The short-time and long-time behaviors of non-Markovianity measure using two-time correlation functions in open quantum systems

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We investigate non-Markovianity measure using two-time correlation functions for open quantum systems. We define non-Markovianity measure as the difference between the exact two-time correlation function and the one obtained in the Markov limit. Such non-Markovianity measure can easily be measured in experiments. We found that the non-Markovianity dynamics in different time scale crucially depends on the system-environment coupling strength and other physical parameters such as the initial temperature of the environment and the initial state of the system. In particular, we obtain the short-time and long-time behaviors of non-Markovianity for different spectral densities. We also find that the thermal fluctuation always reduce the non-Markovian memory effect. Also, the non-Markovianity measure shows non-trivial initial state dependence in different time scales.

I. INTRODUCTION

Markov approximation [1–4] which completely ignores the memory effects between the system and its environments is widely used in the study of open quantum systems. However, in quantum information processing, many quantum devices exhibit *non-Markovian* (memory) behaviors which prohibits the use of Markov approximation. In particular, non-Markovian memory effects are typically significant for structured environments at low temperature in the strong system-environment coupling regime. Various concepts of non-Markovianity were introduced recently to define the border between Markovian and non-Markovian quantum evolution, although the very definition of non-Markovianity is still a debated issue. Many different measures of *non-Markovianity* have been proposed in the literature to quantify memory effects in open systems, based on, for examples, the divisibility of dynamical map [5–8], the distinguishability of states [9–12], quantum entanglement [13–17], quantum Fisher information [18–21], mutual information [22–25], geometrical characterization [8, 26] and the decay rate of the master equation itself [27, 28]. Predominately, almost all these measures of non-Markovianity are introduced in terms of mathematical quantities. Most of them do not directly related to physically measurable observables, although some of them are experimentally measured through reconstruction of the output state in terms of density matrix, using quantum state tomography [11, 29]. Also, some of these measures even quantitatively diverge from each other [25, 30, 31].

Recently, a general theory of non-Markovian dynamics is developed for open quantum systems of bosons (fermions) interacting with a general bosonic (fermionic) environments through particle-tunneling processes [32], where exact dynamics of open quantum systems is explored through the nonequilibrium Green's functions which account all the information of non-Markovian back-action memory effects. The relation between the exact master equation and the nonequilibrium Green's functions manifests the fundamental importance of correlation functions in the description of the non-Markov dynamics for open systems. Therefore, non-Markovianity measure extracted from correlation functions in time-domain has a special significance. In particular, two-time correlation functions are experimentally measurable. For examples, the two-time correlation functions of the electromagnetic field emitted by an atom can be measured through fluorescence spectrum [1–4]; the two-time correlation functions of the number of emitted photons give the information about the photon statistics and describe the behavior of photon bunching and antibunching [1–4]; two-time correlation functions have also been measured in optical measurements and light-harvesting photosynthesis for testing the non-Markovian memory effects [33, 34]; the two-time correlation functions of the electric current through nanostructure devices are used in quantum transport to study the current fluctuations and noise spectrum [35–37]. Intuitively, two-time correlation functions correlating a past event with its future provide useful information about the system-environment back-action processes revealing the non-Markovian memory effects.

In this paper, we introduce a non-Markovianity measure using two-time correlation functions. In open quantum systems, when the Markov approximation [1–4] is valid, two-time correlation functions are usually calculated using quantum regression theorem where the reservoirs are assumed to be always in an equilibrium state. Our non-Markovianity measure in terms of two-time correlation functions is defined as the departure of the exact dynamics of the two-time correlation function from the one obtained through quantum regression theorem in the Markov limit. Our definition of non-Markovianity measure is valid for arbitrary finite temperature of the

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environment. We also show that the non-Markovianity defined in this way is a function of time and therefore the non-Markovianity measure is a dynamical quantity, which is physically expected. The non-Markovianity defined in this method also depends crucially on various system and environment parameters and on the nature of system-environment coupling. As a specific example, we take an open system with one dynamical degree of freedom coupled to a general non-Markovian environment containing infinite degrees of freedom described by the famous Fano-Anderson model [38, 39]. Then the two-time correlation function we used to define the non-Markovianity measure is just the first-order coherence function in quantum optics, which can be easily measured in experiments [1–4, 35–37].

The rest of the paper is organized as follows. In Sec. II A, we consider our open system described by the Fano-Anderson model, where we show that the exact two-time correlation functions of system operators can be obtained through exact master equation associated with the nonequilibrium Green's functions. In Sec. II B, we calculate the two-time correlation function in Markov limit using the quantum regression theorem. We define our non-Markovianity measure in Sec. II C using two-time correlation functions, where we also include a comparative discussion of our definition with other relevant non-Markovianity measure. In Sec. III, we present our numerical results and discussions, the dynamics of our non-Markovianity measure for different system-environment coupling, different initial temperature of the environment, and different initial states of the system. Our results also show non-trivial interesting non-Markovian features on the dependence of the system and environmental parameters, in particular, the existence of both the short-time and long-time behavior of the non-Markovianity. Finally, a conclusion is given in Sec. IV.

II. THE NON-MARKOVIANITY MEASURE BASED ON TWO-TIME CORRELATION FUNCTION

A. Two-time correlation function obtained from the exact master equation

We consider an open system described by the famous Fano-Anderson Hamiltonian [38, 39]

$$H = \hbar\omega_0 a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + \sum_k \hbar V_k \left(a^\dagger b_k + a b_k^\dagger \right), \quad (1)$$

where $a^\dagger$ and a are the creation and annihilation operators of a single frequency mode. A general non-Markovian environment is modeled as a collection of infinite modes, and $b_k^\dagger$ and b_k are the corresponding creation and annihilation operators of the k-th mode with frequency ω_k. The parameter V_k is the tunneling amplitude between the system and its environment. The

model Hamiltonian (1) has wide applications in atomic physics and condensed matter physics [40, 41].

We consider initially the environment be at thermal equilibrium. Tracing over all the environmental degrees of freedom using the influence functional approach [42, 43] in the coherent state representation [44], the exact master equation of such open systems has been derived [45–47]:

$$\begin{aligned}
\frac{d}{dt}\rho(t) = &- i\omega_0'(t) \left[a^\dagger a, \rho(t) \right] \\
&+ \gamma(t) \left[2a\rho(t)a^\dagger - a^\dagger a \rho(t) - \rho(t) a^\dagger a \right] \\
&+ \widetilde{\gamma}(t) \left[a\rho(t)a^\dagger + a^\dagger \rho(t)a - a^\dagger a \rho(t) - \rho(t)aa^\dagger \right],
\end{aligned} \quad (2)$$

The time-dependent coefficients in the master equation are fully determined by the nonequilibrium Green's functions $u(t, t_0)$ and $v(t, t)$, which satisfy the following Dyson equation of motion and the non-equilibrium fluctuation-dissipation theorem [32], respectively,

$$\dot{u}(t, t_0) + i\omega_0 u(t, t_0) + \int_{t_0}^t d\tau g(t, \tau) u(\tau, t_0) = 0, \quad (3)$$

$$v(t, t + \tau) = \int_{t_0}^t d\tau_1 \int_0^{t+\tau} d\tau_2 u(t, \tau_1) \tilde{g}(\tau_1, \tau_2) u^*(t + \tau, \tau_2). \quad (4)$$

The integral kernels in Eqs. (3) and (4) characterize all the non-Markovian back-action between the system and the reservoir, and can be determined by the spectral density $J(\omega)$ of the reservoir through the relations: $g(t, \tau) = \int_0^\infty \frac{d\omega}{2\pi} J(\omega) e^{-i\omega(t-\tau)}$, and $\tilde{g}(\tau_1, \tau_2) = \int_0^\infty \frac{d\omega}{2\pi} J(\omega) \bar{n}(\omega, T) e^{-i\omega(\tau_1-\tau_2)}$, where the spectral density is defined by $J(\omega) = 2\pi \sum_k |V_k|^2 \delta(\omega - \omega_k)$ and $\bar{n}(\omega, T) = \frac{1}{e^{\hbar\omega/k_B T}-1}$ is the particle number distribution of the bosonic reservoir.

The functions $u(t, t_0)$ and $v(t, t + \tau)$ are indeed related to the two-time correlation functions $\langle [a(t), a^\dagger(t_0)] \rangle$ and $\langle a^\dagger(t + \tau)a(t) \rangle$ respectively, which are the two basic nonequilibrium Green's functions in the Schwinger-Keldysh theory [32]. These two-time correlation functions can also be calculated directly from equation of motion approach [48, 49]. In fact, all two-time correlation functions of system operators can be constructed in terms of the above nonequilibrium Green's functions. For example, as we have shown in our previous works [46, 50], the exact two-time correlation function $\langle a^\dagger(t)a(t+\tau) \rangle$ can be expressed in terms of $u(t, t_0)$ and $v(t, t + \tau)$ as follows,

$$\langle a^\dagger(t)a(t + \tau) \rangle_{\mathcal{E}} = u^*(t, t_0) n(t_0) u(t + \tau, t_0) + v^*(t, t + \tau). \quad (5)$$

The exact analytic solution of the integro-differential equation (3) is recently given in [32],

$$u(t, t_0) = \mathcal{Z} e^{-i\omega_b(t-t_0)} + \int_0^\infty \frac{J(\omega) e^{-i\omega(t-t_0)}}{[\omega - \omega_0 - \Delta(\omega)]^2 + \gamma^2(\omega)} \frac{d\omega}{2\pi}, \quad (6)$$

where $\Delta(\omega) = \mathcal{P} \int_0^\infty \frac{J(\omega')}{\omega-\omega'} \frac{d\omega'}{2\pi}$ is a principal-value integral and $\gamma(\omega) = J(\omega)/2$, which are the real and imaginary parts of the self-energy correction to the system induced by the system-environment coupling,

$$\Sigma(\omega \pm i0^+) = \int_0^\infty \frac{d\omega'}{2\pi} \frac{J(\omega')}{\omega - \omega' \pm i0^+} = \Delta(\omega) \mp i\gamma(\omega). \,(7)$$

The first term in Eq. (6) is the contribution of the dissipationless localized mode, which exists only when the environment has band-gap structures. The localized mode frequency ω_b is determined by the pole condition $\omega_b - \omega_0 - \Delta(\omega_b) = 0$, and $\mathcal{Z} = [1 - \Sigma'(\omega_b)]^{-1}$ corresponds to the residue at the pole, which gives the amplitude of the localized mode. Thus, the two-time correlation functions can be exactly calculated from the exact solution of Eq. (6) through Eq. (4).

B. Two-time correlation functions in Markov limit using quantum regression theorem

Two-time correlation functions can easily be obtained through the *quantum regression theorem* defined in the Markov limit [51] when the back-action or memory effect is totally negligible. Quantum regression theorem states that in the Markov limit, evolution equations of the two-time correlation functions of system observables is the same as the evolution equation for single-time expectation values of the observables [2, 51]. The single time expectation values of a complete set of system operators $\hat{O}_i$, can be calculated using the master equation (2)

$$\frac{\partial}{\partial t} \langle \hat{O}_i(t) \rangle = \mathrm{Tr}_S \left[\hat{O}_i(0) \frac{\partial}{\partial t} \rho(t) \right] = \sum_j M_{ij} \langle \hat{O}_j(t) \rangle, \,(8)$$

where the evolution matrix elements M_{ij} are determined from the master equation (see Appendix-A).

According to quantum regression theorem, the evolution equation (8) for single-time expectation values is also valid for the two-time correlation functions in the Markov limit:

$$\frac{\partial}{\partial \tau} \langle \hat{O}_1(t) \hat{O}_i(t+\tau) \rangle = \sum_j M_{ij} \langle \hat{O}_1(t) \hat{O}_j(t+\tau) \rangle. \,(9)$$

The operator $\hat{O}_1$ can be any system operator, not necessarily one of the $\{\hat{O}_i\}$, and M_{ij} is the same matrix given in Eq. (8).

The Markov master equation for the open system of Eq. (1) is well known in the literature [2] and it has the same form as the exact master equation (2). However, the coefficients in the Markov master equation become time-independent, and can be derived from the exact master equation (2) using a perturbation expansion up to the second order and then taking the Markov limit [45]. The result is,

$$\gamma(t) \rightarrow \gamma = J(\omega_0)/2, \,(10)$$
$$\widetilde{\gamma}(t) \rightarrow \widetilde{\gamma} = J(\omega_0)\bar{n}(\omega_0, T), \,(11)$$
$$\omega_0'(t) \rightarrow \omega_0 + \delta\omega_0, \,(12)$$

with $\delta\omega_0 = \mathcal{P} \int_0^\infty \frac{d\omega}{2\pi} \frac{J(\omega)}{(\omega-\omega_0)}$. Using this Markov master equation, one can obtain the evolution matrix M_{ij} in the complete set of system operators $(a, a^\dagger, a^\dagger a, \mathbb{1})$:

$$\mathbf{M} = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta^* & 0 & 0 \\ 0 & 0 & -2\gamma & \tilde{\gamma} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \,(13)$$

where $\beta = -(\gamma + i\omega_0')$. By choosing the operator $\hat{O}_1 = a^\dagger$, the evolution equation for the two-time correlation function $\langle a^\dagger(t)a(t+\tau) \rangle$ based on quantum regression theorem is given by

$$\frac{\partial}{\partial \tau} \langle a^\dagger(t)a(t+\tau) \rangle_{\mathcal{M}}$$
$$= -(\gamma + i\omega_0') \langle a^\dagger(t)a(t+\tau) \rangle_{\mathcal{M}}, \,(14)$$

which can be solved explicitly with the solution

$$\langle a^\dagger(t)a(t+\tau) \rangle_{\mathcal{M}}$$
$$= \left[n(t_0)e^{-2\gamma t} + \bar{n}(\omega_0, T)\left(1 - e^{-2\gamma t}\right) \right] e^{-(\gamma + i\omega_0')\tau}. \,(15)$$

Here "$\mathcal{M}$" means that the correlation function is calculated in the Markov limit through the quantum regression theorem. In the next subsection, we will use the exact two-time correlation functions and the one in the Markov limit to define the non-Markovianity measure.

C. Definition of non-Markovianity and its measure

Our definition of the non-Markovianity measure quantifies the departure of the exact dynamics of two-time correlation function from the one obtained in the Markov limit. Thus, non-Markovianity measure in terms of two-time correlation functions is defined by

$$\mathcal{N}(t, \tau) = |g_\mathcal{E}(t, \tau) - g_\mathcal{M}(t, \tau)|, \,(16)$$

where $g_\mathcal{E}(t, \tau)$ and $g_\mathcal{M}(t, \tau)$ are the first order coherence functions in terms of two-time correlation functions for the *exact* and *Markov* dynamics, respectively,

$$g_\mathcal{E}(t, \tau) = \frac{\langle a^\dagger(t)a(t+\tau) \rangle_\mathcal{E}}{\sqrt{\langle a^\dagger(t)a(t) \rangle_\mathcal{E} \langle a^\dagger(t+\tau)a(t+\tau) \rangle_\mathcal{E}}}, \,(17)$$

and

$$g_\mathcal{M}(t, \tau) = \frac{\langle a^\dagger(t)a(t+\tau) \rangle_\mathcal{M}}{\sqrt{\langle a^\dagger(t)a(t) \rangle_\mathcal{M} \langle a^\dagger(t+\tau)a(t+\tau) \rangle_\mathcal{M}}}. \,(18)$$

Here $\langle a^\dagger(t)a(t) \rangle_\mathcal{E}$ and $\langle a^\dagger(t)a(t) \rangle_\mathcal{M}$ can be calculated simply from Eqs. (5) and (15) by setting $\tau = 0$. In other words, we take $g_\mathcal{M}(t, \tau)$, which is obtained in the Markov limit, as a reference point to account the non-Markovianity of the open system dynamics.

Based on this definition (16), it is clear that such a non-Markovianity measure can be directly measured in

experiments through the measurement of the exact two-time correlation function or more precisely the first order coherence function $g_\mathcal{E}(t,\tau)$. The two-time correlation functions are defined for arbitrary finite temperature of the environment. Hence the non-Markovianity measure defined in Eq. (16) take into account the proper temperature dependence, which will be explicitly described in Sec. III. Our non-Markovianity measure is also an explicit function of time. Physically, this must be the case because non-Markovianity characterizes the memory effect which must be time-dependent in general. It is also clear from the analytical expression (16) through Eq. (5) that this non-Markovianity measure $\mathcal{N}(t,\tau)$ depends on the initial state of the system. This initial state dependence is indeed expected as a typical non-Markovian memory effect.

Here, we should point out that a similar definition of non-Markovianity measure has been recently proposed by Guarnieri *et al.* for a pure dephasing spin-boson model at zero temperature [31]. In their work, the non-Markovianity was defined as

$$\mathcal{N}' = 1 - \frac{\langle A(t_2)B(t_1)\rangle_{qrt}}{\langle A(t_2)B(t_1)\rangle_{exc}}, \tag{19}$$

where $\langle A(t_2)B(t_1)\rangle_{qrt}$ is the two-time correlation function obtained from quantum regression theorem by naively using the exact master equation, and $\langle A(t_2)B(t_1)\rangle_{exc}$ is the exact two-time correlation function. We find that this definition cannot account all the non-Markovianity. This is because quantum regression theorem is based on a crucial assumption that the total density operator of the system plus environment must be approximately factorized and the environment must always remain in the initial state at an arbitrary time, namely

$$\rho_{tot}(t) = \rho(t) \otimes \rho_E(t_0). \tag{20}$$

The detail derivation of this condition is given in Appendix-A). For non-Markov evolution based on the exact master equation, the condition of Eq. (20) cannot be satisfied in general and therefore the two-time correlation functions calculated from such a naive quantum regression theorem is inadequate. Thus, non-Markovianity defined in Ref. [31] cannot account the true non-Markovianity degree due to the improper use of the quantum regression theorem.

To show this inadequacy explicitly, let us apply this naive quantum regression theorem defined in Ref. [31] to our exact master equation (2). It is straightforward to show (see Appendix-B) that the two-time correlation function based on such a quantum regression theorem is given by

$$\langle a^\dagger(t)a(t+\tau)\rangle_{qrt} = u^*(t,t_0)n(t_0)u(t,t_0)u(\tau,t_0)$$
$$+ v(t,t)u(\tau,t_0). \tag{21}$$

This result is different from Eq. (5) which is exact, it is also different from Eq. (15) which is the two-time correlation function in the Markov limit. Thus, the two-time correlation function of Eq. (21), if it is applicable,

contain partial non-Markovianity. As a result, the non-Markovianity measure given in Ref. [31] cannot account all the non-Markovianity degree.

On the other hand, in the literature, there are some works [52–54] that attempted to generalize quantum regression theorem to non-Markovian regime where the system and the reservoirs are only weakly coupled to each other so that the master equation can be obtained perturbatively under the condition of Eq. (20). In this case Eq. (20) is actually the well-known Born-approximation. Thus, under this condition, quantum regression theorem is valid so that one can calculate two-time correlation functions from the Born-approximated master equation. But, the result obtained from such a quantum regression theorem cannot be used as a reference point to define non-Markovianity measure, because such a quantum regression theorem-based two-time correlation functions may contain partial non-Markovianity. In conclusion, our definition of non-Markovianity properly accounts all the non-Markovianity degree in terms of a physically measurable quantity.

III. RESULTS AND DISCUSSION

In order to evaluate explicitly the two-time correlation functions and the non-Markovianity measure $\mathcal{N}(t,\tau)$, we need to specify the spectral density $J(\omega)$ of the environment. In the following, we consider the Ohmic-type spectral density which can simulate a large class of thermal bath [55]

$$J(\omega) = 2\pi\eta\,\omega\left(\frac{\omega}{\omega_c}\right)^{s-1}e^{-\omega/\omega_c}, \tag{22}$$

where η is the coupling strength between the system and the environment, and ω_c is the frequency cutoff of the environmental spectra. When $s = 1, < 1$, and > 1, the corresponding environments are called as Ohmic, sub-Ohmic, and super-Ohmic in the literature, respectively. With the spectral density specified, the Green's function $u(t,t_0)$ and $v(t,t+\tau)$ are determined by Eqs. (3) and (4). The non-Markovianity measure can then be calculated from Eq. (16) through the two-time correlation functions of Eqs. (5) and (15).

In Fig. 1, we plot the non-Markovianity measure $\mathcal{N}(t,\tau)$ as a function of time τ for different system-environment coupling strengths η at a fixed initial temperature $T = 0.5$K. For simplicity, we also set $t = 0$. The non-Markovianity measure is examined for different spectral densities with $s = 1/2$ (Fig. 1a for sub-Ohmic), $s = 1$ (Fig. 1b for Ohmic), $s = 2$ (Fig. 1c for super-Ohmic), and $s = 3$ (Fig. 1d for super-Ohmic). Different curves represent different coupling strengths, namely $\eta = 0.1\eta_c$ (black dotted), $\eta = 0.5\eta_c$ (blue dashed), $\eta = 1.0\eta_c$ (dashed double dotted red), $\eta = 1.2\eta_c$ (dashed dotted violet), and $\eta = 1.5\eta_c$ (solid green), where $\eta_c = \omega_0/\omega_c\Gamma(s)$ is the critical coupling strength [32] for the occurrence of the localized mode given in Eq. (6). We observed that

for spectral densities with $s \leq 2$ and when the coupling strength is below the critical value (see black-dotted and blue-dashed curves in Figs. 1a-c for $\eta < \eta_c$), the non-Markovianity measure increases very fast and approaches a maximum value in a very short time scale, and then decays asymptotically to zero. This indicate that when the system-environment coupling is weak ($\eta < \eta_c$), the non-Markovianity dynamics only shows a short-time behavior, as was pointed first by Caldeira and Leggett in quantum Brownian motion [43]. For stronger coupling strengths ($\eta > \eta_c$), the non-Markovianity dynamics is significantly different, and we find that the non-Markovianity measure tends to approach a saturated value in the long time limit after few slow oscillations (see dashed-dotted-violet, and solid-green lines in Figs. 1a-c). This behavior of non-Markovianity dynamics is contributed from the localized mode given by the first term in Eq. (6). This corresponds to the localized-mode-induced dissipationless dynamics discovered recently in [32], as a long-time non-Markovian behavior.

For super-Ohmic case with $s > 2$, the above short-time and long-time behavior of non-Markovianity dynamics is slightly changed. In this case, for weak coupling $\eta < \eta_c$, the decay rate of the non-Markovianity measure becomes very slow (see Fig. 1d) where the non-Markovianity measure is calculated up to $\omega_0\tau \sim$ a few thousands. This result is quite different from the case of $s < 2$ where the non-Markovianity measure already decays to zero at $\omega_0\tau \sim$ few hundreds or less. This slow decay for $s > 2$ was discussed long time ago in Ref. [56], but in that work [56], such a delayed damping was interpreted as an undamped process and they claimed that the system would never go to its equilibrium for $s > 2$. This interpretation is not fully correct. As we can see from the black-dotted curve and the blue-dashed curve in Fig. 1d, it shows that the non-Markovianity measure damps very very slowly but will eventually decays to zero, so that the system will approach to equilibrium after a very long time. However, the system will not approach to equilibrium only in the strong-coupling regime ($\eta > \eta_c$) when the localized mode has significant contribution [32, 57–59]. In this case, the non-Markovianity dynamics will show the true long-time behavior, namely the memory effect will be kept forever. This is given by the dashed-dotted-violet curve ($\eta = 1.2\ \eta_c$) and solid-green curve ($\eta = 1.5\ \eta_c$) of Fig. 1d, where the non-Markovianity measure reaches a finite value in the long-time limit. To sum up, in Fig. 1, we show for the first time the existence of both the short time and long time features of non-Markovianity dynamics for different system-environment coupling strength.

In Fig. 2, we discuss the temperature dependence of the non-Markovianity dynamics using two different type of coupling strengths: a weak coupling $\eta = 0.5\eta_c$ (see Figs. 2 a, c, e, and g) and a strong coupling $\eta = 1.5\eta_c$ (see Figs. 2 b, d, f, and h). The temperature dependence of the non-Markovianity originates from the particle number distribution $\bar{n}(\omega, T)$ and non-equilibrium thermal fluctuation $v(t, t+\tau)$ involved in the two-time correlation functions of Eq. (5). We plot again the non-Markovianity measure for different spectral densities with $s = 1/2$ (Figs. 2 a,b for sub-Ohmic), $s = 1$ (Figs. 2 c,d for Ohmic), $s = 2$ (Figs. 2 e,f for super-Ohmic), and $s = 3$ (Figs. 2 g,h for super-Ohmic), where the temperature dependence of non-Markovianity is given for $T = 0.05$K (solid blue), $T = 0.5$K (red dashed), and $T = 5$K (green dotted). For spectral densities with $s \leq 2$ and the coupling strength is weak ($\eta < \eta_c$), non-Markovianity dynamics still shows a short-time behavior as expected but it takes longer time to decay to zero at low temperature (see solid-blue curves in Figs. 2 a, c, and e for $T = 0.05$K). For a higher temperature, the non-Markovianity decays much faster (see green-dotted curves in Figs. 2 a, c, and e for $T = 5$K). This indicates that the non-Markovian memory effect is lost faster when temperature is higher. In stronger coupling regime ($\eta > \eta_c$), non-Markovianity measure also tends to approach a steady value in the long time limit due to the localized mode contribution but the magnitude of this stationary values of non-Markovianity become higher for lower temperature of the environment (see Figs. 2 b,d). This shows that in both the weak and strong coupling regimes, thermal fluctuation always reduce the non-Markovian degree for $s < 2$.

For super-Ohmic spectral densities with $s > 2$, the situation is quite different. For weak coupling $\eta < \eta_c$ and low temperature, the non-Markovianity shows a large oscillation and then decay extremely slow (see Fig. 2g for $T = 0.05$K). The damping becomes faster with a higher temperature of the environment (see the red-dashed and green-dotted curves of Fig. 2g for $T = 0.5$K and $T = 5$K). In the strong coupling regime, the non-Markovianity takes much longer time to reach a steady value which is significantly different from the case for $s < 2$. In this case, strong oscillations of non-Markovianity is found at low temperature (solid-blue curve of Fig. 2h) before approaching its long-time steady value. The large oscillations persists in the beginning even at higher temperatures (red-dashed and green-dotted lines of Fig. 2h). We also notice that the long time stationary values of the non-Markovianity become less and less sensitive to the temperature, contrary to the case for $s < 2$. To explicitly see this steady state behavior (see Fig. 2h), we check the analytic expression of the non-Markovianity measure, which is greatly simplified for $t = 0$ and $\tau \to \infty$ as

$$\mathcal{N}(0, \tau \to \infty) = \left| \frac{u(\tau \to \infty, 0)\sqrt{n(t_0)}}{\sqrt{n(t_0)\left|u(\tau \to \infty, 0)\right|^2 + v(\tau, \tau \to \infty)}} \right|. \tag{23}$$

This asymptotic solution shows that the steady value of the non-Markovianity is determined through a competition between the temperature-dependent thermal fluctuation $v(\tau, \tau \to \infty)$ and the temperature-independent localized mode contribution $n(t_0)|u(\tau \to \infty, 0)|^2$. We find that the steady value $u(\tau \to \infty, 0)$ increases as we increase the value of s due to an enhanced contribution of the localized mode amplitude, while the steady value

of thermal fluctuation $v(\tau, \tau \to \infty)$ decreases with the increase of s. This is why the influence of thermal fluctuation in the steady value of non-Markovianity is less significant for $s > 2$ since the localized mode contribution dominate over the thermal fluctuation $v(\tau, \tau \to \infty)$ under this situation.

In Fig. 3, we show the initial state dependence of non-Markovianity measure. Here the different initial state is characterized by different initial particle number $n(t_0)$. We take $n(t_0) = 1$ (solid blue), $n(t_0) = 10$ (red dashed) and $n(t_0) = 50$ (green dotted) in Fig. 3, with a fixed temperature ($T = 0.5$K) of the environment for weak coupling $\eta = 0.5\eta_c$ (see Figs. 3 a, c, e, and g) and strong coupling $\eta = 1.5\eta_c$ (see Figs. 3 b, d, f, and h). The non-Markovianity dynamics shows an explicit $n(t_0)$-dependence. We find that the magnitude of non-Markovianity is hiked as the initial particle number $n(t_0)$ is increased in the weak coupling regime ($\eta < \eta_c$), although the decay time scale of the non-Markovianity looks similar for different $n(t_0)$ (see Figs. 3a, c, and e for spectral densities with $s \leq 2$). But this initial state dependence is only a short-time behavior in the weak system-environment coupling regime. However, for strong coupling ($\eta > \eta_c$) with $s < 2$, we find that the non-Markovianity shows an explicit long-time dependence of the initial state, due to the existence of the localized modes in the strong coupling regime (see Figs. 3b,d for $\eta = 1.5\eta_c$). Thus, the magnitude of non-Markovianity measure increases in general, as the initial particle number $n(t_0)$ increases.

On the other hand, for super-Ohmic case with $s > 2$ in the weak coupling regime $\eta < \eta_c$, a very slow decay of the $n(t_0)$-dependent non-Markovianity degree is observed (see Fig. 3g). In strong coupling regime, short-time dynamics of non-Markovianity shows strong oscillations which is $n(t_0)$-dependent (see Fig. 3h). Whereas the long-time steady value of non-Markovianity does not significantly depend on $n(t_0)$. Such a steady state behavior (see Fig. 3h) is a consequence of the fact that the localized mode contribution dominate over the thermal fluctuation for $s > 2$, and the non-Markovianity approaches to unity according to Eq. (23). Physically, the initial state dependence of the non-Markovianity manifests different non-Markovian memory effects in different time scales. Such non-trivial non-Markovian behaviors are worthwhile for further experimental tests.

IV. CONCLUSION

To summarize, we have introduced a new definition of non-Markovianity measure using two-time correlation functions which play a fundamental role in the exact non-Markovian dynamics of open quantum systems. Our non-Markovianity measure quantifies the difference between the exact two-time correlation function and the one obtained in the Markov limit, namely we take Markov dynamics of two-time correlation function as a reference
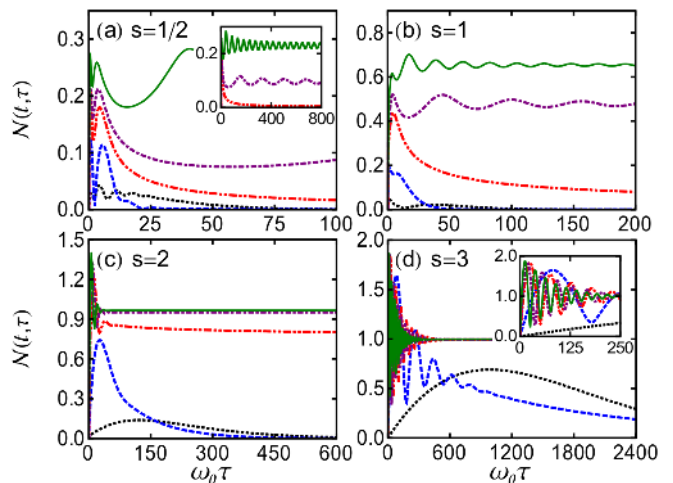

FIG. 1. (Color online) The non-Markovianity measure $\mathcal{N}(t, \tau)$ is plotted as a function of τ with $t = 0$ at a temperature ($T = 0.5$K) for different system-environment coupling strengths: $\eta = 0.1\eta_c$ (black dotted), $\eta = 0.5\eta_c$ (blue dashed), $\eta = 1.0\eta_c$ (dashed double dotted red), $\eta = 1.2\eta_c$ (dashed dotted violet), and $\eta = 1.5\eta_c$ (solid green). Four different spectral densities are considered (a) sub-Ohmic ($s = 1/2$) (b) Ohmic ($s = 1$) (c) super-Ohmic ($s = 2$) and (d) super-Ohmic ($s = 3$) for the bosonic reservoir. The other parameters are taken as $\omega_0 = 10$ GHz, $\omega_c = 5\omega_0$, and $n(t_0) = 1$.

point for measuring non-Markovianity. Such a definition of non-Markovianity take into account the proper temperature dependence given naturally in two-time correlation functions, and can be measured directly in experiments. We also find that the degree of non-Markovianity is reduced in general due to thermal fluctuations for hotter environments. Also, our non-Markovianity measure is an explicit function of time. We discover that in general non-Markovianity shows both the short-time and long-time behavior, which crucially depends on the system-environment coupling, and thereby the significant contribution of the localized mode induced by the strong system-environment coupling. The non-Markovianity measure also depends on the initial state of the open system in different time scales, as a non-trivial non-Markovian memory effect. Our approach of quantifying non-Markovianity is applicable to the more general class of open quantum systems interacting with a general environment through particle tunneling processes. Further experimental investigations are required to explore these non-trivial dynamics of the non-Markovianity measure through the measurement of two-time correlation functions.

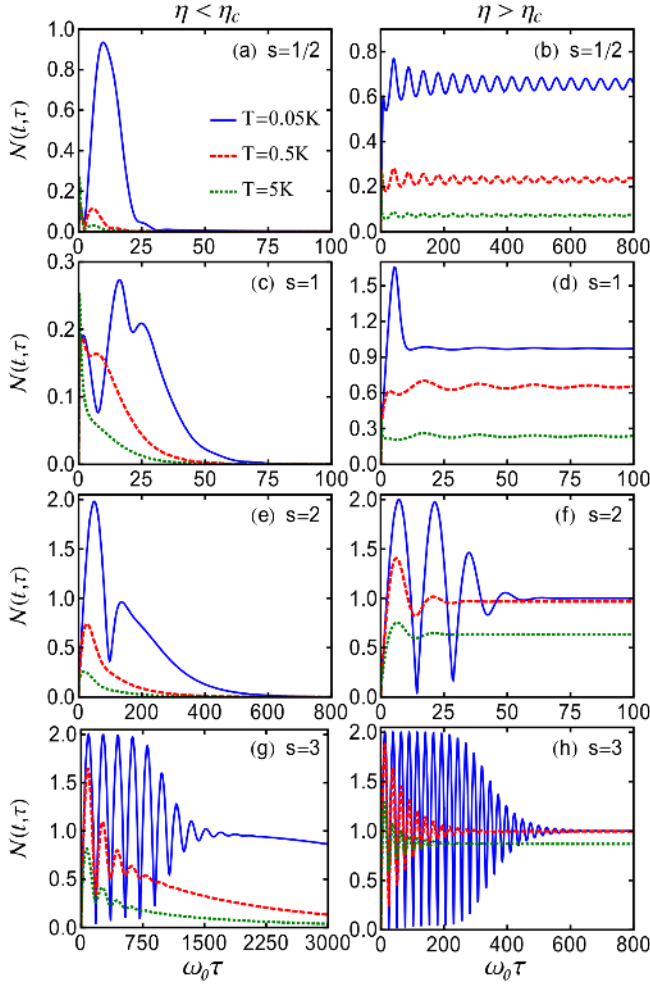FIG. 2. (Color online) Temperature dependence of non-Markovianity measure is shown as a function of τ at three different temperatures: $T = 0.05K$ (solid blue), $T = 0.5K$ (red dashed), and $T = 5K$ (dotted green) with two type of coupling strengths: (a,c,e,g) weak coupling ($\eta = 0.5\eta_c$) and (b,d,f,h) strong coupling ($\eta = 1.5\eta_c$). Four different spectral densities for the bosonic environment are considered: (a,b) sub-Ohmic ($s = 1/2$), (c,d) Ohmic ($s = 1$), (e,f) super-Ohmic ($s = 2$), and (g,h) super-Ohmic ($s = 3$). The other parameters are taken as $\omega_0 = 10$ GHz, $\omega_c = 5\omega_0$, and $n(t_0) = 1$ with a fixed value of $t = 0$.

FIG. 3. (Color online) Initial state dependence of non-Markovianity measure is shown as a function of τ for three different $n(t_0)$: $n(t_0) = 1$ (solid blue), $n(t_0) = 10$ (red dashed), and $n(t_0) = 50$ (green dotted) with two type of coupling strengths: (a,c,e,g) weak coupling ($\eta = 0.5\eta_c$) and (b,d,f,h) strong coupling ($\eta = 1.5\eta_c$). Four different spectral densities for the bosonic environment is investigated: (a,b) sub-Ohmic ($s = 1/2$), (c,d) Ohmic ($s = 1$), and (e,f) super-Ohmic ($s = 2$), and (g,h) super-Ohmic ($s = 3$). The other parameters are taken as $\omega_0 = 10$ GHz, $\omega_c = 5\omega_0$, and $T = 0.5K$ with a fixed value of $t = 0$.

Appendix A: Generalized quantum regression theorem under Born approximation

Here we present a generalized quantum regression theorem. The total density operator of the system plus environment is governed by the quantum Liouville equation in the Schrödinger picture
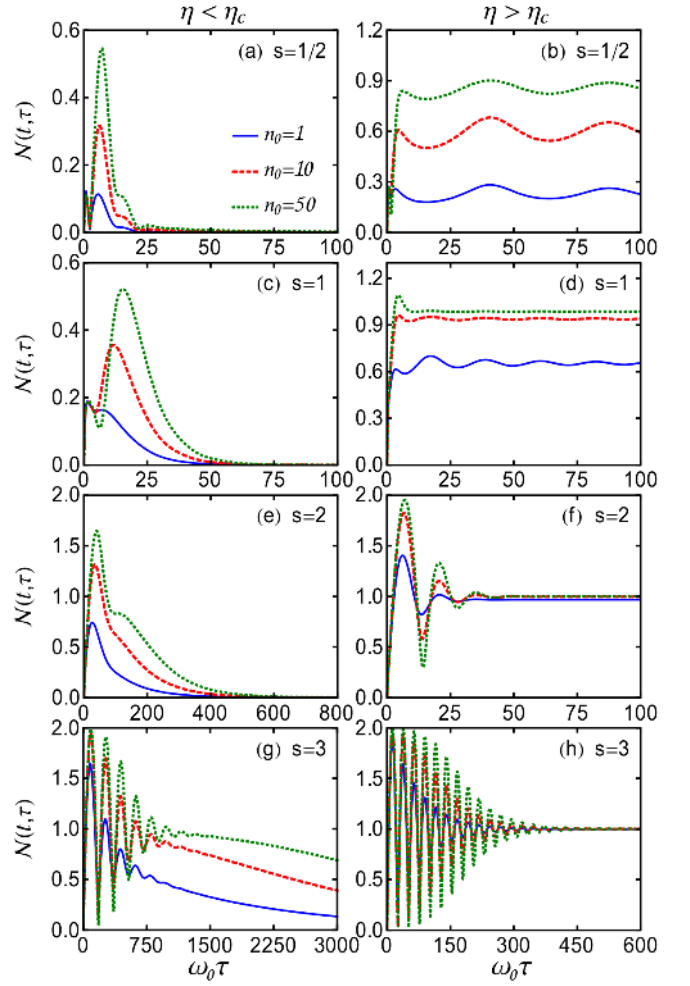
$$\frac{\partial}{\partial \tau}\rho_{tot}(\tau) = \frac{1}{i\hbar}\left[H, \rho_{tot}(\tau)\right], \quad (A1)$$

with a formal solution

$$\rho_{tot}(\tau) = e^{-\frac{i}{\hbar}H\tau}\rho_{tot}(0)e^{\frac{i}{\hbar}H\tau}. \quad (A2)$$

Consider an initial state $\rho_{tot}(0)$ of the total system to be uncorrelated and the reservoir to be in a thermal equilibrium state, that is,

$$\rho_{tot}(0) = \rho(0) \otimes \rho_E(0), \quad \rho_E(0) = \frac{e^{-\beta H_E}}{\text{Tr}\, e^{-\beta H_E}}, \quad (A3)$$

where $H_E = \sum_k \hbar\omega_k b_k^\dagger b_k$ and $\beta = 1/k_B T$ is the initial temperature of the reservoir. In this case, if no non-linear dynamics is involved, tracing over all the environmental degrees of freedom is easily carried out to obtain a time-local master equation in terms of the reduced density operator

$$\frac{\partial}{\partial\tau}\rho(\tau) = \mathcal{L}(\tau)\rho(\tau), \tag{A4}$$

where $\rho(\tau) = Tr_E[\rho_{tot}(\tau)]$. One can then calculate the mean values of a complete set of system operators $\hat{O}_i$, $i = 1, 2, ...$, using Eq. (A4). The expectation value of an operator $\hat{O}_i$ obey the relation

$$\frac{\partial}{\partial\tau}\langle\hat{O}_i(\tau)\rangle = \langle\hat{O}_i(0)\dot{\rho}(\tau)\rangle = \langle\hat{O}_i(0)\mathcal{L}(\tau)\rho(\tau)\rangle. \tag{A5}$$

For each $\hat{O}_i$, one can define

$$\langle\hat{O}_i(0)\mathcal{L}(\tau)\rho(\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_j(\tau)\rangle, \tag{A6}$$

Then Eq. (A5) becomes

$$\frac{\partial}{\partial\tau}\langle\hat{O}_i(\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_j(\tau)\rangle. \tag{A7}$$

On the other hand, given two system operators $\hat{O}_1$ and $\hat{O}_2$, their two-time correlation function is defined more conveniently in the Heisenberg picture,

$$\langle\hat{O}_1^H(t_1)\hat{O}_2^H(t_2)\rangle = \mathrm{Tr}_{S\oplus E}\left[\rho_{tot}^H\hat{O}_1^H(t_1)\hat{O}_2^H(t_2)\right]. \tag{A8}$$

Then going back to the Schrödinger picture through the transformations

$$\hat{O}_1^H(t_1) = e^{\frac{iHt_1}{\hbar}}\hat{O}_1 e^{-\frac{iHt_1}{\hbar}}, \tag{A9}$$

$$\hat{O}_2^H(t_2) = e^{\frac{iHt_2}{\hbar}}\hat{O}_2 e^{-\frac{iHt_2}{\hbar}}, \tag{A10}$$

$$\rho_{tot}^H = e^{\frac{iHt}{\hbar}}\rho_{tot}(t)e^{-\frac{iHt}{\hbar}}, \tag{A11}$$

also taking $t_1 = t$ and $t_2 = t + \tau$, and using the cyclic property of trace, we obtain

$$\langle\hat{O}_1^H(t)\hat{O}_2^H(t+\tau)\rangle = \mathrm{Tr}_{S\oplus E}\left[\hat{O}_2\ e^{-\frac{iH\tau}{\hbar}}\rho_{tot}(t)\hat{O}_1 e^{\frac{iH\tau}{\hbar}}\right]$$

$$= \mathrm{Tr}_{S\oplus E}\left[\hat{O}_2\ \rho_{tot,\hat{O}_1}(t,\tau)\right], \tag{A12}$$

where we have used the fact that the state of the total system $\rho_{tot}(t)$ in the Schrödinger picture acted by the operator $\hat{O}_1$ becomes a new state denoted by $\rho_{tot,\hat{O}_1}(t,0) \equiv \rho_{tot}(t)\hat{O}_1$ such that

$$\rho_{tot,\hat{O}_1}(t,\tau) = e^{-\frac{iH\tau}{\hbar}}\rho_{tot,\hat{O}_1}(t,0)e^{\frac{iH\tau}{\hbar}}. \tag{A13}$$

Now, if $\rho_{tot}(t)$ is factorized as

$$\rho_{tot}(t) = \rho(t) \otimes \rho_E(0), \tag{A14}$$

then

$$\rho_{tot,\hat{O}_1}(t,0) = \rho_{\hat{O}_1}(t,0) \otimes \rho_E(0), \tag{A15}$$

where $\rho_{\hat{O}_1}(t,0) \equiv \rho(t)\hat{O}_1$ is the state of the system acted by $\hat{O}_1$ at time t. Thus, Eq. (A12) can be reduced to

$$\langle\hat{O}_1^H(t)\hat{O}_2^H(t+\tau)\rangle = \mathrm{Tr}_S\left[\hat{O}_2\,\mathrm{Tr}_E\{\rho_{tot,\hat{O}_1}(t,\tau)\}\right]$$

$$= \mathrm{Tr}_S\left[\hat{O}_2\,\rho_{\hat{O}_1}(t,\tau)\right], \tag{A16}$$

with $\rho_{\hat{O}_1}(t,\tau) = \mathrm{Tr}_E\{\rho_{tot,\hat{O}_1}(t,\tau)\}$.

As one can see, Eq. (A15) for $\rho_{tot,\hat{O}_1}(t,0)$ obeys the same initially factorized condition of Eq. (A3) for $\rho_{tot}(0)$, and the formal solution (A13) for $\rho_{tot,\hat{O}_1}(t,\tau)$ has the same form of Eq. (A2) for $\rho_{tot}(\tau)$. Thus, the master equation for $\rho_{\hat{O}_1}(t,\tau)$ with respect to τ must be the same as that for $\rho(\tau)$, namely

$$\frac{\partial}{\partial\tau}\rho_{\hat{O}_1}(t,\tau) = \mathcal{L}(\tau)\rho_{\hat{O}_1}(t,\tau). \tag{A17}$$

This gives

$$\frac{\partial}{\partial\tau}\langle\hat{O}_1^H(t)\hat{O}_2^H(t+\tau)\rangle = \langle\hat{O}_2\mathcal{L}(\tau)\rho_{\hat{O}_1}(t,\tau)\rangle. \tag{A18}$$

For $\hat{O}_2$, we choose a complete set of system operators $\hat{O}_i$ so that for each $\hat{O}_i$

$$\langle\hat{O}_i(0)\mathcal{L}(\tau)\rho_{\hat{O}_1}(t,\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_j\rho_{\hat{O}_1}(t,\tau)\rangle. \tag{A19}$$

Thus, we reach

$$\frac{\partial}{\partial\tau}\langle\hat{O}_1^H(t)\hat{O}_i^H(t+\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_1^H(t)\hat{O}_j^H(t+\tau)\rangle. \tag{A20}$$

This is the generalized quantum regression theorem namely, the form of the evolution equation (A7) for single-time expectation values is the same as that for the two-time correlation functions (A20). Notice that Eq. (A15) is the well-known Born approximation in the derivation of the perturbative master equation up to the second order for system-environment coupling. Thus, the above generalized quantum regression theorem is only valid under Born approximation.

Appendix B: Calculating $\langle a^\dagger(t)a(t+\tau)\rangle_{qrt}$

Using the exact master equation (2), one can obtain the evolution matrix $M_{ij}(\tau)$ of Eq. (A7) by calculating the single-time expectation values of a complete set of system operators (a, $a^\dagger$, n and $\mathbb{1}$), where $n = \langle a^\dagger a\rangle$:

$$\mathbf{M}(\tau) = \begin{pmatrix} \alpha(\tau) & 0 & 0 & 0 \\ 0 & \alpha^*(\tau) & 0 & 0 \\ 0 & 0 & -2\gamma(\tau) & \widetilde{\gamma}(\tau) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B1}$$

where $\alpha(\tau) = -(\gamma(\tau) + i\omega_0'(\tau))$. The evolution matrix elements of $\mathbf{M}(\tau)$ are now time-dependent, which is different from the Markov evolution where the matrix elements are time-independent, see Eq. (13). The time-dependent coefficients in the exact master equation (2) are explicitly given by

$$\omega_0'(\tau) = -Im\left[\frac{\dot{u}(\tau,t_0)}{u(\tau,t_0)}\right], \tag{B2}$$

$$\gamma(\tau) = -Re\left[\frac{\dot{u}(\tau,t_0)}{u(\tau,t_0)}\right], \tag{B3}$$

$$\widetilde{\gamma}(\tau) = \dot{v}(\tau,\tau) + 2v(\tau,\tau)\gamma(\tau). \tag{B4}$$

Now, naively applying generalized quantum regression theorem as done in Ref. [31] (this is indeed incorrect because the exact master equation does not involve the Born approximation), the evolution equation for the two-time correlation function $\langle a^\dagger(t)a(t+\tau)\rangle$ is given by

$$\frac{d}{d\tau}\langle a^\dagger(t)a(t+\tau)\rangle_{qrt}$$
$$= -(\gamma(\tau) + i\omega_0'(\tau))\langle a^\dagger(t)a(t+\tau)\rangle_{qrt}$$
$$= \frac{\dot{u}(\tau,t_0)}{u(\tau,t_0)}\langle a^\dagger(t)a(t+\tau)\rangle_{qrt}. \tag{B5}$$

The exact solution of this equation is

$$\langle a^\dagger(t)a(t+\tau)\rangle_{qrt} = n(t)u(\tau,t_0), \tag{B6}$$

where $n(t) = \langle a^\dagger(t)a(t)\rangle$ which is given by [45]

$$n(t) = v(t,t) + u^*(t,t_0)n(t_0)u(t,t_0). \tag{B7}$$

Thus, the two-time correlation function naively using the generalized quantum regression theorem is given by

$$\langle a^\dagger(t)a(t+\tau)\rangle_r = u^*(t,t_0)n(t_0)u(t,t_0)u(\tau,t_0)$$
$$+ v(t,t)u(\tau,t_0). \tag{B8}$$

This solution is not the exact two-time correlation function given by Eq. (5) nor the result of two-time correlation function in the Markov limit, *i.e.* Eq. (15).

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