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Non-minimum phase switched systems: HOSM-based fault detection and fault identification via Volterra integral equation[‡,§]

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SUMMARY

In this paper, the problem of continuous and discrete state estimation for a class of linear switched systems with additive faults is studied. The class of systems under study can contain non-minimum phase zeroes in some of their 'operating modes'. The conditions for exact reconstruction of the discrete state are given using structural properties of the switched system. The state space is decomposed into the strongly observable part, the non-strongly observable part, and the unobservable part, to analyze the effect of the unknown inputs. State observers based on high-order sliding mode to exactly estimate the strongly observable part and Luenberger-like observers to estimate the remaining parts are proposed. For the case when the exact estimation of the state cannot be achieved, the ultimate bounds on the estimation errors are provided. The proposed strategy includes a high-order sliding-mode-based fault detection and a fault identification scheme via the solution of a Volterra integral equation. The feasibility of the proposed method is illustrated by simulations. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: switched systems; fault detection and identification; non-minimum phase; high-order sliding mode

1. INTRODUCTION

1.1. Nomenclature

The following nomenclature is used in the paper.

- FD = fault detection (*FD indicates that something is wrong in the system*)
- FI = fault isolation (*FI determines the location and the fault type*)
- FId = fault identification (*FId determines the magnitude and shape of the fault*)
- FDI = fault detection and isolation
- FDId = fault detection and identification
- FTC = fault tolerant control
- HOSM = high-order sliding mode

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- HS = hybrid systems
- LSS = linear switched systems
- NSS = nonlinear switched systems
- NMP = non-minimum phase
- SMC = sliding-mode control
- SS = switched systems
- VIE = Volterra integral equation

1.2. Antecedents and motivation

The HS, whose behavior can be represented by the interaction of continuous and discrete dynamics, have been widely studied during the last decades. This kind of systems can be used to describe a wide range of physical and engineering systems. Much attention has been focused on the problems of stability and stabilization (see, e.g., [2–4]). An important class of HS is composed of SS whose dynamics consists of a finite number of subsystems and a logical rule that drives the switchings between these subsystems.

Observation of SS (i.e., the estimation of the continuous and discrete states) is also of great interest in many control areas. This problem has been studied by many authors using different approaches. The main difference is related to the knowledge of the active discrete state: Some approaches consider only continuous state uncertainty with known discrete state, while others assume that both of them are unknown. In [5] and [6], a Luenberger observer approach and an HOSM observer for LSS are proposed for the known discrete state case. In [7], the problem of the simultaneous state and input estimation for SS subject to input disturbances is addressed by an algorithm based on the moving horizon estimation method. For unknown discrete state, based on strong detectability and using an LMI approach, in [8], two state observers are designed for some classes of LSS with unknown inputs. To Markovian jump singular systems, another class of SS, in [9], an integral sliding-mode observer is designed to estimate the system states, and an SMC scheme is synthesized for the reaching motion based on the state estimates. Considering that the output and an initial state are available, in [10], necessary and sufficient conditions for SS to be invertible are proposed, that is, condition for recovering the switching signal and the input uniquely. In the same context, a nonlinear finite time observer to estimate the capacitor voltage for multicellular converters, which have a switched behavior, is proposed by Defoort *et al*. [11]. In [12], a hybrid adaptable observer is proposed that is able to estimate the state for locally Lipschitz systems with application on mechanical oscillators. In [13], based on the non-homogeneous HOSM approach, a robust observer for the unknown and exogenous switching signal is proposed to solve the problem of continuous and discrete state estimation for a class of NSS. A nonlinear observer synthesis based on second-order SMC for autonomous SS with jumps is proposed in [14]. In [15] and [16], observability conditions for some classes of SS and a design of hybrid observers to reconstruct both continuous and discrete states are presented. Considering that the continuous state is known, an algorithm for the discrete state reconstruction in uncertain NSS is presented in [17] based on the SMC theory. Sliding-mode observers producing suitable residual signals for the problem of simultaneous discrete and continuous state reconstruction in LSS are proposed in [18]. A robust observer is proposed in [19] that is able to estimate the continuous and discrete states and unknown inputs for autonomous NSS based on HOSM observers.

The issue of model-based FDI in dynamic systems has been an active research area during the last three decades (see, e.g., [20] and [21]). A typical FD process includes residual generation and evaluation. A huge number of publications about FDI problem exist (see, e.g., [22–24] and the references therein). Residual generation schemes, where the output error between the system and the observer is analyzed to form the residuals, have been extensively studied (see, e.g., [25, 26] and [27]). In the last decades, observer-based FDI schemes that incorporate different control approaches have been developed [28]. In the SMC theory, the most recent contributions have been presented in [29–32] and [33]. Most of these works formulate actuator or sensor FId as additional unknown input and exploit the inherent robustness properties of SMC to a certain class of uncertainties, including its ability to directly handle actuator faults.

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An FDI methodology that uses structured parity residuals for SS is proposed in [34]. For the feasibility of the proposed methodology, the authors propose the definition of discernability that is similar to the concept of observability in LTI systems. In [35] based on a hybrid automata model that parameterizes abrupt and incipient faults, an approach for a diagnoser design is presented. The diagnostic system architecture that integrates the modeling, prediction, and diagnosis components is described. A methodology for detection and isolation of faults to control SS using a diagnoser is presented in [36]. The notion and conditions of diagnosability of SS in the hybrid input output automata framework is addressed, based on the previous diagnosability notion. The work [37] developed techniques for fault diagnosis in SS based on a knowledge-based and bond graphs approach. In [38], a robust hybrid observer is proposed for a class of uncertain nonlinear SS with unknown mode transition functions, model uncertainties, and unknown disturbances. The transition detectability and mode identifiability conditions are studied. Based on a hybrid observer, a robust fault diagnosis scheme is presented for faults modeled as discrete modes with unknown transition functions. In [39], a model-based FDI using multiple hypothesis testing is proposed for stochastic linear SS. A residual generation filter is proposed that generates a residual vector with zero mean and a known covariance when the stochastic linear SS match the system dynamics. For a class of nonlinear SS with faults and parametric uncertainties, in [40], an observer is designed whose estimation error is not affected by faults, and an observer-based fault tolerant tracking controller is proposed to make the outputs asymptotically track the reference signals for bounded states. In [41], the FD and reconstruction is formulated as an unknown input problem, and an algorithm for the invertibility of NSS is proposed. A robust hybrid observer for LSS with known active mode, unknown inputs, and modeling error is presented in [42]. The proposed observer is synthesized for the task of robust FD as a H_∞ model-matching problem. Recently, in [43], a residual generator-based robust FD filter is used for the problem of designing an FD system for SS with unknown inputs and known switching signal. The residual generator design is formulated as an optimization problem and solved iteratively by LMI. In [44], algorithms for robust estimation and FDI for a class of stochastic SS are presented. The robust hybrid estimation algorithm is designed for two kinds of discrete state transition models: the Markov-jump transition model whose discrete transition probabilities are constant and the state-dependent transition model whose discrete state transitions are determined by some guard conditions.

1.3. Main contributions

In most of the aforementioned schemes, faults are modeled as a discrete state, that is, as SS as the result of the interaction between a system and possible faults (system + faults = SS) and not as SS interacting with faults (SS + faults) that is more complicated.

In this paper, a solution of the problem of state estimation for LSS subject to additive faults and possibly unstable invariant zeroes (NMP systems with respect to unknown inputs or additive faults)[‖] is presented. State observers based on HOSM to exactly estimate the strongly observable part and Luenberger-like observers to estimate the remaining parts are proposed. The exact estimation of the continuous state allows to realize a finite time and exact estimation of the discrete state in the presence of additive faults. Moreover, the proposed strategy includes a HOSM-based FD composed by a residual generator helped by a bank of observers and an FId scheme via the numerical solution of a VIE.

1.4. Structure of the paper

The paper has the following structure. Section 2 deals with the problem statement, and some preliminaries are recalled. In Section 3, the system transformation is proposed. The observer design is presented in Section 4. The problem of discrete state estimation is analyzed in Section 5. The FDId problem is carried out in Section 6. The simulation results are shown in Section 7. Finally, some concluding remarks are given in Section 8.

[‖]See, for example, [45] for the observation problem for a class of NMP causal nonlinear systems.

2. PROBLEM STATEMENT

Consider the following LSS with faults:

$$\begin{aligned} \dot{x}(t) &= A_{j(x(t))}x(t) + B_{j(x(t))}u(t) + E_{j(x(t))}f(t), \\ y(t) &= C_{j(x(t))}x(t), \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subseteq \Re^n$ is the state vector, $u \in \mathcal{U} \subseteq \Re^p$ is the known input vector, $y \in \mathcal{Y} \subseteq \Re^m$ is the output, and $f \in \mathcal{F} \subseteq \Re^m$ is the fault vector, which is bounded, that is, $\|f(t)\| \leqslant f^+ < \infty$. The so-called discrete state $j(x(t)) : \Re \to \mathcal{Q} = \{1, \dots, q\}$ determines the current system dynamics among the possible q 'operating modes', that is, $\{A_1, B_1, C_1, E_1\}, \{A_2, B_2, C_2, E_2\}, \dots, \{A_q, B_q, C_q, E_q\}$. The discrete state is generated by a scalar function of the system states (the switching signal) defined as

$$j(x(t)) = \begin{cases} 1, & \forall x \mid \quad Hx \in \mathcal{H}_1, \\ 2, & \forall x \mid \quad Hx \in \mathcal{H}_2, \\ \vdots & \\ q, & \forall x \mid \quad Hx \in \mathcal{H}_q, \end{cases} \tag{2}$$

where H is a known matrix and $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_q \subseteq \mathcal{H} \in \Re$ are known convex disjoint subsets, respectively. Notice that for every value of the continuous state $x(t)$, there is only one single value of the discrete state $j(x(t))$; that is, the discrete state is distinguishable by definition.

In this paper, the studied problems are the following:

- estimation of the continuous state $x(t)$,
- estimation of the discrete state $j(x(t))$,
- fault detection,
- fault identification.

2.1. *Preliminaries*

2.1.1. Notation. The following notation is used. The pseudoinverse matrix of $F \in \Re^{n \times m}$ is defined as $F^+ = (F^T F)^{-1} F^T \in \Re^{m \times n}$. For a matrix $J \in \Re^{n \times m}$ with $n \geqslant m$ and $\text{rank}(J) = r$, $J^\perp \in \Re^{(n-r) \times n}$ is defined as a matrix such that $\text{rank}(J^\perp) = n - r$ and $J^\perp J = 0$. The matrix $J^{\perp\perp} \in \Re^{r \times n}$ corresponds to one of the full row rank matrices such that $\text{rank}(J^{\perp\perp}) = r$ and $J^\perp (J^{\perp\perp})^T = 0$. It is clear that the matrices $J^\perp$ and $J^{\perp\perp}$ are not unique and that $\text{rank}\begin{bmatrix} (J^\perp)^T & (J^{\perp\perp})^T \end{bmatrix}^T = n$. Denote by $\mathcal{L}_\infty$ the set of all inputs ν that satisfy $\|\nu\| < \infty$. Finally, $\lceil \nu \rfloor^r = |\nu|^r \text{sign}(\nu)$.

2.1.2. Definitions. Some basic definitions for strong observability, strong detectability, invariant zeroes, relative degree, and dwell time are introduced in this section (see, e.g., [46–48] and [2]).

Consider an LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ef(t), \\ y &= Cx(t), \end{aligned} \tag{3}$$

where $x \in \Re^n$ is the state, $y \in \Re^m$ is the output, $f \in \Re^m$ is an 'unknown input' or a fault, and the known matrices A, C, and E have corresponding dimensions. In this case, it can be assumed, without loss of generality, that the known input $u(t)$ is equal to 0.

Definition 1

System (3) is called strongly observable if for any initial condition $x(0)$ and for all unknown inputs $f(t)$, the identity $y(t) \equiv 0 \ \forall t \geqslant 0$ implies that also $x(t) \equiv 0 \ \forall t \geqslant 0$.

Definition 2

System (3) is called strongly detectable if for any initial condition $x(0)$ and for all unknown inputs $f(t)$, the identity $y(t) \equiv 0 \ \forall t \geqslant 0$ implies that $x(t) \to 0$ as $t \to \infty$.

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Definition 3
The complex number $s_0 \in \mathcal{C}$ is called an invariant zero of the triple (A, E, C) if $\operatorname{rank}(R(s_0)) < n + \operatorname{rank}(E)$, where $R(s)$ is the Rosenbrock matrix of system (1), that is,

$$R(s) = \begin{bmatrix} sI - A & -E \\ C & 0 \end{bmatrix}.$$

Fact 1
The following statements are equivalent:

(i) System (3) is strongly observable.
(ii) The triple (A, E, C) has no invariant zeroes.

Fact 2
The following statements are equivalent:

(i) System (3) is strongly detectable.
(ii) The system is minimum phase (i.e., the invariant zeroes of the triple (A, E, C) satisfy $Re\{s\} < 0$).

Now, basing on the given statements, we can define the set of unstable invariant zeroes of system (3) as the set of invariant zeroes of the triple (A, E, C) satisfy $Re\{s\} \geqslant 0$. Moreover, notice that if there exist unstable invariant zeroes, then system (3) is not strongly detectable and NMP either.

In the case when $E = 0$, the notions of strong observability and strong detectability coincide with observability and detectability, respectively. Now, introduce the observability matrix

$$\mathrm{O} = \begin{bmatrix} C^T, & (CA)^T, & \cdots, & (CA^{n-1})^T \end{bmatrix}^T.$$

Notice that system (3) is observable, independently of the unknown inputs, if and only if $\operatorname{rank}(\mathrm{O}) = n$. The unobservable subspace of the pair (C, A) is denoted by $\mathcal{N}$, and it is defined as $\mathcal{N} = \ker(\mathrm{O})$.

Definition 4
The scalar value $\lambda_0 \in \Im$ is said to be an (C,A)-unobservable eigenvalue if

$$\operatorname{rank}\begin{pmatrix} \lambda_0 I - A \\ C \end{pmatrix} < n.$$

Definition 5
A point x_0 is called weakly unobservable if there exists an input $w(t)$, such that the corresponding output satisfies $y_w(t, x_0) \equiv 0 \ \forall t \geqslant 0$. The set of all weakly unobservable points of (3) is denoted by $\mathcal{V}^*$ and is called the weakly unobservable subspace of (3).

The weakly unobservable subspace satisfies the following relations:

$$A\mathcal{V}^* \subset \mathcal{V}^* \oplus \mathcal{E}, \quad C\mathcal{V}^* = 0, \tag{4}$$

where $\mathcal{E}$ denotes the image of E. For any null-output (A, E)-invariant subspace, there exists a map $\bar{K} : \mathcal{X} \to \mathcal{W}$ such that

$$\left(A + E\bar{K}\right)\mathcal{V}^* \subset \mathcal{V}^*, \quad C\mathcal{V}^* = 0. \tag{5}$$

Definition 6
The output $y(t)$ is said to have a relative degree vector $(r_1, \ldots, r_m)$ with respect to the unknown input $w(t)$ if

$$c_i A^k E = 0, \ \forall k < r_i - 1, \tag{6}$$

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$$c_i A^{r_i-1} E \neq 0, \forall i = 1, \ldots, m. \quad (7)$$

and

$$\det Q \neq 0,$$
$$Q = \begin{bmatrix} c_1 A^{r_1-1} E_1 & \cdots & c_1 A^{r_1-1} E_m \\ & \ddots & \\ c_m A^{r_m-1} E_1 & \cdots & c_m A^{r_m-1} E_m \end{bmatrix}, \quad (8)$$

where c_i is the i-*th* row of matrix C, and E_j is the j-*th* column of matrix E.

Definition 7 ([2]*)*
The minimal dwell time is a number $T_\delta > 0$ such that the class of admissible switching signals satisfy the property that the switching times $t_1,\ t_2,\ \ldots$ fulfill the inequality $t_{j+1} - t_j \geqslant T_\delta$ for all j.

In this paper, we study the systems whose hybrid time trajectories satisfy the minimal dwell time definition. Moreover, it is assumed that the dwell time is sufficiently large, or it is possible to estimate it (see, e.g., [49] for the algebraic estimation of the switching times for LSS).

3. SYSTEM TRANSFORMATION

Based on the previous definitions, the following assumption ensures the possibility for state estimation:

Assumption 1
All the $(A_{j(x(t))}, C_{j(x(t))})$-unobservable eigenvalues satisfy $Re\{\lambda\} < 0$, for all $j \in \mathcal{Q}$ and $t \geqslant 0$.

It is clear that, as a consequence of Assumption 1, each subsystem is detectable. Even more, the satisfaction of the aforementioned assumption ensures that the set of unstable invariant zeroes does not belong to the set of $(A_{j(x(t))}, C_{j(x(t))})$-unobservable eigenvalues, for all $j \in \mathcal{Q}$.

Now, a suitable transformation to decompose the system into the strongly observable part, the non-strongly observable part, and the unobservable part is applied to each operating mode of system (1) (see, e.g., [50] and [48]). For simplicity, there will be omitted the index of the discrete state. However, the transformation is applied for each dynamics generated by the discrete state.

Firstly, let us calculate a basis of the weakly unobservable subspace $\mathcal{V}^*$ by means of the computation of the matrices M_i defined by the following recursive algorithm[**]:

$$M_{i+1} = N_{i+1}^{\perp\perp} N_{i+1}, \qquad M_1 = C,$$

$$N_{i+1} = G_i \begin{pmatrix} C \\ M_i A \end{pmatrix}, \quad G_i = \begin{pmatrix} 0_{p\times q} \\ M_i E \end{pmatrix}^{\perp}.$$

The algorithm ends when $\text{rank}(M_{i+1}) = \text{rank}(M_i)$. Therefore, it is possible to define $M_n = M_{n-1} = \ldots = M_i$. It was proven in [50] that $\mathcal{V}^* = \ker(M_n)$. Now, define $n_{\mathcal{V}} := \text{rank}(M_n)$ with $M_n \in \Re^{n_{\mathcal{V}} \times n}$. Then, form the matrix $V \in \Re^{n \times (n-n_{\mathcal{V}})}$ whose columns form a basis of $\mathcal{V}^*$.

Secondly, assume that the following assumption is satisfied:

Assumption 2
The output of system (1) has a relative degree vector $(r_1, \ldots, r_m)$ such that $r_1 + \cdots + r_m = n_{\mathcal{V}}$.

[**]The matrix $N_{i+1}^{\perp\perp}$ is introduced to exclude the linearly dependent terms of N_{i+1}. Therefore, M_{i+1} has full row rank (see [48]).

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Remark 1
Assumption 2 is satisfied for all system, with full relative degree [51]. If it is not satisfied, it is possible to apply alternative methods such as Molinari's algorithm [48].

According to Definition 6 and Assumption 2, it is possible to form the following matrix $U \in \Re^{n_{\mathcal{V}} \times n}$

$$U = \begin{bmatrix} c_1^T, & (c_1 A)^T, & \cdots, & \left(c_1 A^{r_1 - 1}\right)^T, & \cdots, & c_m^T, & (c_m A)^T, & \cdots, & \left(c_m A^{r_m - 1}\right)^T \end{bmatrix}^T. \tag{9}$$

It is easy to see that $\text{rank}(U) = n_{\mathcal{V}}$. Now, from the matrix U, form the following matrices $U_1 \in \Re^{n_{\mathcal{V}} - m \times n}$ and $U_2 \in \Re^{m \times n}$

$$U_1 = \begin{bmatrix} c_1^T, & (c_1 A)^T, & \cdots, & \left(c_1 A^{r_1 - 2}\right)^T, & \cdots, & c_m^T, & (c_m A)^T, & \cdots, & \left(c_m A^{r_m - 2}\right)^T \end{bmatrix}^T, \tag{10}$$

$$U_2 = \begin{bmatrix} \left(c_1 A^{r_1 - 1}\right)^T, & \cdots, & \left(c_m A^{r_m - 1}\right)^T \end{bmatrix}^T. \tag{11}$$

Finally, form the matrix N whose columns form a basis of the unobservable subspace $\mathcal{N}$. It is clear by Definition 5 that $\mathcal{N} \subset \mathcal{V}^*$. Therefore, it is possible to select a full column rank matrix V forming a basis of $\mathcal{V}^*$ adapted to $\mathcal{N}$, that is,

$$V = \begin{bmatrix} V_{\bar{\mathcal{N}}}, & N \end{bmatrix}. \tag{12}$$

Define $n_{\mathcal{N}} = \dim(\mathcal{N})$. Then, $V_{\bar{\mathcal{N}}} \in \Re^{n \times (n - n_{\mathcal{V}} - n_{\mathcal{N}})}$ and $N \in \Re^{n \times n_{\mathcal{N}}}$. Moreover, matrix V satisfied the following equalities

$$AV + EK^* = VQ \;\; \Leftrightarrow \;\; \left(A + E\bar{K}^*\right) V = VQ, \tag{13}$$

$$CV = 0, \tag{14}$$

for some matrices $\bar{K}^* \in \Re^{m \times n}$, $K^* \in \Re^{m \times (n - n_{\mathcal{V}})}$, and $Q \in \Re^{(n - n_{\mathcal{V}}) \times (n - n_{\mathcal{V}})}$. Notice that (13)–(14) are the matrix representations of the map (5), and that $V^+ V = I$ implies $\bar{K}^* = K^* V^+$, satisfying (13).

The following non-singular transformation matrix can be defined:

$$T := \begin{bmatrix} U_1^T, & U_2^T, & \left(V_{\bar{\mathcal{N}}}{}^+\right)^T, & (N^+)^T \end{bmatrix}^T. \tag{15}$$

The transformation $\bar{x}(t) = Tx(t)$, with matrix T designed according to (15), transforms system (1) into the following form:

$$\begin{bmatrix} \dot{\bar{x}}_{11}(t) \\ \dot{\bar{x}}_{12}(t) \\ \dot{\bar{x}}_{21}(t) \\ \dot{\bar{x}}_{22}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \bar{x}_{11}(t) \\ \bar{x}_{12}(t) \\ \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ E_{12} \\ E_{21} \\ E_{22} \end{bmatrix} \bar{f}(t), \tag{16}$$

$$y(t) = C_1 \begin{bmatrix} \bar{x}_{11}(t)^T, & \bar{x}_{12}(t)^T \end{bmatrix}^T, \tag{17}$$

$$\bar{f}(t) = f(t) - K_1^* \bar{x}_{21}(t), \tag{18}$$

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where $\bar{x}_{11}(t) \in \Re^{n_\nu - m}$, $\bar{x}_{12}(t) \in \Re^{m}$, $\bar{x}_{21}(t) \in \Re^{n - n_\nu - n_\mathcal{N}}$, $\bar{x}_{22}(t) \in \Re^{n_\mathcal{N}}$, $K_1^* \in \Re^{m \times (n - n_\nu - n_\mathcal{N})}$, and

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} = T\left(A + E\bar{K}^*\right)T^{-1}, \quad \begin{bmatrix} 0 \\ E_{12} \\ E_{21} \\ E_{22} \end{bmatrix} = TE,$$

$$C_1 = C\begin{bmatrix} \left(U_1^+\right)^T, & \left(U_2^+\right)^T \end{bmatrix}^T,$$

$$K^* = \begin{bmatrix} K_1^*, & 0 \end{bmatrix}, \begin{bmatrix} B_1^T, & B_2^T, & B_3^T, & B_4^T \end{bmatrix}^T = TB.$$

For systems (16)–(18), it is possible to demonstrate that the set of invariant zeroes that do not belong to unobservable subspace $\mathcal{N}$ is equal to the set of eigenvalues of the matrix A_{33}, and the set of invariant zeroes that belong to unobservable subspace $\mathcal{N}$ is equal to the set of eigenvalues of the matrix A_{44} (see, e.g., [52]). Notice that this transformation should be computed for each operating mode $j \in \mathcal{Q}$.

4. OBSERVER DESIGN

The continuous and discrete state observers have been previously presented in [1]. However, in this version, the detailed proofs of the theorems are described.

Consider the state estimation problem for a constant index $j(x(t)) = j^* = \text{const}$. Let us describe the observer design for each partition of the state $\bar{x}(t)$.

4.1. *State observer for $\bar{x}_{11}(t)$ and $\bar{x}_{12}(t)$*

Consider the first two parts of systems (16)–(18) for $j(x(t)) = j^*$ with the partition $\bar{x}_1(t) = \begin{bmatrix} \bar{x}_{11}(t)^T, & \bar{x}_{12}(t)^T \end{bmatrix}^T$, that is,

$$\begin{aligned} \dot{\bar{x}}_1(t) &= A_{1j^*}\bar{x}_1(t) + E_{1j^*}\bar{f}_1(t) + B_{12j^*}u(t), \\ y(t) &= C_{1j^*}\bar{x}_1(t), \end{aligned} \tag{19}$$

where

$$A_{1j^*} = \begin{bmatrix} A_{11j^*} & A_{12j^*} \\ A_{21j^*} & A_{22j^*} \end{bmatrix}, \quad E_{1j^*} = \begin{bmatrix} 0 \\ E_{12j^*} \end{bmatrix}, \quad B_{12j^*} = \begin{bmatrix} B_{1j^*} \\ B_{2j^*} \end{bmatrix}.$$

In accordance with the structure of the transformation, the states $\bar{x}_1(t)$ form the strongly observable subspace. Then, the following observer for $\bar{x}_1(t)$ could be designed (see [31])

$$\hat{x}_{1_{j^*}}(t) = z_{1_{j^*}}(t) + P_{1_{j^*}}^{-1}\nu_{j^*}(t), \tag{20}$$

$$\dot{z}_{1_{j^*}}(t) = A_{1j^*}z_{1j^*}(t) + B_{12j^*}u(t) + L_{1j^*}\left(y(t) - C_{1j^*}z_{1j^*}(t)\right), \tag{21}$$

$$\dot{\nu}_{j^*}(t) = W_{j^*}\left(y(t) - C_{1j^*}z_{1j^*}(t), \nu_{j^*}(t)\right), \tag{22}$$

where $z_{1_{j^*}}(t)$, $\hat{x}_{1_{j^*}}(t) \in \Re^{n_{\nu_{j^*}}}$, and the matrix $L_{1j^*} \in \Re^{n_{\nu_{j^*}} \times m}$ is selected such that matrix $\left(A_{1j^*} - L_{1j^*}C_{1j^*}\right) = A_{L_{1_{j^*}}}$ is Hurwitz[††]. The distribution matrix P_{1j^*} takes the following structure

[††]Because of Assumption 2 and Definition 6, such matrix $L_{1_{j^*}}$ always exist.

$$P_{1j^*} = \Big[\ c_{1j^*}^T,\ \ \left(c_{1j^*}A_{L_1j^*}\right)^T,\ \ \cdots,\ \ \left(c_{1j^*}A_{L_1j^*}^{r_{1j^*}-1}\right)^T,\ \ \cdots,\\ c_{mj^*}^T,\ \ \left(c_{mj^*}A_{L_1j^*}\right)^T,\ \ \cdots,\ \ \left(c_{mj^*}A_{L_1j^*}^{r_{mj^*}-1}\right)^T\ \Big]^T. \tag{23}$$

According to Assumption 2 the condition $\operatorname{rank}\left(P_{1j^*}\right) = n_{\nu_{j^*}}$ is satisfied. The vector $\nu_{j^*}(t) = \left(\nu_{1_{1j^*}},\ldots,\nu_{1_{r_{1j^*}}},\nu_{2_{1j^*}},\ldots,\nu_{2_{r_{2j^*}}},\ldots,\nu_{m_{1j^*}},\ldots,\nu_{m_{r_{mj^*}}}\right)$ and the nonlinear function W_{j^*} are chosen using the HOSM differentiator (for more details, see [53])

$$\begin{aligned}
\dot{\nu}_{k_{1j^*}} &= \nu_{k_{2j^*}} - \alpha_{k_{1j^*}} M_{kj^*}(t)^{\frac{1}{r_{kj^*}}} \left\lceil \nu_{k_{1j^*}} - e_{y_{kj^*}} \right\rfloor^{\frac{r_{kj^*}-1}{r_{kj^*}}},\\
\dot{\nu}_{k_{ij^*}} &= \nu_{k_{(i+1)j^*}} - \alpha_{k_{ij^*}} M_{kj^*}(t)^{\frac{1}{r_{kj^*}-i+1}} \left\lceil \nu_{k_{ij^*}} - \dot{\nu}_{k_{(i-1)j^*}} \right\rfloor^{\frac{r_{kj^*}-i}{r_{kj^*}-i+1}},\ i = 2,\ldots,r_{kj^*}-1,\\
\dot{\nu}_{k_{r_{kj^*}}} &= -\alpha_{k_{r_{kj^*}}} M_{kj^*}(t) \left\lceil \nu_{k_{r_{kj^*}}} - \dot{\nu}_{k_{r_{kj^*}-1}} \right\rfloor^{0},\ \forall k = 1,\ldots,m,
\end{aligned} \tag{24}$$

where $e_{y_{kj^*}}(t) = y_k(t) - c_k z_{1j^*}(t)$, and the constants $\alpha_{k_{ij^*}}$ are chosen recursively and sufficiently large. In particular, according to [54], one possible choice is $\alpha_{k_{6j^*}} = 1.1$, $\alpha_{k_{5j^*}} = 1.5$, $\alpha_{k_{4j^*}} = 2$, $\alpha_{k_{3j^*}} = 3$, $\alpha_{k_{2j^*}} = 5$, and $\alpha_{k_{1j^*}} = 8$, which is enough for the case that $r_{kj^*} \leqslant 6$, $\forall k = 1,\ldots,m$.

Remark 2

According to [54], if $y_k(t)$ is a $\left(r_{kj^*}\right)$-times continuously differentiable signal with a bounded Lebesgue measurable noise, $\upsilon_k(t) \in \mathcal{L}_\infty$. Then, there exist $0 \leqslant t_{j^*} < \infty$ and constants $\mu_{k_{ij^*}}$, only dependent on $\alpha_{k_{ij^*}}$ and M_{kj^*}, such that for all $t \geqslant t_{j^*}$, the $\nu_{k_{ij^*}}$ in (24) satisfy

$$\left|\nu_{k_{(i+1)j^*}} - e_{y_{kj^*}}^{(i)}\right| \leqslant \mu_{k_{ij^*}} \left\|\upsilon_k\right\|_\infty^{\frac{r_{kj^*}-i+1}{r_{kj^*}+1}},\ \forall i = 0,\ldots,r_{kj^*},\ \forall k = 1,\ldots,m.$$

The continuous functions $M_{kj^*}(t)$ are known locally as Lipschitz constants, at time t, for each $e_{y_{kj^*}}^{\left(r_{kj^*}\right)}$, and they can be computed in the following way.

Proposition 1

There exist a time $\tilde{t}$ and known positive functions $\beta_{k_{1j^*}}(t)$, and constants $\beta_{k_{2j^*}}$, $\beta_{k_{3j^*}}$, $k_{L_{1j^*}}$, and $\lambda_{L_{1j^*}}$ such that

$$\left|e_{y_{kj^*}}^{(r_{kj^*})}(t)\right| \leqslant M_{kj^*}(t),\ \ \forall t > \tilde{t}, \tag{25}$$

with

$$M_{kj^*}(t) = \beta_{k_{1j^*}}(t) + \beta_{k_{2j^*}} \int_0^t \exp\left(-\lambda_{L_{1j^*}}(t-\tau)\right) \left\|\bar{f}(\tau)\right\| d\tau + \beta_{k_{3j^*}},\ \forall k = 1,\ldots,m. \tag{26}$$

Proof

Because matrix $A_{L_1j^*}$ is Hurwitz, there exist known positive constants $k_{L_{1j^*}}$, $\lambda_{L_{1j^*}}$ such that $\left\|\exp\left(A_{L_1j^*}t\right)\right\| \leqslant k_{L_{1j^*}} \exp\left(-\lambda_{L_{1j^*}}t\right)$. Therefore, from the solution of the estimation error for observer (21), that is, $\bar{e}_{1j^*}(t) = \bar{x}_1(t) - z_{1j^*}(t)$, see (27), it is obtained that

$$\left\|\bar{e}_{1j^*}(t)\right\| \leqslant k_{L_{1j^*}} \exp\left(-\lambda_{L_{1j^*}}t\right) \left\|\bar{e}_{1j^*}(0)\right\| + \left\|E_{1j^*}\right\| \int_0^t \exp\left(-\lambda_{L_{1j^*}}(t-\tau)\right) \left\|\bar{f}(\tau)\right\| d\tau.$$

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For an arbitrary constant $\lambda_{L_{1j*}}$, there exists a time $\tilde{t}$ such that $k_{L_{1_{j*}}} \exp\left(-\lambda_{L_{1j*}} t\right) \left\|\bar{e}_{1_{j*}}(0)\right\|$ is less than $\Gamma_{j*}(t),\ \forall t \geqslant \tilde{t}$. Thus,

$$\left\|\bar{e}_{1_{j*}}(t)\right\| \leqslant \Gamma_{j*}(t) + k_{L_{1_{j*}}} \left\|E_{1_{j*}}\right\| \int_0^t \exp\left(-\lambda_{L_{1j*}}(t-\tau)\right) \left\|\bar{f}(\tau)\right\| d\tau.$$

From the earlier inequality and the output error dynamics $e_{y_{k_{j*}}}^{\left(r_{k_{j*}}\right)}(t)$, see (30), the following inequality is obtained:

$$\left|e_{y_{k_{j*}}}^{\left(r_{k_{j*}}\right)}(t)\right| \leqslant |c_k| \left\|A_{L_{1_{j*}}}^{r_{k_{j*}}}\right\| \left(\Gamma_{j*}(t) + k_{L_{1_{j*}}} \left\|E_{1_{j*}}\right\| \int_0^t \exp\left(-\lambda_{L_{1j*}}(t-\tau)\right) \left\|\bar{f}(\tau)\right\|\right)$$
$$+ |c_k| \left\|A_{L_{1_{j*}}}^{r_{k_{j*}}-1}\right\| \left\|E_{1_{j*}}\right\| \left\|\bar{f}(t)\right\|.$$

Thus, Proposition 1 is proven with $\beta_{k_{1j*}}(t) = |c_k| \left\|A_{L_{1_{j*}}}^{r_{k_{j*}}}\right\| \Gamma_{j*}(t)$, $\beta_{k_{2j*}} = k_{L_{1_{j*}}} |c_k| \left\|A_{L_{1_{j*}}}^{r_{k_{j*}}}\right\| \left\|E_{1_{j*}}\right\|$, and $\beta_{k_{3j*}} = |c_k| \left\|A_{L_{1_{j*}}}^{r_{k_{j*}}-1}\right\| \left\|E_{1_{j*}}\right\| \left\|\bar{f}(t)\right\|$. □

Taking into account the previous explanations, the following theorem can be stated.

Theorem 1
Let $j(x(t)) = j^* =$ const, and observers (20)–(22) with the correction terms (24) be applied to system (19). Let Assumptions 1–2 be satisfied. Then, provided that constants $\alpha_{k_{i_{j*}}}$ are chosen properly and $M_{k_{j*}}(t)$ are selected as in Proposition 1, the state estimation error for $\bar{x}_1(t)$ converges to 0 exactly and in a finite time, that is, $e_{1_{j*}}(t) = \bar{x}_1(t) - \hat{x}_{1_{j*}}(t) = 0\ \forall t \in \left[t_{j*}, t_1\right)$.[‡‡]

Proof
Let us define the state estimation error $\bar{e}_{1_{j*}}(t) = \bar{x}_1(t) - z_{1j*}(t)$. The dynamics of $\bar{e}_{1_{j*}}(t)$ is given by

$$\dot{\bar{e}}_{1_{j*}}(t) = A_{L_{1j*}}\bar{e}_{1_{j*}}(t) + E_{1j*}\bar{f}(t), \tag{27}$$

$$e_{y_{j*}}(t) = C_{1j*}e_{y_{j*}}(t). \tag{28}$$

Because of Assumption 2, it is always possible to compute matrix L_{1j*} such that the matrix $A_{L_{1j*}}$ is Hurwitz.

Applying the transformation P_{1j*} to the estimation error $\bar{e}_{1_{j*}}$, system (27) can be transformed into

$$P_{1j*}\dot{\bar{e}}_{1_{j*}}(t) = P_{1j*}A_{L_{1j*}}\bar{e}_{1_{j*}}(t) + P_{1j*}E_{1j*}\bar{f}(t). \tag{29}$$

According to Assumption 2 and the structure of the transformation, (29) can be rewritten as

$$\begin{bmatrix} \dot{e}_{y_{1_{j*}}}(t) \\ \vdots \\ e_{y_{1_{j*}}}^{\left(r_{1_{j*}}\right)}(t) \\ \vdots \\ \dot{e}_{y_{m_{j*}}}(t) \\ \vdots \\ e_{y_{m_{j*}}}^{\left(r_{m_{j*}}\right)}(t) \end{bmatrix} = \begin{bmatrix} c_{1j*}A_{L_{1j*}} \\ \vdots \\ c_{1j*}A_{L_{1_{j*}}}^{r_{1_{j*}}} \\ \vdots \\ c_{mj*}A_{L_{1j*}} \\ \vdots \\ c_{mj*}A_{L_{1_{j*}}}^{r_{m_{j*}}} \end{bmatrix} \bar{e}_{1_{j*}}(t) + \begin{bmatrix} 0 \\ \vdots \\ c_{1j*}A_{L_{1_{j*}}}^{r_{1_{j*}}-1}E_{1j*} \\ \vdots \\ 0 \\ \vdots \\ c_{mj*}A_{L_{1_{j*}}}^{r_{m_{j*}}-1}E_{1j*} \end{bmatrix} \bar{f}(t). \tag{30}$$

[‡‡] t_{j*} is the time when observers (20)–(22) has converged to 0, and t_1 is the first switching time.

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Notice that the derivatives of $e_{y_{k_{j^*}}}(t)$ are estimated by the HOSM differentiator (24). Thus, if $M_{k_{j^*}}(t)$ are selected as in Proposition 1, the differentiator converges (for more details, see [53]), therefore

$$P_{1j^*}\bar{e}_{1_{j^*}}(t) = \nu_{j^*}(t). \tag{31}$$

The estimation of the variable $\bar{x}_1(t)$ is obtained by algebraic manipulation as

$$\hat{x}_{1_{j^*}}(t) = z_{1j^*}(t) + P_{1j^*}^{-1}\nu_{j^*}(t). \tag{32}$$

Therefore, the exact convergence to 0 for $e_{1_{j^*}}(t) = \bar{x}_1(t) - \hat{x}_{1_{j^*}}(t)$ is obtained. Notice that if there exists measurement noise, the state estimation error order will be directly proportional to order of the noise, as Remark 2 shows. □

Remark 3
Notice that it is possible to consider, in addition to the faults $f(t)$, disturbances with the same structural properties in its distribution matrix, that is, $\bar{f}(t) = f(t) + d(t) - K_1^*\bar{x}_{21}(t)$. Observers (20)–(22) will keep on estimating exactly and in finite time the state $\bar{x}_1$ provided that Proposition 1 is satisfied (taking into account bounded disturbances, i.e., $\|d(t)\| \leqslant d^+ < \infty$, with a known constant d^+).

4.2. State observer for $\bar{x}_{21}(t)$

Let $\hat{x}_{21_{j^*}}(t)$ be the state observer for $\bar{x}_{21}(t)$ defined by

$$\hat{x}_{21_{j^*}}(t) = z_{2j^*}(t) + L_{2j^*}\hat{x}_{12_{j^*}}(t), \tag{33}$$

$$\dot{z}_{2j^*}(t) = \bar{A}_{1_{j^*}}\hat{x}_{11_{j^*}}(t) + \bar{A}_{2_{j^*}}\hat{x}_{12_{j^*}}(t) + A_{L_{2j^*}}\hat{x}_{21_{j^*}}(t) + \bar{B}_{2_{j^*}}u(t), \tag{34}$$

where the estimations of $\hat{x}_{11_{j^*}}(t)$ and $\hat{x}_{12_{j^*}}(t)$ are provided by observers (20)–(22). The matrices in (34) are defined by

$$\bar{A}_{1_{j^*}} = A_{31j^*} - L_{2j^*}A_{21j^*}, \tag{35}$$

$$\bar{A}_{2_{j^*}} = A_{32j^*} - L_{2j^*}A_{22j^*}, \tag{36}$$

$$A_{L_{2j^*}} = A_{33j^*} - E_{21j^*}K_{1\,j^*}^* + L_{2j^*}E_{12j^*}K_{1\,j^*}^*, \tag{37}$$

$$\bar{B}_{2_{j^*}} = B_{3j^*} - L_{2j^*}B_{2j^*}. \tag{38}$$

Then, the following theorem can be stated.

Theorem 2
Let $j(x(t)) = j^* = \text{const}$ be satisfied. Then, provided that the matrix L_{2j^*} is selected such that the matrix $A_{L_{2j^*}}$ is Hurwitz, the state estimation error for $\bar{x}_{21}(t)$ is ultimately bounded by a constant $\gamma_{j^*}f^+$, that is, $\left\|e_{21_{j^*}}(t)\right\| = \left\|\bar{x}_{21}(t) - \hat{x}_{21_{j^*}}(t)\right\| \leqslant \gamma_{j^*}f^+$ as $t \to \infty$.

Proof
From the dynamics $\dot{\bar{x}}_{12}(t)$ in (16), it is obtained that

$$E_{12j^*}K_{1\,j^*}^*\bar{x}_{21}(t) = \left(A_{21j^*}\bar{x}_{11}(t) + A_{22j^*}\bar{x}_{12}(t) + B_{2j^*}u(t) + E_{12j^*}f(t)\right) - \dot{\bar{x}}_{12}(t). \tag{39}$$

Adding and subtracting $L_{2j^*}E_{12j^*}K_{1\,j^*}^*\bar{x}_{21}(t)$ in the dynamics $\dot{\bar{x}}_{21}(t)$, it is obtained that

$$\dot{\bar{x}}_{21}(t) = \bar{A}_{1_{j^*}}\bar{x}_{11}(t) + \bar{A}_{2_{j^*}}\bar{x}_{12}(t) + A_{L_{2j^*}}\bar{x}_{21}(t) + \bar{B}_{2_{j^*}}u + E_{L_{2j^*}}f(t) + L_{2j^*}\dot{\bar{x}}_{21}(t), \tag{40}$$

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where $E_{L_2 j^*} = E_{21 j^*} - L_{2 j^*} E_{12 j^*}$.

The dynamics error $e_{21_{j^*}}(t) = \bar{x}_{21}(t) - \hat{x}_{21_{j^*}}(t)$ is governed by

$$\dot{e}_{21_{j^*}}(t) = A_{L_2 j^*} e_{21_{j^*}}(t) + E_{L_2 j^*} f(t). \tag{41}$$

Thus, $L_{2 j^*}$ has to be selected such that the matrix $A_{L_2 j^*}$ is Hurwitz. Finally, it is clear that $e_{21_{j^*}}(t)$ is bounded by

$$\left\| e_{21_{j^*}}(t) \right\| \leqslant \gamma_{j^*} f^+, \text{ as } t \to \infty, \tag{42}$$

with $\gamma_{j^*} = \frac{k_{L_2 j^*} \left\| E_{L_2 j^*} \right\|}{\lambda_{L_2 j^*}}$, and the positive constants $k_{L_2 j^*}, \lambda_{L_2 j^*}$ such that $\left\| \exp\left(A_{L_2 j^*} t \right) \right\| \leqslant k_{L_2 j^*} \exp\left(-\lambda_{L_2 j^*} t \right)$.

Such matrix $L_{2 j^*}$ always exists if the pair $\left(E_{12 j^*} K^*_{1\, j^*}, A_{33 j^*} - E_{21 j^*} K^*_{1\, j^*} \right)$ is detectable. To prove this, consider the Rosenbrock matrix of system (16)–(17), without faults, that is,

$$R(s) = \left[\begin{array}{cc|c|c} sI - A_{11j^*} & -A_{12j^*} & 0 & 0 \\ C_{11j^*} & C_{12j^*} & 0 & 0 \\ -A_{21j^*} & sI - A_{22j^*} & E_{12j^*} K^*_{1j^*} & 0 \\ -A_{31j^*} & -A_{32j^*} & sI - A_{33j^*} + E_{21j^*} K^*_{1j^*} & 0 \\ -A_{41j^*} & -A_{42j^*} & -A_{43j^*} & sI - A_{44j^*} \end{array} \right].$$

Assumption 1 implies that A_{44j^*} is Hurwitz, and the rank of matrix $R(s)$ is full except for s being an eigenvalue of A_{44j^*}. Thus, it is clear that the rank of the first column is equal to $n_{\mathcal{V}_{j^*}}$ because the pair $\left(C_{1j^*}, A_{1j^*} \right)$ is observable; the rank of the third column is equal to $n_{\mathcal{N}_{j^*}}$ because of Assumption 1; then, the rank of the second column is equal to $n - n_{\mathcal{V}_{j^*}} - n_{\mathcal{N}_{j^*}}$ implying that the pair $\left(E_{12 j^*} K^*_{1\, j^*}, A_{33 j^*} - E_{21 j^*} K^*_{1\, j^*} \right)$ is detectable and that matrix $L_{2 j^*}$ always exists. □

Remark 4

If there exist disturbances $E_{L_2 j^*} d(t)$, the state estimation error $e_{21_{j^*}}(t)$ will be bounded by $\left\| e_{21_{j^*}}(t) \right\| \leqslant \gamma_{j^*} f^+ + \varsigma_{j^*} d^+$, as $t \to \infty$, with ς_{j^*} positive known constant.

Remark 5

It is possible to formulate a constrained linear optimization problem in order to select the matrix $L_{2 j^*}$ in such a way that the estimation error $e_{21_{j^*}}(t)$ is minimized.

4.3. State observer for $\bar{x}_{22}(t)$

Without loss of generality, it is assumed that

$$\text{rank} \left[\begin{array}{c} E_{12 j^*} \\ E_{21 j^*} \end{array} \right] = \text{rank}(E_{2 j^*}) = m.$$

Let $\hat{x}_{22_{j^*}}(t)$ be the state observer for $\bar{x}_{22}(t)$ defined by

$$\hat{x}_{22_{j^*}}(t) = z_{3 j^*}(t) + E_{22 j^*} E_{2\, j^*}^{+} \left[\begin{array}{c} \hat{x}_{12}(t) \\ \hat{x}_{21}(t) \end{array} \right], \tag{43}$$

$$\begin{aligned} \dot{z}_{3_{j^*}}(t) = {} & A_{41_{j^*}} \hat{x}_{11_{j^*}}(t) + A_{42_{j^*}} \hat{x}_{12_{j^*}}(t) + A_{43_{j^*}} \hat{x}_{21_{j^*}}(t) + A_{44_{j^*}} \hat{x}_{22_{j^*}} + B_{4 j^*} u_{j^*}(t) \\ & - E_{22 j^*} E_{2\, j^*}^{+} \left[\begin{array}{c} \left(A_{21_{j^*}} \hat{x}_{11_{j^*}}(t) + A_{22_{j^*}} \hat{x}_{12_{j^*}}(t) + B_{2_{j^*}} u(t) \right) \\ \left(A_{31_{j^*}} \hat{x}_{11_{j^*}}(t) + A_{32_{j^*}} \hat{x}_{12_{j^*}}(t) + A_{33_{j^*}} \hat{x}_{21_{j^*}}(t) + B_{3_{j^*}} u(t) \right) \end{array} \right], \end{aligned} \tag{44}$$

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where the estimations of $\hat{x}_{11_{j*}}(t) - \hat{x}_{12_{j*}}(t)$, and $\hat{x}_{21_{j*}}(t)$ are provided by observers (20)–(22) and (33)–(34), respectively.

Then, the following theorem can be stated.

Theorem 3

Let $j(x(t)) = j^* =$const and rank$(E_{2j^*}) = m$ be satisfied. Then, the state estimation error for $\bar{x}_{22}(t)$ is ultimately bounded by a constant $\delta_{j^*} f^+$, that is, $\left\| e_{22_{j*}}(t) \right\| = \left\| \bar{x}_{22}(t) - \hat{x}_{22_{j*}}(t) \right\| \leqslant \delta_{j^*} f^+$ as $t \to \infty$.

Proof

Because rank$(E_{2j^*}) = m$, from (16), $\bar{f}(t)$ can be rewritten as

$$\bar{f}(t) = E_{22j^*} E_{2j^*}^+ \begin{bmatrix} \dot{\bar{x}}_{12}(t) - \left(A_{21_{j*}} \bar{x}_{11}(t) + A_{22_{j*}} \bar{x}_{12}(t) + B_{2_{j*}} u(t)\right) \\ \dot{\bar{x}}_{21}(t) - \left(A_{31_{j*}} \bar{x}_{11}(t) + A_{32_{j*}} \bar{x}_{12}(t) + A_{33_{j*}} \bar{x}_{21}(t) + B_{3_{j*}} u(t)\right) \end{bmatrix}.$$

The dynamics error $e_{22_{j*}}(t) = \bar{x}_{22}(t) - \hat{x}_{22_{j*}}(t)$, substituting $\bar{f}(t)$, is governed by

$$\dot{e}_{22_{j*}}(t) = A_{44j^*} e_{22_{j*}}(t) + A_{43j^*} e_{21_{j*}}(t). \tag{45}$$

Because A_{44j^*} is Hurwitz, using (42), the error $e_{22_{j*}}(t)$ is bounded by

$$\left\| e_{22_{j*}}(t) \right\| \leqslant \delta_{j^*} f^+, \text{ as } t \to \infty, \tag{46}$$

with $\delta_{j^*} = \frac{k_{44_{j*}} \|A_{43j^*}\|}{\lambda_{44_{j*}}} \gamma_{j^*}$, and the positive constants $k_{44_{j*}}, \lambda_{44_{j*}}$ such that $\left\| \exp\left(A_{44j^*} t\right) \right\| \leqslant k_{44_{j*}} \exp(-\lambda_{44_{j*}} t)$. □

Remark 6

If there exist disturbances $E_{22j^*} d(t)$, the state estimation error $e_{22_{j*}}(t)$ will be bounded by $\left\| e_{22_{j*}}(t) \right\| \leqslant \delta_{j^*} f^+ + \varrho_{j^*} d^+$, *as* $t \to \infty$, with ϱ_{j^*} positive known constant.

4.4. *Bank of observers*

To solve the continuous state estimation problem for the NMP system (1), the following bank of observers is proposed

$$\tilde{x}_{1_\lambda}(t) = z_{1\lambda}(t) + P_{1_\lambda}^{-1} \nu_\lambda(t), \tag{47}$$

$$\tilde{x}_{21_\lambda}(t) = z_{2\lambda}(t) + L_{2\lambda} \tilde{x}_{12_\lambda}(t), \tag{48}$$

$$\tilde{x}_{22_\lambda}(t) = z_{3\lambda}(t) + E_{22\lambda} E_{2\lambda}^+ \begin{bmatrix} \tilde{x}_{12}(t) \\ \tilde{x}_{21}(t) \end{bmatrix}, \tag{49}$$

$$\tilde{y}(t) = C_{1\lambda} \tilde{x}_1(t), \; \forall \lambda = 1, \ldots, q, \tag{50}$$

where $\tilde{x}_{1_\lambda}(t)$, $\tilde{x}_{21_\lambda}(t)$, $\tilde{x}_{22_\lambda}(t)$, and their components are designed according to Section 4. Now, the following assumption is stated.

Assumption 3

Assume that the initial discrete state is known.

Theorems 1–3 establish that the j^* observer reconstructs the continuous state correctly, and according to Assumption 3, we know the j^* observer has made it on the interval $[0, t_1)$. According

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to [55], to know when the j^* observer has converged, it is sufficient to verify that the following inequality is satisfied:

$$\left\| e_{\tilde{y}_{j^*}}(t) \right\| \leqslant \xi_{j^*} M_{j^*}^{+} h^{n\nu_{j^*}}, \ \forall t \in \left[t_1 - \xi_{j^*_t} h, t_1\right), \tag{51}$$

where $e_{\tilde{y}_{j^*}} = y - \tilde{y}_{j^*}$, ξ_{j^*}, and $\xi_{j^*_t}$ are positive constants; h is the sampling time, and $M_{j^*}^{+} = \max_{\forall k=1,\dots,m}(M_{k_{j^*}}(t))$. It is natural to estimate the constants ξ_{i^*} and $\xi_{i^*_t}$ through simulation. Thus, in this way, it is possible to determine when the j^**-th* observer has converged to the correct continuous state during the time interval $t \in [0, t_1)$.

Then, the real estimated state $\hat{x}$ is defined as follows:

$$\hat{x} = \tilde{x}_{j^*}, \ \ \forall t \in [t_{j^*}, t_1).^{\S\S} \tag{52}$$

Because of the transformation $\bar{x}(t) = Tx(t)$, the bank of observers for the original state vector has to be designed as follows:

$$\hat{x}(t) = T_{\lambda}^{-1} \left[\ \tilde{x}_{11_\lambda}^{T}(t), \ \ \tilde{x}_{12_\lambda}^{T}(t), \ \ \tilde{x}_{21_\lambda}^{T}(t), \ \ \tilde{x}_{22_\lambda}^{T}(t) \ \right]^{T}. \tag{53}$$

Theorem 4
The original state estimation error generated by observer (53) and system (1) is ultimately bounded by a positive constant $\rho_\lambda f^{+}$, that is,

$$\|\hat{x}(t) - x(t)\| \underset{t\to\infty}{\leqslant} \rho_\lambda f^{+}. \tag{54}$$

Proof
It is clear from Theorems (1), (2), and (3), and the coordinates combination that the error generated by the estimation properties in the transformed coordinates is propagated in the original coordinates, producing $\|\hat{x}(t) - x(t)\| \leqslant \rho_\lambda f^{+}$ with a positive constant ρ_λ defined by δ_λ. □

5. DISCRETE STATE ESTIMATION

Once the continuous state is estimated, the following discrete state observer is proposed:

$$\lambda(\hat{x}) = \begin{cases} 1, & \forall \hat{x} \mid \quad H\hat{x} \in \mathcal{H}_1 \\ 2, & \forall \hat{x} \mid \quad H\hat{x} \in \mathcal{H}_2 \\ \vdots & \\ q, & \forall \hat{x} \mid \quad H\hat{x} \in \mathcal{H}_q \end{cases}. \tag{55}$$

For each discrete state j, define the observability matrix of the discrete state as

$$Q_j = \left[\ H^T, \ \ (HA_j)^T, \ \ \cdots, \ \ \left(HA_j^{n-1}\right)^T \ \right]^T.$$

The exactly and finite time discrete state estimation is described by the following lemma.

Lemma 1
Under the statement of Theorem 1, the discrete state $j(x(t))$ is estimated exactly and in finite time if the following condition is satisfied

$$\mathcal{V}_j^* \subset ker(Q_j), \ \forall j = 1, \dots, q, \tag{56}$$

where $\mathcal{V}_j^*$ is the weakly unobservable subspace for each j.

§§Notice that it is always possible to design the gain $M_{j^*}(t)$ in such a way that inequality $t_{j^*} < t_1$ is satisfied.

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Proof
If (56) is satisfied, then there exists a matrix $P_{Q_j} \in \Re^{n\times n\nu_j}$ such that $Hx(t) = HT_jP_{Q_j}\bar{x}_1(t)$. According to exact and finite time convergence properties of $\hat{x}_1(t)$ described in Theorem 1 of Section 4.1, the discrete state can be estimated exactly and in finite time by means of (55) with $Hx(t) = HT_jP_{Q_j}\bar{x}_1(t)$. □

Remark 7
Condition (56) implies that an exact estimation of the discrete state is achieved only if the scalar function $j(x(t))$ can be represented as a combination of the strongly observable states, that is, $\bar{x}_{11}(t)$ and $\bar{x}_{12}(t)$, as a consequence of the observer properties described by Theorem 1.

5.1. State estimation on switching times

Let $t_i{}^+$ be the time instants after the switching times t_i. In order to maintain the state estimation on the switching times, the following algorithm is proposed.

Proposition 2
The state estimation of system (1) is maintained in spite of the switchings if the following reset equations are implemented in the bank of observers (47)–(50) for all $\lambda \neq j^*$

$$\nu_\lambda\left(t_i{}^+\right) = 0, \tag{57}$$

$$z_{1_\lambda}\left(t_i{}^+\right) = \tilde{x}_{1_{j*}}(t_i), \tag{58}$$

$$z_{2_\lambda}\left(t_i{}^+\right) = \tilde{x}_{21_{j*}}(t_i) - L_{2_{j*}}\tilde{x}_{12_{j*}}(t_i), \tag{59}$$

$$z_{3_\lambda}\left(t_i{}^+\right) = \tilde{x}_{22_{j*}}(t_i) - E_{22j^*}E_{2_{j^*}}^+\begin{bmatrix}\tilde{x}_{12}(t_i)\\ \tilde{x}_{21}(t_i)\end{bmatrix}. \tag{60}$$

Proof
In accordance with Theorem 1, $\bar{x}_1$ is estimated exactly and in finite time by means of $\tilde{x}_{1_{j*}}$. Thus, the following equation is stated:

$$\nu_{j^*}\left(t_i{}^+\right) = P_{1j^*}\left(\tilde{x}_{1_{j*}}(t_i) - z_{1j^*}(t_i{}^+)\right). \tag{61}$$

If reset equation (57) is applied to (24) on each switching time, then the trajectories always remain in the sliding surface, and the following equality is established:

$$\nu_{j^*}\left(t_i{}^+\right) = P_{1j^*}\left(\tilde{x}_{1_{j*}}(t_i) - z_{1j^*}(t_i{}^+)\right) = 0. \tag{62}$$

Applying the reset equation (58) to each dynamics (21) for all $\lambda \neq j^*$, (62) is satisfied in each switching time t_i. Thus, the state estimation for $\bar{x}_1$ is maintained in spite of the switchings.

From (33), to maintain the estimation for $\bar{x}_{21}(t)$, the following equality has to be satisfied:

$$\tilde{x}_{21_{j*}}(t_i) = z_{2_{j*}}(t_i{}^+) + L_{2_{j*}}\tilde{x}_{12_{j*}}(t_i). \tag{63}$$

Applying reset equation (59) to (33) for all $\lambda \neq j^*$, (63) is satisfied, and e_{21_λ} remains bounded as in (42).

From (43), to maintain the estimation for $\bar{x}_{22}(t)$, the following equality has to be satisfied:

$$\tilde{x}_{22_{j*}}(t_i) = z_{3_{j*}}(t_i{}^+) + E_{22j^*}E_{2_{j^*}}^+\begin{bmatrix}\tilde{x}_{12}(t_i)\\ \tilde{x}_{21}(t_i)\end{bmatrix}. \tag{64}$$

Applying reset equation (60) to (43) for all $\lambda \neq j^*$, (64) is satisfied, and e_{22_λ} remains bounded as in (46). □

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6. FAULT DETECTION AND IDENTIFICATION

The FDId scheme is based on the bank of observers (47)–(50) and discrete state observer (55). Once the whole state has been estimated, it is possible to make a decision test on the occurrence and a possible FId.[¶¶]

6.1. Fault detection scheme

From dynamics $\bar{x}_{12}(t)$ in (16), it is obtained that

$$\dot{\bar{x}}_{12}(t) - \left(A_{21_{j(\bar{x})}}\bar{x}_{11}(t) + A_{22_{j(\bar{x})}}\bar{x}_{12}(t) + B_{2_{j(\bar{x})}}u + E_{12_{j(\bar{x})}}\bar{f}(t)\right) = 0. \tag{65}$$

Substituting $\bar{f}(t)$ and the estimated state in (65), in the fault-free case, the following expression is obtained

$$\begin{aligned}\left(\dot{\hat{x}}_{12}(t) + \dot{e}_{12}(t)\right) - \Big[A_{21_{\lambda(\hat{x})}}\left(\hat{x}_{11}(t) + e_{11}(t)\right) + A_{22_{\lambda(\hat{x})}}\left(\hat{x}_{12}(t) + e_{12}(t)\right)\\ + B_{2_{\lambda(\hat{x})}}u - E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}\left(\hat{x}_{21}(t) + e_{21}(t)\right)\Big] = 0,\end{aligned} \tag{66}$$

where $e_{ij} = \bar{x}_{ij} - \hat{x}_{ij},\ i,j = 1,2$, are the estimation errors.

Let us define the following residual signal:

$$r_{\lambda(\hat{x})}(t) = \dot{\hat{x}}_{12}(t) - \left[A_{21_{\lambda(\hat{x})}}\hat{x}_{11}(t) + A_{22_{\lambda(\hat{x})}}\hat{x}_{12}(t) + B_{2_{\lambda(\hat{x})}}u - E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}\hat{x}_{21}(t)\right]. \tag{67}$$

Notice that all the variables of the right-hand expression in (67) are available except $\dot{\hat{x}}_{12}(t)$ that can be estimated using the differentiator (24) for $n_{\mathcal{V}} = 1$. In this way, it can be shown that

$$\left\|r(t)_{\lambda(\hat{x})}\right\| \leqslant \|\dot{e}_{12}(t)\| + \left\|A_{21_{\lambda(\hat{x})}}e_{11}(t)\right\| + \left\|A_{22_{\lambda(\hat{x})}}e_{12}(t)\right\| + \left\|E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}e_{21}(t)\right\|, \tag{68}$$

$$\left\|r(t)_{\lambda(\hat{x})}\right\| \leqslant r^{+}_{\lambda(\hat{x})}(t), \text{ a.e.,} \tag{69}$$

where $r^{+}_{\lambda(\hat{x})}(t)$ is a constant for each $\lambda(\hat{x})$, which depends on the sampling time h, the system, and observer parameters. It is possible to estimate each $r^{+}_{\lambda(\hat{x})}(t)$ just calculating, by simulation, the value of the right-hand expression in (67).

Remark 8

If there exist disturbances, it would be necessary to take into account the bound of the disturbances in the calculation of the threshold, that is, $\left\|r(t)_{\lambda(\hat{x})}\right\| \leqslant r^{+}_{\lambda(\hat{x})}(t) + \eta_{\lambda(\hat{x})}d^{+}$, with $\eta_{\lambda(\hat{x})}$ positive known constant for each $\lambda(\hat{x})$.

Then, the decision on the occurrence of a fault $f(t)$ is carried out when the norm of $r_{\lambda(\hat{x})}(t)$ exceeds its corresponding threshold $r^{+}_{\lambda(\hat{x})}(t)$.

Remark 9

Notice that (69) is related to the smallest detectable fault and the fault detectability condition. It is clear that those faults which magnitude is less than $r^{+}_{\lambda(\hat{x})}(t)$ will not be detectable.

Thus, the time instant t_f such that $\left\|r_{\lambda(\hat{x})}(t_f)\right\| > r^{+}_{\lambda(\hat{x})}(t)$ for $t_f > t_{j^*}$ with t_{j^*}, the time when the observer has converged, is defined as the FD time. Of course, this implies that it is only possible to detect the faults after the bank of observers (47)–(50) and (55) have converged.

The following theorem establishes the class of faults that can be detected by the proposed FD scheme.

[¶¶]The identification of faults implies to know the value and shape of the fault signal.

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Theorem 5
Let the bank of observers (47)–(50) and (55), and the FD decision scheme (69) be applied to system (16). For all time instant $t_f > t_{j^*}$ such that $f(t)$ satisfies

$$E_{12_{\lambda(\hat{x})}} f(t) \neq E_{12_{\lambda(\hat{x})}} K^*_{1_{\lambda(\hat{x})}} \int_{t_{j^*}}^{t_f} \exp\left(A_{L_{2_{\lambda(\hat{x})}}} \left(t_f - t_{j^*} - \tau \right) \right) E_{L_{2_{\lambda(\hat{x})}}} f(\tau) d\tau, \qquad (70)$$

then the fault $f(t)$ will be detected, that is, $\left\| r_{\lambda(\hat{x})}(t_f) \right\| > r^+_{\lambda(\hat{x})}(t)$, at time $t = t_f$.

Proof
In the faulty case, from (66), it is obtained that

$$\begin{aligned} \left(\dot{\hat{x}}_{12}(t) + \dot{e}_{12}(t) \right) - \Big[& A_{21_{\lambda(\hat{x})}} \left(\hat{x}_{11}(t) + e_{11}(t) \right) + A_{22_{\lambda(\hat{x})}} \left(\hat{x}_{12}(t) + e_{12}(t) \right) \\ & + B_{2_{\lambda(\hat{x})}} u - E_{12_{\lambda(\hat{x})}} K^*_{1_{\lambda(\hat{x})}} \left(\hat{x}_{21}(t) + e_{21}(t) \right) \Big] = E_{12_{\lambda(\hat{x})}} f(t). \end{aligned} \qquad (71)$$

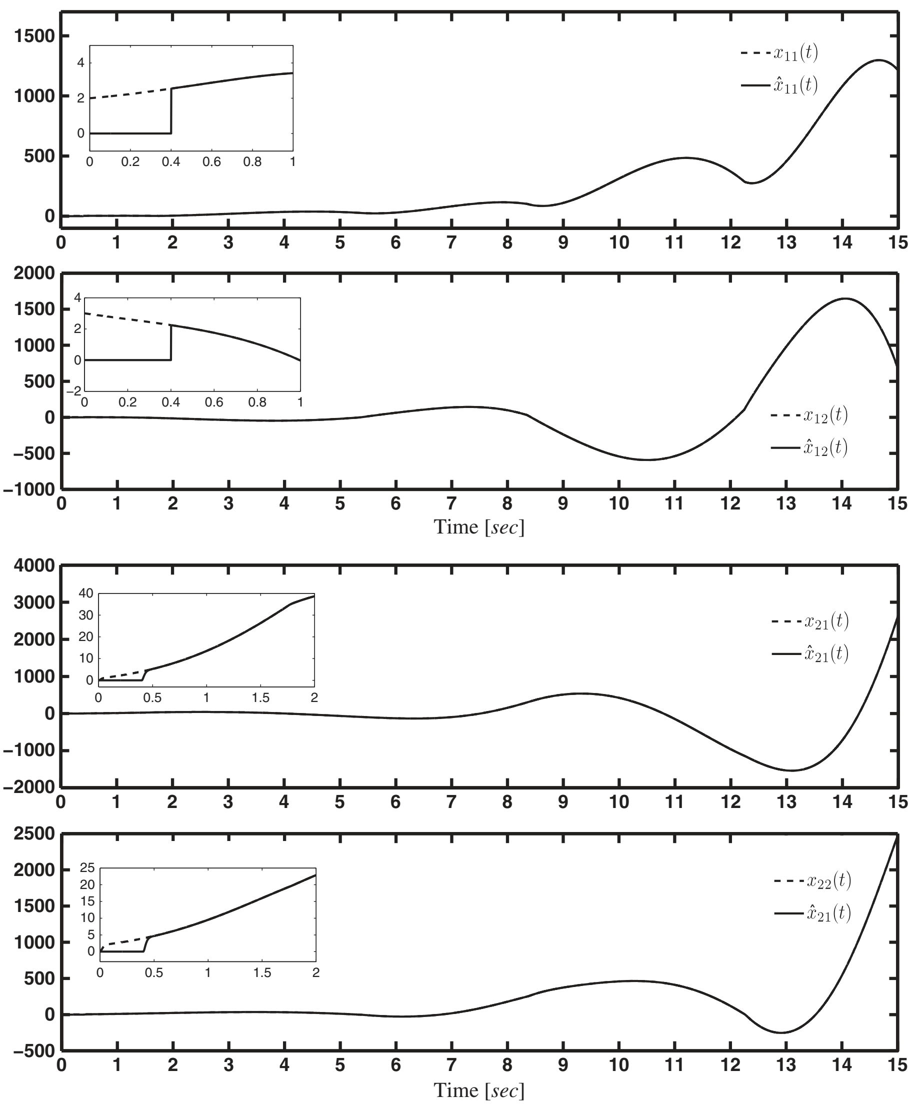

Figure 1. Continuous state estimation: fault-free case.

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Substituting the solution of (41) in (71),

$$\left(\dot{\hat{x}}_{12}(t)+\dot{e}_{12}(t)\right)-\Bigg[A_{21_{\lambda(\hat{x})}}\left(\hat{x}_{11}(t)+e_{11}(t)\right)+A_{22_{\lambda(\hat{x})}}\left(\hat{x}_{12}(t)+e_{12}(t)\right)+B_{2_{\lambda(\hat{x})}}u$$
$$-E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}\Bigg(\hat{x}_{21}(t)+e_{21}(t_{j^*})\exp(A_{L_{2_{\lambda(\hat{x})}}}t)+$$
$$\int_{t_{j^*}}^{t_f}\exp\left(A_{L_{2_{\lambda(\hat{x})}}}\left(t_f-t_{j^*}-\tau\right)\right)E_{L_{2_{\lambda(\hat{x})}}}f(\tau)d\tau\Bigg)\Bigg]=E_{12_{\lambda(\hat{x})}}f(t). \tag{72}$$

From (72), the FD decision (69) is stated, that is, $\left\|r_{\lambda(\hat{x})}(t)\right\|\leqslant r^{+}_{\lambda(\hat{x})}$. Notice that in the faulty case, from (72), it is obtained that

$$\begin{aligned}\left\|r_{\lambda(\hat{x})}(t)\right\|>\|\dot{e}_{12}(t)\|+\left\|A_{21_{\lambda(\hat{x})}}e_{11}(t)\right\|+\left\|A_{22_{\lambda(\hat{x})}}e_{12}(t)\right\|\\+\left\|E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}e_{21}(t)\right\|+\left\|E_{12_{\lambda(\hat{x})}}f(t)\right\|,\end{aligned} \tag{73}$$

$$\begin{aligned}\left\|r_{\lambda(\hat{x})}(t)\right\|>r^{+}_{\lambda(\hat{x})}+\left\|E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}\int_{t_{j^*}}^{t_f}\exp\left(A_{L_{2_{\lambda(\hat{x})}}}(t_f-t_{j^*}-\tau)\right)E_{L_{2_{\lambda(\hat{x})}}}f(\tau)d\tau\right\|\\+\left\|E_{12_{\lambda(\hat{x})}}f(t)\right\|,\end{aligned} \tag{74}$$

$$\left\|r_{\lambda(\hat{x})}(t)\right\|>r^{+}_{\lambda(\hat{x})}(t)+\left(\gamma_{\lambda(\hat{x})}+\left\|E_{12_{\lambda(\hat{x})}}\right\|\right)f^{+},\ \text{ a.e.} \tag{75}$$

Based on (72) and (75), if there exists a time instant $t_f>t_{j^*}$ such that (70) is satisfied, then it is concluded that $\left\|r_{\lambda(\hat{x})}(t_f)\right\|>r^{+}_{\lambda(\hat{x})}(t)$ at time $t=t_f$, and therefore, the fault $f(t)$ is detected. □

The proposed scheme ensures the robustness of the observer w.r.t. bounded additive faults in the state estimation, maintaining the estimation error bounded by a bound of $f(t)$, leading to FD, in the noise-free case. In this sense, the condition (69), as it was mentioned in Remark 9, and (70) are called structural detectability conditions. Therefore, all the faults that satisfy the structural detectability conditions will be detected. In the noisy case, it is necessary to extend these conditions to obtain a fault detectability conditions from a sensitivity point of view.

Remark 10

Notice that (69) and (75) are not satisfied during the switching times. However, because it is possible to estimate the discrete state in the presence of faults (Lemma 1), it is possible to ignore the time instants in which these conditions are not satisfied.

6.2. Fault identification scheme

In the following, the FId problem is studied. Notice that, from (72), the following 'VIE of second kind' could be established:

$$\Lambda_{\lambda(\hat{x})}f(t)+\Gamma_{\lambda(\hat{x})}\int_{t_f}^{t}K_{\lambda(\hat{x})}(t,\tau)f(\tau)d\tau=\Phi_{\lambda(\hat{x})}(t), \tag{76}$$

where

$$\begin{aligned}\Lambda_{\lambda(\hat{x})}&=E_{12_{\lambda(\hat{x})}},\\\Gamma_{\lambda(\hat{x})}&=E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}},\\K_{\lambda(\hat{x})}(t,\tau)&=\exp(A_{L_{2_{\lambda(\hat{x})}}}(t-t_f-\tau))E_{L_{2_{\lambda(\hat{x})}}},\\\Phi_{\lambda(\hat{x})}(t)&=\dot{\hat{x}}_{12}(t)-A_{21_{\lambda(\hat{x})}}\hat{x}_{11}(t)-A_{22_{\lambda(\hat{x})}}\hat{x}_{12}(t)-B_{2_{\lambda(\hat{x})}}u\\&\quad+E_{12_{\lambda(\hat{x})}}K^{*}_{1_{\lambda(\hat{x})}}\hat{x}_{21}(t)+e_{21}(t_{j^*})\exp\left(A_{L_{2_{\lambda(\hat{x})}}}t\right)+\epsilon(t).\end{aligned}$$

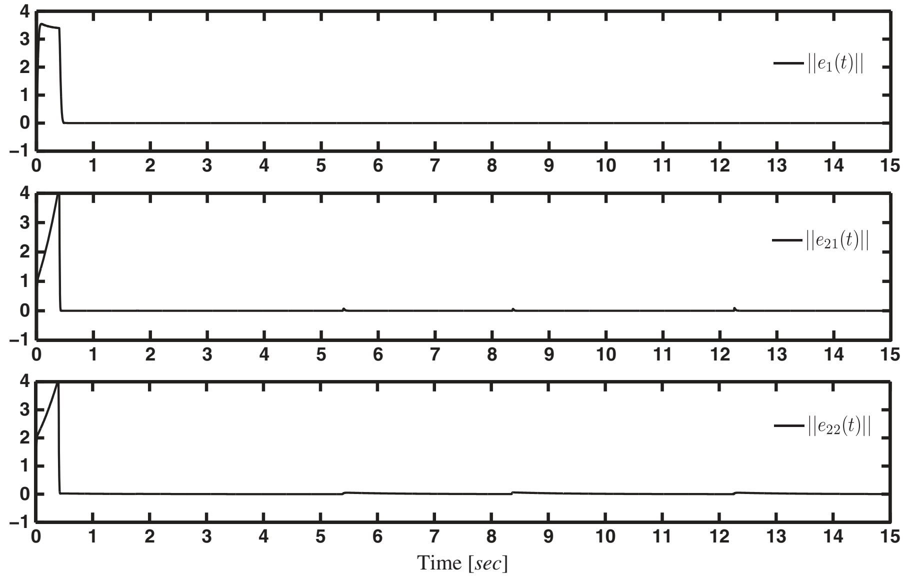

Figure 2. State estimation error: fault-free case.

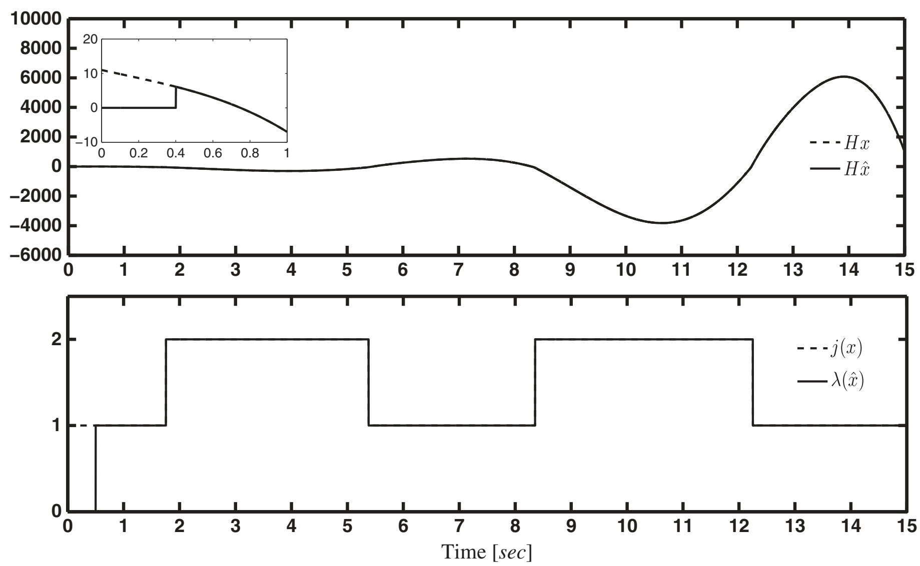

Figure 3. Switching function and discrete state estimation: fault-free case.

The problem is to find the solution of (76), that is, $f(t)$, taking into account that the parameters $\Lambda_{\lambda(\hat{x})}$, $\Gamma_{\lambda(\hat{x})}$, $K_{\lambda(\hat{x})}(t,\tau)$, and $\Phi_{\lambda(\hat{x})}(t)$[‖‖] are known. It is well-known that to find an analytical solution for this type of equations is difficult, sometimes impossible, and the numerical solution requires an important computational effort. In this paper, the following numerical solution is given.

Consider (76) and its representation in discrete time in a time interval $(t_f, t_f + T]$, that is,

$$\Lambda_{\lambda(\hat{x})} f(t_i) + \zeta \Gamma_{\lambda(\hat{x})} \sum_{k=1}^{N} K_{\lambda(\hat{x})}(t_i, t_k) f(t_k) = \Phi_{\lambda(\hat{x})}(t_i), \ \forall i = 1, \ldots, N, \qquad (77)$$

[‖‖]The term $\epsilon(t) = \dot{e}_{12}(t) - A_{21_{\lambda(\hat{x})}} e_{11}(t) - A_{22_{\lambda(\hat{x})}} e_{12}(t)$ contains all the computational errors produced by the state estimation process and due to sample time. Notice that, theoretically, an exact state estimation is achieved for the states $\bar{x}_{11}(t)$ and $\bar{x}_{12}(t)$.

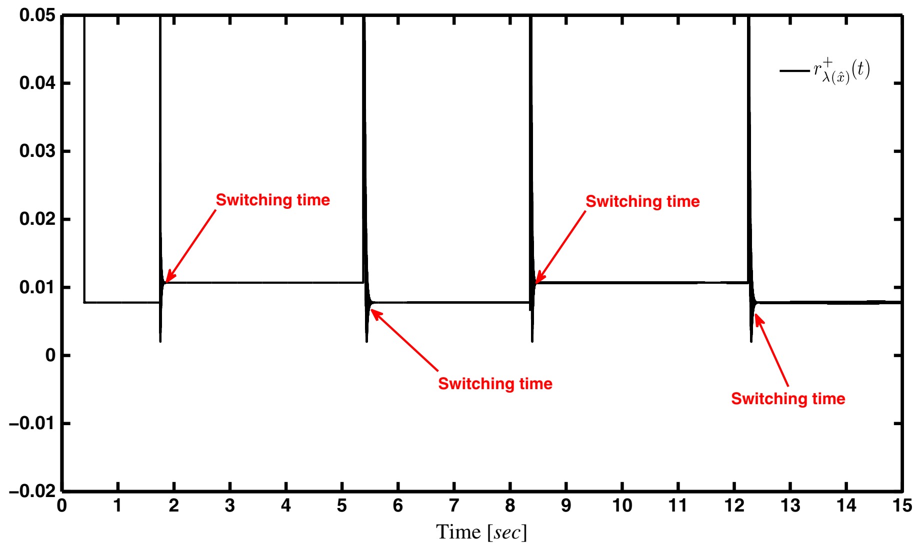

Figure 4. Fault detection decision: fault-free case.

where $t_i = t_f + \zeta i$, $\forall i = 1, \dots, N$, and $\zeta = \frac{T}{N}$. In this way, (77) can be rewritten as

$$\mathbf{K}_{\lambda(\hat{x})}(\Lambda, \delta\Gamma, K) \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_N) \end{bmatrix} = \begin{bmatrix} \Phi_{\lambda(\hat{x})}(t_1) \\ \Phi_{\lambda(\hat{x})}(t_2) \\ \vdots \\ \Phi_{\lambda(\hat{x})}(t_N) \end{bmatrix}, \tag{78}$$

where

$$\mathbf{K}_{\lambda(\hat{x})}(\Lambda, \delta\Gamma, K) = \begin{bmatrix} \Lambda_{\lambda(\hat{x})} + \zeta\Gamma_{\lambda(\hat{x})}K_{\lambda(\hat{x})}(t_1, t_1) & \cdots & \zeta\Gamma_{\lambda(\hat{x})}K_{\lambda(\hat{x})}(t_1, t_N) \\ \zeta\Gamma_{\lambda(\hat{x})}K_{\lambda(\hat{x})}(t_2, t_1) & \cdots & \zeta\Gamma_{\lambda(\hat{x})}K_{\lambda(\hat{x})}(t_2, t_N) \\ \vdots & \ddots & \vdots \\ \zeta\Gamma_{\lambda(\hat{x})}K_{\lambda(\hat{x})}(t_N, t_1) & \cdots & \Lambda_{\lambda(\hat{x})} + \zeta\Gamma_{\lambda(\hat{x})}K_{\lambda(\hat{x})}(t_N, t_N) \end{bmatrix}.$$

The FId problem is reduced to that one of solving the matrix equation of the form $\mathbf{K}f = \Phi$ for f. However, the problem can be ill-conditioned if the condition number*** associated with (78) is high, because the condition number gives a bound on how inaccurate the solution f will be after approximation. In particular, it is possible to think of the condition number as being (very roughly) the rate at which the solution, f, will change with respect to a change in Φ. Thus, if the condition number is large, even a small error in Φ may cause a large error in f. On the other hand, if the condition number is small, then the error in f will not be much bigger than the error in Φ.

Remark 11
The condition number is related to fault identifiability condition. It is clear that a well condition number, that is, near 1, will allow us to solve the FId problem.

In order to reduce the condition number of $\mathbf{K}$, it is possible to obtain a preconditioner $\mathbf{P}$ of matrix $\mathbf{K}$ such that $\mathbf{P}^{-1}\mathbf{K}$ has a smaller condition number than $\mathbf{K}$. The preconditioned matrix $\mathbf{P}^{-1}\mathbf{K}$ is not explicitly formed. The action of applying the preconditioner operation to a given vector needs to be computed in iterative methods (see, e.g., [56]).

In what follows, simulation results are presented in order to show the workability of the proposed methods.

***The condition number of a matrix measures the sensitivity of the solution of a system of linear equations to errors in the data. It gives an indication of the accuracy of the results from matrix inversion and the linear equation solution. Values of the condition number near 1 indicate a well-conditioned matrix.

7. SIMULATION RESULTS

Consider the following switched linear system to illustrate the proposed approach. The discrete state is given by

$$j(x(t)) = \begin{cases} 1, & \forall x \mid Hx \in [-50, 2000), \\ 2, & \forall x \mid Hx \in (-\infty, -50) \cup [2000, \infty), \end{cases}$$

with $Hx = -2x_1(t) + 5x_2(t)$. The following matrices correspond to the dynamics equations (1)

$$A_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & -1 \end{bmatrix},$$

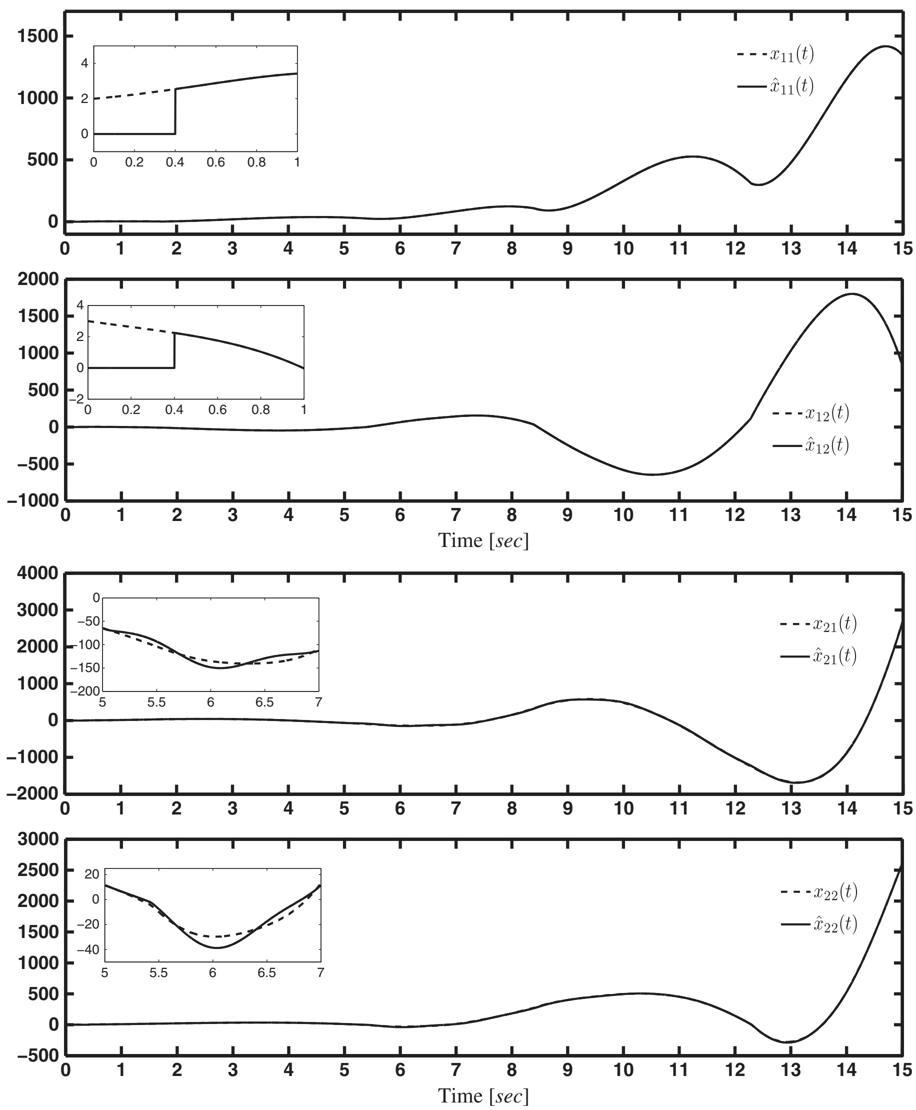

Figure 5. Continuous state estimation: faulty case.

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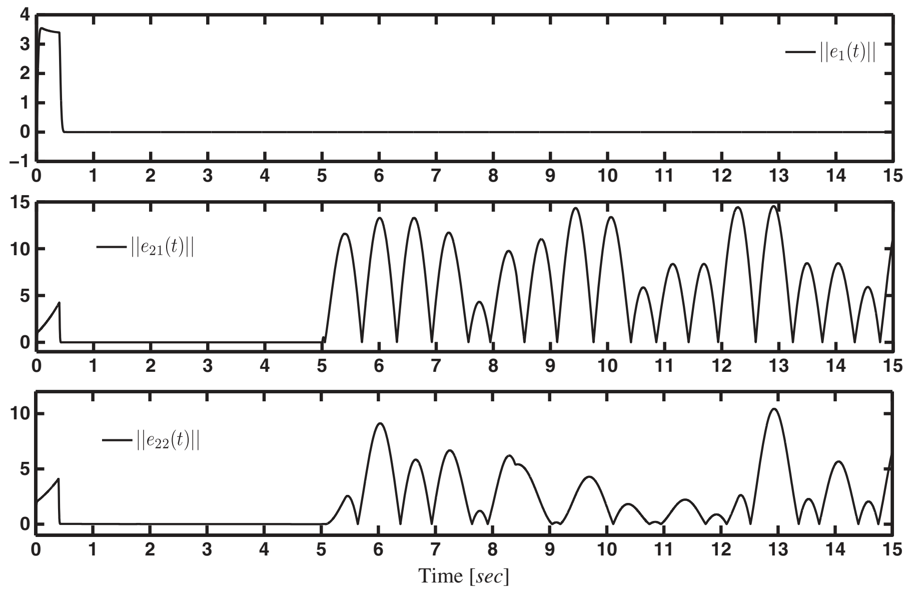

Figure 6. State estimation error: faulty case.

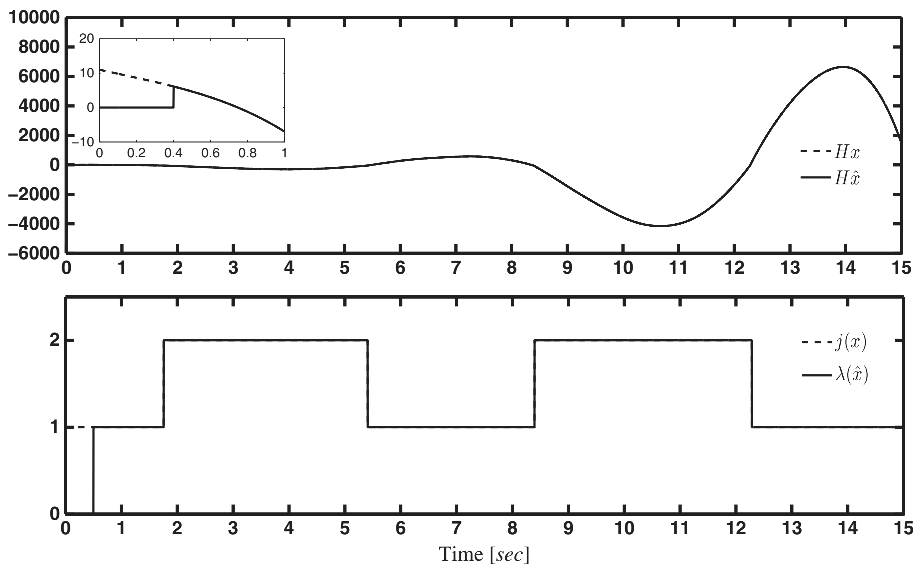

Figure 7. Switching function and discrete state estimation: faulty case.

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},\ E_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 1.5 & 0 & 0 & 0 \end{bmatrix}.$$

The output and known input are given by $y = C_{j(x(t))}x_1(t)$ and $u(t) = 5\sin(t)$, respectively. The system initial conditions are set to $x(0) = \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix}^T$ and $j(x(0)) = 1$. Simulations have been performed in the MATLAB Simulink (Natick, MA, USA) environment, with the Euler discretization method and sampling time $h = 0.0001$ s. It is possible to show that all assumptions stated along the paper are satisfied. The values of the designed matrices are as follows:

$$T_1 = I_{4\times4},\ T_2 = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ K^*_{1\,1} = K^*_{1\,2} = \begin{bmatrix} 1 \end{bmatrix},\ \bar{K}^*_1 = \bar{K}^*_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

$$L_{11} = \begin{bmatrix} 33 & 267 \end{bmatrix}^T,\ L_{12} = \begin{bmatrix} 22 & -178 \end{bmatrix}^T,\ L_{21} = -5,\ L_{22} = -10,$$

$$P_{11} = \begin{bmatrix} 1 & 0 \\ -34 & 1 \end{bmatrix},\ P_{12} = \begin{bmatrix} 1.5 & 0 \\ -51 & -1.5 \end{bmatrix}.$$

The HOSM differentiators in (24) are designed for $n_{\mathcal{V}} = 2$ with M_1 and M_2 according to Proposition 1. The reset equations in Proposition 2 are implemented. The results for the fault-free case are depicted in Figures 1–3.

It is possible to see that the main features of the proposed observer are clearly illustrated, that is, finite time estimation for $x_{11}(t)$, $x_{12}(t)$, and $j(x(t))$ and exponential estimation for $x_{21}(t)$ and $x_{22}(t)$. The decision signal $r^+_{\lambda(\hat{x})}(t)$ is computed by simulation, and the estimation of $\dot{x}_{21}(t)$ is provided by (24) with $n_{\mathcal{V}} = 1$. The behavior of the decision signal $r^+_{\lambda(\hat{x})}(t)$ for the fault-free case is shown in Figure 4. Notice that in the switching times, condition (69) is not satisfied as it is mentioned in Remark 10.

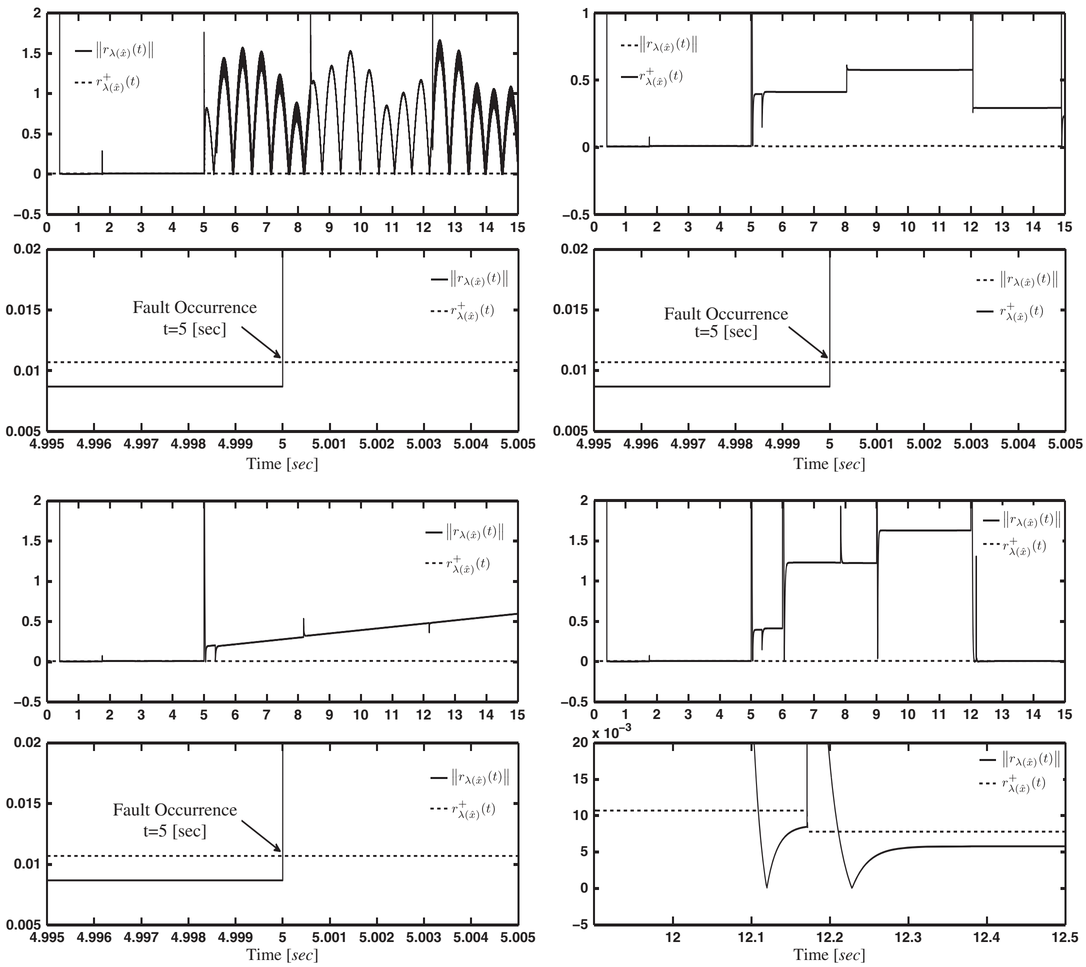

Figure 8. Fault detection decision: faulty case. Notice that, in the intermittent fault, when the fault disappears, $\left\|r_{\lambda(\hat{x})}(t)\right\|$ remains less than the threshold $r^+_{\lambda(\hat{x})}(t)$ indicating that fault is already not present.

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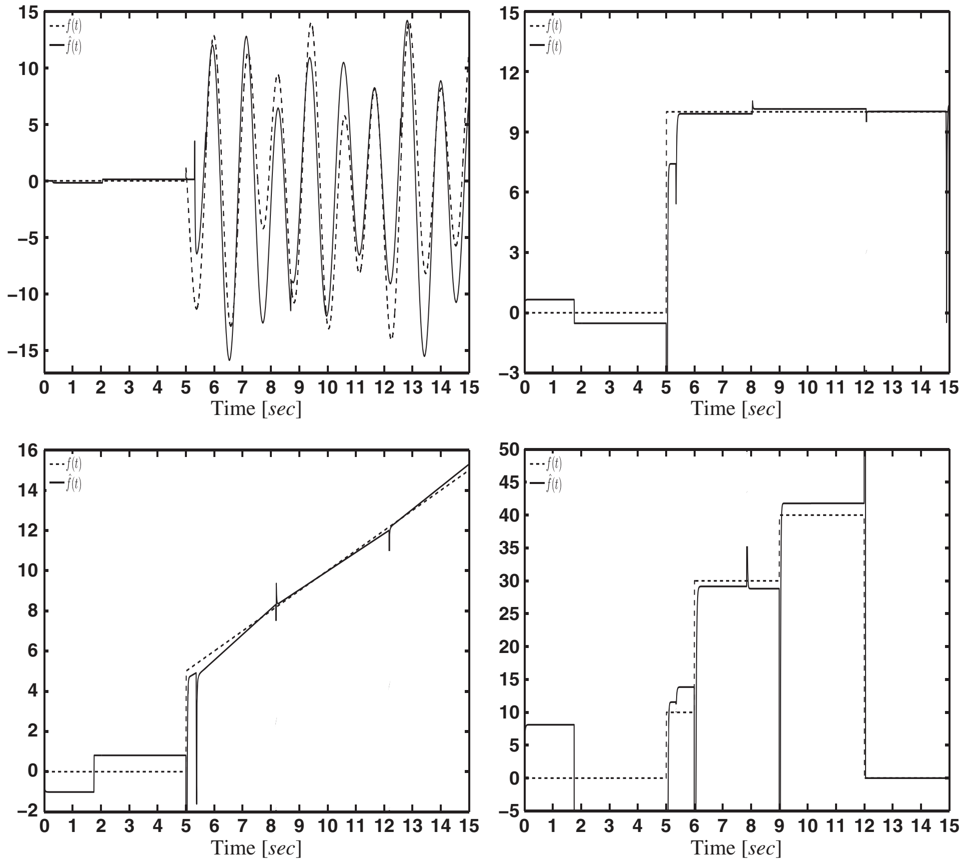

Figure 9. Fault identification.

Now, consider the following faulty scenario. The fault $f(t) = 10\sin(5.5t) + 4\sin(3.5t) + 2\sin(t)$ appears in $t = 5$ s. The results for the faulty case are depicted in Figures 5–7.

It is easy to see that the finite time estimation for $x_{11}(t)$ and $x_{12}(t)$ is maintained, and the estimation error for $x_{21}(t)$ and $x_{22}(t)$ remains bounded, in spite of the switchings on the system and the fault. Notice that the fault $f(t)$ affects the behavior of the trajectories of the whole system implying changes in the discrete state; however, the continuous and discrete state estimation is maintained.

The norm of the decision signal $r^+_{\lambda(\hat{x})}(t)$ for different kinds of faults (oscillatory, abrupt, incipient, and intermittent) is shown in Figure 8. The term $\left\| r_{\lambda(\hat{x})}(t) \right\|$ indicates when the fault has occurred; that is, once the value of $\left\| r_{\lambda(\hat{x})}(t) \right\|$ exceeds the threshold $r^+_{\lambda(\hat{x})}(t)$, the fault is detected.

Finally, the FId scheme proposed in (78) is shown in Figure 9 for each kind of fault. Clearly, the proposed scheme provides an approximate identification of the fault $f(t)$. Nevertheless, it is possible to improve this one reducing the condition number of the matrix $\mathbf{K}_{\lambda(\hat{x})}(\Lambda, \delta\Gamma, K)$.

8. CONCLUSIONS

A solution of the problem of state estimation for NMP SS with additive faults is presented. A robust observer-based scheme for this kind of systems is proposed. The proposed state observers are based on HOSM to exactly estimate the strongly observable part and Luenberger-like observers to estimate the remaining parts. The exact estimation of the continuous state allows us to realize a finite time and exact estimation of the discrete state in the presence of additive faults. The HOSM-based FD is composed by a residual generator accompanied by a bank of observers, and the numerical solution of a VIE allows to establish an FI scheme. Simulation results support the proposed approaches for different kind of faults.

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