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Non-Linear Resonance in Nearly Geodesic Motion in Low-Mass X-Ray Binaries

Marek A. ABRAMOWICZ,[1,2] Vladimir KARAS,[3,*] Włodzimierz KLUŹNIAK,[2,4,5,†]
William H. LEE,[2,6] and Paola REBUSCO[2,7]
[1]*Astrophysics Department, Chalmers University, S-412 96 Göteborg, Sweden*
[2]*Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Beirut 2-4, I-34 014 Trieste, Italy*
[3]*Astronomical Institute, Charles University Prague, V Holešovičkách 2, CZ-180 00 Praha, Czech Republic*
[4]*Institute of Astronomy, Zielona Góra University, Lubuska 2, P-65 265 Zielona Góra, Poland*
[5]*CESR, 9, ave. Colonel-Roche, F-31028 Toulouse Cedex 4, France*
[6]*Instituto de Astronomía, UNAM, Apdo. Postal 70-264 Cd. Universitaria, DF 04510, Mexico*
[7]*Department of Physics, Trieste University, I-34 127 Trieste, Italy*

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Abstract

We have explored the ideas that parametric resonance affects nearly geodesic motion around a black hole or a neutron star, and that it may be relevant to the high-frequency (twin) quasi-periodic oscillations that occur in some low-mass X-ray binaries. We have assumed the particles or fluid elements of an accretion disc to be subject to an isotropic perturbation having a hypothetical but rather general form. We find that the parametric resonance is indeed excited close to the radius where epicyclic frequencies of the radial and meridional oscillations are in a 2 : 3 ratio. The location and frequencies of the highest amplitude excitation vary with the strength of the perturbation. These results agree with actual frequency ratios of twin kHz QPOs that have been reported in some black hole candidates, and they may be consistent also with correlation of the twin peaks in Sco X-1.

Key words: accretion — general relativity — QPOs — X-rays: binaries — X-rays: individual (Scorpio X-1, J1655-40, J1550-564)

1. Introduction

Pairs of high frequencies, known as kilohertz quasi-periodic oscillations (kHz QPOs), have been detected in the X-ray emission of low-mass X-ray binaries (LMXBs) for more than 20 neutron stars and several black holes. It has been suggested that a non-linear resonance within an accretion disc in a general-relativistic space-time metric plays a role in the excitation of the two oscillations (Abramowicz, Kluźniak 2001; Kluźniak, Abramowicz 2001).[1]

It is now recognized that the frequencies of the two peaks in the power spectrum of the X-ray variability in black holes are in rational ratios, $\omega_1 : \omega_2 = m : n$, with $m : n = 2 : 3$ for two sources, and $m : n = 3 : 5$ in a third source (Abramowicz, Kluźniak 2001; Kluźniak, Abramowicz 2002; Remillard et al. 2002; Abramowicz et al. 2002a). These black-hole QPOs are thought to have stable frequencies, and until the discovery of the puzzling rational ratios they have been thought to correspond to a trapped g-mode or c-mode of disc oscillation in the Kerr metric (Okazaki et al. 1987; Wagoner 1999; Wagoner et al. 2001; Kato 2001).

In neutron-star sources, the twin kHz QPO frequencies are known to vary considerably and to be mutually correlated. In the context of disc oscillations, two frequencies varying in this fashion can be explained as the fundamental and the first harmonic of the non-axisymmetric ($m = 1$) g-mode (Kato 2002, 2003). In Sco X-1, a prototypical Z-source, the slope of the correlation line is somewhat steeper than 2/3, but it is not uncommon to find the source in a state when $\omega_1 \approx 600$ Hz and $\omega_2 \approx 900$ Hz, i.e., at about a 2 : 3 ratio (Abramowicz et al. 2002b).

It would appear that the 2 : 3 frequency ratio is common to neutron-star and black-hole systems. Kluźniak and Abramowicz (2002), and Abramowicz and Kluźniak (2003) suggest that the 2 : 3 ratio follows from the properties of the parametric resonance between two eigenfrequencies, of which one is always lower than the other. Specifically, two modes of disc oscillation are thought to be possible with frequencies close to the two epicyclic frequencies of the free orbital motion. Within general relativity, the radial epicyclic frequency is generically lower than the meridional one, their ratio varying from one to zero as the radius of the circular orbits decreases from infinity down to that of the marginally stable orbit.

The observed rational ratios of frequencies may follow from the fundamental properties of both strong gravity and a non-linear resonance between radial and vertical oscillations in accretion discs with fluid lines that are nearly geodesic, nearly circular, and nearly planar. In this paper we investigate a simple mathematical model of such motions. To this aim, we discuss the properties of parametric resonance in the Paczyński and Wiita's (1980) model of the Schwarzschild metric.

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† On leave of absence from the Copernicus Astronomical Center of the Polish Academy of Sciences.
1 Non-linear resonance has been favored as an explanation of QPOs also by Titarchuk (2002) in a rather different ("transition layer") model. In another context, a tidally forced, Newtonian, non-linear vertical resonance in accretion discs has been discussed for close binary systems (Lubow 1981; Goodman 1993).

2. Nearly Geodesic Motion

Let us consider a fluid in which the flow lines are only slightly non-circular, and slightly off the $\theta = \pi/2$ symmetry plane. In spherical coordinates,

$$r(t) = r_0 + \delta r(t), \quad \theta(t) = \frac{1}{2}\pi + \delta\theta(t), \quad \phi(t) = \Omega t. \tag{1}$$

With accuracy to the third order in $\delta r \ll r_0$ and $\delta\theta \ll \pi/2$, the equations of fluid motion are:

$$\delta\ddot{\theta} + \omega_\theta^2 \left[1 + \frac{(\omega_\theta^2)'}{\omega_\theta^2}\delta r + \frac{1}{2}\frac{(\omega_\theta^2)''}{\omega_\theta^2}\delta r^2\right]\delta\theta + \frac{2}{r}\left(1 - \frac{\delta r}{r}\right)\delta\dot{\theta}\,\delta\dot{r} + \frac{1}{6r^2}\left(\frac{\partial^4\mathcal{U}}{\partial\theta^4}\right)_\ell \delta\theta^3 = f_\theta, \tag{2}$$

$$\delta\ddot{r} + \omega_r^2\,\delta r + \frac{1}{2}(\omega_r^2)'\delta r^2 + \frac{1}{6}(\omega_r^2)''\delta r^3 - r\left(\delta\dot{\theta}\right)^2 + \delta r\left(\delta\dot{\theta}\right)^2 = f_r, \tag{3}$$

$$\dot{\ell} = r^2 \sin^2\theta\, f_\phi. \tag{4}$$

Here, the time derivative is denoted by a dot and the radial derivative by a prime; f_i are components of a small force of non-gravitational origin (pressure, viscous, magnetic, or other). The epicyclic eigenfrequencies, ω_θ and ω_r, are defined in terms of the effective potential $\mathcal{U} = \Phi(r,\theta) + \ell^2/(2r^2\sin^2\theta)$, and the specific angular momentum, $\ell = \dot{\phi} r^2 \sin^2\theta$, by

$$\omega_\theta^2 = \frac{1}{r^2}\left(\frac{\partial^2\mathcal{U}}{\partial\theta^2}\right)_\ell, \quad \omega_r^2 = \left(\frac{\partial^2\mathcal{U}}{\partial r^2}\right)_\ell. \tag{5}$$

The derivation of these equations assumes equatorial plane symmetry for the gravitational potential, Φ. Here, we adopt a spherically symmetric potential,[2]

$$\Phi(r) = -\frac{GM}{r - r_G}, \quad r_G = \frac{2GM}{c^2}. \tag{6}$$

From equations (5) and (6) it follows that

$$\omega_r^2 = \frac{GM}{(r - r_G)^3}\left(1 - \frac{r_{ms}}{r}\right) < \omega_\theta^2 = \frac{GM}{r(r - r_G)^2}, \tag{7}$$

where $r_{ms} = 3r_G$ is the marginally stable orbit.

3. Strong Gravity's 2 : 3 Resonance

A parametric resonance instability occurs near $\omega_r = 2\omega_\theta/n$ for $n = 1, 2, 3, \ldots$ in an oscillator that obeys a Mathieu-type equation of motion (Landau, Lifshitz 1976),

$$\delta\ddot{\theta} + \omega_\theta^2[1 + h_1\cos(\omega_r t)]\delta\theta + \lambda\,\delta\dot{\theta} = 0. \tag{8}$$

When the coupling is weak (h_1 small), the strongest instability occurs for the lowest value of n possible (see figure 1).

With this textbook mathematics in mind, let us consider a simplified version of equations (2)–(4), with $f_i = 0$, and some of the higher order terms neglected. With accuracy up to linear terms in (3) one obtains, obviously, a solution of the form $\delta r(t) \propto \cos(\omega_r t)$. Substituting this solution in (2) brings this equation close to the standard form (8). We have solved these equations for $n\omega_r = 2\omega_\theta$, with n being a positive real parameter. The instability regions for a lower-order version of equations (2)–(3) resemble the well-known tongues of instability of the Mathieu equation. Since $\omega_r < \omega_\theta$ in general relativity, the lowest value for which the resonance can occur is $n = 3$, and the first two tongues of instability in figure 1 would be absent; i.e., the ratio of the two eigenfrequencies in (8) is 2 : 3 for the strongest resonance (Kluźniak, Abramowicz 2002).

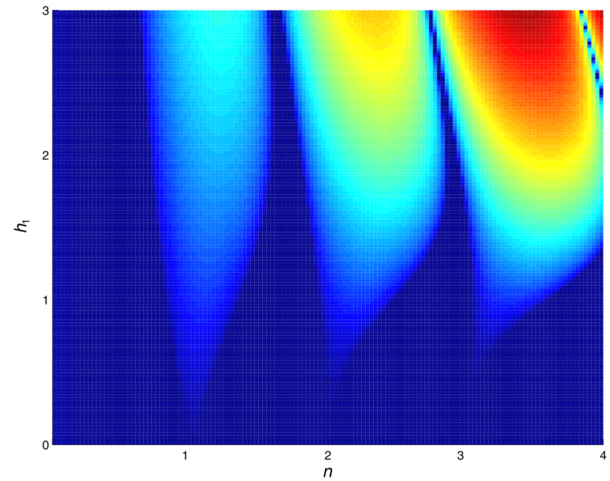

Fig. 1. Instability regions of the Mathieu equation (8) with the amplitude of variation of the eigenfrequency h_1, $n \equiv 2\omega_\theta/\omega_r$, and $\lambda = 0$. The levels of shading indicate different values of the growth rate of $\delta\theta(t)$. Three tongues of instability, of the order n = 1, 2, and 3, are clearly visible. Equations (2)–(3) considered in this paper have similar regions of instability. Note that the range of the frequencies for which the instability develops increases with h_1.

However, the behaviour of the resonance can only be properly studied if the third-order terms in (2) and the second-order terms in (3) are retained. These terms provide the nonlinearity necessary to saturate the amplitude at a finite value and the damping which affects the frequency of oscillations. Using the standard analytic method of successive approximations (Landau, Lifshitz 1976), we verified that, as expected, no parametric resonance occurs for strictly geodesic motion. We also checked numerically that resonance disappears when approaching the geodesic case. However, resonance does occur (as anticipated in Kluźniak, Abramowicz 2002) for even very slightly non-geodesic motion, when the higher order terms are influenced by non-geodesic effects of a rather general form.

4. Non-Geodesic Coupling

In general, it is not possible to specify the form of the very small force f_i exactly, because the nature of non-geodesic forces in accretion discs is not yet very accurately known. Here, we only explore the basic mathematical form of the solutions, not the physical origin of the f_i force. We assume *ad hoc* that the possible non-geodesic effects have the form of a

[2] See Paczyński and Wiita (1980). This form is known to be a convenient model for the external gravitational field of a non-rotating black hole or a neutron star; it captures the essential feature of Einstein's strong gravity relevant here, i.e., the fact that $\omega_r(r) < \omega_\theta(r)$, and $\omega_r = 0$ at the marginally stable orbit.

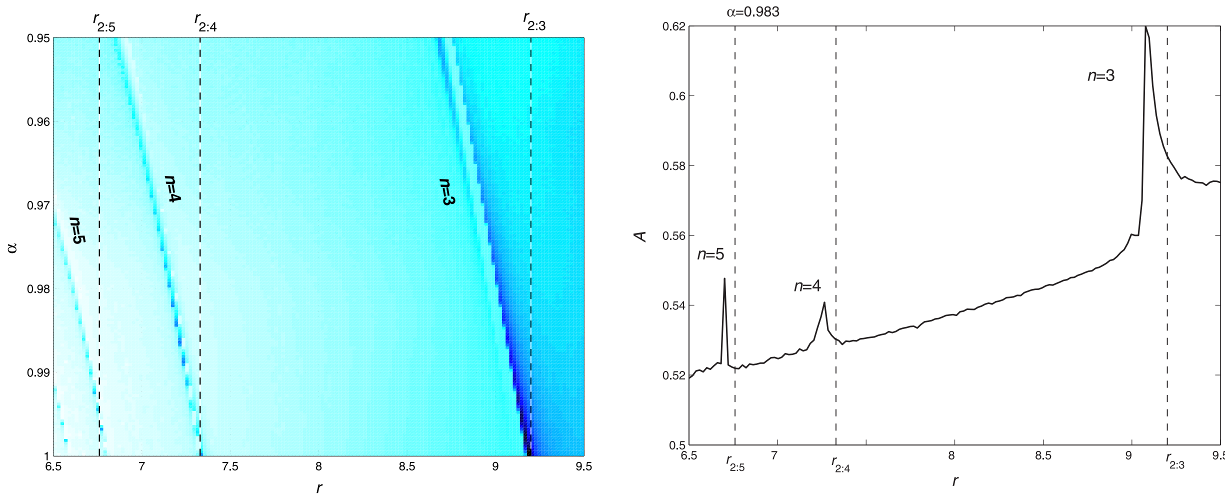

Fig. 2. (Left:) Location of the first three resonances shown, for a particular choice of initial amplitude, $\delta\theta(0) = 0.5$, by encoding the growth rate of $|\delta\theta|$ from equation (2) with different levels of shading in the (r, α)-plane (radius versus strength of perturbation). Note that α decreases along the y-axis. For $\alpha = 1.00$, the resonant orbits are located at radii $r_{m:n}$ (indicated by vertical lines), where ω_r and ω_θ are in rational ratios, $m : n$. In particular, the trace of the 2 : 3 resonance ($n = 3$) starts from $r_0 = 9.2GM/c^2$ in the Paczyński–Wiita potential. The frequencies and radius of the resonant orbits vary when α is changed. (Right:) The maximum amplitude A of $\delta\theta(t)$ for a fixed value of $\alpha = 0.983$ as a function of the radius of the unperturbed circular orbit.

non-linear isotropic coupling between the two components of the geodesic deviation:

$$f_\theta = +\alpha_\theta \left[\frac{1}{2}(\omega_\theta^2)''\delta r^2 + \frac{1}{6r^2}\left(\frac{\partial^4 \mathcal{U}}{\partial\theta^4}\right)_\ell \delta\theta^2\right]\delta\theta, \tag{9}$$

$$f_r = -\alpha_r \left[1 - \frac{\delta r}{r}\right] r(\delta\dot{\theta})^2. \tag{10}$$

Here, α_r and α_θ measure the strength of non-geodesic forces in comparison with the geodesic terms of the corresponding form and order. We first assumed $\alpha_r = \alpha_\theta \equiv \alpha$ for the sake of simplicity.[3]

After choosing a particular value of α, we solved equations (2)–(3) numerically by using the Runge–Kutta adaptive step-size routine. The integration was performed over a wide range of starting radii, $6 \leq r_0(c^2/GM) \lesssim 15$. For the initial conditions, we took $\delta\theta(0) = 0.5$, $\delta\dot{\theta}(0) = 0.01$, $\delta r(0) = 0.5$, and $\delta\dot{r}(0) = 0.01$. We then repeated the same procedure for another value of α, and in this way we constructed a sequence of solutions that is parameterized by α and r_0.

One can locate radii $r_{\mathrm{P}}(\alpha)$ for which $|\delta\theta(t; r_{\mathrm{P}}; \alpha)|$ grows in time up to a maximum amplitude. The saturation amplitude, A, is shown in figure 2 (right panel) as a function of radius. Clearly, the largest amplitudes correspond to parametric resonance. Close to those radii the test-particle epicyclic frequencies are in rational ratios $m : n$; $r_{\mathrm{P}} = r_{2:3}$, $r_{2:4}$, and $r_{2:5}$. Note that the growth rate of the amplitudes $|\delta\theta(t)|$, $|\delta r(t)|$ decreases with decreasing α. The 2 : 3 resonance is the strongest one, as expected.

[3] The perturbations considered here may be consistent with the formation of non-axially symmetric coherent structures in accretion discs ("planets", "vortices", "magnetic flux tubes", or "spiral waves"), as proposed by several authors: Abramowicz et al. (1992); Goodman, Narayan, and Goldreich (1987); Adams and Watkins (1995); Bracco et al. (1998); Karas (1999); Kato (2002, 2003).

The exact slope of the 2 : 3 resonance trace $r_{\mathrm{P}}(\alpha)$, starting from the point $(r_0, \alpha) = (9.2, 1)$, depends on the choice of coupling parameters in equations (9)–(10), so the above assumption of identical values for $\alpha_\theta = \alpha_r$ is not universal. For example, assuming $\alpha_r = \kappa\alpha_\theta$ and setting $\kappa = 0.2$, one finds that the end of the trace is moved to the point (9.0, 0.95). It is thus shifted in the (r, α)-plane with respect to the case $\kappa = 1$, for which the trace ends at the point (8.8, 0.95), as shown in the left panel of figure 2. Also the initial conditions for $\delta\theta$ and δr have an influence on the trace slope. Therefore, the corresponding frequencies that are present in the solutions for $\delta r(t)$, $\delta\theta(t)$ depend on the details of the solution. Nevertheless, the ratio of the frequencies stays at 2 : 3 at the maximum amplitude. This fact is in agreement with the intuition about the properties of equations (2)–(3), and we checked it by Fourier-analyzing their solution.

In this work we assumed the gravitational potential of equation (6) for the sake of simplicity. When a more accurate, full formulation in general relativity is developed together with a physical mechanism responsible for the non-geodesic coupling term (9)–(10), one will be able to constrain the absolute value of the mass of the accreting body.

5. Frequency Correlation in Sco X-1

As noted in section 1, the frequency ratio of the QPOs in some black hole sources is accurately 2 : 3. Specifically, in GRO J1655-40 the frequency ratio is that of 300 Hz and 450 Hz, while in XTE J1550-564 it is that of 184 Hz and 276 Hz (Abramowicz, Kluźniak 2001; Remillard et al. 2002). The properties of parametric resonance, discussed above, provide a natural explanation for this ratio.

Can the same mechanism be responsible for the observed frequencies of the kHz QPOs in neutron stars? In neutron star sources the two kHz QPOs vary considerably in frequency, and

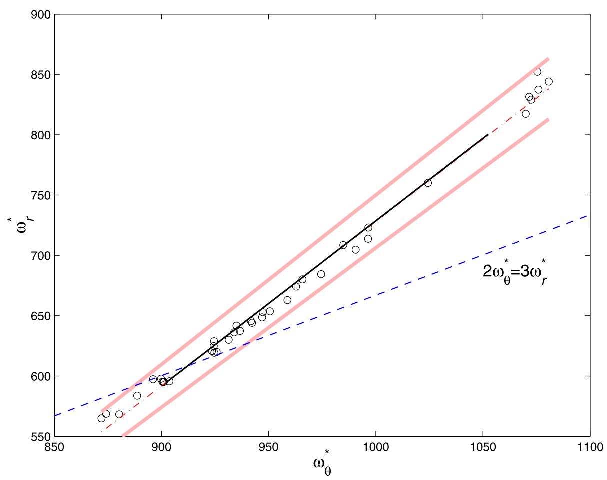

Fig. 3. Pairs of oscillation frequencies at points along the $n = 3$ trace of figure 2 were obtained by integrating equations (2)–(3) with a different initial value $\delta(0) = 0.8$, yielding a frequency ratio differing from 2 : 3. The solid line is a result of the calculation described in the text. For a comparison with observations, we have scaled all computed frequencies with one arbitrary multiplicative factor. The frequencies then depend only on the strength of the perturbation, α, running in the figure from $\alpha = 1.0$ at $\omega^*_\theta = 900\,$Hz to $\alpha = 0.95$ at about 1050 Hz. Note the agreement between the slope of the computed line and the observed kHz QPO frequencies in Sco X-1, also shown (as circles). The dot-dashed line is the least-squares best-fit to the data points (two lines with a 3% offset from the best-fit line are also plotted). The dashed line gives a reference slope of 2/3 for a comparison. The Sco X-1 data (van der Klis et al. 1997) has been kindly provided by Michiel van der Klis.

their ratio is not always 2 : 3. A case in point is Sco X-1, whose two kHz frequencies are linearly correlated (figure 3), but not directly proportional to each other (van der Klis et al. 1997). However, the distribution of points along the line of correlation is not uniform, and the resulting distribution of frequency ratios has a prominent peak at about 2 : 3 value, strongly suggesting the presence of a non-linear resonance (Abramowicz et al. 2002b).

We note that QPOs are not coherent oscillations, and that the typical integration time (minutes) greatly exceeds the coherence time of the signal (fraction of a second). We find that the slope of Sco X-1 frequency correlation can be reproduced if we assume that the signal observed at a given time corresponds to the frequencies originating at a particular radius, for example the radius $r_{\rm P}(\alpha)$ discussed in the previous section. If this is *not* strictly equal to the radius where the maximum amplitude is attained, the ratio of the frequencies departs from 2/3. Specifically, taking an arbitrary initial amplitude $\delta\theta(0)$ we constructed Fourier power spectra of $\delta\theta(t; r_{\rm P}, \alpha)$, and $\delta r(t; r_{\rm P}, \alpha)$. These power spectra peak at frequencies $\omega^*_\theta(\alpha)$ and $\omega^*_r(\alpha)$, which we interpret here as the observed pair of oscillations.

Figure 3 shows the correlation between ω^*_θ and ω^*_r along the α-trace from figure 2. We assumed $\kappa = 1$ as before. The actual value of $\delta\theta(0)$ has been selected to match the slope of Sco X-1 data (van der Klis et al. 1997). Because α measures the deviation from geodesicity, its value is likely to be a function of the accretion rate in the disc, $\alpha(\dot{M})$. This notion agrees with the reported correlation between the frequencies and the position of the source along the Z curve in the color–color diagram. Figure 3 illustrates that the simple scheme that we introduced in the present paper may, indeed, explain the slope of $\omega^*_r(\omega^*_\theta)$ relation.

6. Conclusions

We have expanded, through the third order, relativistic equations of test motion about a circular geodesic, and have shown that the lowest order expansion corresponds to Mathieu's equation with non-standard damping. As expected, the solutions of these (unperturbed) equations do not show any resonant phenomenon — they simply describe geodesic motion.

When a sufficiently large non-geodesic perturbation is imposed, the deviations grow rapidly at certain resonant radii and the motion departs from circular geodesics. This is caused by parametric resonance between the meridional and radial epicyclic motions, and the strongest resonance occurs when the two dominant frequencies are near ratio 2 : 3.

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