Non-Linear Theory of Gravitational Instability in the Expanding Universe. III

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First it is indicated that, at the radiation-dominated stage of the expanding universe, two ways should be distinguished for the evolution of the second-order density waves most naturally associated with the first-order rotational and gravitational waves, and the behavior of an isolated eddy is examined for a comparison with those of periodically distributed eddies.

Next the second-order waves associated with rotational, density and their mixed waves at the intermediate stage (before and after the decoupling epoch t_D) are analyzed and the rapidity of compression due to the inertial force appearing soon after t_D is calculated.

Finally the possibility of galaxy formation is examined on the basis of the above analysis. Moreover a possibility for comparatively small galaxies to be formed through the cascade due to non-linear process is considered.

§ 1. Introduction

In two previous papers[1],[2] (referred to as [I] and [II]), we have extended Lifshitz's[3] linearized theory of gravitational instability so as to deal with non-linear processes up to second-order terms, and applied the non-linear theory to the problems of evolution of local inhomogeneities in the expanding universe. However, the applications have been limited to the problems at a later stage and at a radiation-dominated early stage. It is, therefore, necessary that we analyze continuously the behavior of inhomogeneities at an intermediate stage which includes the stage I from $t = t_*$ (the beginning of a matter-dominated stage) to $t = t_D$ (the decoupling epoch of matter from radiation) and the stage II after t_D.

The most striking change in a physical state at the intermediate stage is a rapid compression of gaseous matter within inhomogeneous regions soon after t_D (for instance, $t_D < t < 2t_D$). This arises because the balance between inertial force and radiation pressure gradient is broken in the inhomogeneities at t_D and gaseous matter is compressed till the compression is prevented by the effect of background expansion (much stronger than the gradient of gas pressure). The qualitative analyses for galaxy formation due to this hydrodynamical instability have been given by Ozernoi and Chernin,[4] Ozernoi and Chibisov,[5] Satō et al.[6] and Satō,[7] and it has been shown that an inhomogeneous region with mass of a large galaxy (or a cluster of galaxies) can survive against dissipation effects.

In this paper quantitative analyses for the motion of inhomogeneities are

carried out by the use of hydrodynamical equations derived in [II], and how and to what extent the perturbations of density and velocity are amplified at the intermediate stage are shown. Moreover the lower limits for initial perturbations leading to galaxy formation are evaluated and also a possibility for the formation of comparatively small galaxies are indicated.

In § 2 it is shown at a radiation-dominated stage that two types should be distinguished for the evaluation of density waves associated with rotational and gravitational waves, and the motion of an isolated eddy is studied in a comparison with that of periodically distributed eddies. In §§ 3 and 4, the second-order density waves associated with the first-order rotational, density and mixed waves are derived at the intermediate stages I ($t < t_D$) and II ($t > t_D$), respectively, and in the latter section the junction between two stages is dealt with. In § 5 possibilities of galaxy formation are examined on the basis of the above results. Moreover, Appendices A and B are given over to deriving hydrodynamical equations for the second-order quantities at the relevant stages, and to showing the solutions of Lifshitz's linearized equations.

Notation

The notation in this paper is the same as in the previous one. But $w \equiv \varepsilon_m/\varepsilon_r = a/a_*$ is employed in place of η for convenience. Here ε_m and ε_r are the unperturbed matter and radiation mass densities, and the asterisk denotes an epoch of $\varepsilon_m = \varepsilon_r$.

Assuming that ε_m and ε_r are conserved separately and the matter pressure p_m is negligible (i.e., $\varepsilon_m \propto a^{-3}$, $\varepsilon_r \propto a^{-4}$), we can represent the equations describing the unperturbed state of the universe as

$$w'^2 \equiv (dw/d\eta)^2 = (w+1)/\eta_*{}^2, \qquad (1 \cdot 1)$$

$$(w+1)^{1/2} - 1 = \tfrac{1}{2}\eta/\eta_* , \qquad (1 \cdot 2)$$

$$\eta_*{}^2 \equiv 3\varepsilon_r a^4/(\varepsilon_m a^3)^2 = \text{const.}$$

In this paper all relativistic models are taken into consideration through a parameter $\Omega \equiv \varepsilon(t_0)/\varepsilon_c$ ($\varepsilon_c = 10^{-29}$ g cm^{-3}), while spatial curvature is neglected. Suffix 0 stands for the present epoch.

In order to represent the time scale of expansion, we use the Hubble parameter $H \equiv a'/a^2 = \varepsilon/3$, where units $c = 8\pi G = 1$ are used.

§ 2. Radiation-dominated stage

(i) *Two types of the second-order waves associated with the first-order ones*

In [II], we have derived the second-order perturbation $\phi \equiv \underset{2}{\delta}\varepsilon/\varepsilon$ associated with rotational and gravitational waves regarded as the first-order quantities. This ϕ is expressed as a sum of a special solution of the relevant inhomogeneous differential equation and its homogeneous solutions, and is uniquely deter-

mined by setting initial and boundary conditions. Those conditions should be chosen in such a way that the second-order waves associate most reasonably with the first-order ones. For this purpose, we must carefully deal with the homogeneous part, as will be shown below.

If we take a very early stage such as $n\eta \ll 1$ as a setting epoch, the condition ① must be imposed such that the homogeneous part ($\propto \eta$, η^2) should be discarded. On the other hand, if we take a stage $n\eta \gg 1$ (although before t_D) as a setting epoch, another condition ② is necessary such that acoustic waves with constant amplitudes (the homogeneous part at that stage) should be discarded. These two conditions ① and ② are, however, not compatible with each other. In fact, the incompatibility is shown by inserting the following homogeneous solutions (h.s.) into the expressions for ϕ (see the Appendix of [III]):

$$\text{h.s.} = C_1[y^{-1}\cos y - y^{-2}(1-y^2/2)\sin y]Q$$

$$+ C_2[y^{-1}\sin y + y^{-2}(1-y^2/2)\cos y - y^{-2}]Q$$

$$\simeq \begin{cases} (\tfrac{1}{6}C_1 y + \tfrac{1}{8}C_2 y^2)Q & \text{for } y \ll 1, \\ (\tfrac{1}{2}C_1 \sin y - \tfrac{1}{2}C_2 \cos y)Q & \text{for } y \gg 1, \end{cases}$$

where $y \equiv (2/\sqrt{3})n\eta$ and Q is a scalar harmonics specified by $\Delta Q = -n^2 Q$. Because of this situation, the associated waves with no typical h.s. at the stage $n\eta \ll 1$ have acoustical characteristics of h.s. at the stage $n\eta \gg 1$, and inversely the associated waves with no acoustic properties show typical properties such as
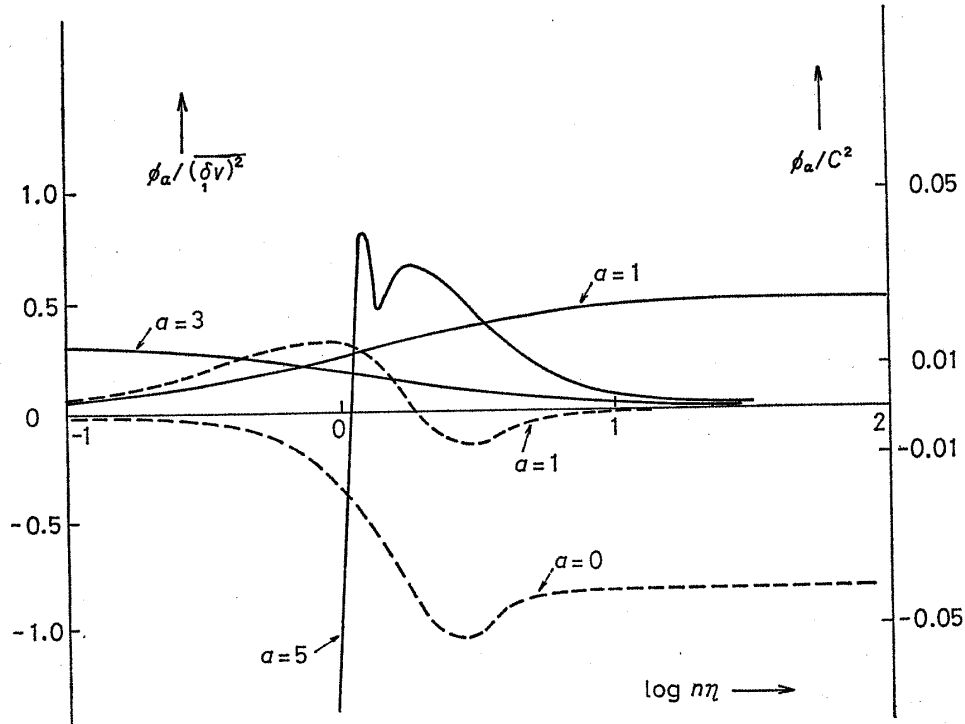

Fig. 1. Density perturbations of the associated type 2 waves. The solid lines denote rotational waves and the broken lines denote gravitational waves.

density perturbations at the stage $n\eta \ll 1$. Accordingly it seems to be difficult to decide which condition of ① or ② is superior, and so we had better adopt both conditions separately. In Figs. 2 and 3 of [II], we have shown the middle case (between ① and ②), such that the contribution from the terms $\propto \eta$ is discarded at $n\eta \ll 1$. The behavior of ϕ in this case is analogous to that in the case of ① (type 1 waves). Here we shall show the behavior in the case of ② (type 2 waves) in Fig. 1 for a comparison. It is found in this figure that the parts representing non-uniformity of density distribution damp out with time, except the part with a suffix $a=1$ in the rotational waves.

(ii) *Behavior of an isolated eddy motion*

In [II] we have considered the eddies which are periodically distributed, and it has been shown that at $n\eta \ll 1$ the peaks of the second-order density perturbations appear at the corners among the eddies and at $n\eta \gg 1$ the type 1 waves have the peaks at the corners and centers oscillating alternately. Moreover, it is shown that the type 2 waves have stationary peaks at the corner. However, since eddies are not necessarily distributed periodically, it is important to examine whether and where the peaks appear in a quite different distribution of eddies.

In what follows, let us consider an extreme case of an isolated eddy for a comparison. Its simple model with symmetry around x^3 axis can be described by Eq. (3·5) of [II] and the velocity field is given by

$$\underset{1}{\delta}v^1 = (C/8)(\Delta F)_{,2}, \qquad \underset{1}{\delta}v^2 = -(C/8)(\Delta F)_{,1}, \qquad \underset{1}{\delta}v^3 = 0.$$

Here F may be a function of $r \equiv (x^\mu x^\mu)^{1/2}$ and $\varpi \equiv \{(x^1)^2 + (x^2)^2\}^{1/2}$ in general, but we shall treat a simple case specified by $F = F(r)$. (The generalization to the case of $F = F(r, \varpi)$ can be done easily but with lengthy calculation.) Then $\underset{1}{\delta}v^\alpha$ is compactly written as

$$\underset{1}{\delta}v^\alpha = \omega(r)(x^2, -x^1, 0) \qquad \text{for } \alpha = (1, 2, 3), \text{ respectively,} \qquad (2\cdot1)$$

where

$$\omega \equiv (C/8)\left(F_{,rrr} + \frac{2}{r}F_{,rr} - \frac{2}{r^3}F_{,r}\right)\bigg/r$$

plays a role of angular velocity. From Eq. (3·2) of [II], we obtain

$$f'' - \frac{1}{3}\Delta f = -\frac{16}{3}\eta^{-1}\left[\omega^2 + \frac{1}{2}(\omega^2)_{,r}r(\varpi/r)^2\right]$$

$$-8\eta^{-3}\omega^2\varpi^2 - 128\eta^{-5}\left(r^{-3}\int_0^r drr^4\omega\right)^2(\varpi/r)^2. \qquad (2\cdot2)$$

Now let us represent by n_c a wave number characteristic of the eddy with length $L(n_c \equiv 2\pi a/L)$, and show its respective behaviors at $n_c\eta \ll 1$ and $\gg 1$.

For $n_c\eta \ll 1$, we get

$$f \simeq -\frac{32}{3}\eta^{-3}\varPhi_1 + \text{h.s.}$$

with

$$\varPhi_1 = \left(r^{-3}\int_0^r dr r^4 \omega\right)^2 (\varpi/r)^2,$$

so that (cf. Eq. (2·8) of [III])

$$\phi \simeq \frac{32}{3}\eta^{-2}(1-\ln n\eta)\,\varPhi_1 + \text{h.s.} \tag{2·3}$$

For $n_c\eta \gg 1$, we get

$$-\frac{1}{3}\varDelta f \simeq -\frac{16}{3}\eta^{-1}\left[\omega^2 + \frac{1}{2}(\omega^2)_{,r}\,r\,(\varpi/r)^2\right] \tag{2·2'}$$

for the part of type 2 and solving this equation

$$f \simeq 8\eta^{-1}\varPhi_2 + \text{h.s.}$$

with

$$\varPhi_2 = \left\{(\varpi/r)^2 - \frac{2}{3}\right\}r^{-3}\int_0^r dr r^4 \omega^2 + \frac{2}{3}\int_0^r dr r\omega^2 + \text{const.}$$
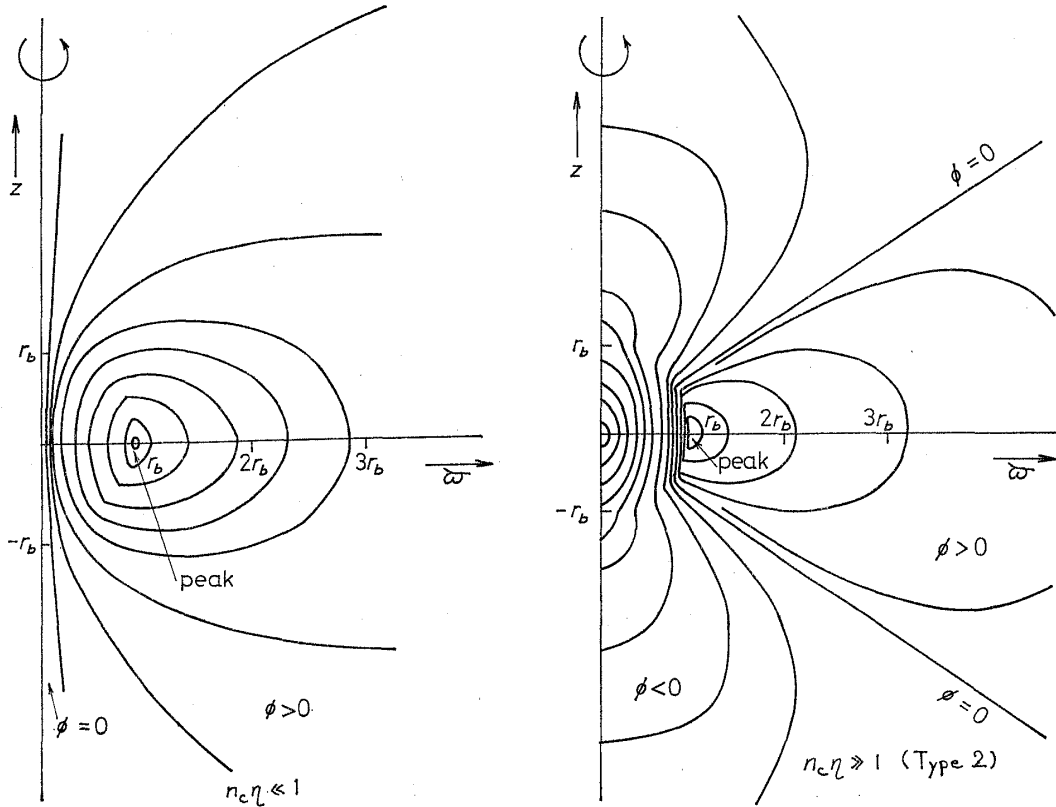

Fig. 2. Density distributions associated with an isolated eddy at $n_c\eta \ll 1$ and $n_c\eta \gg 1$ (type 2). Curves denote equidensity shells. It is remarkable that a ring appears around the center of the eddy.

where const is taken in such a way that $\Phi_2 \to 0$ for $r \to \infty$. Therefore we obtain

$$\phi \simeq 4\Phi_2 + \text{h.s.} \tag{2.4}$$

The solutions of type 1 is obtained by superposing on the solution (2.4) the acoustic waves, for which Eq. (2.2') cannot be used as an approximate equation of (2.2).

As long as we are concerned with the main parts other than h.s., the general properties of these solutions are that (a) at $n_c\eta \ll 1$, ϕ vanishes *along z axis* and $\phi \to 0$ at $\varpi = 0$, ∞ and $\phi > 0$ at the middle points *on ϖ plane*, and (b) at $n_c\eta \gg 1$, ϕ (the type 2) is kept constant in time, $\phi < 0$ *along z axis* and $\phi \to 0$ at $\varpi \to \infty$, $\phi \to \phi_0$ (<0) at $\varpi \to 0$ *on ϖ plane*. Here we assumed that $\omega(0)$ is const ($\neq 0$) and $\underset{i}{\delta}v^\alpha$ vanishes at $r \to \infty$. In Fig. 2, we show the density distributions in the case where $\omega = \text{const}$ ($\neq 0$) for $r \leq r_b$ and $\omega = 0$ for $r > r_b$.

In the above, we have treated the cases of $n_c\eta \ll 1$ and $n_c\eta \gg 1$ (type 2 only) separately. For the junction at $n_c\eta \simeq 2$ and a derivation of the part of type 1, the direct integration of Eq. (2.2) is necessary, but their systematic treatments are omitted in this paper.

§ 3. Intermediate stage I. ($t_* < t < t_D$)

At the stage from the epoch t_* to t_D, we shall derive the second-order waves associated with rotational, density and their mixed waves. Radiation and ionized matter are dealt with as one fluid at this stage and the equation of state is expressed by $p = \varepsilon_r/3$ and $\varepsilon = \varepsilon_r + \varepsilon_m$. Then the sound velocity c_s is given by

$$c_s = (\delta p/\delta\varepsilon)_s^{1/2} = \left(\frac{4}{3}\frac{p}{\varepsilon+p}\right)^{1/2} = \frac{2}{3}\left(w + \frac{4}{3}\right)^{-1/2}, \tag{3.1}$$

where $s = $ entropy and $w \equiv a(t)/a(t_*)$. In the following the condition $c_s n w/w'$ ($= 2\pi c_s H^{-1}/L) \gg 1$ is assumed, as has been assumed at the end of radiation-dominated stage ($n\eta \gg 1$), and our analysis is limited to adiabatic perturbations.

(i) *Second-order density wave associated with rotational waves*

The behavior of this density wave is different according as we adopt the viewpoint ① or ② in (i) of § 2.

First let us adopt the viewpoint ①. Then, after an epoch $n\eta \sim 2$ when ϕ reaches $\sim (\underset{i}{\delta}v)_i^2/c^2$, the density wave behaves like a free acoustic wave whose density has a constant amplitude $\sim (\underset{i}{\delta}v)_i^2/c^2$ at the radiation-dominated stage. At the intermediate stage I, the amplitude decreases slowly as

$$(w+1)^{-1}(w+\tfrac{4}{3})^{3/2}, \tag{3.2}$$

as shown in Eq. (B.8) of Appendix B.

On the other hand, if we adopt the viewpoint ②, the density is kept at the value $\sim (\underset{i}{\delta}v)_i^2/c^2$ after $n\eta = 2$ at the radiation-dominated stage. The second-order

density and velocity at the intermediate stage I are shown in the following (cf. Eqs. (3·10) and (3·11) below).

For that purpose, the differential equations for ϕ are derived in Appendix A and the first-order perturbations are described in Appendix B as the solutions of Lifshitz's linearized equation. For the first-order rotational perturbation, it is to be noticed under the condition of $c_s nw/w' \gg 1$ that the dominant term on the right-hand side of Eq. (A·5) is only that including $\overset{\alpha}{\underset{1}{\delta v_{,\beta}}}\overset{\beta}{\underset{1}{\delta v_{,\alpha}}}$. Therefore we have

$$Y_{,ww} + \frac{7/2}{w+1}Y_{,w} - \frac{1}{w(w+1)^2}Y - \Delta Y \Big/ \left\{3\eta_*^{-2}(1+w)\left(1+\frac{3}{4}w\right)\right\}$$

$$= \eta_*^2 w^{-2}(w+1)^{-3/2}\left\{w^2(w+1)^{-1/2}\left(w+\frac{4}{3}\right)\overset{\alpha}{\underset{1}{\delta v_{,\beta}}}\overset{\beta}{\underset{1}{\delta v_{,\alpha}}}\right\}_{,w}, \qquad (3·3)$$

where

$$\phi = \left(\frac{w^2\sqrt{w+1}}{w+4/3}\right)^{-1}\int Y\frac{w^2\sqrt{w+1}}{w+4/3}dw. \qquad (3·4)$$

Inserting Eq. (B·16), we can rewrite the right-hand side of Eq. (3·3) as

$$\frac{C_r^2\eta_*^4}{18}\frac{w^2+6w+16/3}{w(w+1)^3(w+4/3)^2}\Delta S_{,\beta}^\alpha\Delta S_{,\alpha}^\beta. \qquad (\equiv R_1\Delta S_{,\beta}^\alpha\Delta S_{,\alpha}^\beta) \qquad (3·5)$$

Here, for simplicity, we shall assume S^α to be a vector harmonics satisfying $\Delta S^\alpha = -n^2 S^\alpha$ and $S_{,\alpha}^\alpha = 0$. Then we can separate $\Delta S_{,\beta}^\alpha\Delta S_{,\alpha}^\beta = n^4 S_{,\beta}^\alpha S_{,\alpha}^\beta$ into $n^6(Q_0+Q_{2n})$, where Q_0 is a constant and Q_{2n} is a scalar harmonics satisfying $\Delta Q_{2n} = -(2n)^2 Q_{2n}$. (For example, in the case of periodic eddies where S^α is defined by $\varepsilon^{\alpha\mu\nu}F_{\mu,\nu}$ with $F_1 = F_2 = 0$, $F_3 = \sin nx^1 \sin nx^2$, we have $Q_0 = 0$ and $Q_{2n} = \cos 2nx^1 + \cos 2nx^2$). Then we can separate the differential equation (3·3) into two parts as follows:

$$L(Y_0) = n^6 R_1, \qquad (3·6)$$

$$L(Y_{2n}) + \frac{4}{9}(2n\eta_*)^2\frac{Y_{2n}}{(w+1)(w+4/3)} = n^6 R_1, \qquad (3·7)$$

where

$$Y \equiv Y_0 Q_0 + Y_{2n}Q_{2n} \qquad (3·8)$$

and the operator L is defined by

$$L \equiv \partial_{ww} + \frac{7/2}{w+1}\partial_w - w^{-1}(w+1)^{-2}.$$

The approximate solutions of Eq. (3·7) can be easily obtained, if we notice $c_s nw/w' \gg 1$. That is to say, on the left-hand side of Eq. (3·7), the last term is dominant and its solution is expressed as

$$Y_{2n} = \frac{9}{16}n^4\eta_*^{-2}(w+1)(w+4/3)R_1.$$

Hence, if we separate ϕ into two parts as

$$\phi = \phi_0 Q_0 + \phi_{2n} Q_{2n} \qquad (3\cdot 9)$$

and integrate Eq. (3·4), we get

$$\phi_{2n} = \frac{1}{16} C_r^{\,2} n^4 \eta_*^{\,2} / (w+1).$$

Using Eqs. (3·1) and (B·16) again, we can express ϕ_{2n} as

$$\phi_{2n} = \frac{1}{4} \frac{\overline{(\underset{1}{\delta v})^2}}{c_s^{\,2}} \left(\frac{w+4/3}{w+1} \right) \Big/ \overline{(S^\alpha S^\alpha)}, \qquad (3\cdot 10)$$

where the bars denote spatial average.

On the other hand, the solution of Eq. (3·6) is estimated to be of the order of $n^6 w^2 R_1$, which is $\sim (c_s n w / w')^2 Y_{2n}$. Therefore, ϕ_{2n} is negligibly small compared with ϕ_0 which is of the order of $(n w / w')^2 \overline{(\underset{1}{\delta v})^2} = (n \eta_* w / \sqrt{w+1})^2 \overline{(\underset{1}{\delta v})^2}$. In fact, for $w \gg 1$, we get from Eqs. (3·6) and (B·16)

$$\phi_0 = -\tfrac{1}{9} C_r^{\,2} n^6 \eta_*^{\,2} / w = - \overline{(n \eta_* \sqrt{w})^2 (\underset{1}{\delta v})^2} / \overline{S^\alpha S^\alpha}.$$

Because of its spatial constancy, however, ϕ_0 does not contribute to the growth or decay of non-uniformity at all. Hence, we should pay attention to ϕ_{2n} given by Eq. (3·10). (In the above case of periodic eddies, ϕ_0 does not appear, because of $Q_0 = 0$).

For $\underset{2}{\delta v^\alpha}$, we obtain from Eqs. (A·2) and (B·16)

$$\underset{2}{\delta v^\alpha} / \overline{(\underset{1}{\delta v})^2} = 2\sqrt{3}\, n \eta_* (w+4/3) \{ \tan^{-1}(\sqrt{3(w+1)}) + h_1 \} b_0, \qquad (3\cdot 11)$$

where $b_0 \equiv n^{-1}(\tfrac{1}{4} Q_{2n,\alpha} + S^\alpha_{,\beta} S^\beta) / \overline{(S^\mu S^\mu)} \sim 1$ and h_1 is an integration constant. If we take $h_1 = -\pi/3$ so as to exclude homogeneous terms in $\underset{2}{\delta v^\alpha}$ for $w \ll 1$, $\underset{2}{\delta v^\alpha} / \sqrt{\overline{(\underset{1}{\delta v})^2}}$ for $w \gtrsim 1$ becomes nearly constant in time.

The condition that our second-order approximation is applicable can be given by $\underset{2}{\delta v} < \underset{1}{\delta v}$, i.e.,

$$(\underset{1}{\delta v}) n (w/w') (w+4/3) \sqrt{w+1} \{ \tan^{-1}(\sqrt{3(w+1)}) + h_1 \} < 1 . \qquad (3\cdot 12)$$

On the other hand, the condition that non-linear inertial force is ineffective is given by

$$(\underset{1}{\delta v}) n (w/w') (= 2\pi \underset{1}{\delta v} H^{-1}/L) < 1, \qquad (3\cdot 12')$$

The discrepancy between conditions (3·12) and (3·12′) is raised by the inevitable mixing of homogeneous terms at $w \gtrsim 1$.

(ii) *Second-order density wave associated with a density wave (acoustic wave)*

The first-order density wave before t_D is given by Eqs. (B·8) $\sim$ (B·10) and the associated second-order density wave is described by Eqs. (A·4) and (A·5). It should be noticed here that, under the condition $c_s n w / w' \gg 1$, their terms including K and $\underset{0}{\delta v^\alpha}$ are dominant on the right-hand side of Eq. (A·5).

Inserting Eqs. (B·8) and (B·9) into Eq. (A·5), we obtain

$$L(Y) - \Delta Y \Big/ \Big[3\eta_*^{-2}(1+w)\Big(1+\frac{3}{4}w\Big)\Big] = \frac{C_a^2}{27}(n\eta_*)^5(w+1)^{-5/2}\Big(w+\frac{4}{3}\Big)^{-2}$$

$$\times \sin 2\Phi \Big[-\frac{1}{3}\Big(w-\frac{4}{3}\Big)\cos 2\theta + w + \frac{28}{9}\Big], \qquad (3\cdot13)$$

where we put $Q \equiv \sin\theta$, $P_\alpha = Q_{,\alpha}/n^2 = n^\alpha/n^2 \cdot \cos\theta$, $\theta \equiv n^\alpha x^\alpha + \theta_0$ (θ_0: a constant phase factor). The definition of a phase factor Φ is given in Eq. (B·7) of Appendix B.

By separating Y into two parts, i.e.,

$$Y = Y_0 + Y_1 \cos 2\theta ,$$

Eq. (3·13) is reduced to

$$L(Y_0) = \frac{C_a^2}{27}(n\eta_*)^5(w+1)^{-5/2}(w+4/3)^{-2}(w+28/9)\sin 2\Phi , \qquad (3\cdot14)$$

$$L(Y_1) + \frac{16}{9}(n\eta_*)^2(w+1)^{-1}(w+4/3)^{-1}Y_1$$

$$= -\frac{C_a^2}{81}(n\eta_*)^5(w+1)^{-5/2}(w+4/3)^{-2}(w-4/3)\sin 2\Phi . \qquad (3\cdot15)$$

Equation (3·14) is integrated as

$$Y_0 = -\frac{C_a^2}{48}(n\eta_*)^3(w+1)^{-3/2}(w+4/3)^{-1}(w+28/9)\sin 2\Phi .$$

On the other hand, if we insert $Y_1 = F \cos 2\Phi$ into Eq. (3·15), we get

$$F = \frac{C_a^2}{216}(n\eta_*)^4(w+1)^{-3/2}\Big(w+\frac{4}{3}\Big)^{1/4}(I+\text{const}),$$

where

$$I(w) \equiv \int_0^w dw \Big(w-\frac{4}{3}\Big)(w+1)^{-1/2}\Big(w+\frac{4}{3}\Big)^{-7/4}.$$

From these results, Y_0 proves to be negligible compared with Y_1, so that we have approximately $Y = Y_1 \cos 2\theta$. Therefore, from Eq. (A·4), we obtain

$$\phi = \frac{C_a^2}{288}(n\eta_*)^3(w+1)^{-1}\Big(w+\frac{4}{3}\Big)^{3/4}I(w)\cos 2\theta \sin 2\Phi + \text{h.s.} , \qquad (3\cdot16)$$

where homogeneous solutions (h.s.) take the form (B·8).

Here let us consider the spatial averages $\overline{K^2}$ and $\overline{\overset{}{\underset{i}{\delta}v^2}}$ for the squares of amplitudes (the coefficients of $\cos\Phi$) of K and $\overset{}{\underset{i}{\delta}}v^\alpha$ given by Eqs. (B·8) and (3·9). Then ϕ is rewritten as

$$\phi = \frac{1}{36}n\eta_*\overline{K^2}(w+1)(w+4/3)^{-3/4}I(w)\cos 2\theta \sin 2\Phi , \qquad (3\cdot17)$$

where we have $\overline{K^2} = \overline{(\underset{1}{\delta}v)^2}/c_s^2$. Moreover we get using $nw/w' = 2\pi/(LH)$

$$\phi/(\overline{K^2})^{1/2} = \frac{H^{-1}\{\overline{(\underset{1}{\delta}v)^2}\}^{1/2}}{L}J(w)\cos 2\theta \sin 2\Phi , \qquad (3\cdot18)$$

where

$$J(w) \equiv \frac{2\pi}{9}w^{-1}(w+1)^{1/2}(w+4/3)^{3/4}I(w) . \qquad (3\cdot19)$$

Because J is of the order of unity, the condition for applicability of the second-order approximation is defined by $\phi/(\overline{K^2})^{1/2} < 1$ and expressed as

$$H^{-1}(\overline{\underset{1}{\delta}v^2})^{1/2} < L . \qquad (3\cdot20)$$

The regions where this condition $(3\cdot20)$ and the condition $(3\cdot12)$ for rotational waves are satisfied are represented in Fig. 3.

Next we shall examine the second-order velocity field. From Eq. $(A\cdot2)$ we obtain

$$\underset{2}{\delta}v^\alpha = -\eta_*\left(w+\frac{4}{3}\right)^{-1}\left[\frac{4}{9}\int\sqrt{w+1}\,(w+4/3)^{-1}\phi_{,\alpha}dw\right.$$

$$\left. + \int\frac{w+4/3}{\sqrt{w+1}}(\underset{1}{\delta}v^\alpha\underset{1}{\delta}v^\beta)_{,\beta}dw + \eta_*^{-1}(w+1)(w+16/9)(w+4/3)^{-1}K\underset{1}{\delta}v^\alpha\right] , $$
$$(3\cdot21)$$

where only dominant terms are included. The first term including $\phi_{,\alpha}$ is gradient and irrotational. If we are concerned with monochronous waves (wave number n^α), the second term is also gradient, because $(P^\alpha P^\beta)_{,\beta} = \{(Q_{,\beta})^2 - n^2 Q^2\}_{,\alpha}/(2n^4)$. Even if we consider complex waves with many wave numbers, the situation is unchanged. From Eqs. $(3\cdot21)$ and $(B\cdot9)$, we get

$$\underset{2}{\delta}v^\alpha/\overline{(\underset{1}{\delta}v)^2} = -2n^\alpha\eta_*(w+4/3)^{1/2}\{\text{sh}^{-1}\sqrt{3(w+1)} - \text{sh}^{-1}\sqrt{3}\}\sin^{-2}\Phi \sin 2\theta ,$$

where an integration constant has been taken so as to exclude homogeneous terms in $\underset{2}{\delta}v^\alpha$ for $w \ll 1$. The condition of applicability given by $\underset{2}{\delta}v < \underset{1}{\delta}v$ is consistent with that in $(3\cdot20)$.

(iii) *Mixed waves*

We shall treat the case when in the first-order smallness there coexist rotational and acoustic waves which are interacting with each other. In this case, if we are concerned with one wave number n, the velocity field $\underset{1}{\delta}v^\alpha$ is given by

$$\underset{1}{\delta}v^\alpha = \frac{1}{3}n^2\eta_*[C_r(w+4/3)^{-1}S^\alpha - C_a(w+\tfrac{4}{3})^{-3/4}\sin \Phi P^\alpha] , \qquad (3\cdot22)$$

where constants C in the rotational and acoustic cases are discriminated by C_r, C_a, respectively. The contributions of this velocity field to the non-linear coupling consist of pure rotational parts, pure acoustic parts and the parts for the interaction between a rotational wave and an acoustic wave. The first two parts

are included in the treatments of (i), (ii), and so here we shall confine ourselves to the analysis of the last part which is indicated by a suffix ra in the following. Here the dominant ra term on the right-hand side of Eq. (A·5) is given by the term with

$$(\underset{1}{\delta} v^{\alpha}_{,\beta} \underset{1}{\delta} v^{\beta}_{,\alpha})_{ra} = -\tfrac{4}{9} n^4 \eta_*^2 C_r C_a (w+4/3)^{-7/4} \sin \varPhi (S^{\beta}_{,\alpha} P^{\beta}_{,\alpha}). \qquad (3\cdot23)$$

From Eq. (A·5) we get

$$L(Y_{ra}) - \varDelta Y_{ra} \Big/ \Big\{ 3\eta_*^{-2}(1+w)\Big(1+\tfrac{3}{4}w\Big)\Big\}$$

$$= -\frac{8}{27}(n\eta_*)^5 C_r C_a \{(w+1)^2(w+4/3)\}^{-5/4} \cos \varPhi S^{\alpha}_{,\beta} P^{\beta}_{,\alpha}, \qquad (3\cdot24)$$

where Y_{ra} denotes the ra part of Y and the right-hand side of the above equation has been derived approximately taking $c_s n \eta_* \gg 1$ into account.

When we separate $S^{\alpha}_{,\beta} P^{\beta}_{,\alpha}$ into $n(Q_0 + Q_{2n})$, Eq. (3·24) is solved as follows:

$$Y_{ra} = Y_{ra0} Q_0 + Y_{ra1} Q_{2n}, \qquad (3\cdot25)$$

$$\begin{cases} Y_{ra0} = \tfrac{1}{3}(n\eta_*)^3 C_r C_a (w+1)^{-2}(w+4/3)^{-3/4} \cos \varPhi, \\ Y_{ra1} = -\tfrac{1}{3} Y_{ra0}. \end{cases}$$

For ϕ_{ra}, we have from Eq. (A·4)

$$\phi_{ra} = \tfrac{1}{2}(n^3 \eta_*^2) C_r C_a (w+1)^{-3/2}(w+4/3)^{-1/4} \sin \varPhi (Q_0 - \tfrac{1}{3} Q_{2n}). \qquad (3\cdot26)$$

Moreover, we shall examine the second-order velocity field which consists of rotational part $(\underset{2}{\delta} v^{\alpha})_r$ and gradient part $(\underset{2}{\delta} v^{\alpha})_a$. From Eq. (A·2), we get

$$\Big\{\Big(w+\frac{4}{3}\Big)(\underset{2}{\delta} v^{\alpha})_a\Big\}_{,w} = -\frac{4}{9}\Big(\frac{w+1}{w+4/3}\Big)\phi_{,\alpha}/w', \qquad (3\cdot27)$$

$$\{(w+\tfrac{4}{3})(\underset{2}{\delta} v^{\alpha})_r\}_{,w} = (w+4/3) \{\text{rotational part of}$$

$$(\underset{1}{\delta} v^{\alpha} \underset{1}{\delta} v^{\beta})_{,\beta}\}/w' + (w+1)(w+16/9)(w+4/3)^{-1} \underset{1}{\delta} v^{\alpha} K_{,w}, \qquad (3\cdot28)$$

where only dominant terms are written on the right-hand sides. Here we shall derive the expression for $(\underset{2}{\delta} v^{\alpha})_r$. Inserting Eq. (3·22) into Eq. (3·28), we obtain

$$(\underset{2}{\delta} v^{\alpha})_r = -\frac{1}{3}(n^3 \eta_*^2) C_r C_a (w+4/3)^{-5/4} \cos \varPhi \Big\{(S^{\alpha} P^{\beta})_{,\beta} - \frac{1}{2}\Big(\frac{w+16/9}{w+4/3}\Big)S^{\alpha}Q\Big\}, \qquad (3\cdot29)$$

where we used $P^{\alpha} = Q_{,\alpha}/n^2$.

The contribution from only $(\underset{1}{\delta} v^{\alpha})_r$ to $(\underset{2}{\delta} v^{\alpha})_r$ is of higher-order smallness.

§ 4. Intermediate stage II. $(t \geq t_D)$

In this section we shall derive the expressions for the second-order density waves associated with rotational, density and their mixed waves after t_D and deal

with their junction with the counterparts before t_D.

Since matter and radiation have moved together before t_D, there remain the radiative perturbations soon after t_D in the perturbed region of matter. But the radiative perturbations have influence on matter through gravitational attraction which is negligibly small compared with hydrodynamical non-linear force. Therefore we shall confine our perturbation analysis to the matter part. Moreover, as an initial condition of the intermediate stage II, we shall assume that at t_D the density and velocity perturbations of matter have the same values as those of the matter part at the end of stage I. At stage I, the velocity of matter is the same as that of radiation, but the density of matter is related (by virtue of the assumption of adiabatic change) to the total density as follows (cf. Appendix A):

$$\underset{1}{\delta}\varepsilon_m/\varepsilon = (w/(w+4/3))\underset{1}{\delta}\varepsilon/\varepsilon\,,$$

$$\underset{2}{\delta}\varepsilon_m/\varepsilon = (w/(w+4/3))\underset{2}{\delta}\varepsilon/\varepsilon - \frac{2}{9}\frac{w(w+1)}{(w+4/3)^3}(\underset{1}{\delta}\varepsilon/\varepsilon)^2.$$

(i) *Second-order density wave associated with rotational waves*

As long as we adopt the viewpoint ①, rotational motions have associated acoustic waves almost free from the original rotational motions. Hence their behavior can be analyzed in the next subsections (ii) and (iii). We shall therefore deal with the second-order quantities from the viewpoint ②.

Inserting the expressions for the first-order rotational perturbation (B·14) $\sim$ (B·16) into Eq. (A·9) and taking into account that only the term including $\underset{1}{\delta}v^\alpha_{,\beta}\underset{1}{\delta}v^\beta_{,\alpha}$ is dominant under the condition $c_s nw/w'$ ($\equiv c_s n\eta_* w/\sqrt{w+1}$)$\gg 1$, we get

$$L(Y) = n^6 R_1(Q_0 + Q_{2n}),\qquad(4\cdot1)$$

where R_1 is defined in Eq. (3·5) and $S^\alpha_{,\beta}S^\beta_{,\alpha}$ is replaced by $n^2(Q_0 + Q_{2n})$. Hence, contrary to the case before t_D (§ 3 (i)), Y_{2n} satisfies the same differential equation as that for Y_0, where $Y = Y_0 Q_0 + Y_{2n}Q_{2n}$. As a result, Y_{2n} and ϕ_{2n} will increase promptly after t_D by a factor $\sim(c_s nw/w')^2$, so as to become comparable with Y_0 and ϕ_0, respectively. In fact, we have the following solutions for ϕ_{2n}:

$$\phi_{2n}Q_{2n} = \begin{cases} -(n\eta_* w)^2\overline{(\underset{1}{\delta}v)^2}b_1 + \text{h.s.} & \text{for } w \gg 1\,, \\ -\frac{1}{2}(n\eta_* w)^2\overline{(\underset{1}{\delta}v)^2}(\frac{5}{4} - \ln w)b_1 + \text{h.s.} & \text{for } w \ll 1\,, \end{cases}\qquad(4\cdot2)$$

where h.s. represents homogeneous solutions for density waves given by Eq. (B·12) and $b_1 = Q_{2n}/\overline{S^\alpha S^\alpha} \sim 1$.

For $\underset{2}{\delta}v^\alpha$, we obtain from Eqs. (A·7) and (B·16)

$$\underset{2}{\delta}v^\alpha/\overline{(\underset{1}{\delta}v)^2} = 2\sqrt{3}n\eta_*(w+4/3)\tan^{-1}(\sqrt{3(w+1)})S^\alpha_{,\beta}S^\beta/(n\overline{S^\mu_{,\mu}S^\mu}) + \text{h.s.}\,,$$

$$(4\cdot3)$$

where h.s. is a homogeneous part determined by joining the above $\underset{2}{\delta}v^\alpha$ with $\underset{2}{\delta}v^\alpha$ before t_D. A comparison with Eq. (3·11) shows that the amount of $\underset{2}{\delta}v^\alpha$ does

not much change after t_D.

Now let us consider the junction of perturbed motions before and after t_D. Linearized solutions for rotational perturbations are common before and after t_D and continuous at t_D. For the second-order quantities, the part of Q_0 is continuous at t_D as it is, and so we have only to treat the part of Q_{2n}. Therefore we equate at t_D the values of the matter part of ϕ_{2n} given by Eqs. (3·10) and (4·2), and the values of $\underset{2}{\delta}v^\alpha$ given by Eqs. (3·11) and (4·3). These two equations determine the additive homogeneous solutions (h.s.), or the constants B and C_α appearing in Eqs. (B·12) and (B·13). Another constant D is related to the definition of density perturbation itself and cannot be determined uniquely. Here and in the following, D is taken to be zero. As a result of the junction, we get

$$
\phi_{2n} =
\begin{cases}
\left(\dfrac{\overline{(\underset{1}{\delta}v)^2}}{c_s^2}\right)_D \left[\dfrac{1}{4} + (c_s n \eta_* \sqrt{w})_D^2 (w/w_D - 1) + \dfrac{8\sqrt{3}}{15}(c_s n \eta_* w)_D \right. \\
\qquad \left. \times \{\tan^{-1}\sqrt{3(w+1)} + h_1\}(\sqrt{w_D/w} - w/w_D)\right] b_1 \quad \text{for } w_D(\equiv 12.8\Omega) \gg 1, \\[4mm]
\left(\dfrac{\overline{(\underset{1}{\delta}v)^2}}{c_s^2}\right)_D \left[\dfrac{1}{4} w_D\left(\dfrac{w_D}{w}\right) + \dfrac{1}{2}(c_s n \eta_* w)_D^2 \right. \\
\qquad \left. \times \left\{\dfrac{w_D}{w}\left(\dfrac{5}{4} - \ln w_D\right) - \left(\dfrac{w}{w_D}\right)^2\left(\dfrac{5}{4} - \ln w\right)\right\}\right] b_1 \quad \text{for } w_D \ll 1,
\end{cases}
$$

$$
\underset{2}{\delta}v^\alpha / \overline{(\underset{1}{\delta}v)^2} = 2\sqrt{3}(w + 4/3)\left[\{\tan^{-1}\sqrt{3(w+1)} + h_1\}S^{\alpha}_{,\beta}S^\beta \right.
$$
$$
\left. + \{\tan^{-1}\sqrt{3(w_D+1)} + h_1\}Q_{2n,\alpha}\right]/(n\overline{S^\mu S^\mu}).
$$

From this expressions, we find that during an interval between $w = w_D$ and $2w_D$, ϕ increases by a factor $\sim (c_s n w/w')_D^2$.

(ii) *Second-order density wave associated with a density wave (acoustic wave)*

Let us consider the second-order density waves associated with the first-order wave given by Eqs. (B·11)~(B·13). In this case, the dominant term on the right-hand side of Eq. (A·10) is that including $\underset{1}{\delta}v^\alpha_{,\beta}\underset{1}{\delta}v^\beta_{,\alpha}$, so that Eq. (A·9) leads to

$$
L(Y) = \dfrac{1}{2}\eta_*^2\left(w_D + \dfrac{4}{3}\right)\dfrac{w^3 + 6w + 16/3}{w(w+1)^3(w+4/3)^2}(\underset{1}{\delta}v^\alpha_{,\beta}\underset{1}{\delta}v^\beta_{,\alpha})_D, \qquad (4·4)
$$

where we have put

$$
\underset{1}{\delta}v^\alpha = (\underset{1}{\delta}v^\alpha)_D(w_D + 4/3)/(w + 4/3). \qquad (4·5)
$$

Now we must solve Eq. (4·4) so as to satisfy the junction condition at t_D. In the case of $w_D \gg 1$, we have

$$
Y = -\tfrac{1}{2}\eta_*^2 w_D w^{-2}(\underset{1}{\delta}v^\alpha_{,\beta}\underset{1}{\delta}v^\beta_{,\alpha})_D + \text{h.s.},
$$

so that

$$
\phi = -\eta_*^2(w_D/w)(\underset{1}{\delta}v^\alpha_{,\beta}\underset{1}{\delta}v^\beta_{,\alpha})_D + \text{h.s.} \qquad (4·6)
$$

Here, from Eq. (B·9), we get

$$(\underset{1}{\delta} v^{\alpha}_{,\beta} \underset{1}{\delta} v^{\beta}_{,\alpha})_D = n^2 \overline{(\underset{1}{\delta} v)}_D^2 b_2 ,$$

where $b_2 \equiv P^{\alpha}_{,\beta} P^{\beta}_{,\alpha} / (n^2 \overline{P^{\alpha} P^{\alpha}})$ is a function of spatial variables of the order of unity.
For $\underset{2}{\delta} v^{\alpha}$, we get from Eqs. (A·7), (B·12) and (B·13)

$$\underset{2}{\delta} v^{\alpha} = \frac{1}{162} \widetilde{C}_a^{\,2} (n^2 n^{\alpha} \eta_*^3) \left(w + \frac{4}{3} \right)^{-1} \Big[2\sqrt{3} \tan^{-1} \sqrt{3(w+1)}$$

$$- \frac{1}{30} \left(w + \frac{4}{3} \right)^{-1} \Big\{ \frac{11}{w\sqrt{w+1}} - 18\sqrt{w+1} + \frac{16}{\sqrt{w+1}} + 2\check{K} \ln (\sqrt{w+1} + 1)$$

$$- \Big(\check{K} + \frac{16}{w\sqrt{w+1}} \Big) \ln w + 2(B/\widetilde{C}_a)\check{K} \Big\} \Big] \sin 2(n^{\alpha} x^{\alpha} + \theta_0) + \text{h.s.} \qquad (4·7)$$

Now let us deal with the junction at $w = w_D \gg 1$. First we must join the linearized solutions. By equating the values of K and $\underset{1}{\delta} v^{\alpha}$ at both stages, we get

$$B/\widetilde{C}_a = -\tfrac{1}{6} w_D^{-3/2} [1 + 15(n\eta_*)^{-1} \cot \Phi_D], \qquad (4·8)$$

$$\widetilde{C}_a/C_a = -3 w_D^{1/4} \sin \Phi_D .$$

In the same way, by equating the values of ϕ and $\underset{2}{\delta} v^{\alpha}$ at both stages, we obtain the following expressions.

The first-order wave:

$$K = - \frac{\sqrt{2}}{10} (\sqrt{\overline{(\underset{1}{\delta} v)^2}} / c_s)_D \Big[(c_s n \eta_* \sqrt{w})_D \Big\{ \sqrt{\frac{w_D}{w}} - \Big(\frac{w}{w_D} \Big) \Big\} - 10 \Big(\frac{w}{w_D} \Big) \cot \Phi_D \Big] Q ,$$

$$\underset{1}{\delta} v^{\alpha} = (\underset{1}{\delta} v^{\alpha})_D w_D / w . \qquad (\overline{(K)}_D^2 = (\overline{(\underset{1}{\delta} v)^2}/c_s^2)_D \cot^2 \Phi_D) \qquad (4·9)$$

The second-order wave:

$$\phi = \frac{1}{3} (c_s n \eta_* \sqrt{w})_D \Big(\frac{(\underset{1}{\delta} v)^2}{c_s^2} \Big)_D w_D^{-1} \Big[\sin 2\Phi_D \cos 2\theta$$

$$+ 3 (c_s n \eta_* \sqrt{w})_D \Big[\frac{w_D}{10} \{ (\text{sh}^{-1} \sqrt{3(w_D+1)} - \text{sh}^{-1} \sqrt{3}) / \sin^2 \Phi$$

$$+ \sqrt{3} \tan^{-1} \sqrt{3(w_D+1)} \cdot \sqrt{w_D} \} \Big(\sqrt{\frac{w_D}{w}} - \frac{w}{w_D} \Big) + \Big(\frac{w}{w_D} - \frac{w_D}{w} \Big) \Big] b_2 \Big] ,$$

$$\underset{2}{\delta} v^{\alpha} = 2 \overline{(\underset{1}{\delta} v)}_D^2 (n^{\alpha} \eta_* w_D) \Big[\frac{w_D}{w} \tan^{-1} \sqrt{3(w+1)} - \tan^{-1} \sqrt{3(w_D+1)}$$

$$- w_D \Big(\frac{w_D}{w} \Big) \text{sh}^{-1} \sqrt{3(w_D+1)} / \sin^2 \Phi_D \Big] \sin 2\theta . \qquad (4·10)$$

From these results we find that, during an interval between $w = w_D$ and $2w_D$, both density contrasts increase by a factor $\sim c_s n w / w'$. For $\underset{2}{\delta} v^{\alpha}$, the amount does

not much change.   For $w_D \ll 1$ also, the analogous results are derived, as in the case of rotational waves, but they are omitted here.

(iii) *Mixed waves*

If rotational and acoustic waves coexist in the first-order smallness, the velocity field $\overset{\alpha}{\underset{1}{\delta}v}{}^{\alpha}$ is given by

$$\overset{}{\underset{1}{\delta}v}{}^{\alpha} = \tfrac{1}{3}(n^2\eta_*)(w+4/3)^{-1}(C_r S^{\alpha} - \tfrac{1}{3}C_a P^{\alpha})$$

$$= (\overset{}{\underset{1}{\delta}v}{}^{\alpha})_D(w_D + 4/3)/(w + 4/3), \tag{4·11}$$

where C_r, C_a denote the constants C in the rotational and acoustic cases, respectively.   Similarly to the previous cases (i) and (ii), ϕ and $\overset{}{\underset{2}{\delta}v}{}^{\alpha}$ can be derived.

Here we shall examine the second-order velocity field.   From Eq. (A·7), we obtain

$$\overset{}{\underset{2}{\delta}v}{}^{\alpha} = -\eta_*(w+4/3)^{-1}\int dw\,(w+4/3)\,(w+1)^{-1/2}(\overset{}{\underset{1}{\delta}v}{}^{\alpha}\overset{}{\underset{1}{\delta}v}{}^{\beta})_{,\beta}$$

$$= -\frac{1}{27}n^4\eta_*{}^3(w+4/3)^{-1}\int dw\,(w+4/3)^{-1}(w+1)^{-1/2}$$

$$\times \left[\frac{1}{3}C_a{}^2(P^{\alpha}P^{\beta})_{,\beta} - C_r C_a(S^{\alpha}P^{\beta})_{,\beta}\right]. \tag{4·12}$$

The rotational part $(\overset{}{\underset{2}{\delta}v}{}^{\alpha})_r$ of $\overset{}{\underset{2}{\delta}v}{}^{\alpha}$ comes from the term including $(S^{\alpha}P^{\beta})_{,\beta}$:

$$(\overset{}{\underset{2}{\delta}v}{}^{\alpha})_r = \frac{2\sqrt{3}}{27}n^4\eta_*{}^3(w+4/3)^{-1}\tan^{-1}(\sqrt{3(w+1)})C_r C_a(S^{\alpha}P^{\beta})_{,\beta} + \text{h.s.} \tag{4·13}$$

In order to determine h.s. $(\propto (w+4/3)^{-1})$, let us join $(\overset{}{\underset{2}{\delta}v}{}^{\alpha})_r$ before and after t_D. From Eqs. (3·29) and (4·14), we have

$$\text{h.s.} = -\frac{1}{3}n^3\eta_*{}^2 C_r C_a\left(w + \frac{4}{3}\right)^{-1}(S^{\alpha}P^{\beta})_{,\beta} \tag{4·14}$$

$$\times \left[(w_D+4/3)^{-1/4}\cos\varPhi_D + \frac{2\sqrt{3}}{9}n\eta_*\tan^{-1}(\sqrt{3(w_D+1)})\right],$$

so that

$$\frac{(\overset{}{\underset{2}{\delta}v}{}^{\alpha})_r}{(\overset{}{\underset{1}{\delta}v})_r{}^2} = n\eta_*\left[\frac{2\sqrt{3}}{9}n\eta_*(\tan^{-1}\sqrt{3(w+1)} - \tan^{-1}\sqrt{3(w_D+1)})\right.$$

$$\left. - (w_D+4/3)^{-1/4}\cos\varPhi_D\right]C_a b_3,$$

$$= \frac{\sqrt{(\overset{}{\underset{2}{\delta}v})_a{}^2}}{c_s}(w+4/3)^{1/2}\left[-\frac{4}{\sqrt{3}}n\eta_*(\tan^{-1}\sqrt{3(w+1)} - \tan^{-1}\sqrt{3(w_D+1)})\right.$$

$$\left. + 6(w_D+3/4)^{-1/4}\cos\varPhi_D\right]b_4, \tag{4·15}$$

where $b_3 \equiv (S^\alpha P^\beta)_{,\beta}/\sqrt{S^\alpha S^\alpha} \sim 1$, $b_4 \equiv b_3/\sqrt{n^2 P^\alpha P^\alpha} \sim 1$. From this it follows that $\underset{2}{\delta} v_r \sim \underset{1}{\delta} v_a (nw/w') \cdot \underset{1}{\delta} v_r$ after t_D, and we find from Eqs. (3·29) and (4·13) that by a factor $\sim c_s nw/w'$ rotational motion can be amplified by an acoustic wave.

§ 5. Clues to galaxy formation

The deviations of the universe from homogeneity and isotropy consist of three kinds of perturbations which are of the forms of density (acoustic), rotational and gravitational waves. In this section we shall examine the possibility of galaxy formation from these waves on the basis of the results in the previous sections.

(i) *From rotational waves*

First let us follow the course of formation from rotational waves. As indicated in § 2, the behavior of the second-order density waves at $n\eta \sim 2$ are distinguished in two ways. For type 1 waves, their treatment is reduced to that in the next case for acoustic waves. For type 2 waves, ϕ does not oscillate with time after $n\eta = 2$, and decreases slowly in proportion to $(w+4/3)/(w+1)$ till t_D.

After t_D, the densities of the surviving associated density waves are amplified during the interval $w = w_D \sim 2w_D$ by a factor $\sim c_s nw/w'$ for ϕ of type 1 and by a factor $\sim (c_s nw/w')^2$ for ϕ of type 2. Here this factor has the following value:

$$(c_s nw/w')_D^2 = 5.0 \times 10^4 \frac{w_D^2 \Omega^{-4/3}}{(w_D+1)(w_D+4/3)} \left(\frac{M}{10^{10} M_\odot}\right)^{-2/3}$$

$$\simeq (5.0 \times 10^4 \Omega^{-4/3}, 6.1 \times 10^6 \Omega^{2/3}) \left(\frac{M}{10^{10} M_\odot}\right)^{-2/3} \qquad (5\cdot1)$$

for $\Omega >, < 10^{-1.1}$, respectively, where $w_D = 12.8\Omega$. In the mass range $M < 10^{9.8} \Omega^{-11/4} M_\odot$, $10^{12.2} \Omega^{-1/2} M_\odot (\Omega >, < 10^{-1.1}$, respectively), the rotational waves dissipate at t_D, so that the maximum value of amplification factor for type 2 is given by $6.8 \times 10^4 \Omega^{1/2}$, $2.4 \times 10^5 \Omega$ for $\Omega >, < 10^{-1.1}$. For type 1, the maximum value is given by $8.0 \times 10^2 \Omega^{-1/2}$, $6.0 \times 10^4 \Omega$, because the acoustic waves dissipate at t_D in the range $M < 10^{12.7} \Omega^{-5/4} M_\odot$, $10^{13.1} \Omega^{-1/2} M_\odot$, respectively. In Table I, the values of $\phi \equiv \underset{2}{\delta}\varepsilon/\varepsilon$ directly before t_D and after the amplification are shown by means of the parameters of M, Ω, $(\underset{1}{\delta} v/c)_i$.

On the other hand, the value of $\underset{2}{\delta} v^\alpha$ does not change its order of magnitude after t_D, but a translational velocity develops which is comparable with the rotational part of $\underset{2}{\delta} v^\alpha$. This translational velocity corresponds to the increase of density.

Now let us consider what conditions are imposed on the velocities of inhomogeneities from observational evidence.

a) If we assume the rotational and peculiar velocities of existing galaxies to be $v_g = 10^{7.5}$ cm/sec and the formation time to be $\varepsilon_m = 10^{-24}$ g/cm³, their velocities
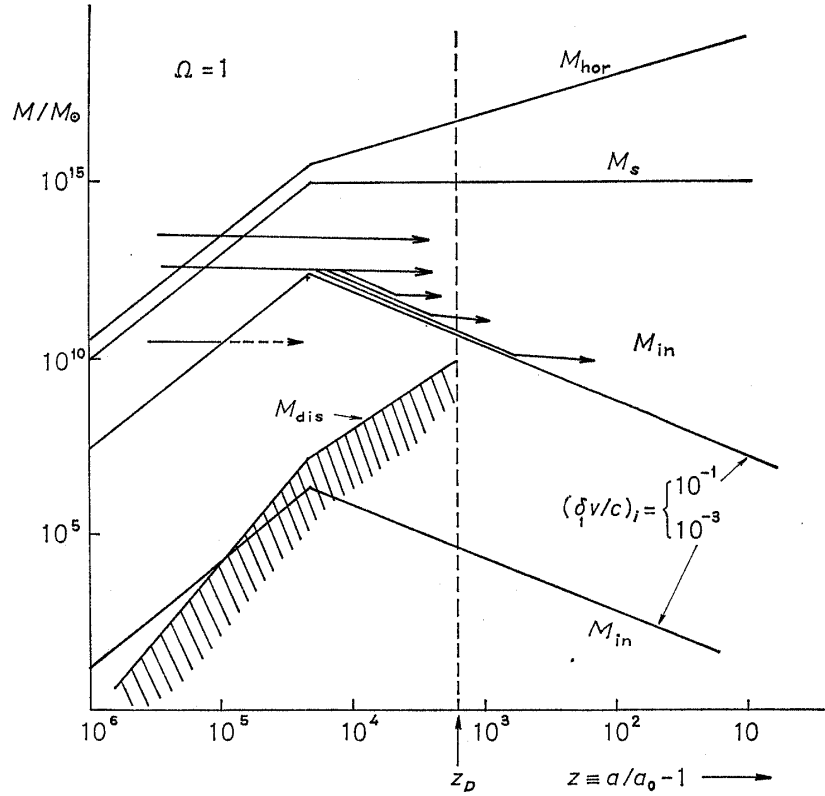
Fig. 3(a). Characteristic masses for rotational waves. M_{hor}, M_s, M_{in} are the masses of matter within the spheres with radius $L=ct$, $c_s t$, $(\delta v)_i t$, respectively, and M_{dis} is the mass for dissipation. Vertical lines denote the decoupling epoch. Horizontal lines with arrows are the evolutionary paths of several masses with $(\delta v/c)_i = 1/10$.

Table I.

	ϕ (type 1)	ϕ (type 2)
$n\eta=2$	$(\delta v/c)_i^2$	$(\delta v/c)_i^2$
Directly before t_D	$(w_D+1)^{-1}(w_D+4/3)^{3/4}(\delta v/c)_i^2$	$(w_D+1)^{-1}(w_D+4/3)\,(\delta v/c)_i^2$
After amplification	$2.2\times10^2\left(\dfrac{w_D\sqrt{w_D+4/3}}{(w_D+1)^3}\right)^{1/2}$ $\times\left(\Omega^2\dfrac{M}{10^{10}M_\odot}\right)^{-1/3}(\delta v/c)_i^2$	$5.0\times10^4(1+1/w_D)^{-2}$ $\times\left(\Omega^2\dfrac{M}{10^{10}M_\odot}\right)^{-2/3}(\delta v/c)_i^2$

v_t at t_D lead[6] to

$$(v_t)_i/c = ((v_t)_D/c)\,3w_D/4 \simeq 0.44\,\Omega^{4/3}. \tag{5·2}$$

Since, in the above calculation, we have not taken into account the increase of velocities due to the contraction, the value is an upper limit.

b) The motions with high speed at t_D disturb the background temperature T_r of cosmic microwave radiation. Considering the upper limit from the measurements,[12] we have the condition

$$(v_t)_D/c \lesssim \mathrm{Max}(\Delta T_r/T_r) = 10^{-3},$$

or

$$(v_t)_i/c \lesssim (9.6\Omega, 1) \times 10^{-3} \qquad (5\cdot3)$$

for $\Omega >, < 10^{-1.1}$, respectively.

Under these conditions, what behavior of rotational waves and associated waves will be allowed? We shall analyze them in the following alternative cases such as $\Omega > 10^{-1.1}$ or $< 10^{-1.1}$. In the case of $\Omega > 10^{-1.1}$, their behavior is shown on the diagram in Fig. 3(a). The rotational waves which survive till t_D are those passing over the mountain, which are specified by $H^{-1} \underset{i}{\delta}v < L$. If they enter into the mountain, they cascade owing to full non-linear effect of inertial force and further are reduced to thermal motion. In particular, the waves which enter into the neighbourhood of the top of the mountain will not cascade completely and may be able to run away from the right hand of the mountain. Moreover, if the following condition is satisfied, i.e.,

$$(\underset{i}{\delta}v/c)_i \gtrsim 0.05\Omega^{1/4}, \qquad (5\cdot4)$$

the right skirt of the mountain is over the barrier of dissipation. Then some eddies will cascade not only till t_D, but also after t_D continuously and run away from the skirt towards the outside of the mountain. Comparing the above condition (5·4) with (5·3) for $v_t = \underset{i}{\delta}v$, we find that such a situation is on the critical point of compatibility. In that situation, therefore, the effectiveness of dissipation comes into severe question. If dissipation is not effective in the top region of the barrier and the above situation has been realized, much interesting results are yielded. That is to say, if the eddies cascading till t_D at the skirt are turbulent, the size spectrum will be close to the characteristic turbulent spectrum which can be approximated by Kolmogorov's one.[8],[9] That spectrum is frozen after running away. The spectrum of the eddies which cascade after t_D may be $v \propto L^{1/3+n}$ ($n = 0 \sim 1$), as was estimated by Weizsäcker,[9] and Ozernoi and Chibisov[5] for supersonic turbulence. Thus, under the condition (5·4), even the eddies with masses smaller than the characteristic mass limit given by the dissipation barrier may survive and dwarf galaxies will be formed from the associated density waves of such eddies.

In the case of $\Omega < 10^{-1.1}$, the analogous situation arises when the top of the mountain is higher than the barrier of dissipation, i.e., some eddies can arrive at the top of the mountain without being dissipated. This condition is given by

$$(\underset{i}{\delta}v/c)_i \gtrsim 0.1\Omega^{1/2}. \qquad (5\cdot5)$$

In this case, even if the eddies survived till t_D and were amplified after t_D, they

may or may not enter into the mountain and cascade completely. In order to pass over the top or enter into the neighbourhood of the top, they must have the mass

$$M \gtrsim 1.3 \times 10^{16} \Omega^{-2} (\overset{}{\underset{1}{\delta}} v/c)_i^3 M_\odot . \tag{5·6}$$

If the condition (5·4) or (5·5) is not satisfied, the situation is quite different, since the birth of comparatively small galaxies at $t \gtrsim t_D$ is impossible. However, let us consider here under what condition the second-order density wave can lead to a gravitationally bound system at least. Because the wave must have at least $\phi \gtrsim 10^{-2}$ in order to be bound at $t < 10^{-1} t_0$, we find from Table I that

$$(\overset{}{\underset{1}{\delta}} v/c)_i \gtrsim \begin{cases} 6.7 \times 10^{-3} \left(\dfrac{(w_D+1)^3}{w_D{}^2 \sqrt{w_D+3/4}} \right)^{1/4} \Omega^{1/3} \left(\dfrac{M}{10^{10} M_\odot} \right)^{1/6} & \text{for type 1}, \\[3mm] 4.5 \times 10^{-4} (1+1/w_D) \Omega^{2/3} \left(\dfrac{M}{10^{10} M_\odot} \right)^{1/3} & \text{for type 2}. \end{cases} \tag{5·7}$$

As for the upper limit of masses, we have no clear cut. However, in order that the jump at t_D is effective, we must have $L < c_s H^{-1}$, i.e., $M < 1.8 \times 10^{15} \Omega^{-2} M_\odot$, $8.1 \times 10^{16} \Omega M_\odot$ for $\Omega >$, $< 10^{-1,1}$, and the amplification factor is larger, if the mass is smaller.

(ii) *From acoustic waves*

For acoustic (density) waves we can examine the possibility of galaxy formation in parallel with the case of rotational waves. In the present case, however, the first-order waves themselves have density perturbation $K(\sim \overset{}{\underset{1}{\delta}} v/c)$ and they are amplified by a factor $\sim c_s n w / w'$ soon after t_D as well as the second-order waves. Moreover, if weak rotational waves coexist with acoustic waves, they are amplified at the same time as the acoustic waves by interacting each other, and as a result large rotational motions may be raised. Such rotational velocities increase by a factor $\sim c_s n w / w'$ after t_D. In Table II, the values of K, ϕ and $\overset{}{\underset{2}{\delta}} v_r{}^\alpha$ are shown.

The behavior of acoustic waves is explained on the diagram in Fig. 3(b). In the case of $\Omega > 10^{-1,1}$, the first-order acoustic waves (with $c_s n w / w' \gg 1$ before t_D) can survive till t_D only when they satisfy the condition $\overset{}{\underset{1}{\delta}} v H^{-1} < L$, i.e., they pass over the mountain. If they enter into the mountain, they will cascade to smaller waves owing to non-linear inertial force. Only when they arrive at the neighbourhood of the top, they will cascade partially and run away partially from the right skirt of the mountain. Moreover, if the right skirt is above the dissipation barrier $M = 10^{12·7} \Omega^{-5/4} M_\odot$, i.e.,

$$(\overset{}{\underset{1}{\delta}} v/c)_i \gtrsim 0.2 \Omega^{1/2}, \tag{5·8}$$

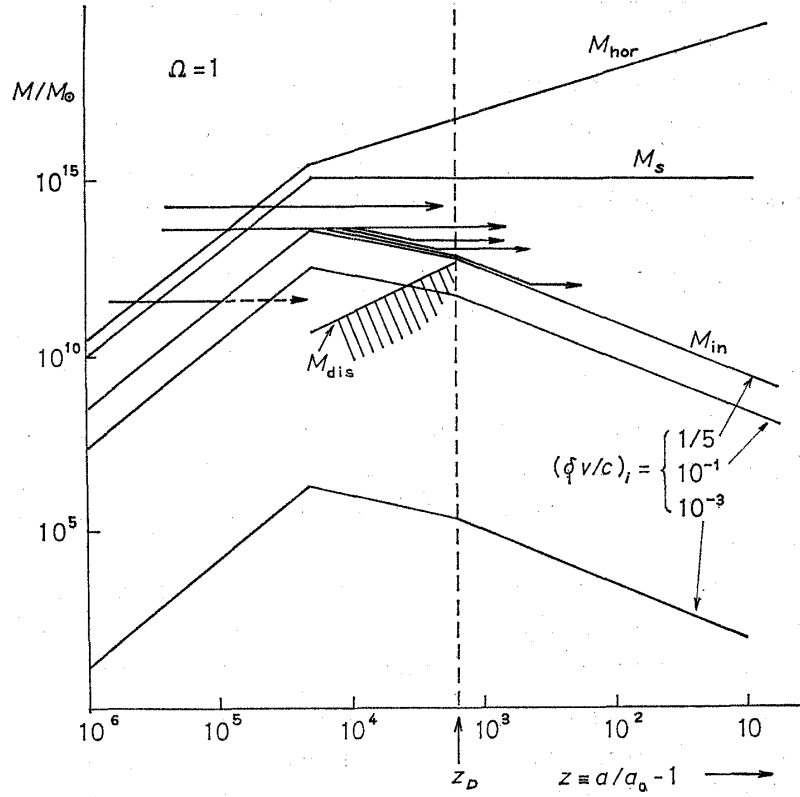it is possible that the acoustic waves are cascading from the neighbourhood of

Fig. 3(b). Characteristic masses for acoustic waves. The evolutionary paths for $(\delta v/c)_i = 1/5$ are represented by the horizontal lines with arrows. The definitions of M_{hor}, M_s, M_{in} and M_{dis} are the same as those in Fig. 3(a).

Table II.

	K	$\dfrac{\phi}{K(\delta v/c)_i}$	δv_r
$n\eta = 2$	$(\delta v/c)_i$	1	$(\delta v_r)_i$
Directly before t_D	$(w_D+1)^{-1}(w_D+4/3)^{3/4}(\delta v/c)_i$	$I(w_D)^{a)}$	$(w_D+4/3)^{-5/4}(\delta v_r)_i$
After amplification	$2.2\times10^2\left(\dfrac{w_D\sqrt{w_D+4/3}}{(w_D+1)^3}\right)^{1/2}$ $\times\left(\Omega^2\dfrac{M}{10^{10}M_\odot}\right)^{-1/3}(\delta v/c)_i$	$I(w_D)$	$2.2\times10^2 w_D(w_D+1)^{-1/2}$ $\times(w_D+4/3)^{-7/4}$ $\times\left(\Omega^2\dfrac{M}{10^{10}M_\odot}\right)^{-1/3}(\delta v_r)_i$

a) Cf. § 3 (ii).

the top beyond the dissipation barrier at t_D and run away in the supersonic region. This situation may explain the formation of comparatively small galaxies.

In the case of $\Omega < 10^{-1.1}$, the top of the mountain is on right-hand side of a line for decoupling. If the top is higher than the dissipation barrier, i.e., the same condition as (5·8) is satisfied, the waves which arrive at the neighbourhood

of the top can thereafter cascade partially and run away partially from the right skirt of the mountain.

However, the above possibilities seem to be more difficult than for rotational waves, when we compare Eq. (5·8) with Eq. (5·3). Under another situation such that the condition (5·8) is not satisfied, the amplified value of K soon after t_D should exceed 10^{-2} at least. Then we get from Table II

$$(\overset{\circ}{_1}v/c)_i > 5.7 \times 10^{-6} \left(\frac{(w_D+1)^3}{w_D\sqrt{w_D+4/3}} \right)^{1/2} \Omega^{1/6} \left(\frac{M}{10^{10}M_\odot} \right)^{1/3}. \qquad (5·9)$$

(iii) *From gravitational waves*

The second-order density perturbation due to gravitational waves leads to $\simeq 0.05 \langle h_\alpha{}^\beta h_\beta{}^\alpha \rangle_i$ at $n\eta = 2$, as was shown in § 4 of [II]. Now the behavior of the associated density waves at $n\eta \simeq 2$ is distinguished in two ways, as in the case of rotational waves.

For type 1 waves, their density waves behave like free acoustic waves and the amplitudes decrease slowly as $(w+1)^{-1}(w+4/3)^{3/4}$. Soon after t_D, the amplitudes increase promptly by a factor $\sim c_s nw/w'$. Therefore the amplified value of ϕ is about $10^2 \{w_D\sqrt{w_D+4/3}/(w_D+1)^3\}^{1/2} \Omega^{-1/6} (M/10^{10}M_\odot)^{-1/3} \langle h_\alpha{}^\beta h_\beta{}^\alpha \rangle_i$. It should be noticed here that the associated waves with $M = 10^{12.7}\Omega^{-5/4}M_\odot$, $10^{13.1}\Omega^{-1/2}M_\odot$ (for $\Omega >$, $< 10^{-1.1}$) are cut out due to dissipation. Accordingly, in order that the amplified value leads to 10^{-2}, we must have $\langle h_\alpha{}^\beta h_\beta{}^\alpha \rangle_i > 0.074$ at least for $\Omega = 1$.

For type 2 waves, the amplitude of the associated wave decreases rapidly after $n\eta = 2$, as has been shown in § 2. However, as was concluded in § 4 of [II], the possibility of the formation through the latter course is not excluded.

(iv) *After being bound gravitationally*

Under the initial condition that the fluid motions at an early stage are at an extremely high speed (so that for instance the condition (5·4), (5·5) or (5·8) is satisfied), the relative density perturbations soon after t_D will reach unity and non-linear force of gravitation will act on them strongly. As a result, the gas in those perturbed regions is bound gravitationally and tends to contract soon. However, gas clouds thus formed cannot collapse to a final state. As they contract, the gas temperature rises owing to heating due to compression and dissipation of waves, so that they are ionized and opaque to radiation confined within themselves. Then the gradient of radiation pressure prevents them from contracting and makes them expand again to the maximum size. Thereafter such gravitationally bound gas clouds will continue to oscillate till an epoch t_1 or t_2 (defined below) when the pressure gradient becomes ineffective. The epoch t_1 is given by

$$z = a/a_* = 100 \, (M/10^{10}M_\odot)^{1/6}\Omega^{1/3},$$

for which Jeans's wavelength of ionized gas and radiation with $T_m = 10^4$ °K is equal to the radius of clouds with mass M. Another epoch t_2 is given by $z = 200 (M/10^{10} M_\odot)^{-1/4} \Omega^{-1/2}$, for which the mean free path $m_p/(\varepsilon_m \sigma_T)$ at $T_m = 10^4$ °K is equal to the radius of clouds (σ_T is Thomson's scattering cross section).

Acknowledgements

Appendix A

Hydrodynamical equations at the intermediate stage

(i) *Before the decoupling epoch t_D*

For adiabatic disturbance, we get

$$\delta \varepsilon_r / \varepsilon_r = \tfrac{4}{3} \delta \varepsilon_m / \varepsilon_m$$

from the first law of thermodynamics, so that

$$\underset{1}{\delta} p = (dp/d\varepsilon)_s \underset{1}{\delta} \varepsilon = \tfrac{1}{3}(1 + \tfrac{3}{4} w)^{-1} \underset{1}{\delta} \varepsilon ,$$

$$\underset{2}{\delta} p = (dp/d\varepsilon)_s \underset{2}{\delta} \varepsilon + \tfrac{1}{2}(d^2 p/d\varepsilon^2)_s (\underset{1}{\delta} \varepsilon)^2$$

$$= \tfrac{1}{3}(1 + \tfrac{3}{4} w)^{-1} \underset{2}{\delta} \varepsilon + \tfrac{1}{32} w (1 + \tfrac{3}{4} w)^{-3}(1 + w)(\underset{1}{\delta} \varepsilon)^2 / \varepsilon .$$

Inserting these into Eqs. (2·2) $\sim$ (2·4) in [II], we obtain

$$(w+1)\phi_{,w} - \frac{1/3}{w+4/3}\phi + (w+4/3)\left\{(\underset{2}{\delta} v^\alpha)_{,\alpha}/w' + \frac{1}{2} l_{,w}\right\} + A = 0, \qquad \text{(A·1)}$$

$$\left\{(w+4/3)\underset{2}{\delta} v^\alpha\right\}_{,w} + \frac{4(w+1)}{9(w+4/3)}\phi_{,\alpha}/w' + B^\alpha = 0, \qquad \text{(A·2)}$$

$$l_{,ww} + \left(\frac{1}{2(w+1)} + \frac{1}{w}\right) l_{,w} + \frac{3}{w^2}\left(\frac{w+8/3}{w+4/3}\right)\phi + C = 0, \qquad \text{(A·3)}$$

where

$$A \equiv \frac{2}{9}\frac{(w+1)^2}{(w+4/3)^3}K^2 + \left\{(w+4/3)(\underset{1}{\delta} v)^2\right\}_{,w} + \frac{(w+16/9)(w+1)}{w+4/3}K_{,\alpha}\underset{1}{\delta} v^\alpha/w'$$

$$+ \frac{1}{2}(w+1)\frac{(w+16/9)}{w+4/3}K h_{,w} + \frac{1}{2}(w+4/3)\underset{1}{\delta} v^\alpha h_{,\alpha}/w' - \frac{1}{4}(w+4/3)(h_\mu{}^\nu h_\nu{}^\mu)_{,w} ,$$

$$B^\alpha \equiv \frac{w+4/3}{w'}(\underset{1}{\delta} v^\alpha \underset{1}{\delta} v^\beta)_{,\beta} + \frac{2}{27}\frac{w(w+1)^2}{(w+4/3)^3}(K^2)_{,\alpha}/w'$$

$$+ \left\{\frac{(w+1)(w+16/9)}{w+4/3}\underset{1}{\delta} v^\alpha K + (w+4/3)h_\nu{}^\alpha \underset{1}{\delta} v^\nu\right\}_{,w}$$

$$+ \frac{1}{2}(w+4/3)h_{,w}\underset{1}{\delta}v^{\alpha} - \frac{2(w+1)}{9(w+4/3)}h_{,w}K/w',$$

$$C \equiv -h_{\mu}{}^{\nu}\left\{(h_{\nu}{}^{\mu})_{,ww} + \left(\frac{1}{2(w+1)} + \frac{1}{w}\right)(h_{\nu}{}^{\mu})_{,w}\right\} - \frac{1}{2}(h_{\mu}{}^{\nu})_{,w}(h_{\nu}{}^{\mu})_{,w}$$

$$+ \frac{3}{w^2(w+1)}\left\{3(w+4/3)(\underset{1}{\delta}v)^2 + \frac{2w(w+1)^2}{9(w+4/3)^2}K^2\right\}.$$

Here ϕ, K stand for $\underset{2}{\delta}\varepsilon/\varepsilon$, $\underset{1}{\delta}\varepsilon/\varepsilon$.

Eliminating $\underset{2}{\delta}v^{\alpha}$, l from Eqs. (A·1)~(A·3), we obtain

$$\phi = \left(\frac{w^2\sqrt{w+1}}{w+4/3}\right)^{-1}\int Y\frac{w^2\sqrt{w+1}}{w+4/3}dw, \tag{A·4}$$

where Y satisfies an inhomogeneous second-order differential equation

$$L(Y) - \Delta Y/\{3(w')^2(1+\tfrac{3}{4}w)\} = w^{-2}(w+1)^{-3/2}(w^2\psi_1)_{,w} + \psi_2/w \tag{A·5}$$

with

$$L \equiv \partial_{ww} + \frac{7/2}{w+1}\partial_w - \frac{1}{w(w+1)^2},$$

$$\psi_1 \equiv (w+1)^{1/2}\Bigg[(w+4/3)\underset{1}{\delta}v^{\alpha}_{,\beta}\underset{1}{\delta}v^{\beta}_{,\alpha}/(w')^2 + \frac{2w(w+1)^2}{27(w+4/3)^3}\frac{\Delta(K^2)}{(w')^2}$$

$$+ \left\{\frac{(w+1)(w+16/9)}{w+4/3}K\underset{1}{\delta}v^{\alpha} + (w+4/3)h_{\nu}{}^{\alpha}\underset{1}{\delta}v^{\nu}\right\}_{,w\alpha}/w'$$

$$+ \frac{1}{2}(w+4/3)(h_{,w}\underset{1}{\delta}v^{\alpha})_{,\alpha}/w' - \frac{2}{9}\left(\frac{w+1}{w+4/3}\right)(h_{,\alpha}K)_{,\alpha}/(w')^2$$

$$- \Bigg[(w+1)^{1/2}\Bigg[\{(w+4/3)(\underset{1}{\delta}v)^2\}_{,w} + \frac{1}{2}(w+4/3)h_{,\alpha}\underset{1}{\delta}v^{\alpha}/w'$$

$$+ \frac{(w+1)(w+16/9)}{2(w+4/3)}Kh_{,w} - \frac{1}{4}(w+4/3)(h_{\mu}{}^{\nu}h_{\nu}{}^{\mu})_{,w}$$

$$+ \frac{2(w+1)^2}{9(w+4/3)^3}K^2 + \left\{\frac{(w+1)(w+16/9)}{w+4/3}K\underset{1}{\delta}v^{\alpha}\right\}_{,\alpha}/w'\Bigg]\Bigg]_{,w}$$

$$- (w+1)^{3/2}(w+4/3)\psi_2,$$

$$\psi_2 \equiv \frac{3/4}{w+1}\Bigg[\frac{1}{2}h_{\mu}{}^{\nu}\left\{(h_{\nu}{}^{\mu})_{,ww} + \frac{3(w+2/3)}{2w(w+1)}(h_{\nu}{}^{\mu})_{,w}\right\} + \frac{1}{4}(h_{\mu}{}^{\nu})_{,w}(h_{\nu}{}^{\mu})_{,w}$$

$$- \frac{3(w+4/3)}{2w^2(w+1)}\left\{(\underset{1}{\delta}v)^2 + \frac{w(w+1)^2}{27(w+4/3)^3}K^2\right\}\Bigg].$$

(ii) *After the decoupling epoch t_D*

We consider only matter part of the disturbances, so that $\underset{1}{\delta}\varepsilon = \underset{1}{\delta}\varepsilon_m$, $\underset{2}{\delta}\varepsilon = \underset{2}{\delta}\varepsilon_m$

and $\underset{1}{\delta}p = \underset{2}{\delta}p = 0$ are assumed in the following.

From Eqs. $(2 \cdot 2) \sim (2 \cdot 4)$ in [II], we obtain

$$(w+1)\phi_{,w} - \phi/w + (w+4/3)\left[(\underset{2}{\delta}v^{\alpha})_{,\alpha}/w' + \tfrac{1}{2}l_{,w}\right] + A = 0, \qquad (A \cdot 6)$$

$$\{(w+4/3)\underset{2}{\delta}v^{\alpha}\}_{,w} + B^{\alpha} = 0, \qquad (A \cdot 7)$$

$$l_{,ww} + \tfrac{3}{2}(w+2/3)w^{-1}(w+1)^{-1}l_{,w} + 3w^{-2}\phi + C = 0, \qquad (A \cdot 8)$$

where

$$A \equiv \{(w+4/3)(\underset{1}{\delta}v)^{2}\}_{,w} + (w+1)K_{,\alpha}\underset{1}{\delta}v^{\alpha}/w' + \tfrac{1}{2}(w+1)Kh_{,w}$$
$$+ \tfrac{1}{2}(w+4/3)\underset{1}{\delta}v^{\alpha}h_{,\alpha}/w' - \tfrac{1}{4}(w+4/3)(h_{\mu}{}^{\nu}h_{\nu}{}^{\mu})_{,w},$$

$$B^{\alpha} \equiv (w+4/3)(\underset{1}{\delta}v^{\alpha}\underset{1}{\delta}v^{\beta})_{,\beta}/w' + \{(w+1)K\underset{1}{\delta}v^{\alpha} + (w+4/3)h_{\nu}{}^{\alpha}\underset{1}{\delta}v^{\nu}\}_{,w}$$
$$+ \tfrac{1}{2}(w+4/3)h_{,w}\underset{1}{\delta}v^{\alpha},$$

$$C \equiv -h_{\mu}{}^{\nu}\{(h_{\nu}{}^{\mu})_{,ww} + \tfrac{3}{2}(w+2/3)w^{-1}(w+1)^{-1}(h_{\nu}{}^{\mu})_{,w}\} - \tfrac{1}{2}(h_{\mu}{}^{\nu})_{,w}(h_{\nu}{}^{\mu})_{,w}$$
$$+ 9(\underset{1}{\delta}v)^{2}(w+4/3)/\{w^{2}(w+1)\}.$$

Eliminating $\underset{2}{\delta}v^{\alpha}$, l from Eqs. $(A \cdot 6) \sim (A \cdot 8)$, we get

$$\phi = (w\sqrt{w+1})^{-1}\int Yw\sqrt{w+1}\,dw, \qquad (A \cdot 9)$$

where Y satisfies

$$L(Y) = w^{-2}(w+1)^{-3/2}(w^{2}\psi_{1})_{,w} + \psi_{2}/3 \qquad (A \cdot 10)$$

with

$$\psi_{1} \equiv (w+1)^{1/2}\Big[(w+4/3)\underset{1}{\delta}v^{\alpha}_{,\beta}\underset{1}{\delta}v^{\beta}_{,\alpha}/w' + \{(w+1)K\underset{1}{\delta}v^{\alpha}$$
$$+ (w+4/3)h_{\nu}{}^{\alpha}\underset{1}{\delta}v^{\nu}\}_{,w\alpha} + \tfrac{1}{2}(w+4/3)(h_{,w}\underset{1}{\delta}v^{\alpha})_{,\alpha}/w'$$
$$- \big[(w+1)^{1/2}\big[\{(w+4/3)(\underset{1}{\delta}v)^{2}\}_{,w} + \tfrac{1}{2}(w+4/3)h_{,\alpha}\underset{1}{\delta}v^{\alpha}/w'$$
$$+ \tfrac{1}{2}(w+1)h_{,w}K - \tfrac{1}{4}(w+4/3)(h_{\mu}{}^{\nu}h_{\nu}{}^{\mu})_{,w}\big]\big]_{,w} - (w+1)^{3/2}(w+4/3)\psi_{2},$$

$$\psi_{2} \equiv \tfrac{3}{8}(w+1)^{-1}\Big[h_{\mu}{}^{\nu}\{(h_{\nu}{}^{\mu})_{,ww} + \tfrac{3}{2}(w+2/3)w^{-1}(w+1)^{-1}(h_{\nu}{}^{\mu})_{,w}\}$$
$$+ \tfrac{1}{2}(h_{\mu}{}^{\nu})_{,w}(h_{\nu}{}^{\mu})_{,w} - 9(w+4/3)w^{-2}(w+1)^{-1}(\underset{1}{\delta}v)^{2}\Big].$$

If we use $\tilde{Y} \equiv w^{-1}(w+1)^{1/2}Y$, $y \equiv w+1$ in place of Y, w, the homogeneous equation corresponding to Eq. $(A \cdot 10)$ is reduced to

$$y(1-y)\tilde{Y}_{,yy} + \tfrac{1}{2}(5-9y)\tilde{Y}_{,y} - \tfrac{3}{2}\tilde{Y} = 0,$$

which was derived and solved by Nariai, Tomita and Kato.[13]

Appendix B

Perturbations in the linear approximation at the intermediate stage

In this Appendix we shall present at the relevant stage the solutions of Lifshitz's equations for the first-order perturbations, which are used as a basis

to our derivation of the second-order non-linear quantities.

(i) *Acoustic (density) perturbations*

Following Lifshitz's procedure, put the metric perturbation as

$$h_\alpha{}^\beta = \lambda P_\alpha{}^\beta + \mu Q_\alpha{}^\beta, \tag{B·1}$$

where

$$\varDelta Q = -n^2 Q, \qquad Q_\alpha{}^\beta \equiv \tfrac{1}{3}\delta_\alpha{}^\beta Q, \qquad P_\alpha{}^\beta \equiv n^{-2}Q_{,\alpha\beta} + \tfrac{1}{3}\delta_\alpha{}^\beta Q.$$

Then we have equations for λ and μ:

$$\lambda_{,ww} + \frac{2(w+4/5)}{w(w+1)}\lambda_{,w} - \frac{1}{3}(n\eta_*)^2(\lambda+\mu)/(w+1) = 0,$$

$$\mu_{,ww} + \left\{\frac{3(w+2/3)}{2w(w+1)} + \frac{2}{w}\left(1 + \frac{3}{2}(dp/d\varepsilon)_s\right)\right\}\lambda_{,w}$$

$$+ \frac{1}{3}(n\eta_*)^2(\lambda+\mu)(1+3(dp/d\varepsilon)_s)/(w+1) = 0. \tag{B·2}$$

If we define ξ, ζ by $\lambda+\mu = (\lambda_0+\mu_0)\int\xi d\eta$, $\lambda'-\mu' = (\lambda_0'-\mu_0')\int\xi d\eta + \zeta/a$ with $\lambda_0 \equiv -n^2\int d\eta/a$, $\mu_0 \equiv n^2\int d\eta/a - 3a'/a^2$, we have the following equations for ξ and ζ:

$$\xi_{,w} + \xi\left\{\frac{1}{w+1} - \frac{2}{w}\left(1 - \frac{3}{2}(dp/d\varepsilon)_s\right)\right\} + \frac{1}{2}\eta_*\zeta(dp/d\varepsilon)_s/\sqrt{w+1} = 0, \tag{B·3}$$

$$\zeta_{,w} + \frac{\zeta}{w}\left(1 + \frac{3}{2}(dp/d\varepsilon)_s\right) - \frac{2}{\eta_*}\sqrt{w+1}\left[(n\eta_*)^2/(w+1)\right.$$

$$\left. - \frac{3/4}{w(w+1)} + 3w^{-2}\left\{1 - \frac{3}{4}(dp/d\varepsilon)_s\right\}\right] = 0,$$

where $(dp/d\varepsilon)_s = \frac{4}{9}(w+4/3)^{-1}$, 0 before and after the decoupling epoch t_D, respectively. Moreover, we have

$$K \equiv \underset{1}{\delta}\varepsilon/\varepsilon = \tfrac{1}{9}w^2\{(n\eta_*)^2(\lambda+\mu)/(w+1) + 3\mu_{,w}/w\}Q, \tag{B·4}$$

$$\underset{1}{\delta}v^\alpha = \tfrac{1}{9}n^2\eta_*(w+1)^{1/2}(w+4/3)^{-1}(\mu+\lambda)_{,w}P^\alpha, \tag{B·5}$$

where $P^\alpha \equiv Q_{,\alpha}/n^2$ and the definition of w is given in § 1.

Before t_D

By canceling ζ from Eq. (B·1), we obtain

$$\xi_{,ww} + \tfrac{3}{2}(w+1)^{-1}\xi_{,w} + \left[\tfrac{4}{9}(w+1)^{-1}(w+4/3)^{-1}\{(n\eta_*)^2 + 9/16\} - \tfrac{3}{4}w^{-2}\right.$$

$$+ \tfrac{1}{4}(w+4/3)^{-2} - \tfrac{1}{2}(w+1)^{-2} - \tfrac{3}{2}w^{-1}(w+4/3)^{-1} + \tfrac{3}{4}w^{-1}(w+1)^{-1}$$

$$\left. - \tfrac{1}{3}\{w(w+1)(w+4/3)\}^{-1} + \tfrac{4}{3}w^{-2}(w+4/3)^{-1} - \tfrac{4}{9}w^{-2}(w+4/3)^{-2}\right]\xi = 0. \tag{B·6}$$

If we assume $c_s nw/w' \gg 1$, we get an approximate solution of Eq. (B·6):

$$\xi = C_a a_* (w+1)^{-1/2} (w+4/3)^{1/4} \sin \Phi ,$$

$$\Phi = \tfrac{4}{3} n\eta_* \ln\{\sqrt{w+1} + \sqrt{w+4/3}\} + \text{const} , \tag{B·7}$$

and ζ is given by the first line of Eq. (B·3).

Using Eq. (B·2), therefore, we obtain

$$\lambda + \mu = \frac{9}{2} \frac{C_a}{n\eta_*} w^{-2} (w+4/3)^{3/4} \cos \Phi ,$$

$$\mu/(\lambda+\mu) = \tfrac{1}{8}(12w^2 + 29w + 16)/(w+1)$$

so that we obtain

$$K = \tfrac{1}{2} C_a n\eta_* (w+1)^{-1} (w+4/3)^{3/4} \cos \Phi \, Q , \tag{B·8}$$

$$\underset{1}{\delta} v^\alpha = -\tfrac{1}{3} C_a n^2 \eta_* (w+4/3)^{-3/4} \sin \Phi \, P^\alpha , \tag{B·9}$$

$$h_\alpha{}^\beta = \frac{3}{2} C_a (n\eta_*)^{-1} w^{-2} (w+4/3)^{3/4} \cos \Phi \left[\delta_\alpha{}^\beta Q - \frac{3(12w^2 + 21w + 8)}{8(w+1)} Q_{,\alpha\beta}/n^2 \right] . \tag{B·10}$$

After t_D

Directly solving Eq. (B·2), we obtain

$$\lambda + \mu = -\widetilde{C}_a \left[\sqrt{w+1}/w + \frac{1}{2} \ln \frac{\sqrt{w+1}-1}{\sqrt{w+1}+1} \right] + B , \tag{B·11}$$

$$\mu = \frac{1}{45} (n\eta_*)^2 \left[\widetilde{C}_a \left\{ (3w - 8/w - 4\ln w - 9/2) \ln \frac{\sqrt{w+1}-1}{\sqrt{w+1}+1} \right. \right.$$

$$-4(\ln w)^2 + 8(\ln w + 1)/w + \frac{1}{6}(37w + 100/w)\sqrt{w+1}$$

$$\left. -8 \int \ln w/(w+1) \cdot dw \right\} + 2B(3w - 8/w - 4\ln w)$$

$$\left. + 15D \left[\sqrt{w+1}/w + \frac{1}{2} \ln\{(\sqrt{w+1}-1)/(\sqrt{w+1}+1)\} \right] \right] ,$$

where B, $\widetilde{C}_a$ and D are integration constants. Inserting these expressions into Eqs. (B·4) and (B·5), we get

$$K = \frac{1}{270} (n\eta_*)^2 Q \left[\widetilde{C}_a \{(26 + 11/w)/\sqrt{w+1} - 18\sqrt{w+1} \right.$$

$$+ 2\check{K} \ln(\sqrt{w+1}+1) - (\check{K} + 16/(w\sqrt{w+1}))\ln w\}$$

$$+ 2B\check{K} - 30D/(w\sqrt{w+1})] , \tag{B·12}$$

$$\underset{1}{\delta} v^\alpha = \tfrac{1}{9} n^2 \eta_* \widetilde{C}_a (w+4/3)^{-1} P^\alpha , \tag{B·13}$$

where

$$\check{K} = 9w - 7 + 15/(w+1) - 16/w .$$

For $w \gg 1$, we have

$$K = \frac{1}{270}(n\eta_*)^2 Q \left[\frac{3}{\sqrt{w}} (1 + 0(w^{-1})) \tilde{C}_a + 18w(1 + 0(w^{-1})) B \right.$$

$$\left. - 30w^{-3/2}(1 + 0(w^{-1})) D \right].$$

(ii) *Rotational perturbations*

The metric perturbation is expressed by means of a vector harmonic as

$$h_\alpha{}^\beta = \sigma S_\alpha{}^\beta \qquad\qquad (B \cdot 14)$$

with

$$\sigma = C_r \left[2\sqrt{w+1}/w + \ln\left(\frac{\sqrt{w+1}-1}{\sqrt{w+1}+1}\right) \right], \qquad (B \cdot 15)$$

and the velocity perturbation is given by

$$\underset{1}{\delta} v^\alpha = -\tfrac{1}{3} C_r \eta_* \varDelta S^\alpha / (w + 4/3). \qquad\qquad (B \cdot 16)$$

This results are independent of whether the epoch is before or after t_D.

In the place of the above harmonics we can express S^α by $S^\alpha \equiv \varepsilon^{\alpha\mu\nu} F_{\mu,\nu}$ $(S_\alpha{}^\beta \equiv S_{,\beta}^\alpha + S_{,\alpha}^\beta)$, where F_μ is arbitrary functions of spatial variables.

(iii) *Gravitational waves*

The metric perturbation is expressed by means of a tensor harmonics as

$$h_\alpha{}^\beta = \nu G_\alpha{}^\beta, \qquad\qquad (B \cdot 17)$$

where ν satisfies the wave equation

$$\nu_{,ww} + \frac{5(w+4/5)}{2w(w+1)} \nu_{,w} + \frac{(n\eta_*)^2}{w+1} \nu = 0.$$

The approximate solution of this equation for $n(w+1)/w' \gg 1$ is given by

$$\nu = C_g w^{-2} \sin[2n\eta_* \sqrt{w+1} + \text{const.}], \qquad\qquad (B \cdot 18)$$

which is independent of whether the epoch is before or after t_D.

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Note added in proof:

After the completion of this manuscript, we have seen Peebles' paper in Astrophys. and Space Science **11** (1971), 443. There Peebles has asserted that, at the stage $z \simeq 1000$, the contraction of compressed clouds cannot be stopped by pressure gradient because of the free-free cooling process. However, it should be noticed that, after the matter temperature T_m reaches collisional ionization temperature 10^4 °K, the mean free path of photons in fully ionized plasma becomes smaller than the radius of clouds for $M = 10^{10} M_\odot$ (say). Therefore these clouds become opaque to radiation and the pressure gradient of trapped radiation can be effective. Further discussions will be given in a forthcoming paper.