(1/r^{N-\epsilon})$, with $0 < \epsilon \ll 1$ taking into account the fact that the expansion involves powers of $\ln r$. Finally the logarithms in equation (5.1) are conveniently rescaled by means of an arbitrary constant b , it being understood that the functions $F_{L(n)kp}$ are themselves dependent on this constant b . Restricting our attention to the leading coefficient of $1/r$ at infinity, we write

$$h_{(n)}^{\alpha\beta} = \frac{1}{r} z_{(n)}^{\alpha\beta}(\mathbf{n}, \ln r, t_r) + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right), \quad (5.2a)$$

$$\text{with } z_{(n)}^{\alpha\beta} = \sum_{\ell} \hat{n}_L \sum_{p=0}^{n-1} \left(\ln \frac{r}{b}\right)^p F_{L(n)p}^{\alpha\beta}(t_r). \quad (5.2b)$$

We will refer to the logarithms generated in the far-zone expansion of the metric at infinity as the *IR logarithms*. They have their root in the famous logarithmic deviation of the retarded cones $t_r = \text{const}$ in harmonic coordinates with respect to the true null cones $u = \text{const}$, where u is a null coordinate satisfying $g^{\mu\nu} \partial_\mu u \partial_\nu u = 0$. The IR logarithms can be removed by a coordinate transformation order by order in the MPM expansion. We have already seen in equations (3.1)–(3.2) that one can construct radiative coordinates that are free of any such IR logarithms. Radiative coordinates are such that $T_R = u + \mathcal{O}(1/R)$.

In this subsection we consider a particular class of IR logarithms generated by tails. Recall that in the MPM algorithm the IR logarithms are produced by source terms behaving like $1/r^2$ when $r \rightarrow +\infty$ with $t_r = \text{const}$ (where we include in the coefficient the usual dependence on powers of $\ln r$). Although source terms behaving like $1/r^k$ with $k \geq 3$ do generate $1/r$ contributions after application of the retarded integral (i.e., $\text{FP}_{B=0} \square_{\text{ret}}^{-1} \tilde{r}^B$ in the

¹⁴ Recall that $t_r \equiv t - r$. In this section we pose $G = c = 1$.

notation of equation (2.6)), those are in the form of source-free retarded waves which do not contain IR logarithms; this is proved by lemma 7.2 in [14].

Using the leading order behavior $1/r$ of $h_{(n)}$, equation (5.2a), we see that IR logarithms can come only from that part of the gravitational source term Λ in equation (2.1a) which is quadratic in h , say, $\Lambda = N(h, h) + \mathcal{O}(h^3)$ where $N(h, h)$, is bilinear in h as well as in its space-time derivatives ∂h and $\partial^2 h$. Thus IR logarithms come only from solving the equation

$$\square h_{(n)}^{\alpha\beta} = \sum_{m=1}^{n-1} N^{\alpha\beta}(h_{(m)}, h_{(n-m)}) + \mathcal{O}\left(\frac{1}{r^{3-\epsilon}}\right). \quad (5.3)$$

Writing the Einstein equations in harmonic coordinates in terms of the gothic metric and using the asymptotic formula $\partial_\mu h_{(n)}^{\alpha\beta} = -k_\mu \partial_t h_{(n)}^{\alpha\beta} + \mathcal{O}(1/r^{2-\epsilon})$, where $t_r = \text{const}$ and $k_\mu = (-1, n^i)$ denotes a null Minkowskian vector, yields [21, 22]

$$\square h_{(n)}^{\alpha\beta} = \frac{1}{r^2} \left[4M \partial_t^2 z_{(n-1)}^{\alpha\beta} + k^\alpha k^\beta \sigma_{(n)} \right] + \mathcal{O}\left(\frac{1}{r^{3-\epsilon}}\right). \quad (5.4)$$

The second term in that expression is due to the re-radiation of gravitational waves by the stress-energy tensor of gravitational waves themselves. This term is responsible for the non-linear memory effect [22, 26–32] (see section 3.1). The energy density $\sigma_{(n)}$ is proportional to the total gravitational-wave flux emitted to order n and reads explicitly

$$\sigma_{(n)} = \frac{1}{2} \sum_{m=1}^{n-1} \left(\eta_{\mu\rho} \eta_{\nu\sigma} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \partial_t z_{(m)}^{\mu\nu} \partial_t z_{(n-m)}^{\rho\sigma}. \quad (5.5)$$

As proved in [21] (see lemma 2.1 there) the IR logarithms coming from the second term in equation (5.4) can be removed, order by order in the MPM iteration, by means of the gauge transformation with gauge vector

$$\lambda_{(n)}^\alpha = \square_{\text{ret}}^{-1} \left[\frac{k^\alpha}{2r^2} \int_{-\infty}^{t_r} d\tau \sigma_{(n)}(\mathbf{n}, \ln r, \tau) \right]. \quad (5.6)$$

Because the retarded integral is convergent there is no need to include the $\text{FP}_{\beta=0}$ operation.

In this appendix we are interested in the IR logarithms generated by the tail term associated with backscatter onto the static space-time curvature generated by the total ADM mass M of the system—the first term on the right-hand side of equation (5.4). The mass is introduced in the formalism as the constant monopole moment $I \equiv M$ in the ‘canonical’ linearized metric (2.2a). To prove that M enters the source term at any MPM order n in the way shown in (5.4), we invoke our assumption that the matter system is stationary before some instant $-\mathcal{T}$ in the past. Under this assumption the relaxed Einstein equation (2.1a) for the quadratic metric in the stationary epoch $t \leq -\mathcal{T}$ tells us that $\Delta h_{(2)}^{\alpha\beta} = \mathcal{O}(1/r^4)$, hence $h_{(2)}^{\alpha\beta} = \mathcal{O}(1/r^2)$. By immediate recurrence, we conclude that $h_{(n)}^{\alpha\beta} = \mathcal{O}(1/r^n)$ when $t \leq -\mathcal{T}$. From this we infer that $k_\mu k_\nu z_{(n)}^{\mu\nu} = 0$ at any time for $n \geq 2$ since it is constant due to the asymptotic form of the harmonic-gauge condition, and vanishes at early time $t \leq -\mathcal{T}$, whereas we have $k_\mu k_\nu z_{(1)}^{\mu\nu} = -4M$. Thus, for any $n \geq 2$ the first term of equation (5.4) comes only from the coupling between the static part of $h_{(1)}$ and $h_{(n-1)}$.

We will solve recursively and look for IR logarithms in the equation

$$\square \bar{h}_{(n)}^{\alpha\beta} = \frac{4M}{r^2} \partial_t^2 z_{(n-1)}^{\alpha\beta} + \mathcal{O}\left(\frac{1}{r^{3-\epsilon}}\right). \quad (5.7)$$

We add an overbar to emphasize that we are considering an approximation of the harmonic-coordinate MPM algorithm in which we neglect all the IR logarithms generated by the second

term of equation (5.4). The iteration of equation (5.7) is achieved most easily in the frequency space. The time Fourier transform of a function $F(t)$ will be denoted by $\tilde{F}(\Omega) \equiv \mathcal{F}(F)(\Omega)$ with the convention

$$\tilde{F}(\Omega) = \int_{-\infty}^{+\infty} dt F(t) e^{i\Omega t}, \quad F(t) = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \tilde{F}(\Omega) e^{-i\Omega t}. \quad (5.8)$$

The integration of the relevant $1/r^2$ source terms will be achieved in the Fourier domain with the help of the following formula:

$$\begin{aligned} \square_{\text{ret}}^{-1} \left[\frac{\hat{n}_L}{r^2} \left(\ln \frac{r}{b} \right)^p F(t_r) \right] &= \frac{\hat{n}_L}{2(p+1)r} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \frac{\tilde{F}(\Omega) e^{-i\Omega t_r}}{(-i\Omega)} \\ &\quad \times \left[\gamma_\ell^{(p+1)}(0, \Omega b) - \left(\ln \frac{r}{b} \right)^{p+1} \right] \\ &\quad + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right), \end{aligned} \quad (5.9)$$

in which the function of two variables $\gamma_\ell(B, x)$ is defined by (with Γ the Eulerian function)

$$\gamma_\ell(B, x) = \frac{\Gamma(\ell + 1 + B)\Gamma(1 - B)}{(-2ix)^B \Gamma(\ell + 1 - B)}. \quad (5.10)$$

In equation (5.9) this function is differentiated $(p+1)$ times with respect to the first variable B and then evaluated at $B=0$ and for $x = \Omega b$, defining thus

$$\gamma_\ell^{(p+1)}(0, \Omega b) \equiv \left(\frac{\partial^{p+1} \gamma_\ell}{\partial B^{p+1}} \right) (0, \Omega b). \quad (5.11)$$

Again notice that the retarded integral (5.9) is convergent, so there is no need to invoke a finite part operation as in equation (2.6). For the reader's convenience we provide the proof of the elementary formula (5.9) in appendix B.

We can now obtain, thanks to the elementary formula (5.9)¹⁵, the leading order waveform $\bar{z}_{(n)}$ starting from the preceding one $\bar{z}_{(n-1)}$, both having general structures similar to equation (5.2a). This gives some recursion relations for the functions $F_{L(n)p}$ parameterizing their general structures. These are given in the Fourier domain for any $n \geq 2$ by

$$\tilde{F}_{L(n)p}(\Omega) = \frac{2iM\Omega}{p} \tilde{F}_{L(n-1)p-1}(\Omega) \quad \text{for } 1 \leq p \leq n-1, \quad (5.12a)$$

$$\tilde{F}_{L(n)0}(\Omega) = - \sum_{q=0}^{n-2} \frac{2iM\Omega}{q+1} \gamma_\ell^{(q+1)}(0, \Omega b) \tilde{F}_{L(n-1)q}(\Omega) \quad \text{for } p=0. \quad (5.12b)$$

Such recursion formulas are easily iterated, with the result that

$$\tilde{F}_{L(n)p} = \frac{(2iM\Omega)^p}{p!} \tilde{F}_{L(n-p)0} \quad \text{for } 1 \leq p \leq n-1, \quad (5.13a)$$

¹⁵ Here we consider only the retarded integral of the source term, corresponding to the part $\bar{u}_{(n)}^{\alpha\beta}$ of the MPM algorithm (see equations (2.5)–(2.6)), since the part $\bar{v}_{(n)}^{\alpha\beta}$ does not contain IR logarithms.

$$\tilde{F}_{L(n)0} = - \sum_{q=0}^{n-2} \frac{(2iM\Omega)^{q+1}}{(q+1)!} \gamma_\ell^{(q+1)}(0, \Omega b) \tilde{F}_{L(n-q-1)0} \quad \text{for } p = 0. \quad (5.13b)$$

These results yield immediately the complete resummed waveform in the Fourier domain as follows. For convenience we denote by \tilde{z}_L the STF piece with multipolarity ℓ in the full waveform. From the first result (5.13a) we then determine

$$\tilde{z}_L^{\alpha\beta}(\ln r, \Omega) = \sum_{n=1}^{+\infty} \tilde{z}_{L(n)}^{\alpha\beta}(\ln r, \Omega) = e^{2iM\Omega \ln(\frac{r}{b})} \tilde{F}_{L0}^{\alpha\beta}(\Omega), \quad (5.14)$$

in which $\tilde{F}_{L0}(\Omega)$ refers to the logarithmic-free part of the full waveform and is defined by $\tilde{F}_{L0} = \sum_{n=1}^{+\infty} \tilde{F}_{L(n)0}$. Next, the second result (5.13b) gives the logarithmic-free part of the waveform in terms of the linearized approximation, which is nicely factorized out as

$$\tilde{F}_{L0}(\Omega) = \frac{\tilde{F}_{L(1)0}(\Omega)}{\gamma_\ell(2iM\Omega, \Omega b)}, \quad (5.15)$$

where the denominator is made of the function $\gamma_\ell(B, x)$ now evaluated at $B = 2iM\Omega$ and $x = \Omega b$, and where we have used the fact that $\gamma_\ell(0, x) = 1$. Note that $\tilde{z}_{L(1)}(\Omega) = \tilde{F}_{L(1)0}(\Omega)$ is the waveform at the linear order $n = 1$. Finally, combining these two findings with the expression of the function (5.10) we obtain the final expression for our resummed tail-modified waveform:

$$\begin{aligned} \tilde{z}_L(\ln r, t_r) = & \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \frac{\Gamma(\ell + 1 - 2iM\Omega)}{\Gamma(\ell + 1 + 2iM\Omega)\Gamma(1 - 2iM\Omega)} \\ & \times \tilde{z}_{L(1)}(\Omega) e^{-i\Omega t_r + 2iM\Omega \ln(2|\Omega|b) + M|\Omega|\pi}. \end{aligned} \quad (5.16)$$

We gladly notice that the scale b has been canceled out from the final result (5.16), in agreement with the fact that the iteration of the equation (5.7) does not make any reference to an arbitrary scale such as b . However, the dependence on b is restored in radiative coordinates through the redefinition (3.1) of the retarded time T_R at future null infinity, and the phase factor in equation (5.16) becomes $\exp[-i\Omega T_R + 2iM\Omega \ln(2|\Omega|b) + M|\Omega|\pi]$. We observe that all powers of $\ln r$ have been absorbed into T_R , but powers of $\ln(2|\Omega|b)$ are still left. This motivates the change of phase variable introduced in equation (4.16), the constant frequency ω_0 defined in equation (4.17) being chosen for convenience to minimize the number of terms in the waveform. Recall also that, as indicated by an overbar, the resummed waveform (5.16) does not constitute the complete resummed waveform in harmonic coordinates because we have systematically neglected the second term in equation (5.4). However, it motivates the introduction of *factorized* resummed waveforms [40], which we shall shortly review and complete in the next subsection for the case of the mass octupole waveform.

In fact, the solution (5.16) can be obtained without MPM iteration and resummation by computing directly the solution of a ‘scattering’ problem,

$$\square \bar{h} - \frac{4M}{r} \partial_t^2 \bar{h} = \bar{S}, \quad (5.17)$$

where the scattering barrier is the usual potential M/r , and where \bar{S} is some effective source. To recover the solution (5.16), it suffices to posit that \bar{S} decreases like $\mathcal{O}(1/r^{3-\varepsilon})$ at infinity. The most general solution of (5.17) can be constructed formally by convoluting the source \bar{S} with the retarded Green function $G_{\text{ret}}(x - x')$ of the differential operator $\square - (4M/r)\partial_t^2$. In [41], $G_{\text{ret}}(x - x')$ is computed explicitly using standard techniques of scattering theory [42].

In the limit $r \rightarrow +\infty$ at $t_r = \text{const}$, it is essentially proportional to the retarded solution of the modified Whittaker equation and to the normalization coefficient of the solution regular at the origin, for $\Omega r \rightarrow 0$.

5.2. Application to the octupole resummed waveform

We conclude the paper with an application of our foregoing computation of the 3.5PN accurate modes h_{33} and h_{31} to the effective-one-body (EOB) approach [43, 44] to analytically blend PN approximants and numerical-relativity (NR) results. Factorized resummed waveforms were introduced in [40] and consist of a physically motivated product of the Newtonian waveform, a relativistic correction coming from an effective source built from the EOB Hamiltonian, the resummed tail effects linked to propagation on a Schwarzschild background (see section 5.1), a residual tail dephasing $i\delta_{\ell m}$ (complex), and finally the ℓ th power of a residual relativistic amplitude correction (thus purely real) denoted $\rho_{\ell m}$. The factorized resummed waveforms achieve better agreement with NR results than the conventional Taylor expanded PN waveforms [39, 40, 45, 46]. For $\ell + m$ even (corresponding to even-parity $c = 0$) we have

$$H_{\ell m} = H_{\ell m}^{\text{N}} S_{\text{eff}} T_{\ell m} e^{i\delta_{\ell m}} (\rho_{\ell m})^{\ell}. \quad (5.18)$$

The Newtonian approximation to any even-parity mode reads

$$H_{\ell m}^{\text{N}} = \frac{(-)^{(\ell-m+2)/2}}{2^{\ell+1} \left(\frac{\ell+m}{2}\right)! \left(\frac{\ell-m}{2}\right)! (2\ell-1)!!} \times \left(\frac{5(\ell+1)(\ell+2)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(2\ell+1)} \right)^{1/2} s_{\ell}(\nu) (im)^{\ell} x^{\ell/2-1}, \quad (5.19)$$

where we denote $s_{\ell}(\nu) \equiv X_2^{\ell-1} + (-)^{\ell} X_1^{\ell-1}$; see, e.g., equation (9.5) in [12]. The effective source $S_{\text{eff}} = \frac{H_{\text{eff}}}{\mu c^2}$ is given at order 3PN by

$$S_{\text{eff}} = 1 - \frac{x}{2} \left\{ 1 + \left(-\frac{3}{4} - \frac{1}{3}\nu \right) x + \left(-\frac{27}{8} + \frac{11}{4}\nu \right) x^2 + \left(-\frac{675}{64} + \left[\frac{4417}{72} - \frac{205}{96}\pi^2 \right] \nu - \frac{17}{6}\nu^2 + \frac{\nu^3}{81} \right) x^3 + \mathcal{O}\left(\frac{1}{c^8}\right) \right\}. \quad (5.20)$$

For $m \geq 0$, the leading Schwarzschild tail factor reads

$$T_{\ell m} = \frac{\Gamma(\ell+1-2ik_m)}{\Gamma(\ell+1)} e^{k_m[\pi+2i \ln(2m\omega_0 b)]}, \quad (5.21)$$

in which we denote $k_m \equiv GMm\omega/c^3$, with M being the ADM mass to be inserted here at the 1PN order as in equation (4.16) (see equation (5.23) in [12]), and where $\omega_0 b$ is a pure real number to be found from equation (4.17)¹⁶. Note that the EOB tail factor (5.21) is motivated by the factorized resummed waveform (5.16). Indeed, the dominant Fourier component of the mode $h_{\ell m}$ defined by equation (4.15) is obtained by setting $\Omega = m\omega$ in the stationary phase approximation.

¹⁶ Our definition takes into account the modification of the phase given by equation (4.16). The factor used in [40] is thus related to ours by $T_{\ell m}^{\text{DIN}} = T_{\ell m} e^{2ik_m \ln(\omega/\omega_0)}$.

As for the residual dephasings δ_{33} and δ_{31} , we find that they do not receive any finite mass correction $\mathcal{O}(\nu)$ at order 3PN and are therefore given by the sum of the expressions (22) and (24) of [40] truncated at order 2.5PN and the 3PN contributions in the test-mass limit computed in equations (5.8c) and (5.8e) of [39], i.e., up to order 3PN:

$$\delta_{33} = \frac{13}{10}y^{3/2} - \frac{80897}{2430}\nu y^{5/2} + \frac{39}{7}\pi y^3 + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (5.22a)$$

$$\delta_{31} = \frac{13}{30}y^{3/2} - \frac{17}{10}\nu y^{5/2} + \frac{13}{21}\pi y^3 + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (5.22b)$$

with $y = (GM\omega/c^3)^{2/3}$ and M being the ADM mass.

Finally the most important inputs we provide in this application are the finite mass corrections to order 3PN of the amplitude factors ρ_{33} and ρ_{31} that will form the main blocks in the factorized resummation of waveforms [39, 40, 47]. These are straightforwardly computed from equations (4.18); for completeness we report here the full 3PN expressions, extending at the 3PN order equations (52) and (54) in [40]:

$$\begin{aligned} \rho_{33} = & 1 + \left(-\frac{7}{6} + \frac{2}{3}\nu\right)x + \left(-\frac{6719}{3960} - \frac{1861}{990}\nu + \frac{149}{330}\nu^2\right)x^2 \\ & + \left(\frac{3203101567}{227026800} - \frac{26}{7}\gamma_E - \frac{13}{7}\ln(36x) + \left[-\frac{129509}{25740} + \frac{41}{192}\pi^2\right]\nu\right. \\ & \left. - \frac{274621}{154440}\nu^2 + \frac{12011}{46332}\nu^3\right)x^3 + \mathcal{O}\left(\frac{1}{c^8}\right), \end{aligned} \quad (5.23a)$$

$$\begin{aligned} \rho_{31} = & 1 + \left(-\frac{13}{18} - \frac{2}{9}\nu\right)x + \left(\frac{101}{7128} - \frac{1685}{1782}\nu - \frac{829}{1782}\nu^2\right)x^2 \\ & + \left(\frac{11706720301}{6129723600} - \frac{26}{63}\gamma_E - \frac{13}{63}\ln(4x) + \left[-\frac{9688441}{2084940} + \frac{41}{192}\pi^2\right]\nu\right. \\ & \left. + \frac{174535}{75816}\nu^2 - \frac{727247}{1250964}\nu^3\right)x^3 + \mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \quad (5.23b)$$

This completes our application to EOB resummed waveforms.

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Appendix A. Integration of elementary cubic source terms

To compute the tail-of-tail contributions to the waveform in the far zone, we need to control the dominant asymptotic behaviour at future null infinity of the (finite part of the) retarded integrals of the relevant cubic-source piece, i.e., $\Lambda_{(3)}^{a\beta}$ in the notation of equation (2.6). This problem is essentially solved in appendix A of [2], but here we provide some more details and additional material.

For generic elementary source terms with multipolarity ℓ and radial dependence $1/r^k$ ($k \geq 1$) that involve a non-local integral whose kernel is a Legendre function $Q_m(x)$, we consider¹⁷:

$$\Psi_{m,\ell,k}^L = \text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[\tilde{r}^B \hat{n}_L r^{-k} \int_1^{+\infty} dx Q_m(x) F(t - rx) \right]. \quad (\text{A.3})$$

Depending on the values of m , ℓ , and k , the far-zone expansion of $\Psi_{m,\ell,k}^L$ when $r \rightarrow \infty$ (with $t_r = \text{const}$) takes one of the following forms. For $k = 1$ and $m = \ell$ we have

$$\Psi_{\ell,\ell,1}^L = -\frac{\hat{n}_L}{8r} \int_0^{+\infty} d\tau F^{(-1)}(t_r - \tau) \left[\ln^2\left(\frac{\tau}{2r}\right) + 4H_\ell \ln\left(\frac{\tau}{2r}\right) + 4H_\ell^2 \right] + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right), \quad (\text{A.4})$$

where $H_\ell = \sum_{j=1}^{\ell} \frac{1}{j}$ is the ℓ th harmonic number and $F^{(-1)}$ denotes the anti-derivative of F that vanishes at $-\infty$. For $2 \leq k \leq \ell + 2$ and $k \geq \ell + 3$ respectively, we have

$$\Psi_{m,\ell,k}^L = -\alpha_{m,\ell,k} \frac{\hat{n}_L}{r} F^{(k-3)}(t_r) + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right), \quad (\text{A.5a})$$

$$\Psi_{m,\ell,k}^L = -\frac{\hat{n}_L}{r} \int_0^{+\infty} d\tau F^{(k-2)}(t_r - \tau) \left[\beta_{m,\ell,k} \ln\left(\frac{\tau}{2r_0}\right) + \gamma_{m,\ell,k} \right] + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right). \quad (\text{A.5b})$$

The formula (A.4) and the explicit expression of $\alpha_{m,\ell,k}$ for arbitrary m , ℓ , k , are obtained in [2], as well as the explicit expressions of $\beta_{m,\ell,k}$ and $\gamma_{m,\ell,k}$ for specific values of k and ℓ .

Let us first recall the derivation of the coefficient $\alpha_{m,\ell,k}$ which is given for general values of k and ℓ such that $2 \leq k \leq \ell + 2$ by

$$\alpha_{m,\ell,k} = \int_1^{+\infty} dx Q_m(x) \int_x^{+\infty} dz Q_\ell(z) \frac{(z-x)^{k-3}}{(k-3)!}. \quad (\text{A.6})$$

It is convenient to introduce as intermediate notation the $(k-2)$ th anti-derivative of $Q_\ell(x)$ that vanishes at $x = +\infty$ and is given for $k \geq 3$ by

$$Q_\ell^{(-k+2)}(x) = -\int_x^{+\infty} dz Q_\ell(z) \frac{(x-z)^{k-3}}{(k-3)!}. \quad (\text{A.7})$$

For $k = 2$ we naturally pose $Q_\ell^{(0)}(x) = Q_\ell(x)$. Recalling that $Q_\ell(z)$ behaves like $z^{-\ell-1}$ when $z \rightarrow +\infty$ we see that the integral on the right side is convergent when $k \leq \ell + 2$. Anti-derivatives of Legendre functions can be expanded straightforwardly on the basis of Legendre functions themselves by means of the recurrence relation $(d/dz)[Q_{\ell+1}(x) - Q_{\ell-1}(x)] = (2\ell + 1)Q_\ell(x)$. This leads to

¹⁷ The Legendre function of the second kind $Q_m(x)$ considered here has a branch cut from $-\infty$ to 1 and is defined by (the first equality being known as Neumann's formula)

$$Q_m(x) = \frac{1}{2} \int_{-1}^1 dz \frac{P_m(z)}{x-z} = \frac{1}{2} P_m(x) \ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^m \frac{1}{j} P_{m-j}(x) P_{j-1}(x), \quad (\text{A.1})$$

where $P_m(z)$ is the usual Legendre polynomial whose Rodrigues' representation reads

$$P_m(z) = \frac{1}{2^m m!} \frac{d^m}{dz^m} \left[(z^2 - 1)^m \right]. \quad (\text{A.2})$$

$$Q_\ell^{(-k+2)}(x) = (-)^k \sum_{j=0}^{k-2} C_{\ell,j}^{k-2} Q_{\ell+2j-k+2}(x), \tag{A.8}$$

with $C_{\ell,j}^{k-2} = (-)^j \binom{k-2}{j} \frac{(2\ell+2j-2k+3)!!}{(2\ell+2j+1)!!} (2\ell-2k+4j+5),$ (A.9)

where $\binom{k-2}{j}$ is the usual binomial coefficient. Hence the coefficient $\alpha_{m,\ell,k}$ reads

$$\alpha_{m,\ell,k} = (-)^k \int_1^{+\infty} dx Q_m(x) Q_\ell^{(-k+2)}(x) = \sum_{j=0}^{k-2} C_{\ell,j}^{k-2} j_{m,\ell+2j-k+2}, \tag{A.10}$$

where the remaining integral is explicitly given by (see e.g., [48])¹⁸

$$j_{m,p} = \int_1^{+\infty} dx Q_m(x) Q_p(x) = \begin{cases} \frac{H_m - H_p}{(m-p)(m+p+1)} & \text{for } m \neq p, \\ \frac{1}{2m+1} \left(\frac{\pi^2}{6} - H_{m,2} \right) & \text{for } m = p. \end{cases} \tag{A.11}$$

We next tackle the case of the two other coefficients $\beta_{m,\ell,k}$ and $\gamma_{m,\ell,k}$, defined for generic values of k and ℓ such that $k \geq \ell + 3$ as double integrals,

$$\beta_{m,\ell,k} = \int_1^{+\infty} dx Q_m(x) J_{\ell,k}(x), \tag{A.12a}$$

$$\gamma_{m,\ell,k} = \int_1^{+\infty} dx Q_m(x) [(\ln 2 + H_{k-3}) J_{\ell,k}(x) - K_{\ell,k}(x)], \tag{A.12b}$$

with the following definitions:

$$J_{\ell,k}(x) = \frac{1}{2} \int_{-1}^1 dz P_\ell(z) \frac{(z-x)^{k-3}}{(k-3)!}, \tag{A.13a}$$

$$K_{\ell,k}(x) = \frac{1}{2} \int_{-1}^1 dz P_\ell(z) \frac{(z-x)^{k-3}}{(k-3)!} \ln(x-z). \tag{A.13b}$$

Similarly to equation (A.7) we introduce the $(k-2)$ th anti-derivative of the Legendre polynomial $P_\ell(x)$ that vanishes at $x = -1$ and is defined by

$$P_\ell^{(-k+2)}(x) = \int_{-1}^x dz P_\ell(z) \frac{(x-z)^{k-3}}{(k-3)!}, \tag{A.14}$$

together with $P_\ell^{(0)}(x) = P_\ell(x)$. For the Legendre polynomial we have exactly the same expansion as in equation (A.8), namely

$$P_\ell^{(-k+2)}(x) = (-)^k \sum_{j=0}^{k-2} C_{\ell,j}^{k-2} P_{\ell+2j-k+2}(x). \tag{A.15}$$

We start by writing $J_{\ell,k}(x)$ and $K_{\ell,k}(x)$ in a way appropriate for future integration over x with some kernel $Q_m(x)$. Since the following integral is already known [48],

¹⁸ Notice that $\psi(m+1) - \psi(p+1) = H_m - H_p$ and $\psi'(m+1) = \frac{\pi^2}{6} - H_{m,2}$, where $\psi(x)$ is the usual logarithmic derivative of the Euler gamma function $\Gamma(x)$ and $H_{m,2} = \sum_{j=1}^m \frac{1}{j^2}$ is the m th generalized harmonic number of order 2.

$$\int_1^{+\infty} dx Q_m(x)(x-1)^\nu = 2^\nu \frac{[\Gamma(\nu+1)]^2 \Gamma(m-\nu)}{\Gamma(m+\nu+2)}, \quad (\text{A.16})$$

our strategy will consist in expressing the integrals (A.13) as a sum of monomials $(x-1)^\nu$, with $\nu \in \mathbb{N}$ or \mathbb{C} . This is achieved by performing $k-3$ integrations by parts on the expressions (A.13). Concerning $J_{\ell,k}(x)$ this results in

$$J_{\ell,k}(x) = \frac{(-)^{k+1}}{2} \sum_{p=\ell+1}^{k-2} \frac{P_\ell^{(-p)}(1)}{(k-p-2)!} (x-1)^{k-p-2}, \quad (\text{A.17})$$

where the coefficients of each of the monomials are built from the values at 1 of the anti-derivatives of the Legendre polynomial. Adapting some formulas for the anti-derivatives of the Legendre polynomial in [48], we get the values at $z=1$ as

$$P_\ell^{(-p)}(1) = \begin{cases} 0 & \text{for } 1 \leq p \leq \ell, \\ \frac{(-)^\ell 2^p (p-1)!}{(p+\ell)!(p-\ell-1)!} & \text{for } \ell+1 \leq p. \end{cases} \quad (\text{A.18})$$

Note that the case for $1 \leq p \leq \ell$ is a simple consequence of Rodrigues' formula (A.2); accordingly, we have written the sum in equation (A.17) as starting with $p = \ell+1$. Finally, the integral (A.12a) over z can be computed with the help of formula (A.16) with $\nu = k-p-2$.

The treatment of $K_{\ell,k}(x)$ is more involved. It is based on the fact that $\ln(x-z) = \frac{d}{dB} [(x-z)^B]_{B=0}$, where B is a convenient complex parameter. The calculation of $K_{\ell,k}(x)$ is thus similar to that of $J_{\ell,k}(x)$, with one major difference: after the $k-3$ integrations by parts, an x -dependent factor $(z-x)^B$ remains in the source of the integral, which becomes $B(z-x)^{B-1}$ after a last integration by parts. The presence of the prefactor B prevents the appearance of logarithms $\ln(x-z)$ when applying the derivative d/dB and taking the limit $B \rightarrow 0$. We obtain at the end:

$$K_{\ell,k}(x) = \frac{d}{dB} \left[\frac{(-)^{k+1}}{2} \sum_{p=\ell+1}^{k-2} \frac{P_\ell^{(-p)}(1)}{(k-3)! \Gamma(B+k-p-1)} (x-1)^{B+k-p-2} \right]_{B=0} + \frac{(-)^k}{2} \int_{-1}^1 dz \frac{P_\ell^{(-k+2)}(z)}{z-x}. \quad (\text{A.19})$$

The last term in equation (A.19) is expanded by means of relation (A.15), whose validity extends to the case $k \geq \ell+3$, provided we pose $P_{-m-1}(z) \equiv -P_m(z)$ and $(-2m-1)!! \equiv (-)^m/(2m-1)!!$ for any non-negative integer m . Using also Neumann's formula (A.1), we obtain

$$\frac{1}{2} \int_{-1}^1 dz \frac{P_\ell^{(-k+2)}(z)}{z-x} = (-)^{k+1} \sum_{j=0}^{k-2} C_{\ell,j}^{k-2} Q_{\ell+2j-k+2}(z), \quad (\text{A.20})$$

where we have defined $Q_{-m-1}(z) \equiv -Q_m(z)$ for $m \in \mathbb{N}$. As for $J_{\ell,k}(x)$, the next step consists in integrating over x each of the monomials $(x-1)^{B+k-p-2}$ by means of formula (A.16), but this time with a complex parameter $\nu = B+k-p-2$. After integration we apply the derivative with respect to B and take the limit $B \rightarrow 0$ which is performed straightforwardly and yields in particular many terms involving the logarithmic derivative ψ of the Euler gamma function Γ (which relates, for integer arguments, to harmonic numbers through the equality $\psi(k+1) - \psi(1) = H_k$). Note that the integration over x of the last term in (A.19) produces

the coefficient $\alpha_{m,\ell,k}$; compare equation (A.20) with equations (A.8) and (A.10). However, the coefficient $j_{m,p}$ defined in equation (A.11) must now be extended to the case where p is negative by posing $j_{m,-|p|-1} = -j_{m,|p|}$.

Finally we are in a position to write down the expressions for $\beta_{m,\ell,k}$ and $\gamma_{m,\ell,k}$ (besides that given by (A.10) for $\alpha_{m,\ell,k}$) that are effectively used in this work:

$$\begin{aligned} \beta_{m,\ell,k} &= (-)^{k+\ell+1} 2^{k-3} \sum_{j=0}^{k-\ell-3} \frac{(j+\ell)!}{j!(j+2\ell+1)!} \\ &\times \frac{(m-k+\ell+2+j)! [(k-\ell-3-j)!]^2}{(k-\ell-3-j)! (m+k-\ell-2-j)!}, \end{aligned} \quad (\text{A.21a})$$

$$\begin{aligned} \gamma_{m,\ell,k} &= (-)^{k+\ell} 2^{k-3} \sum_{j=0}^{k-\ell-3} \frac{(j+\ell)!}{j!(j+2\ell+1)!} \frac{(m-k+\ell+2+j)! [(k-\ell-3-j)!]^2}{(k-\ell-3-j)! (m+k-\ell-2-j)!} \\ &\times (H_{k-\ell-3-j} - H_{m+\ell+2-k+j} - H_{m+k-\ell-2-j}) + \alpha_{m,\ell,k}. \end{aligned} \quad (\text{A.21b})$$

Appendix B. Proof of an elementary integration formula

We derive the integration formula (5.9) used in section 5.1. We start with the following formula, valid through analytic continuation for any $B \in \mathbb{C}$,

$$\square_{\text{ret}}^{-1} \left[\left(\frac{r}{b} \right)^B \frac{\hat{n}_L}{r^2} F(t_r) \right] = \frac{\hat{n}_L}{2r} \int_0^{+\infty} d\tau F(t_r - \tau) \frac{g_\ell(B)(\tau/2b)^B - (r/b)^B}{B} + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right), \quad (\text{B.1})$$

where the function $g_\ell(B)$ is

$$g_\ell(B) = \frac{\Gamma(\ell+1+B)\Gamma(1-B)}{\Gamma(1+B)\Gamma(\ell+1-B)}. \quad (\text{B.2})$$

This formula (B.1) is deduced from equation (A.2) of [30] by performing the change of variable $s = \tau + r$, expanding the STF derivative operator $\hat{\partial}_L$ (see (A.15) in [30]), and keeping the leading term of order $1/r$. Here we are working in a neighbourhood of the value of interest $B = 0$; in particular, we suppose $\Re(B) < \epsilon$. We point out that the elementary integral (B.1) is convergent for B close to zero. As usual we assume stationarity in the remote past, which implies in this case that $F(t) = 0$ when $t < -\mathcal{T}$. Thus, despite the apparent presence of a pole $1/B$, the explicit expression (B.1) is finite and regular as $B \rightarrow 0$. After differentiating p times with respect to B , the left-hand side of (B.1) acquires p powers of the logarithm of r/b , while the right-hand side is straightforwardly evaluated at $B = 0$ with the help of the Leibniz rule. We find

$$\begin{aligned} \square_{\text{ret}}^{-1} \left[\frac{\hat{n}_L}{r^2} \left(\ln \frac{r}{b} \right)^p F(t_r) \right] &= \frac{\hat{n}_L}{2(p+1)r} \int_0^{+\infty} d\tau F(t_r - \tau) \\ &\times \left(\frac{\partial^{p+1}}{\partial B^{p+1}} \left[g_\ell(B) \left(\frac{\tau}{2b} \right)^B - \left(\frac{r}{b} \right)^B \right] \right)_{B=0} \\ &+ \mathcal{O} \left(\frac{1}{r^{2-\epsilon}} \right). \end{aligned} \quad (\text{B.3})$$

We see that each application of the retarded integral operator $\square_{\text{ret}}^{-1}$ on a source term increases the maximal power of the logarithms by one unit. Since there is no logarithm in the linearized part $h_{(1)}$, we infer by recurrence that the n th MPM coefficient $h_{(n)}$ contains logarithms with maximal power $n - 1$ (the reasoning is in fact valid for any piece $\sim 1/r^k$ in the waveform [14]). Finally the formula [48]

$$\int_0^{+\infty} d\tau \tau^B e^{i\Omega\tau} = \frac{\Gamma(B+1)}{(-i\Omega)^{B+1}} \quad (\text{B.4})$$

achieves the integration of equation (B.3) in the frequency space. This yields

$$\begin{aligned} &\square_{\text{ret}}^{-1} \left[\frac{\hat{n}_L}{r^2} \left(\ln \frac{r}{b} \right)^p F(t_r) \right] \\ &= \frac{\hat{n}_L}{2(p+1)r} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \frac{\tilde{F}(\Omega) e^{-i\Omega t_r}}{(-i\Omega)} \left(\frac{\partial^{p+1}}{\partial B^{p+1}} \left[\frac{g_\ell(B) \Gamma(B+1)}{(-2i\Omega b)^B} - \left(\frac{r}{b} \right)^B \right] \right)_{B=0} \\ &+ \mathcal{O} \left(\frac{1}{r^{2-\epsilon}} \right), \end{aligned} \quad (\text{B.5})$$

which coincides with equation (5.9).

Appendix C. Modal decomposition of the non-linear memory

The expressions for the non-linear memory terms given in the text can be recovered, after appropriate integrations by part, from the general formula known for any ℓ [31, 32]:

$$U_L^{\text{mem}} = \mathcal{U}_L^{\text{mem}} + (\text{instantaneous contributions}), \quad (\text{C.1a})$$

$$\text{with} \quad \mathcal{U}_L^{\text{mem}} = \frac{2c^{\ell-2} (2\ell+1)!!}{(\ell+1)(\ell+2)} \int_{-\infty}^{T_R} dt \int d\Omega \frac{dE}{dt d\Omega} \hat{N}_L, \quad (\text{C.1b})$$

where the instantaneous contributions come from the latter integrations by part. For convenience these instantaneous terms have been transferred to the instantaneous part of the radiative moments (see equations (3.3)). They have been included in our explicit expressions at 3.5PN order in section 3.1.1. Recall in addition that we have $V_L^{\text{mem}} = 0$ for any current moment (at any PN order).

Here $\frac{dE}{dt d\Omega} = \frac{R^2 c^3}{32\pi G} (\partial_t g_{ij}^{\text{TT}})^2$ is the gravitational-wave energy flux per solid angle unit; the other notations are the same as in section 3. The right-hand side may be put in a more explicit form by substituting for g_{ij}^{TT} its asymptotic expansion (3.2) and integrating over the solid angle. The last operation is achieved by means of the useful identity:

$$\int \frac{d\Omega}{4\pi} \hat{n}_{L_1} \hat{n}_{L_2} \hat{n}_L = \frac{\ell_1! \ell_2! \ell!}{\left(\frac{\ell_1 + \ell_2 - \ell}{2}\right)! \left(\frac{\ell_1 + \ell - \ell_2}{2}\right)! \left(\frac{\ell_2 + \ell - \ell_1}{2}\right)!} \text{STF}_{k_1 \dots k_\ell} \times \frac{\delta_{k_1}^{i_1} \dots \delta_{k_s}^{i_s} \delta_{j_1}^{i_{s+1}} \dots \delta_{j_{\ell-s}}^{i_{\ell-s}} \delta_{j_{\ell-s+1}}^{k_{s+1}} \dots \delta_{j_{\ell_2}}^{k_{\ell_2}}}{(\ell + \ell_1 + \ell_2)!!}, \quad (\text{C.2})$$

with $s = (\ell_1 + \ell - \ell_2)/2$ and $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$ (hence $s \leq \ell$ and $s \leq \ell_1$).

After some combinatorics we find

$$\mathcal{U}_L^{\text{mem}} = \frac{2G}{c^3(\ell+1)(\ell+2)} \sum_{p,k} \frac{1}{c^{2p} p! k} \frac{(2\ell+1)!!}{(2p+2\ell+1)!!} \binom{\ell}{k-p} \times \int_{-\infty}^{T_R} d\tau \left[d_{pk}^\ell U_{P\langle K-p}^{(1)} U_{L-[K-P]P}^{(1)} + \frac{e_{pk}^\ell}{c^2} V_{P\langle K-p}^{(1)} V_{L-[K-P]P}^{(1)} + \frac{f_{pk}^\ell}{c^2} \varepsilon_{ab\langle i_1} U_{aPK-p-1}^{(1)} V_{L-[K-P]bP}^{(1)} \right]. \quad (\text{C.3})$$

Note that this expression is implicit because the radiative moments U_L on the right-hand side themselves contains a memory contribution. The coefficient d_{pk}^ℓ reads

$$d_{pk}^\ell = k - 4p \frac{2p+2\ell+1}{2p+\ell-k} + \frac{2p(p-1)(2p+2\ell+1)(2p+2\ell-1)}{(k-1)(2p+\ell-k)(2p+\ell-k-1)} \quad (\text{C.4})$$

if $\max(0, [(5-\ell)/2]) \leq p$ and $\max(p, 2) \leq k \leq \min(p+\ell, 2p+\ell-2)$ (with $[\dots]$ denoting the integer part) and $d_{pk}^\ell = 0$ otherwise. The other coefficients are given by

$$e_{pk}^\ell = \frac{4k(2p+\ell-k)}{(k+1)(2p+\ell-k+1)} d_{pk}^\ell \quad (\text{C.5})$$

and

$$f_{pk}^\ell = \frac{8(k-p)}{(2p+\ell-k+2)} \left[\frac{p(2p+2\ell+1)}{(k-1)(2p+\ell-k)} - 1 \right] \quad (\text{C.6})$$

if $\max(0, [(4-\ell)/2]) \leq p$ and $\max(p+1, 2) \leq k \leq \min(p+\ell, 2p+\ell-1)$ and $f_{pk}^\ell = 0$ otherwise.

For completeness, we will now provide the corresponding formula for the modal decomposition of the non-linear memory term. The complex waveform $h \equiv h_+ - ih_\times$ can be decomposed onto an orthonormal basis of functions with spin-weight -2 defined over the unit sphere. Here we will choose the set of spin-weighted spherical harmonics $Y_{-2}^{\ell m}(\Theta, \Phi)$ and use the same conventions as in [12, 13]:

$$h = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi). \quad (\text{C.7})$$

The coefficients $h_{\ell m}$ may be written in terms of mass-type and current-type radiative components $U_{\ell m}$ and $V_{\ell m}$ (corresponding to a decomposition in even and odd parity modes in the case of non-spinning compact binaries):

$$h_{\ell m} = -\frac{G}{\sqrt{2} R c^{\ell+2}} \left[U_{\ell m} - \frac{i}{c} V_{\ell m} \right]. \quad (\text{C.8})$$

Those are related to the STF radiative moments by [15]:

$$U_{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \alpha_{\ell m}^L U_L, \quad (\text{C.9a})$$

$$V_{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \alpha_{\ell m}^L V_L, \quad (\text{C.9b})$$

where $\alpha_L^{\ell m}$ is the unique (constant) STF tensor such that

$$\hat{N}_L = \sum_{m=-\ell}^{\ell} \alpha_{\ell m}^L Y^{\ell m}(\Theta, \Phi). \quad (\text{C.10})$$

To compute the memory mode $\mathcal{U}_{\ell m}^{\text{mem}} \propto \alpha_{\ell m}^L \mathcal{U}_L^{\text{mem}}$, we insert the angular integral of equation (C.1b) into formula (C.9a) and contract $\alpha_{\ell m}^L$ with \hat{N}_L using the property (C.10) [31]. It remains for us to integrate a product of three harmonic functions. After writing them as Wigner matrices, we obtain an explicit expression for the required integral by means of standard integration formulas:

$$\begin{aligned} & \int \frac{d\Omega}{4\pi} \bar{Y}^{\ell m}(\Theta, \Phi) Y_{-2}^{\ell' m'}(\Theta, \Phi) \bar{Y}_{-2}^{\ell'' m''}(\Theta, \Phi) \\ &= (-)^{m+m'} \left(\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{(4\pi)^3} \right)^{1/2} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & -m'' \end{pmatrix}. \end{aligned} \quad (\text{C.11})$$

This yields for the memory mode (ℓ, m)

$$\begin{aligned} \mathcal{U}_{\ell m}^{\text{mem}} &= \frac{G}{c^3} \left[\frac{(2\ell+1)(\ell-2)!}{8\pi(\ell+2)!} \right]^{1/2} \sum_{k \geq \max(0, 4-\ell)}^{\min(\lfloor \frac{k}{2} \rfloor + \ell, k + \ell - 2)} \sum_{\ell' = \max(\lfloor \frac{k+1}{2} \rfloor, 2)}^{\min(\ell', m + \ell - \ell' + k)} \sum_{m' = \max(-\ell', m + \ell' - \ell - k)}^{\min(\ell', m + \ell - \ell' + k)} \frac{(-)^{m'}}{c^k} \\ &\times [(2\ell'+1)(2k+2\ell-2\ell'+1)]^{1/2} \begin{pmatrix} \ell & \ell' & k + \ell - \ell' \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} \ell & \ell' & k + \ell - \ell' \\ -m & m' & m - m' \end{pmatrix} \\ &\times \left[\int_{-\infty}^{T_R} d\tau U_{\ell' m'}^{(1)} \bar{U}_{k + \ell - \ell' m' - m}^{(1)} + \frac{1}{c^2} \int_{-\infty}^{T_R} d\tau V_{\ell' m'}^{(1)} \bar{V}_{k + \ell - \ell' m' - m}^{(1)} \right. \\ &\left. + \frac{i}{c} \int_{-\infty}^{T_R} d\tau \left(U_{\ell' m'}^{(1)} \bar{V}_{k + \ell - \ell' m' - m}^{(1)} - V_{\ell' m'}^{(1)} \bar{U}_{k + \ell - \ell' m' - m}^{(1)} \right) \right] \end{aligned} \quad (\text{C.12})$$

where the even- k (odd- k) coefficients cancel each other out for memory integrals over products of multipole moments with different (identical) parities, as one can explicitly verify.

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