Progress of Theoretical Physics, Vol. 72, No. 6, December 1984

Non-Linear Realization in Supersymmetric Theories. II

Masako BANDO, Tetsuji KURAMOTO, Toshihide MASKAWA† and Shozo UEHARA*)

Department of Physics, Kyoto University, Kyoto 606
†*Research Institute for Fundamental Physics*
Kyoto University, Kyoto 606

(Received April 23, 1984)

We discuss several problems which were left in the previous paper on Non-Linear Realization in Supersymmetric Theories. The independence of invariant Lagrangians is discussed and the representation of $\hat{H}$ is fully determined.

§ 1. Introduction

This paper complements our previous one entitled "Non-Linear Realization in Supersymmetric Theories"[1] (hereafter referred to as I). First, let us make a brief review of I.

Consider $N=1$ supersymmetric theories with a global internal symmetry group G spontaneously breaking down to its subgroup H. Let $V_{\rm eff}(\phi)$ be an effective potential, which depends on chiral superfield ϕ in ρ-representation of G. Define Σ as

$$\Sigma=\{\phi_0;\ \phi_0 \text{ is a minimal point of } V_{\rm eff}(\phi)\}, \tag{1·1}$$

then of course

$$\rho(H)\phi_0=\phi_0\,, \qquad \rho(G)\Sigma=\Sigma\,. \tag{1·2}$$

The domain of the representation ρ can be extended from G to its complexification G^c ("analytic representation") and the following fact is proved;

$$\rho(H^c)\phi_0=\phi_0\,, \qquad \rho(G^c)\Sigma=\Sigma\,. \tag{1·3}$$

Furthermore, symmetry group at the vacuum point is, in general, larger than H^c. Let $\hat{H}$ be the symmetry group at the vacuum point ϕ_0,

$$\hat{H}=\{g;\ \rho(g)\phi_0=\phi_0,\ g\in G^c\}, \tag{1·4}$$

which includes H^c, of course. In I, we showed that $\hat{H}$ is semidirect product of H^c and a nilpotent group R (generated by a nilpotent ideal) ($\hat{H}$-structure theorem),

$$\hat{H}=H^c * R\,, \tag{1·5}$$

and presented explicitly all the candidates for $\hat{H}$ in every classical group G.

The Goldstone superfields ξ are chiral superfields and representatives of the coset space $G^c/\hat{H}$.[1),2)] They consist of what is called Goldstone bosons whose number is equal to dim G/H, quasi-Goldstone bosons and quasi-Goldstone fermions. The numbers of those quasi-Goldstone bosons N_Q and fermions N_f are given by

*) Fellow of the Japan Society for the Promotion of Science.

$$N_Q=\dim[G^c/\hat{H}]-\dim[G/H]=\dim[G/H]-\dim R\,,$$
$$N_f=\dim[G^c/\hat{H}]/2\,. \tag{1·6}$$

Furthermore, N_Q is equal to the number of M-type Goldstone superfields, while that of P-type ones is equal to dim $R/2$. The transformation law of ξ under G is given by

$$\xi \xrightarrow{g\in G} \xi'=g\xi\hat{h}^{-1}(\xi,g),\qquad \hat{h}\in H\,. \tag{1·7}$$

As to matter chiral superfield N in a ρ_0-representation of $\hat{H}$, it is transformed under G as

$$N \xrightarrow{g\in G} N'=\rho_0(\hat{h}(\xi,g))N\,. \tag{1·8}$$

The linear base ϕ is constructed by taking some representation ρ of G^c whose restriction to $\hat{H}$ is ρ_0,

$$\phi=\rho(\xi)N\,, \tag{1·9}$$

$$\phi \xrightarrow{g\in G} \phi'=\rho(g)\phi\,. \tag{1·10}$$

According to the transformation properties of ξ and N, we showed in I how to construct invariant Lagrangians for Goldstone and matter superfields.

In this paper we prove the statements which were only given without proof in I. In §2 we clarify the representation dependence of B-type and C-type invariant Lagrangians. In §3 we discuss representation of $\hat{H}$. We can show the structure of $\hat{H}$ when H is embedded into G in an unnatural way (for example, using higher representations).

§ 2. Independence of invariant Lagrangians

In I, we presented the following three types of G-invariant Lagrangians of Goldstone superfields:

[A-type][1),3)] When there exists an analytic representation ρ_0 of G^c whose restriction to $\hat{H}$ contains a trivial one, let e_a be a base of the representation ρ_0,

$$\rho_0(\hat{H})e_a=e_a\,, \tag{2·1}$$

then any function of the invariant $e_a{}^\dagger\rho_0(\xi^\dagger\xi)e_b$,

$$[f(e_a{}^\dagger\rho_0(\xi^\dagger\xi)e_b)]_D\,, \tag{2·2}$$

is a candidate for G-invariant Lagrangians.

[B-type][4)] Let η_i be projections (see (2·24) in I), then

$$[\ln \det{}_{\eta_i}[\rho(\xi^\dagger\xi)]]_D \tag{2·3}$$

is a candidate for Lagrangians, where ρ is any representation of G.

[C-type] Define G-covariant projection P_i as

$$P_i=[\rho(\xi)\eta_i][\rho(\xi^\dagger\xi)]_{\eta_i}^{-1}[\eta_i\rho(\xi^\dagger)]\,, \tag{2·4}$$

which satisfies

$$P_i^2 = P_i\,, \qquad \mathrm{Tr}\,P_i = \text{const}\,. \tag{2·5}$$

Then any function of non-trivial invariants $\mathrm{Tr}\,P_iP_j$ $(i \neq j)$, $\mathrm{Tr}\,P_iP_jP_k (i \neq j \neq k)$, etc., is a candidate for Lagrangian,

$$[f(\mathrm{Tr}\,P_iP_j,\ \mathrm{Tr}\,P_iP_jP_k,\ \cdots)]_D. \tag{2·6}$$

Here arises a question: Which of the invariants are independent in B- and C-type recipes?

First, we show the following lemma which is necessary to answer the question.

Lemma 7 Let $\mathfrak{g}$ be a Lie algebra and $\{\mathfrak{g}_\alpha,\ \alpha = 1, \cdots, r\}$ be an arbitrary decomposition into its subspaces,

$$\mathfrak{g} = \sum_{\alpha=1}^{r} \mathfrak{g}_\alpha. \qquad (\dim\,\mathfrak{g} = \sum_\alpha \dim \mathfrak{g}_\alpha) \tag{2·7}$$

Then, for any element X of $\mathfrak{g}$, there exists such $X_\alpha (\in \mathfrak{g}_\alpha)$ that

$$e^X = e^{X_1} e^{X_2} \cdots e^{X_r}\,. \tag{2·8}$$

Proof It is evident that any element $A (\in \mathfrak{g})$ can be written as

$$A = \sum_\alpha A_\alpha\,, \qquad A_\alpha \in \mathfrak{g}_\alpha\,. \tag{2·9}$$

In the following we denote the α-component of an element A as $[A]_\alpha$, i.e.,

$$A_\alpha \equiv [A]_\alpha\,. \tag{2·10}$$

It is sufficient to prove the following: There exist such $X_\alpha(t) \in \mathfrak{g}_\alpha$ that

$$e^{tX} = e^{X_1(t)} e^{X_2(t)} \cdots e^{X_r(t)}\,, \tag{2·11}$$

where t is a parameter. From (2·11), we see

$$X = e^{-tX} \frac{d}{dt} e^{tX}$$

$$= e^{-X_r(t)} e^{-X_{r-1}(t)} \cdots e^{-X_1(t)} \frac{d}{dt} (e^{X_1(t)} e^{X_2(t)} \cdots e^{X_r(t)}). \tag{2·12}$$

Thus

$$[X]_\alpha = \frac{d}{dt} (X_\alpha(t)) + [\Delta(t)]_\alpha\,, \tag{2·13}$$

where $\Delta(t)$ is a polynomial in $X_\alpha(t)$'s higher than quadratic terms and of course an element of $\mathfrak{g}$. Hence the solutions of the equation,

$$X_\alpha(t) = t[X]_\alpha - \int_0^t ds [\Delta(s)]_\alpha\,, \tag{2·14}$$

give our desired $X_\alpha(t)$, which can be obtained by iteration. Q.E.D.*)

*) To be more precise, the proof is not sufficient unless the convergence of the iterated series of (2·13) is shown. See Ref. 7) for an exact proof in the particular case. However we think that the proof is enough for our practical use since the effective Lagrangian theory deals with ξ only in the form of asymptotic expansion.

2.1. *Independence in B-type recipe*

First, let us consider the case of pure realization. As was shown in I, $\mathfrak{g}^c$ is given by

$$\mathfrak{g}^c=\mathfrak{h}^c+\mathfrak{r}+\mathfrak{r}^\dagger . \tag{2·15}$$

We assume for the moment that $\mathfrak{g}$ is a semi-simple algebra. In case when $\mathfrak{g}$ contains $U(1)$ algebras, we can make a similar discussion to a little care about $U(1)$ part. Further $\mathfrak{h}^c$ is decomposed into semi-simple and $U(1)$ parts,

$$H^c=H^c_{\text{s.s.}}+\sum_i(c_i\theta_i),\quad H^c\in\mathfrak{h}^c \text{ and } H^c_{\text{s.s.}}\in\mathfrak{h}^c_{\text{s.s.}}. \tag{2·16}$$

According to lemma 7, $\xi^\dagger\xi(\in G^c)$ can be written as

$$\xi^\dagger\xi=e^{R_\xi{}^\dagger}e^{H_\xi{}^{\text{S.S.}}}e^{\Sigma_i c_i(\xi)\theta_i}e^{R_\xi} \tag{2·17}$$

with

$$R_\xi{}^\dagger\in\mathfrak{r}^\dagger,\quad R_\xi\in\mathfrak{r},\quad H_\xi{}^{\text{S.S.}}\in\mathfrak{h}^c_{\text{s.s.}} \text{ and } c_i(\xi)\theta_i\in\{c_i\theta_i\}. \tag{2·18}$$

Noting that

$$\rho(e^{R_\xi})\eta=\eta\rho(e^{R_\xi})\eta\ , \tag{2·19}$$

we have

$$\begin{aligned}\ln \det{}_\eta\rho(\xi^\dagger\xi)=&\ln \det{}_\eta\rho(e^{R_\xi{}^\dagger})+\ln \det{}_\eta\rho(e^{R_\xi})\\ &+\ln \det{}_\eta\rho(e^{H_\xi{}^{\text{S.S.}}})+\ln \det{}_\eta\rho(e^{\Sigma_i c_i(\xi)\theta_i}).\end{aligned} \tag{2·20}$$

The first two terms on the r.h.s. are zero according to the nilpotency of R_ξ and $R_\xi{}^\dagger$ (see Theorem 7 in §3). The η-projected subspaces of ρ-representation are invariant under $\mathfrak{h}^c_{\text{s.s.}}$ and any representation of semi-simple Lie algebras is traceless. Hence the third term is also zero. So (2·20) gives the following,

$$[\ln \det{}_\eta\rho(\xi^\dagger\xi)]_D=[\ln \det{}_\eta\rho(e^{\Sigma_i c_i(\xi)\theta_i})]_D=\sum_i\alpha_i[c_i(\xi,\xi^\dagger)]_D\ , \tag{2·21}$$

where α_i's are some constants. Thus the number of independent invariants is equal to that of $U(1)$ factors of H.*) For a classical group G, we can get all the independent $c_i(\xi,\xi^\dagger)$'s with only the fundamental representation ρ_f and its associated η_i's.

Next let us consider other (non-pure) cases. Recall the following transformation law,

$$\begin{aligned}\ln \det{}_\eta[\rho(\xi^\dagger\xi)]\xrightarrow{g\in G}&\ln \det{}_\eta[\rho(\xi^\dagger\xi)]+\ln \det{}_\eta[\rho(\hat{h}^{-1}(\xi,g))]\\ &+\ln \det{}_\eta[\rho(\hat{h}^{-1\dagger}(\xi,g))].\end{aligned} \tag{2·22}$$

Similar discussion tells us that

$$\ln \det{}_\eta[\rho(h^{-1}(\xi,g))]=\sum_{i=1}^{r}\alpha_i\cdot c_i(\xi,g), \tag{2·23}$$

where r is equal to the number of $U(1)$ factors contained in H. If we pick up r pieces

*) When G contains $U(1)$ factors, this number is equal to the difference between the number of $U(1)$ factors of H and that of G.

of independent B-type Lagrangians charactered by ρ_i and η_i $(i=1, \cdots, r)$, we have the following G-invariant quantity with some appropriate constants b_i,

$$\varDelta \equiv \ln \det_{\eta}[\rho(\xi^{\dagger}\xi)] - \sum_{i=1}^{r} b_i \ln \det_{\eta_i}[\rho_i(\xi^{\dagger}\xi)]. \tag{2·24}$$

Hence $\varDelta$ is given by A-type or C-type recipes. For a classical group G, it is sufficient to adopt the fundamental representation ρ_f and the associated projections for ρ_i and η_i.

2.2. *Independence in C-type recipe*

Here let us restrict our discussion to classical groups. It is evident that A or C-type invariants are no other than functions of $(\xi^{\dagger}\xi)_{ab}$. Consider the following transformation law,

$$\eta_i \xi^{\dagger} \xi \eta_j \xrightarrow{g \in G} \eta_i \hat{h}^{-1\dagger}(\xi, g) \eta_i \xi^{\dagger} \xi \eta_j \hat{h}^{-1}(\xi, g) \eta_j . \tag{2·25}$$

Thus the factor $\hat{h}^{-1}(\xi, g)$ (or $\hat{h}^{-1\dagger}(\xi, g)$) can be cancelled only in the following three cases:

i) There exists such η_j as

$$\eta_j \hat{h}^{-1} \eta_j = (\text{const}). \tag{2·26}$$

ii) We can construct such a "composite" vector from ξ as supplies the $[\eta_j \hat{h}^{-1} \eta_j]^{-1}$-factor after being transformed under G.

iii) $\eta_i \xi^{\dagger} \xi \eta_j$ is contracted with $[\eta_j \xi^{\dagger} \xi \eta_j]^{-1}$ as $(\eta_i \xi^{\dagger} \xi \eta_j)[\eta_j \xi^{\dagger} \xi \eta_j]^{-1}$.

The first two cases correspond to the A-type recipe. The last case tells that only fundamental representation is adequate to the C-type recipe.*)

2.3. *Absence of A- and C-type invariants in pure realization*

If projections η_i and η_j satisfy the relation,

$$\eta_i \eta_j = \eta_i , \tag{2·27}$$

then a similar relation holds for P_i and P_j;

$$\begin{aligned} P_i P_j &= \rho(\xi)\eta_i[\rho(\xi^{\dagger}\xi)]^{-1}_{\eta_i}\eta_i\rho(\xi^{\dagger}\xi)\eta_j[\rho(\xi^{\dagger}\xi)]^{-1}_{\eta_j}\eta_j\rho(\xi^{\dagger}) \\ &= \rho(\xi)\eta_i[\rho(\xi^{\dagger}\xi)]^{-1}_{\eta_i}\eta_i\eta_j\rho(\xi^{\dagger}\xi)\eta_j[\rho(\xi^{\dagger}\xi)]^{-1}_{\eta_j}\eta_j\rho(\xi^{\dagger}) \\ &= \rho(\xi)\eta_i[\rho(\xi^{\dagger}\xi)]^{-1}_{\eta_i}\eta_i\eta_j\rho(\xi^{\dagger}) = P_i . \end{aligned} \tag{2·28}$$

In the case of pure realization, it can be shown that if ρ is an irreducible representation of G, any pair of η_i and η_j satisfy (2·27).**) Hence there exists such P_i among any set of G-covariant projections that

$$\text{Tr}\, P_{i_1} P_{i_2} \cdots P_{i_r} = \text{Tr}\, P_i = (\text{const}). \tag{2·29}$$

Thus there exist no C-type invariants.

*) Also it is possible to construct invariant Lagrangians according to the hybrid recipe of A- and C-types.

**) For a classical group G it is enough to take fundamental representations ρ_f among ρ. Hence it becomes evident that the corresponding projections satisfy (2·27), if one writes down their explicit form according to the recipe explained in I.

As to A-type invariants, there exists, in general, a representation ρ of G^c whose restriction on $\hat{H}$ contains a one-dimensional representation. In pure realization case, however, it can be shown that the one-dimensional representation becomes a trivial one of $\hat{H}$ if and only if ρ itself is trivial.[5] Thus there exist no A-type invariants.

§ 3. $\hat{H}$-representaion theorem

We have clarified the structure of $\hat{H}$ in I mainly for the case when $\hat{H}$ is naturally embedded in G^c. Even if we pick up some groups G and H, however, there are various ways in practice to embed H in G. For example, H can be embedded in a higher representation, or even in a reducible representation. Here we present the following theorem about representation of $\hat{H}$.

THEOREM 8 For any irreducible representation (ρ, V) of G, the restriction of ρ on $\hat{H}$ can be always expressed in the following form by taking appropriate basis,

$$\rho(\mathfrak{h}^c)=\overset{n_1 n_2 \cdots n_r}{\begin{bmatrix} * & & 0 \\ & * & \\ 0 & & \ddots \\ & & & * \end{bmatrix}} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{matrix}, \quad \rho(\mathfrak{r})=\overset{n_1 n_2 \cdots n_r}{\begin{bmatrix} 0 & & * \\ & 0 & \\ 0 & & \ddots \\ & & & 0 \end{bmatrix}} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{matrix}, \tag{3·1}$$

where the notation $*$ indicates non-zero matrix elements.

Proof G can be assumed to be a compact semi-simple group. Let N_0 be such a subspace of V that

$$N_0=\{e; \rho(\mathfrak{r})e=0, e\in V\}. \tag{3·2}$$

According to Lie's theorem,[6] there exists one-dimensional representation of any nilpotent algebra, and we showed in I that the eigenvalue is zero in this case (Statement 3 in Appendix A). So non-trivial N_0 does necessarily exist. Since

$$[\mathfrak{h}^c, \mathfrak{r}]\subset\mathfrak{r}, \tag{3·3}$$

then

$$\rho(\mathfrak{h}^c)N_0\subset N_0. \tag{3·4}$$

Thus we can naturally define a representation $(\rho_1, V_1\equiv V/N_0)$ of $\hat{\mathfrak{h}}$ from the representation ρ. We can make a similar discussion to (ρ_1, V_1) and have the representation $(\rho_2, V_2 \equiv V_1/N_1\equiv V/\tilde{N}_1)$ of $\hat{\mathfrak{h}}$. In this way we get the following sequence,

$$N_0\subset\tilde{N}_1\subset\tilde{N}_2\cdots\subset\tilde{N}_r=V, \qquad n_a=\dim \tilde{N}_a-\dim \tilde{N}_{a-1}\cdots. \tag{3·5}$$

The basis of $\tilde{N}_{i+1}$ is given by adding such vectors to that of $\tilde{N}_i$ that they are orthogonal to the basis of $\tilde{N}_i$. Q.E.D.

Acknowledgements

One of the authors (S.U.) is indebted to the Japan Society for the Promotion of Science for financial support.

References

1) M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Prog. Theor. Phys. **72** (1984), 313.
2) T. Kugo, I. Ojima and T. Yanagida, Phys. Lett. **135B** (1984), 402.
3) W. Lerche, Munich Preprint MPI-PAE/Pth 59/83, to appear in Nucl. Phys. B.
4) B. Zumino, Phys. Lett. **87B** (1979), 203.
5) A. Borel, Proc. Natl. Acad. Science **40** (1954), 1147.
6) C. Hochshild, *The Structure of Lie Groups* (Holden-Day, Inc., 1965).
J. E. Humphreys, *Introduction to Lie Algebra and Representation Theory* (Springer-Verlag, GTM series).
7) T. Marumori, T. Maskawa, F. Sakata and A. Kuriyama, Prog. Theor. Phys. **64** (1980), 1294.