

Non-linear particle acceleration at non-relativistic shock waves in the presence of self-generated turbulence

E. Amato[★] and P. Blasi[★]

INAF – Osservatorio Astrofisico di Arcetri, Largo E. Fermi, 5, 50125, Firenze, Italy

Accepted 2006 June 27. Received 2006 June 15; in original form 2006 February 27

ABSTRACT

Particle acceleration at astrophysical shocks may be very efficient if magnetic scattering is self-generated by the same particles. This non-linear process adds to the non-linear modification of the shock due to the dynamical reaction of the accelerated particles on the shock. Building on a previous general solution of the problem of particle acceleration with arbitrary diffusion coefficients, we present here the first semi-analytical calculation of particle acceleration with both effects taken into account at the same time; charged particles are accelerated in the background of Alfvén waves that they generate due to the streaming instability, and modify the dynamics of the plasma in the shock vicinity.

Key words: acceleration of particles – shock waves.

1 INTRODUCTION

Soon after the pioneering papers by Krymskii (1977), Blandford & Ostriker (1978) and Bell (1978a,b), introducing the test particle theory of particle acceleration at collisionless shocks, it became clear that the dynamical reaction of the accelerated particles on the plasmas involved in the shock formation may not be negligible. It is now clear that such reaction may in fact make shocks efficient accelerators and change quite drastically the predictions of the test particle theory. The main consequences of the shock modification induced by the accelerated particles can be summarized as follows: (i) a precursor, consisting in a gradual braking of the upstream fluid, is created; (ii) particles with different momenta *feel* different effective compression factors, which reflects in the fact that the spectrum of accelerated particles is no longer a power law, but rather a concave spectrum, as hard as $p^{-3.2}$ at high momenta; (iii) the shock becomes less efficient in heating the background plasma, so that the temperature of the downstream gas is expected to be lower than predicted through the usual Rankine–Hugoniot relations at an unmodified shock front (see Drury 1983; Blandford & Eichler 1987; Jones & Ellison 1991; Malkov & Drury 2001, for reviews on different aspects of the subject). The reaction of the accelerated particles has been calculated within different approaches: the so-called two-fluid models (Drury & Völk 1980, 1981), kinetic models (Malkov 1997; Malkov, Diamond & Völk 2000; Blasi 2002, 2004) and numerical approaches, both Monte Carlo and other simulation procedures (Bell 1987; Ellison, Möbius & Paschmann 1990; Jones & Ellison 1991; Ellison, Baring & Jones 1995, 1996; Kang & Jones 1997, 2005; Kang, Jones & Gieseler 2002). In most of these calculations, the diffusion properties of the plasma upstream and downstream are provided as an input to the problem. This also results in fixing

the value of the maximum momentum of the accelerated particles. However, one of the well-known and most-disturbing problems associated with the mechanism of particle acceleration at shock fronts is that a substantial amount of magnetic scattering of the particles is required (e.g. Lagage & Cesarsky 1983a,b). In the absence of it, the maximum energy of the accelerated particles is exceedingly low and uninteresting for astrophysical applications (e.g. Blasi 2005). Bell (1978a) proposed that the streaming instability of cosmic rays could be responsible for the generation of perturbations in the magnetic field of an amplitude necessary to provide pitch angle scattering (and therefore spatial diffusion) of the accelerated particles. Lagage & Cesarsky (1983b) used this argument to estimate the maximum energy of particles accelerated at shocks in supernova remnants. In all previous works, either the shock was considered unmodified, or the diffusion coefficients were fixed a priori, because a comprehensive theory of particle acceleration was missing. Recently, Amato & Blasi (2005) have found a general exact solution of the system of equations describing the diffusion–convection of accelerated particles, and the dynamics and thermodynamics of plasmas in the shock region, for an arbitrary choice of the spatial and momentum dependence of the diffusion coefficient. In the present paper, we use the formalism proposed by Amato & Blasi (2005) and combine it with calculations of the perturbations created through streaming instability, so that the diffusion coefficient, as a function of spatial location and momentum, is determined from the spectrum and spatial distribution of the accelerated particles. This provides the first combined description of the process of particle acceleration at collisionless shocks in the presence of particle reaction and wave generation. In this approach, the spectrum of accelerated particles, their distribution in the upstream plasma and the diffusion coefficient are outputs of the problem.

The paper is organized as follows. In Section 2, we summarize the findings of Amato & Blasi (2005). In Section 3, we illustrate our treatment of the streaming instability and determine a

[★]E-mail: amato@arcetri.astro.it (EA); blasi@arcetri.astro.it (PB)

relation between the power spectrum of magnetic fluctuations and the diffusion coefficient upstream. In Section 4, we describe the results of our calculations. In Section 5, we shortly discuss how the results presented in the previous section change when the effects of turbulent heating are taken into account. We conclude in Section 6.

2 CALCULATIONS OF THE SPECTRUM FOR ARBITRARY DIFFUSION COEFFICIENT

In this section, we briefly summarize the mathematical procedure proposed by Amato & Blasi (2005) to calculate the spectrum and spatial distribution of particles accelerated at astrophysical shocks, and their dynamical reaction on the shock structure, for an arbitrary diffusion coefficient $D(x, p)$. The reader is referred to the paper by Amato & Blasi (2005), for more details.

The equation for the conservation of the momentum between upstream infinity and a point x in the upstream region can be written as

$$\xi_c(x) = 1 + \frac{1}{\gamma_g M_0^2} - U(x) - \frac{1}{\gamma_g M_0^2} U(x)^{-\gamma_g}, \quad (1)$$

where $\xi_c(x) = P_{\text{CR}}(x)/\rho_0 u_0^2$ and $U(x) = u(x)/u_0$ and we used conservation of mass $\rho_0 u_0 = \rho(x) u(x)$ [here ρ_0 and u_0 refer to as the density and plasma velocity at upstream infinity, while $\rho(x)$ and $u(x)$ are the density and velocity at the location x upstream. M_0 is the sonic Mach number at upstream infinity].

The pressure in the form of accelerated particles is defined as

$$P_{\text{CR}}(x) = \frac{1}{3} \int_{p_{\text{inj}}}^{p_{\text{max}}} dp \, 4\pi p^3 v(p) f(x, p), \quad (2)$$

and $f(x, p)$ is the distribution function of accelerated particles. Here, p_{inj} and p_{max} are the injection and maximum momentum. The function f vanishes at upstream infinity, which implies that there are no cosmic rays infinitely distant from the shock in the upstream region.¹ The distribution function satisfies the following transport equation in the reference frame of the shock:²

$$\frac{\partial}{\partial x} \left[D(x, p) \frac{\partial}{\partial x} f(x, p) \right] - u \frac{\partial f(x, p)}{\partial x} + \frac{1}{3} \left(\frac{du}{dx} \right) p \frac{\partial f(x, p)}{\partial p} + Q(x, p) = 0. \quad (3)$$

The x -axis is oriented from upstream infinity ($x = -\infty$) to downstream infinity ($x = +\infty$), with the shock located at $x = 0$. The injection is introduced here through the function $Q(x, p)$. The diffusion properties are described by the arbitrary function $D(x, p)$, depending on both momentum and space.³

¹ This assumption implies that we are not considering any reacceleration of pre-existing seed particles.

² Since we will be using this equation in Section 4 for the case in which diffusion is due to a strongly amplified turbulent magnetic field, a few comments are in order: rigorously, this equation describes the isotropic part of the distribution function, and as long as the quasi-linear theory holds, the anisotropic part is expected to represent a small perturbation. It is not clear how the equation would generalize to the strongly non-linear case, though it may be reasonable to assume that the anisotropy remains rather small as long as the Alfvén speed in the perturbed field is negligible compared with the fluid speed.

³ In writing equation (3) in this form, we are neglecting the velocity of the scattering centres u_w with respect to the fluid velocity upstream. This is always a good approximation for the cases considered in this paper.

Amato & Blasi (2005) showed that an excellent approximation to the solution $f(x, p)$ has the form

$$f(x, p) = f_0(p) \exp \left[-\frac{q(p)}{3} \int_x^0 dx' \frac{u(x')}{D(x', p)} \right], \quad (4)$$

where $f_0(p) = f(x = 0, p)$ is the cosmic rays' distribution function at the shock and $q(p) = -\frac{d \ln f_0(p)}{d \ln p}$ is its local slope in momentum space.

The function $f_0(p)$ can be written in a very general way as found by Blasi (2002):

$$f_0(p) = \left[\frac{3R_{\text{tot}}}{R_{\text{tot}} U_p(p) - 1} \right] \frac{\eta n_0}{4\pi p_{\text{inj}}^3} \exp \left[-\int_{p_{\text{inj}}}^p \frac{dp'}{p'} \frac{3R_{\text{tot}} U_p(p')}{R_{\text{tot}} U_p(p') - 1} \right]. \quad (5)$$

Here we introduced the function $U_p(p) = u_p/u_0$, with

$$u_p = u_1 - \frac{1}{f_0(p)} \int_{-\infty}^0 dx (du/dx) f(x, p), \quad (6)$$

where u_1 is the fluid velocity immediately upstream (at $x = 0^-$). We used $Q(x, p) = \frac{\eta n_{\text{gas},1} u_1}{4\pi p_{\text{inj}}^2} \delta(p - p_{\text{inj}}) \delta(x)$, with $n_{\text{gas},1} = n_0 R_{\text{tot}}/R_{\text{sub}}$ being the gas density immediately upstream ($x = 0^-$) and η being the fraction of the particles crossing the shock which are going to take part in the acceleration process. In the expressions above, we also introduced the two quantities $R_{\text{sub}} = u_1/u_2$ (compression factor at the subshock) and $R_{\text{tot}} = u_0/u_2$ (total compression factor). If the heating of the upstream plasma takes place only due to adiabatic compression, the two compression factors are related through the following expression (Blasi 2002):

$$R_{\text{tot}} = M_0^{\frac{2}{\gamma_g + 1}} \left[\frac{(\gamma_g + 1) R_{\text{sub}}^{\gamma_g} - (\gamma_g - 1) R_{\text{sub}}^{\gamma_g + 1}}{2} \right]^{\frac{1}{\gamma_g + 1}}, \quad (7)$$

where M_0 is the Mach number of the fluid at upstream infinity and γ_g is the ratio of specific heats for the fluid. The parameter η in equation (5) contains the very important information about the injection of particles from the thermal bath. We adopt here the recipe proposed by Blasi, Gabici & Vannoni (2005) (see also previous work by Ellison 1981; Ellison, Jones & Eichler 1981; Ellison & Eichler 1984; Gieseler, Jones & Kang 2000) that allows us to relate η to the compression factor at the subshock as

$$\eta = \frac{4}{3\pi^{1/2}} (R_{\text{sub}} - 1) \xi^3 e^{-\xi^2}. \quad (8)$$

Here, ξ is a parameter that identifies the injection momentum as a multiple of the momentum of the thermal particles in the downstream section ($p_{\text{inj}} = \xi p_{\text{th},2}$). The latter is an output of the non-linear calculation, since we solve exactly the modified Rankine–Hugoniot relations together with the cosmic rays' transport equation. For the numerical calculations that follow we always use $\xi = 3.5$, that corresponds to a fraction of the order of 10^{-4} of the particles crossing the shock to be injected in the accelerator.

In terms of the distribution function (equation 4), we can also write the normalized pressure in accelerated particles as

$$\xi_c(x) = \frac{4\pi}{3\rho_0 u_0^2} \int_{p_{\text{inj}}}^{p_{\text{max}}} dp \, p^3 v(p) f_0(p) \exp \left[-\int_x^0 dx' \frac{U(x')}{x_p(x', p)} \right], \quad (9)$$

where for simplicity we introduced $x_p(x, p) = \frac{3D(x, p)}{q(p)u_0}$.

By differentiating equation (9) with respect to x , we obtain

$$\frac{d\xi_c}{dx} = \lambda(x) \xi_c(x) U(x), \quad (10)$$

where

$$\lambda(x) = \langle 1/x_p \rangle_{\xi_c} = \frac{\int_{p_{inj}}^{p_{max}} dp p^3 \frac{1}{x_p(x,p)} v(p) f_0(p) \exp \left[- \int_x^0 dx' \frac{U(x')}{x_p(x',p)} \right]}{\int_{p_{inj}}^{p_{max}} dp p^3 v(p) f_0(p) \exp \left[- \int_x^0 dx' \frac{U(x')}{x_p(x',p)} \right]}, \quad (11)$$

and $U(x)$ is expressed as a function of $\xi_c(x)$ through equation (1).

Finally, after integration by parts of equation (6), one is able to express $U_p(p)$ in terms of an integration involving $U(x)$ alone:

$$U_p(p) = \int_{-\infty}^0 dx U(x)^2 \frac{1}{x_p(x,p)} \exp \left[- \int_x^0 dx' \frac{U(x')}{x_p(x',p)} \right], \quad (12)$$

which allows one to easily calculate $f_0(p)$ through equation (5).

Equations (1) and (10) can be solved by iteration in the following way: for a fixed value of the compression factor at the subshock, R_{sub} , the value of the dimensionless velocity at the shock is calculated as $U(0) = R_{sub}/R_{tot}$. The corresponding pressure in the form of accelerated particles is given by equation (1) as $\xi_c(0) = 1 + \frac{1}{\gamma_g M_0^2} - \frac{R_{sub}}{R_{tot}} - \frac{1}{\gamma_g M_0^2} \left(\frac{R_{sub}}{R_{tot}} \right)^{-\gamma_g}$. This is used as a boundary condition for equation (10), where the functions $U(x)$ and $\lambda(x)$ [and therefore $f_0(p)$] on the right-hand side at the k th step of iteration are taken as the functions at the step $(k-1)$. In this way, the solution of equation (1) at the step k is simply

$$\xi_c^{(k)}(x) = \xi_c(0) \exp \left[- \int_x^0 dx' \lambda^{(k-1)}(x') U^{(k-1)}(x') \right], \quad (13)$$

with the correct limits when $x \rightarrow 0$ and $x \rightarrow -\infty$. At each step of iteration, the functions $U(x)$, $f_0(p)$ and $\lambda(x)$ are recalculated (through equations 1, 12, 5 and 11, respectively), until convergence is reached. The solution of this set of equations, however, is also a solution of our physical problem only if the pressure in the form of accelerated particles as given by equation (1) coincides with that calculated by using the final $f_0(p)$ in equation (9). This occurs only for one specific value of R_{sub} , which fully determines the solution of our problem for an arbitrary diffusion coefficient as a function of location and momentum.

3 SELF-GENERATED TURBULENCE AND PARTICLE DIFFUSION

The streaming of cosmic rays at super-Alfvénic speed induces a streaming instability, which has been discussed in the previous literature (e.g. Bell 1978a).

Let us define $\mathcal{F}(x, k)$ as the energy density per unit logarithmic bandwidth of waves with wavenumber k . Neglecting the damping, and assuming a steady state, the following relation holds (see e.g. Lagage & Cesarsky 1983a):

$$u \frac{\partial \mathcal{F}(x, k)}{\partial x} = \sigma(x, k) \mathcal{F}(x, k), \quad (14)$$

where $u = u(x)$ is the fluid velocity upstream of the shock and σ is the growth rate of waves with given wavenumber k , which can be related to the distribution function of the resonant cosmic rays, $f(x, p(k))$, through

$$\sigma = \frac{4\pi}{3} \frac{v_A}{U_M \mathcal{F}} \left[p^4 v \frac{\partial f(x, p)}{\partial x} \right]_{p=\bar{p}(k)}. \quad (15)$$

In equation (15), v and p are the particle velocity and momentum, respectively, and the latter is related to the wavenumber k through the resonance condition $\bar{p}(k) = eB/kmc$, U_M is the energy density of the background magnetic field B_0 ($U_M = B_0^2/8\pi$), while v_A is the local Alfvén velocity.

All these expressions have been actually obtained for shocks that are not modified by the dynamical reaction of cosmic rays. In principle, the Fourier analysis, used to obtain the previous expressions and in fact used to reach the conclusion that there are unstable modes, is not formally applicable since all these calculations assume that the background quantities (the fluid velocity u , in particular) are spatially constant. However, provided that $1/k$ remains much smaller than the spatial extension of the precursor, the conclusions are, in first approximation, still applicable. Clearly, this condition is broken by definition at the maximum momentum p_{max} at least in those cases in which this is determined by the finite size of the accelerator rather than by energy losses. Special care should be taken of the fact that all quantities involved in the equations above depend on the location in the precursor.

It follows that for a cosmic ray modified shock, v_A is not spatially constant since both the upstream plasma density, ρ , and, in general, the background magnetic field, B_0 , are space dependent. However, all previous calculations apply to the case of a parallel shock, for which the strength of the background magnetic field B_0 can be taken to be constant, since there is no adiabatic compression of the magnetic field lines.

Using the equation for conservation of mass $\rho(x) = \rho_0/U(x)$, we can therefore write the local Alfvén velocity as

$$v_A(x) = \frac{B_0}{\sqrt{4\pi\rho_0}} U(x)^{1/2}. \quad (16)$$

An additional warning should be issued in that equation (14) neglects the adiabatic compression of waves in the shock precursor: this reflects in the absence of terms proportional to the gradient of the velocity field. Unfortunately, to our knowledge, discussions of this problem in the literature are limited to integrated quantities (e.g. total energy density and pressure of the waves) while a description of the behaviour of the modes with different wavenumbers is more complex. In fact, in principle even the concept of modes with given k becomes ill defined in a background which has spatial gradients of the quantities to be perturbed.

Once $\mathcal{F}(x, k)$ is known, the diffusion coefficient is known in turn (Bell 1978a):

$$D(x, p) = \frac{4}{\pi} \frac{r_L v}{3 \mathcal{F}}. \quad (17)$$

From the latter equation, where r_L stands for the Larmor radius of particles of momentum p , it is clear that the diffusion coefficient tends to Bohm's expression for $\mathcal{F} \rightarrow 1$. On the other hand, it is also straightforward to check what the expected saturation level for the overall energy density of the perturbed magnetic field is. If we define

$$\frac{\delta B^2}{8\pi} = \mathcal{I} = \frac{B_0^2}{8\pi} \int \frac{dk}{k} \mathcal{F}(k), \quad (18)$$

from equations (14) and (15), we see that

$$\begin{aligned} u \frac{d\mathcal{I}}{dx} &= \frac{B_0^2}{8\pi} \int \frac{dk}{k} \sigma \mathcal{F}(k, x) \\ &= \frac{4\pi}{3} v_A \frac{d}{dx} \int dp v(p) p^3 f(x, p) \\ &= v_A \frac{dP_{CR}}{dx}. \end{aligned} \quad (19)$$

Integration of the latter equation is straightforward when non-linear effects on the fluid are neglected so that u and v_A are both spatially constant. One obtains $\delta B^2/8\pi = (v_A/u) P_{\text{CR}}$, or, in terms of amplification of the ambient magnetic field,

$$\left(\frac{\delta B}{B_0}\right)^2 = 2 M_A \frac{P_{\text{CR}}}{\rho_0 u_0^2}, \quad (20)$$

with $M_A = u_0/v_A$ the Alfvénic Mach number.

It is worth stressing that for $P_{\text{CR}}/\rho_0 u_0^2 \sim 1$ and $M_A \gg 1$, the predicted amplification of the magnetic field exceeds unity. In fact, this result was initially obtained in the context of the so-called quasi-linear theory; therefore it should be taken with caution and checked versus numerical calculations of the non-linear phase of amplification of the waves. It seems clear, however, that the growth may well enter this non-linear regime and lead to turbulent fields in the shock vicinity that exceed the pre-existing background magnetic field.

Let us now go back to equations (14)–(17) with the aim of recasting the relation between the diffusion coefficient and the cosmic ray distribution function in a more compact form. Using equations (14)–(16), we can write

$$\mathcal{F}(x, p) = \frac{8\pi}{3} \frac{v p^4 \Phi(x, p)}{\rho_0 u_0 v_{A0}}, \quad (21)$$

where $v_{A0} = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed at upstream infinity, and

$$\Phi(x, p) = \int_{-\infty}^x \frac{dx'}{U(x')^{1/2}} \frac{\partial f}{\partial x'}(x', p). \quad (22)$$

With this definition of Φ , from equation (17), we obtain

$$D(x, p) = \frac{3}{2\pi^2} D_{B0} \frac{n_0}{p^3 \Phi(x, p)} \frac{v_{A0}}{c} \frac{u_0}{c}, \quad (23)$$

with $D_{B0} = m_p c^3/3eB_0$ being a constant.

It is important to note that since the constant D_{B0} is inversely proportional to the strength of the background field B_0 , and the Alfvén speed v_{A0} is proportional to B_0 , the diffusion coefficient in equation (23) turns out to be independent of B_0 . This result holds only within the context of quasi-linear theory. Even in the context of a quasi-linear theory of the development of magnetic perturbations, a dependence on B_0 could be introduced through the quantity Φ , which is affected by the laws of conservation of momentum and energy in the precursor. However, in the cases of interest for us we will show below that these effects are fully negligible.

4 SPECTRA OF THE ACCELERATED PARTICLES AND SELF-GENERATED DIFFUSION COEFFICIENT

Following the mathematical procedure outlined in Sections 2 and 3, we are able to determine self-consistently the spectrum of accelerated particles and the diffusion coefficient. Within the obvious limitation of using quasi-linear theory to calculate the diffusion coefficient for the non-linear case, this is the first attempt at determining the space and momentum dependence of the diffusion coefficient together with the spectrum of accelerated particles. While in a time-dependent approach to the problem, it would be possible to estimate the maximum energy in a self-consistent way, here we assume for simplicity that the maximum momentum is a given parameter. We chose to carry out the calculations presented in the following for $p_{\text{max}} = 10^5$ mc.

The spectra of the accelerated particles for Mach numbers at upstream infinity ranging from $M_0 = 4$ to 200 are shown in Fig. 1

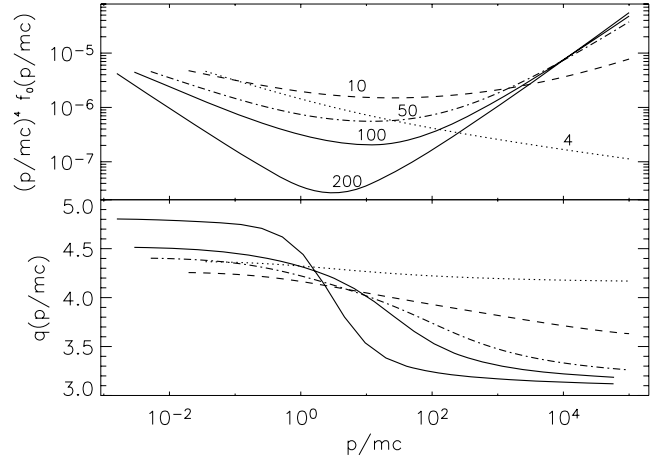


Figure 1. Spectrum and slope at the shock location as functions of energy for $p_{\text{max}} = 10^5$ mc and magnetic field at upstream infinity $B_0 = 1 \mu\text{G}$. The curves refer to Mach numbers at upstream infinity ranging from $M_0 = 4$ to 200: dotted for $M_0 = 4$, dashed for $M_0 = 10$, dot-dashed for $M_0 = 50$, solid for $M_0 = 100$ and dot-dot-dashed for $M_0 = 200$.

for a background magnetic field at upstream infinity $B_0 = 1 \mu\text{G}$. As stressed above the result is, however, expected and actually found to be independent of the strength of the background magnetic field. In the bottom part of the same figure, we plot the slope of the spectrum as a function of momentum.

It is evident that for low Mach numbers and at given p_{max} , the modification of the shock due to the reaction of the accelerated particles is small (see for instance the case $M_0 = 4$). For the strongly modified case (e.g. $M_0 = 200$), the asymptotic spectrum of the accelerated particles is very flat, tending to $p^{-\alpha}$ with $\alpha = 3.1$ – 3.2 for $p \rightarrow p_{\text{max}}$. The momentum at which the spectrum becomes flatter than p^{-4} , the prediction of linear theory depends on the level of shock modification: it is higher (10–20 mc) for relatively low Mach numbers (namely weaker modification) and approaches a few GeV for high Mach numbers and large shock modification. The asymptotic spectrum is reached at $p/mc > 10^2$. These effects might be important in the perspective of reconciling the concave shape of the instantaneous spectra of accelerated particles with observations of the diffuse spectrum of cosmic rays in the Galaxy. Most measurements, mainly related to the abundance of light elements, are in fact limited to relatively low energies, where the spectra predicted in this paper are compatible with power laws softer than p^{-4} . Serious work aimed at predicting the actual spectrum of cosmic rays escaping the sources is urgently needed but still missing in the context of non-linear theories of particle acceleration at shocks.

The diffusion coefficient associated with the self-generated waves is given by equation (23). We plot this diffusion coefficient at the shock location in Fig. 2 for Mach numbers $M_0 = 10$ (dashed lines) and $M_0 = 100$ (solid lines). We fix $B_0 = 1 \mu\text{G}$, but as stressed in the previous section, the diffusion coefficient obtained within the quasi-linear theory of magnetic perturbations is independent of B_0 . For comparison, we also plot the corresponding Bohm diffusion coefficient $D_B(p) \propto v(p)p$ in the unperturbed magnetic field B_0 , for $B_0 = 1$ and $10 \mu\text{G}$. The comparison strikingly shows that for most momenta of the accelerated particles the diffusion takes place at super-Bohm rates (namely the diffusion is slower than predicted by the Bohm coefficient in the unperturbed magnetic field, as could be expected). Moreover, the difference between the self-generated diffusion coefficient and the Bohm coefficient increases at the highest

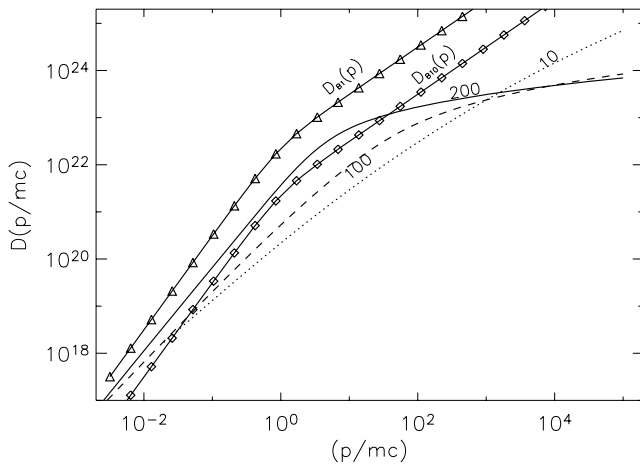


Figure 2. The self-generated diffusion coefficient at the shock location $x = 0^-$ as a function of the particle momentum for Mach numbers $M_0 = 10$ (dotted line), 100 (dashed line) and 200 (solid line). Also plotted is the Bohm diffusion coefficient corresponding to $B_0 = 1 \mu\text{G}$ (solid line with triangles) and $B_0 = 10 \mu\text{G}$ (solid line with diamonds). The y-axis is in units of $\text{cm}^2 \text{s}^{-1}$.

momenta, which might suggest that somewhat higher energies could be achieved if the self-generated turbulence were taken into account. In this respect, it is also important to note that, as one might expect, the more modified the shock, the slower the diffusion.

It is worth keeping in mind that the diffusion coefficient in the amplified magnetic field, as obtained through our calculations, remains larger than the Bohm value in the same field. The latter is in fact considered as a sort of lower limit to the diffusion rate (Casse, Lemoine & Pelletier 2002) even in the case of strong turbulence. The only region in momentum space where this condition may be violated in our calculations is very close to the maximum momentum p_{max} . It is clear, however, that a realistic determination of the diffusion coefficient cannot be achieved in the context of quasi-linear theory and that even numerical approaches to diffusion, such as those of Casse et al. (2002) can only suggest a general trend as long as the turbulent structure of the magnetic field is pre-defined rather than determined by the diffusing particles themselves. In this sense, the limit at the Bohm value in the amplified field should also be taken with caution.

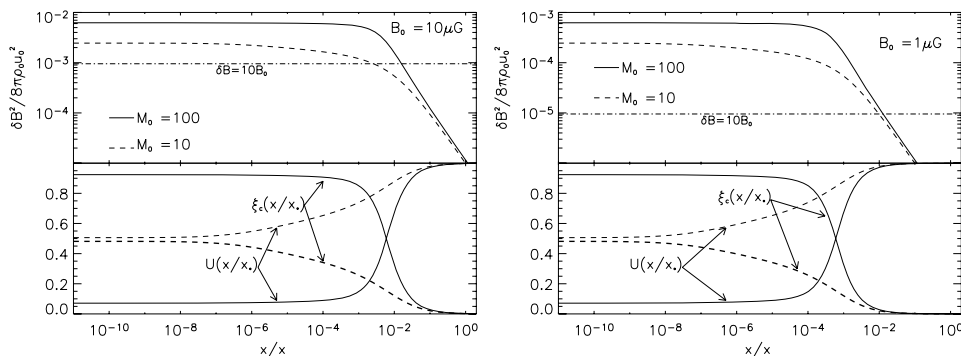


Figure 3. Top panels: the energy density in magnetic field fluctuations $\delta B^2/8\pi\rho_0 u_0^2$ normalized to the fluid ram pressure $\rho_0 u_0^2$ at upstream infinity. Bottom panels: the cosmic ray pressure normalized to $\rho_0 u_0^2$, ξ_c (thick curves), is plotted together with the normalized velocity U (thin curves). All functions are plotted versus spatial location, with the x -coordinate in units of $x_* = -D_B(p_{\text{max}})/u_0$, where $D_B(p)$ stands for the appropriate Bohm diffusion coefficient. The left- and right-hand panels refer to different strengths of the background magnetic field B_0 , as specified in each panel, while the different line types correspond to different Mach numbers: dashed for $M_0 = 10$ and solid for $M_0 = 100$. In the upper panels, we also plot for comparison a dot-dashed curve corresponding to $\delta B = 10 B_0$.

As stressed above, the fact that the diffusion coefficient is smaller than the Bohm coefficient in the background field is the consequence of the fact that the fluctuations in the magnetic field become strongly non-linear, namely $\delta B^2/B_0^2 \gg 1$, at least close to the shock surface. In fact, we find that $\delta B/B_0$ at $x = 0$ is exactly as predicted by equation (20). In these conditions, it is important to check that the dynamical role of the turbulent magnetic field remains small. In Fig. 3, we plot $\delta B^2/8\pi\rho_0 u_0^2$ (top panels) and the cosmic ray normalized pressure $\xi_c(x)$ and velocity $U(x)$ (bottom panels). The curves refer to Mach number $M_0 = 10$ (dashed lines) and 100 (solid lines). The plots on the left-hand (right-hand) panel are obtained for $B_0 = 10 \mu\text{G}$ ($B_0 = 1 \mu\text{G}$). The x -coordinate is in units of $x_* = -D_B(p_{\text{max}})/u_0$, where $D_B(p)$ stands for the Bohm diffusion coefficient appropriate to the considered value of B_0 .

The highest values of $\delta B^2/8\pi\rho_0 u_0^2$, reached close to the shock front, are of the order of $10^{-2} - 10^{-3}$, confirming that even in the extreme non-linear cases the dynamical effect of the magnetic field remains unimportant. This result serves as a justification a posteriori that we could neglect the pressure of the waves and their energy flux in the equations of conservation of momentum and energy, respectively. This result is very specific of the resonant channel of production of Alfvén waves, and is very likely not correct in the case of non-resonant scenarios, such as the one proposed by Bell (2004), where larger amplifications of the magnetic field could be achieved.

The shape of the spectra of accelerated particles is affected in a sizeable way by the adoption of the self-generated diffusion coefficient: Fig. 4 illustrates this point. The continuous lines are for self-generated diffusion (dashed for $M_0 = 10$ and solid for $M_0 = 100$), while the symbols are for Bohm diffusion (diamonds for $M_0 = 10$, almost perfectly superposed on the dashed curve, and filled circles for $M_0 = 100$). While in the weakly modified cases, the spectrum is basically independent of the assumed diffusion coefficient, in the fully non-linear solution self-generated diffusion leads to steeper spectra at low momenta and harder spectra at high momenta.

5 THE ROLE OF TURBULENT HEATING

A major uncertainty in all types of calculations of the non-linear particle acceleration at shock fronts is the effect of *turbulent heating*. This generic expression is used to refer to any process that may determine non-adiabatic gas heating. The two best known examples of this type of processes are Alfvén heating (McKenzie & Völk

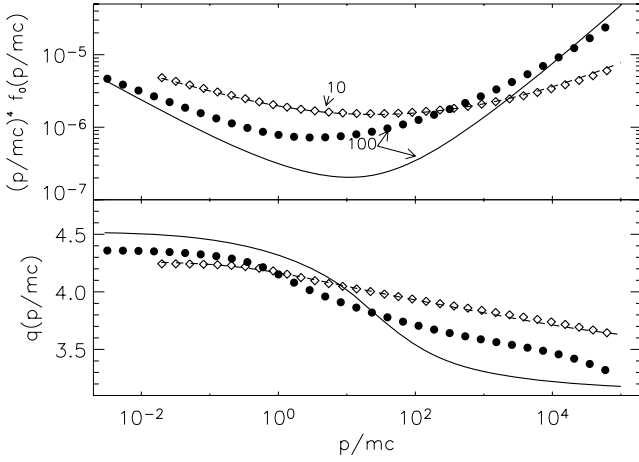


Figure 4. Effects of using the self-generated diffusion coefficient on the particles' distribution functions. In the top and bottom panels, we plot the particles' spectrum and spectrum slope, respectively. The continuous curves are obtained for self-generated $D(x, p)$: dashed curves are for $M_0 = 10$ and solid curves for $M_0 = 100$. The symbols represent the results obtained adopting the Bohm diffusion coefficient: diamonds are for $M_0 = 10$ and filled circles are for $M_0 = 100$.

1982) and acoustic instability (Drury & Falle 1986). Both effects are, however, very hard to implement in a quantitative calculation: in the case of Alfvén heating, the mechanism was originally introduced as a way to avoid the turbulent magnetic field to grow to non-linear levels, while it is usually used even in those cases in which $\delta B/B_0 \gg 1$.

Acoustic instability develops in the pressure gradient induced by cosmic rays in the precursor and results in the development of a train of shock waves that heat the background gas (Drury & Falle 1986). The analysis of the instability is carried out in the linear regime; therefore it is not easy to describe quantitatively the heating effect.

In both cases, the net effect is the non-adiabatic heating of the gas in the precursor, which results in the weakening of the precursor itself and in the reduction of the acceleration efficiency compared with the case in which the turbulent heating is not taken into account.

In order to illustrate this effect, we adopt a phenomenological approach, similar to that of Berezhko & Ellison (1999). We stress that this approach, developed for the case of weak turbulence, is inadequate in principle for the case of interest here, where the turbulence can, in principle, become strong. We adopt it here only for illustration of the main physical effects.

The approach consists in redefining the equation of state of the gas taking into account the heating induced by the accelerated particles (McKenzie & Völk 1982)

$$\frac{\partial}{\partial x} [P_g(x) \rho(x)^{-\gamma_g}] = (\gamma_g - 1) \frac{v_H(x)}{u(x)} \frac{\partial P_{CR}}{\partial x} \rho(x)^{-\gamma_g}, \quad (24)$$

where $u/v_H = M_H$ is the local Mach number of the turbulence relevant for the heating (for instance $v_H = v_A$ for Alfvénic heating). After defining $\tau(x) = P_g(x)/\rho_0 u_0^2$, we replace equation (1) in the set of equations presented in Section 2 with the two following equations:

$$\xi_c(x) + U(x) + \tau(x) = 1 + \frac{1}{\gamma_g M_0^2}, \quad (25)$$

$$\tau(x) = \frac{U(x)^{-\gamma_g}}{\gamma_g M_0^2} \left\{ 1 + \gamma_g(\gamma_g - 1) \frac{M_0^2}{M_{H0}} \left[\frac{1 - U(x)^{\gamma_g+s}}{\gamma_g + s} + \mathcal{I}_\tau(x) \right] \right\}, \quad (26)$$

where the latter is obtained by rewriting equation (24) in terms of the normalized pressures and integrating between upstream infinity and a generic location x in the upstream medium, after expressing $\xi_c(x)$ in terms of $\tau(x)$ and $U(x)$ through equation (25). We have assumed a spatial dependence of the turbulence characteristic velocity v_H in the form of $v_H(x) = v_{H0} U(x)^s$ (with $s = 1/2$ in the case of Alfvén heating, from equation 16), and used as a boundary condition $\tau(-\infty) = 1/(\gamma_g M_0^2)$. The term $\mathcal{I}_\tau(x)$ appearing in equation (26), finally, is defined as

$$\mathcal{I}_\tau(x) = - \int_{-\infty}^x U(x)^{\gamma_g-1+s} \frac{d\tau}{dx'} dx'. \quad (27)$$

The only other changes induced by the inclusion of turbulent heating in our initial set of equations concern the relation between the compression ratios R_{tot} and R_{sub} (equation 7) and the temperature jump between downstream and upstream infinity, that reflects on the minimum cosmic ray momentum p_{inj} . In both cases, the changes can be summarized in the appearance of a factor of $(1 + F_H)$ with

$$F_H = \gamma_g(\gamma_g - 1) \frac{M_0^2}{M_{H0}} \left\{ \frac{1}{\gamma_g + s} \left[1 - \left(\frac{R_{sub}}{R_{tot}} \right)^{\gamma_g+s} \right] + \mathcal{I}_\tau(0) \right\}. \quad (28)$$

With this definition of F_H , we find

$$R_{tot} = M_0^{\frac{2}{\gamma_g+1}} \left[\frac{(\gamma_g + 1)R_{sub}^{\gamma_g} - (\gamma_g - 1)R_{sub}^{\gamma_g+1}}{2(1 + F_H)} \right]^{\frac{1}{\gamma_g+1}}, \quad (29)$$

and

$$T_2 = T_0 \left(\frac{R_{tot}}{R_{sub}} \right)^{\gamma_g-1} (1 + F_H) \frac{(\gamma_g + 1) - (\gamma_g - 1)R_{sub}^{-1}}{(\gamma_g + 1) - (\gamma_g - 1)R_{sub}}. \quad (30)$$

The usual results (adiabatic heating) are recovered when $M_{H0}/M_0^2 \rightarrow \infty$. In spite of the apparent simplicity of these revised relations, there are two complications arising in the solution of the system of equations. A minor difficulty is that the equation relating R_{sub} and R_{tot} now cannot be solved analytically due to the presence of the ratio R_{sub}/R_{tot} in the definition of F_H . A more serious complication, instead, has to do with \mathcal{I}_τ , which requires the knowledge of the complete solution of the problem. However, not even this is too severe a problem to overcome within the framework of an iterative method, although sometimes it results in an appreciable slowing down of the calculation. In fact, it can be seen a posteriori that \mathcal{I}_τ is always negligible compared to $(1 - U^{\gamma_g+s})/(\gamma_g + s)$.

In order to show how our results for particle acceleration may be affected by the inclusion of turbulent heating, we carried out the calculations for the case of Alfvén heating, namely considering the Alfvén velocity as the characteristic turbulence velocity, $v_H = v_A$, which also implies $s = 1/2$ in the equations above, according to equation (16). Results obtained with and without inclusion of Alfvén heating are shown in Fig. 5, where for the background magnetic field we have assumed the largest of the two values so far considered, $B_0 = 10 \mu\text{G}$, with the aim of minimizing the turbulence Mach number and hence maximizing the effects of the heating.

In the left-hand panel of Fig. 5, we show how the turbulent magnetic field strength and cosmic ray pressure are now reduced, for both a mildly ($M_0 = 10$ diamonds versus dashed line) and a strongly ($M_0 = 100$, filled circles versus solid line) modified case. In the strongly modified case, we find that, in the vicinity of the shock,

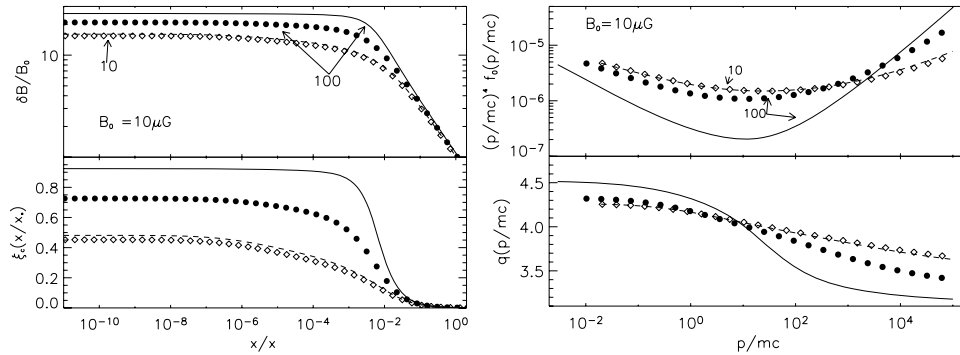


Figure 5. The plots on the left-hand side show: in the upper panel, the ratio between the turbulent and background magnetic field as a function of space for two different values of the Mach number (10 and 100), with and without inclusion of the turbulent heating; in the lower panel the corresponding normalized cosmic ray pressure. The plots on the right-hand side show the particles' spectrum and slope in the same cases. The continuous curves correspond to cases when the turbulent heating is not taken into account: dashed for $M_0 = 10$ and solid for $M_0 = 100$. The symbols correspond to cases including the turbulent heating: diamonds for $M_0 = 10$ and filled circles for $M_0 = 100$.

the turbulent magnetic field strength is decreased by 10 per cent, while the cosmic ray pressure is decreased by 20 per cent. Changes in both quantities are of the order of a few per cent in the weakly modified case. In the right-hand panel of the same figure we show, using the same notation for the different curves, how the particles' spectra are affected: while changes are negligible in the weakly modified case, for $M_0 = 100$ the concavity of the spectrum is appreciably reduced, namely, the spectrum becomes harder towards the low-energy end and softer at high energies.

It is worth stressing once more that this way of including turbulent heating, that is used in many correct approaches to particle acceleration in supernova remnants, is far from self-consistent and the results should only be considered as an indication of a trend. Sometimes, in order to attempt a slightly more realistic approach, one substitutes the Alfvén speed in the background field B_0 with the corresponding quantity in the amplified field $B_0 + \delta B$. Needless to say that such an attempt, though justified by the complete lack of any non-linear theory of turbulent heating, is far from being realistic.

6 CONCLUSIONS

We described the mathematical theory of particle acceleration at non-relativistic shock fronts with dynamical reaction of accelerated particles and self-generated scattering waves. The diffusion coefficient itself is an output of the calculations, though within the limitations imposed by the usage of quasi-linear theory applied to the case of potentially strong magnetic field amplification. The scattering in the upstream plasma is generated through streaming instability, as discussed extensively in the previous literature.

We determined the spectra of accelerated particles, their spatial distribution and the space dependence of the fluid velocity, pressure and temperature. The diffusion coefficient and the strength of the self-generated magnetic perturbations are also calculated, as a function of the distance from the shock front in the precursor. We confirm the general finding that the spectra of accelerated particles are concave, an effect which is particularly evident for strongly modified shocks, namely for large Mach numbers of the moving fluid. However, the shape of the concavity is somewhat affected by the self-determined diffusion coefficient, as visible in Fig. 4.

Having in mind the comparison between the predicted spectra at the sources and the observed cosmic ray spectrum at the Earth, it is worth reminding the reader that what can actually be measured is the combination of the diffusion time, the gas density along the tra-

jectory (responsible for the spallation) and the injection spectrum. In order to infer some conclusions about the spectrum at the source, one has to make assumptions on the diffusion coefficient in the interstellar medium. In alternative it would be a precious step forward if we could measure unambiguously the spectrum of gamma-rays generated by π^0 decays close to the source itself, an evidence that unfortunately is still missing.

The asymptotic slope of the spectra for $p \rightarrow p_{\text{max}}$ may be as flat as ~ 3.1 – 3.2 , but this conclusion is not strongly affected by the fact that the diffusion coefficient is calculated self-consistently.

The most striking new result of our calculations is the energy dependence of the diffusion coefficient and the strength of the amplified turbulent magnetic field. As could be expected, the diffusion coefficient is not Bohm like, and the turbulent component of the magnetic field is amplified so efficiently that the diffusion coefficient is much smaller than the Bohm coefficient in the background magnetic field. This is especially true at the highest momenta, which leads to think that a full non-linear theory might predict higher values of the maximum momentum than expected on naive grounds. Unfortunately, a full, self-consistent calculation of p_{max} for a strongly modified shock has never been carried out, the main difficulty being in accounting for the spatial dependence of all the quantities involved.

When compared with the Bohm diffusion coefficient as calculated in the amplified magnetic field, our diffusion coefficient remains always larger, with the possible exception of a narrow momentum region close to p_{max} .

While the calculation presented here is fully self-consistent in the determination of the shock modification due to the reaction of the accelerated particles, the part related to the amplification of the background field suffers from all the limitations related to the usage of quasi-linear theory for the streaming instability. This approach, initially developed for weakly amplified magnetic fields, is widely applied in the literature to situations that violate this condition. Unfortunately, at the present time this is the only way we have to achieve a (at least partially) self-consistent picture of the process of particle acceleration at cosmic ray modified shocks with self-generated turbulence. This problem is in fact even more serious for those approaches that predict levels of magnetic field amplification which are much higher than those found here (e.g. Bell & Lucek 2001; Bell 2004).

The high acceleration efficiencies obtained in the context of all approaches to particle acceleration at shocks are known to be

reduced by the effect of turbulent heating. Any non-adiabatic heating of the gas in the precursor leads to reducing the energy channelled into non-thermal particles at the shock. This is a serious problem, because the effect of turbulent heating depends dramatically on the type of mechanism that is responsible for the heating: Alfvén heating, often used in the literature, is only one of these mechanisms, and not necessarily the most efficient. For instance, the instability induced by the propagation of acoustic waves in the precursor was shown to lead to the formation of weak shocks in the precursor, which in turn heat the upstream plasma (e.g. Drury & Falle 1986).

These non-linear effects can hardly be taken into account in a credible way. Most notably, the phenomenological expressions proposed in the literature and used also in the present paper have originally been proposed as mechanisms to reduce the amount of magnetic field amplification and remain in the context of small perturbations of the background magnetic field. However, as shown in Fig. 5, even with the Alfvén heating taken into account, the magnetic field can be amplified by a factor in excess of ~ 10 with respect to the background field. This means that a fully non-linear theory of the turbulent heating is required in order to make fully reliable predictions.

From the phenomenological point of view, the best evidences for both magnetic field amplification and efficient particle acceleration come from observations of supernova remnants (see the reviews of Hillas (2005) and Blasi (2005), and references therein). In fact, it has been argued that the amount of field amplification required to explain the thickness of the X-ray bright rims in several remnants is of the order of $\sim 200\text{--}300 \mu\text{G}$ (Völk, Berezhko & Ksenofontov 2005). An important role in explaining this level of amplification could be played by different versions of the streaming instability (Bell & Lucek 2001; Bell 2004, 2005), not requiring resonant interactions of particles and waves. A full non-linear theory including these effects will be described elsewhere (Amato & Blasi, in preparation).

ACKNOWLEDGMENTS

This research was funded through grant COFIN2004-2005. We wish to acknowledge useful conversations with D. Ellison, S. Gabici and M. Vietri. We are also grateful to an anonymous referee for useful comments.

REFERENCES

- Amato E., Blasi P., 2005, *MNRAS*, 364, L76
 Bell A. R., 1978a, *MNRAS*, 182, 147
 Bell A. R., 1978b, *MNRAS*, 182, 443
 Bell A. R., 1987, *MNRAS*, 225, 615
 Bell A. R., 2004, *MNRAS*, 353, 550
 Bell A. R., 2005, *MNRAS*, 358, 181
 Bell A. R., Lucek S. G., 2001, *MNRAS*, 321, 433
 Berezhko E. G., Ellison D. C., 1999, *ApJ*, 526, 385
 Blandford R. D., Eichler D., 1987, *Phys. Rep.*, 154, 1
 Blandford R. D., Ostriker J. P., 1978, *ApJ*, 221, L29
 Blasi P., 2002, *Astropart. Phys.*, 16, 429
 Blasi P., 2004, *Astropart. Phys.*, 21, 45
 Blasi P., 2005, *Mod. Phys. Lett. A*, 20, 1
 Blasi P., Gabici S., Vannoni G., 2005, *MNRAS*, 361, 907
 Casse F., Lemoine M., Pelletier G., 2002, *Phys. Rev. D*, 65, 023002
 Drury L. O'C., 1983, *Rep. Prog. Phys.*, 46, 973
 Drury L. O., Falle S. A. E. G., 1986, *MNRAS*, 223, 353
 Drury L. O'C., Völk H. J., 1980, in Setti G., Spada G., Wolfendale A. W., eds, *Proc. IAU Symp. 94, Origin of Cosmic Rays*. Reidel, Dordrecht, p. 363
 Drury L. O'C., Völk H. J., 1981, *ApJ*, 248, 344
 Ellison D. C., 1981, *Geophys. Res. Lett.*, 8, 991
 Ellison D. C., Eichler D., 1984, *ApJ*, 286, 691
 Ellison D. C., Jones F. C., Eichler D., 1981, *J. Geophys.*, 50, 110
 Ellison D. C., Möbius E., Paschmann G., 1990, *ApJ*, 352, 376
 Ellison D. C., Baring M. G., Jones F. C., 1995, *ApJ*, 453, 873
 Ellison D. C., Baring M. G., Jones F. C., 1996, *ApJ*, 473, 1029
 Gieseler U. D. J., Jones T. W., Kang H., 2000, *A&A*, 364, 911
 Hillas A. M., 2005, *J. Phys. G*, 31, 95
 Jones F. C., Ellison D. C., 1991, *Space Sci. Rev.*, 58, 259
 Kang H., Jones T. W., 1997, *ApJ*, 476, 875
 Kang H., Jones T. W., 2005, *ApJ*, 620, 44
 Kang H., Jones T. W., Gieseler U. D. J., 2002, *ApJ* 579, 337
 Krymskii G. F., 1977, *Sov. Phys.-Dokl.*, 327, 328
 Lagage P. O., Cesarsky C. J., 1983a, *A&A*, 118, 223
 Lagage P. O., Cesarsky C. J., 1983b, *A&A*, 125, 249
 Malkov M. A., 1997, *ApJ*, 485, 638
 Malkov M. A., Drury L. O'C., 2001, *Rep. Prog. Phys.*, 64, 429
 Malkov M. A., Diamond P. H., Völk H. J., 2000, *ApJ*, 533, L171
 McKenzie J. F., Völk H. J., 1982, *A&A*, 116, 191
 Völk H. J., Berezhko E. G., Ksenofontov L. T., 2005, *A&A*, 433, 229

This paper has been typeset from a $\text{\TeX}/\text{\LaTeX}$ file prepared by the author.