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NON-NEGATIVE SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

by D. G. ARONSON

Introduction.

The main results of this paper are several theorems concerning the properties of non-negative solutions of second order linear divergence structure differential equations of parabolic type. One of the main results is the fact that every non-negative solution of the Cauchy problem is uniquely determined by its initial data. Solutions of the Cauchy problem are functions which correspond to a given initial function in some well defined way. There are also solution of the differential equations in question which are non-negative but which do not have initial values in any ordinary sense, for example, the Green's function in a bounded domain or the fundamental solution in an infinite strip. The properties of the Green's function and the fundamental solution are developed in detail, and another of the main results is the fact that these functions are bounded above and below by multiples of the fundamental solution of an equation of the form $\alpha \Delta u = u_t$, where α is a positive constant. In addition, we prove that the Widder rapresentation theorem is valid for the class of equations under consideration. Throughout this paper we work with weak solutions of a very general class of parabolic equations. When specialized to classical solutions of equations with smooth coefficients our results are either new or generalizations of earlier results. All of our results depend ultimately upon the work of Serrin and the author on the local behavior of solutions of general divergence structure equations and upon certain energy type estimates which are derived here.

Let $x = (x_1, \dots, x_n)$ denote points in the n-dimensional Euclidian space E^n with $n \geq 1$ and t denote points on the real line. Let Σ denote an open

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domain in E^n. It is not necessary that Σ be bounded and $\Sigma \equiv E^n$ is not excluded. Let T be a fixed positive number and consider the domain $D = \Sigma \times (0, T]$. For $(x, t) \in D$ we treat the second order linear differential operator

$$Lu = u_t - \{A_{ij}(x, t)\, u_{x_i} + A_j(x, t)\, u\}_{x_j} - B_j(x, t)\, u_{x_j} - C(x, t)\, u$$

where $u_t = \partial u/\partial t$, $u_{x_i} = \partial u/\partial x_i$ and we employ the convention of summation over repeated Latin indices. The coefficients of L are assumed to be defined and measurable in D. Before describing the remaining assumptions on L we introduce some notation.

Let $\mathcal{B}_1(\Sigma)$ denote a Banach space of functions defined on Σ with the norm $|\cdot|_1$, and $\mathcal{B}_2(I)$ denote a Banach space of functions defined on an interval I with the norm $|\cdot|_2$. A function $w = w(x, t)$ defined and measurable on $D = \Sigma \times I$ is said to belong to the class $\mathcal{B}_2[I; \mathcal{B}_1(\Sigma)]$ if $w(\cdot, t) \in \mathcal{B}_1(\Sigma)$ for almost all $t \in I$ and if $||w|_1(t)|_2 < \infty$. When the sets Σ and I are clear from the context we will write $\mathcal{B}_2[\mathcal{B}_1]$ in place of $\mathcal{B}_2[I; \mathcal{B}_1(\Sigma)]$. In reference [6] the classes $L^q[I; L^p(\Sigma)]$ are denoted by $L^{p,q}(D)$ and we shall also use this notation here. Morever, for $w \in L^{p,q}(D)$ with $1 \leq p, q < \infty$ we define

$$\| w \|_{p,q} = \left\{ \int_I \left(\int_\Sigma | w |^p \, dx \right)^{q/p} dt \right\}^{1/q}.$$

In case either p or q is infinite, $\| w \|_{p,q}$ is defined in a similar fashion using L^∞ norms rather than integrals. Let $\mathcal{S}$ denote the set of cylinders of the form $R(\sigma) \times I$, contained $S = E^n \times I$, where $R(\sigma)$ denotes an open cube in E^n of edge length σ and $\sigma = \min(1, \sqrt{|I|})$. A function $w = w(x, t)$ defined and measurable on S is said to belong to the class $L^{p,q}(\mathcal{S})$ if $\sup \| w \|_{q,p} < \infty$, where the norms are taken over cylinders in the family $\mathcal{S}$.

The operator L is defined in a basic domain D which is either a bounded cylinder $\Omega \times (0, T]$ or an infinite strip $E^n \times (0, T]$. It will be convenient in many instances to regard L as being defined throughout the $(n+1)$-dimensional (x, t) space. We therefore adopt the convention that $Lu \equiv u_t - \Delta$ for all $(x, t) \in \complement D$. Throughout the paper it will be assumed that there constants ν, M, M_0 and R_0 such that $0 < \nu$, $M < \infty$, $0 \leq M_0 < \infty$ and $0 \leq R_0 \leq \infty$, and such that the coefficients of L satisfy the following conditions which will be referred to collectively as (H).

(H.1) *For all* $\xi \in E^n$ *and for almost all* (x, t)

$$A_{ij}(x, t)\, \xi_i \xi_j \geq \nu\, |\xi|^2 \text{ and } |A_{ij}(x, t)| \leq M.$$

(H.2) *Let* $Q_0 = (|x| < R_0) \times (0, T]$. *Each of the coefficients* A_j *and* B_j *is contained in some space* $L^{p,q}(Q_0)$, *where* p *and* q *are such that*

$$(*) \qquad 2 < p,\ q \leq \infty \ \textit{and}\ \frac{n}{2p} + \frac{1}{q} < \frac{1}{2},$$

and $|A_j(x, t)|$, $|B_j(x, t)| \leq M_0$ *for almost all* $|x| \geq R_0$ *and* $t \in (0, T]$.

(H.3) $C \in L^{p,q}(\mathcal{S})$, *where* p *and* q *are such that*

$$(**) \qquad 1 < p,\ q \leq \infty \ \text{and}\ \frac{n}{2p} + \frac{1}{q} < 1,$$

and $C(x, t) \leq M_0$ *for almost all* $|x| \geq R_0$ *and* $t \in (0, T]$.

Note that (H.2) implies that the A_j and B_j each belong to some space $L^{p,q}(\mathcal{S})$, where p and q satisfy $(*)$. If $R_0 = \infty$ then Q_0 is the strip $S = E^n \times (0, T]$ and condition (H.2) simply requires that the coefficients A_j and B_j belong to the appropriate spaces $L^{p,q}(S)$. On the other hand, (H) clearly holds if L is uniformly parabolic and all of its coefficients are bounded in S. Let L be an operator defined in a bounded cylinder $Q = \Omega \times (0, T)$, such that (i) $A_{ij}\xi_j\xi_j \geq \nu_1 |\xi|^2$ and $|A_{ij}| \leq M_1$ almost everywhere in Q, (ii) A_j, $B_j \in L^{p,q}(Q)$ where p and q satisfy $(*)$, and (iii) $C \in L^{p,q}(Q)$ where p and q satisfy $(**)$. Then the extension of L according to the convention adopted above satisfies (H) with $\nu = \min(1, \nu_1)$, $M = \max(1, M_1)$, $M_0 = 0$, and any R_0 such that $\Omega \subset \{x;\ |x| < R_0\}$.

Without further hypotheses on L it is not possible, in general, to speak of a classical solution of a differential equation involving the operator L, and it is correspondingly necessary to introduce the notion of a generalized solution. Before doing so, however, we will need several additional definitions. Let Ω denote a bounded open domain in E^n. A function $w = w(x)$ defined and measurable in Ω is said to belong to $H^{1,p}(\Omega)$ if w possesses a distribution derivative w_x and

$$\| w \|_{L^p(\Omega)} + \| w_x \|_{L^p(\Omega)} < \infty.$$

The space $H_0^{1,p}(\Omega)$ is the completion of the $C_0^\infty(\Omega)$ functions in this norm. The space $H^{1,p}(E^n)$ is the completion of the $C_0^\infty(E^n)$ functions in the norm

$$\| \varphi \|_{L^p(E^n)} + \| \varphi_x \|_{L^p(E^n)}.$$

For $(x, t) \in D = \Sigma \times (0, T]$ consider the differential equation

$$Lu = \{F_j(x, t)\}_{x_j} + G(x, t), \tag{1}$$

where the F_j and G are given functions defined and measurable in D. A function $u = u(x, t)$ is said to be a *weak solution* of equation (1) in D if

$$u \in L^\infty[\delta, T;\ L^2_{\mathrm{loc}}(\Sigma)] \cap L^2[\delta, T;\ H^{1,2}_{\mathrm{loc}}(\Sigma)]$$

for all $\delta \in (0, T)$ and u satisfies

$$\iint_D (-u\varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + A_j u\varphi_{x_j} + F_j \varphi_{x_j} - B_j u_{x_j} \varphi - Cu\varphi - G\varphi)\, dx dt = 0 \tag{2}$$

for any $\varphi \in C_0^1(D)$. Clearly every classical solution of equation (1) in D is also a weak solution. We will use the terms weak solution and solution interchangably throughout this paper.

Various known properties of weak solution of equation (1) are summarized for convenient reference in section 1. These results, which are used extensively in the remainder of the paper, include the maximum principle and theorems on local behavior proved by Serrin and the author in reference [6], and theorems on regularity at the boundary due to Trudinger [21]. Moreover, we establish a simple but useful extension principle which permits us, in some cases, to use the local results of reference [6] in the neighborhood of the boundary.

Section 2 is primarily devoted to the derivation of weighted energy type estimates for weak solution of equation (1). In particular, if u is a solution of (1) we derive estimates for the $L^{2,\infty}$ norm of $e^h u$ and the $L^{2,2}$ norm of $e^h u_x$. Here $h = h(x, t)$ has the form

$$-\alpha \frac{|x - \xi|^2}{2\mu - (t - s)} - \beta(t - s),$$

where $\xi \in E^n$ and $s \in [0, T)$ are arbitrary, and α, β, μ are positive constants determined by L and the data. These estimates are related to those obtained by Aronson and Besala in reference [5], and include as special cases estimates due to Il'in, Kalashnikov and Oleinik [12], and Aronson [3].

Let Ω be a bounded open domain in E^n and $Q = \Omega \times (0, T]$. Consider the boundary value problem

$$(3) \qquad \begin{cases} Lu = \{F_j(x, t)\}_{x_j} + G(x, t) \text{ for } (x, t) \in Q \\ u(x, 0) = u_0(x) \text{ for } x \in \Omega, \ u(x, t) = 0 \text{ for } (x, t) \in \partial\Omega \times [0, T], \end{cases}$$

where the F_j are given functions each belonging to a space $L^{p,q}(Q)$ with p, q satisfying (*), G is a given function in $L^{p,q}(Q)$ with p, q satisfyng (**), and u_0 is a given function in $L^2(\Omega)$. In section 2 we define the notion of a weak solution of problem (3) and prove that this problem possesses exactly one weak solution. Moreover, we establish various properties of the solution. Except for certain details, this material is fairly standard. It is included in detail here for the sake of completeness and since we will have occasion to refer to some of the intermediate steps in the proof. Alternate treatments can be found in the work of Ivanov, Ladyženskaja, Treshkunov and Ural'ceva [13], and Ladyženskaja, Solonnikov and Ural'ceva [15].

The Cauchy problem

$$(4) \qquad Lu = \{F_j(x, t)\}_{x_j} + G(x, t) \text{ for } (x, t) \in S, \ u(x, 0) = u_0(x) \text{ for } x \in E^n,$$

where $S = E^n \times (0, T]$, is treated in section 4. Again the F_j, G and u_0 are given functions, and it is assumed that for some constant $\gamma \geq 0$ we have $e^{-\gamma|x|^2} F_j \in L^{p,q}(S)$ with p, q satisfyng (*), $e^{-\gamma|x|^2} G \in L^{p,q}(S)$ with p, q satisfying (**), and $e^{-\gamma|x|^2} u_0 \in L^2(E^n)$. The notion of a weak solution of problem (4) is defined and it is shown that the problem possesses exactly one weak solution in the appropriate class of functions. Various properties of the solution are also established. For example, let u denote the solution of problem (4), where $u_0 = 0$ and where the F_j and G satisfy the hypotheses given above with $\gamma = 0$. Then there exisists a constant $\mathcal{C} > 0$, depending only on T and the quantities in the conditions (H), such that

$$(5) \qquad |u(x, t)| \leq \mathcal{C}\left(\sum_j \|F_j\|_{p,q} + \|G\|_{p,q}\right)$$

for all $(x, t) \in S$. For other work on weak solutions of the Cauchy problem see references [12] and [15].

In section 5 we consider a non-negative solution u of the Cauchy problem

$$(6) \qquad Lu = 0 \text{ for } (x, t) \in S, \ u(x, 0) = u_0(x) \text{ for } x \in E^n,$$

where u_0 is a given non-negative function which belongs to $L^2_{loc}(E^n)$. The main result is that u is uniquely determined by u_0. Specifically, for each $(x, t) \in S$ we have $u(x, t) = \lim_{k \to \infty} u^k(x, t)$, where the u^k are the unique weak solutions of certain boundary value problems involving L and u_0. A special case of this result, with

$$Lu \equiv u_t - \{A_{ij}(x, t)\, u_{x_i}\}_{x_j} \tag{7}$$

was obtained by the author in reference [3]. The first theorem of this type was given in the case $n = 1$ by Serrin [19] for equation

$$u_t - a(x)\, u_{xx} - b(x)\, u_x - c(x)\, u = 0$$

with Hölder continuous coefficients. Our result extends but does not include Serrin's result. In the proof of the main result we use the following generalization of a theorem first proved by Widder [22] for the equation of heat conduction. If u is a non-negative weak solution of the Cauchy problem (6) with $u_0 = 0$, then $u = 0$ in S. This result includes earlier results for classical solutions of equations with smooth coefficients due to Friedman [9], [10] and Krzyżański [14].

The estimate (5) implies that the value at a point of the solution of the Cauchy problem is a bounded linear functional on a certain Banach space. We use this observation in section 6 to establish the existence of the weak fundamental solution $\Gamma(x, t;\ \xi, \tau)$ of $Lu = 0$ in S. In particular, we estimate certain norms of Γ and its derivatives, and show that the solution of the Cauchy problem

$$Lu = F_0(x, t) - \{F_j(x, t)\}_{x_j} \text{ for } (x, t) \in S,\ u(x, 0) = 0 \text{ in } E^n$$

with the $F_j \in L^{p, q}(S)$ for p, q satisfyng (*) is given by the formula

$$u(x, t) = \iint\limits_S \{\Gamma(x, t;\ \xi, \tau)\, F_0(\xi, \tau) + \Gamma_{\xi_j}(x, t;\ \xi, \tau)\, F_j(\xi, \tau)\}\, d\xi\, d\tau. \tag{8}$$

In a similar manner, using the maximum principle for solutions of the boundary value problem. we prove the existence of the weak Green's function for $Lu = 0$ in any bounded cylinder $Q = \Omega \times (0, T]$ and derive a representation formula similar to (8) for solutions of the boundary value problem. The existence and properties of the Green's function for the special case in which L is given by (7) were obtained by the author in reference [1].

The next two sections are devoted to deriving various properties of the weak Green's functions and fundamental solutions. In section 7 we make the temporary assumption that the coefficients of L are smooth and show that the weak Green's functions and fundamental solution coincide with their classical counterparts. Several of the known properties of these functions are described. The main results of this section are upper and lower bounds for these functions which are independent of the smoothness of the coefficients of L. Specifically, we prove that there exist positive constants α_1, α_2 and $\mathcal{C}$ such that

$$\mathcal{C}^{-1} g_1(x-\xi,\ t-\tau) \leq \Gamma(x, t;\ \xi, \tau) \leq \mathcal{C} g_2(x-\xi,\ t-\tau)$$

for all (x, t), $(\xi, \tau) \in S$ with $t > \tau$, where $g_i(x, t)$ is the fundamental solution of $\alpha_i \Delta u = u_t$ for $i = 1, 2$. The constants depend only on T and the quantities in condition (H). In the appendix to reference [16] Nash gives somewhat weaker bounds for Γ in case L is given by (7). A similar result holds for the Green's function γ for L in a bounded cylinder Q. Here, however, for the lower bound x and ξ must be restricted to a convex subdomain Ω' of Ω and the constants depend on the distance from Ω' to $\partial\Omega$. These bounds were announced by the author in reference [4]. In section 8 we remove the assumption of smoothness of the coefficients of L and prove that the function γ and Γ are limits of the corresponding functions for operator obtained from L by regularizing the coefficients. From these considerations it follows that γ and Γ inherit the principal properties of their classical counterparts including the bounds described above.

In section 9 we combine the results obtained in sections 5 and 8 to prove that if u is the non-negative weak solution of the Cauchy problem (6) then

$$u(x, t) = \int_{E^n} \Gamma(x, t;\ \xi, 0)\, u_0(\xi)\, d\xi. \tag{9}$$

This representation theorem includes as special cases earlier results for classical solutions of equations with smooth coefficients due to Friedman [9], [10] and Krzyżański [14]. We also obtain a necessary and sufficient condition for a function defined by a formula such as (9) to be a non-negative solution of the Cauchy problem. Using (9) and the bounds for Γ derived in section 8 we show that if u is a non-negative weak solution of $Lu = 0$ in S then there exist a unique non-negative Borel measure ϱ such that

$$u(x, t) = \int_{E^n} \Gamma(x, t;\ \xi, 0)\, \varrho(d\xi). \tag{10}$$

Moreover, we give a necessary and sufficient condition for a function defined by a formula such as (10) to be a non-negative weak solution of $Lu = 0$ in S. Taken together these two results constitute a generalization of Widder's representation theorem for non-negative solutions of the equation of heat conduction [22]. The representation formula (10) for classical solutions of equations with smooth coefficients was derived by Krżyzański in reference [14]. The uniqueness of the measure ϱ has not been considered by either Krzyżański or Widder. As a corollary to this result we also obtain the following result of Bôcher type concerning non-negative solutions of $Lu = 0$ with an isolated singular point on the hyperplane $t = 0$. Let u be a non-negative weak solution of $Lu = 0$ in S and suppose that

$$\lim_{t \to 0} \int_{E^n} u(x, t)\, \psi(x)\, dx = \int u_0(x)\, \psi(x)\, dx$$

for all $\psi \in C_0^0(E^n \setminus \{0\})$ where $u_0 \geq 0$ is in $L^1_{loc}(E^n)$. Then there exists a constant $\eta \geq 0$ such that

$$u(x, t) = \eta\, \Gamma(x, t; 0, 0) + \int_{E^n} \Gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi.$$

The special case in which $u_0 = 0$ and u is a classical solution of an equation with smooth coefficients is treated by Krżyzański in [14]. A weaker result when L is given by (7) and $u_0 = 0$ was obtained by the author in reference [2].

Results quoted from references [6] and [21] are designated by Theorem A, Theorem B, etc. Theorems and lemmas proved in this paper are numbered consecutively without regard to the sections in which they occur, and we use Corollary $n \cdot m$ to designate the m-th corollary to Theorem n.

The results reported here are a partial summation of research which has spanned several years. Througout this period the author has benefited greatly from countless discussions (and arguments) with Professor James Serrin, and it is a pleasure to thank him here for his interest, encouragement and aid. We also wish to thank Professors Jürgen Moser and Hans Weinberger for their interest in this work and for many stimulating discussions.

1. Preliminary Results.

For $(x, t) \in S$ consider the linear equation

$$Lu = \{F_j(x, t)\}_{x_j} + G(x, t), \tag{1.1}$$

where it is assumed that L satisfies (H), each $F_j \in L^{p,q}(\mathcal{S})$ for some p, q satisfying $(*)$, and $G \in L^{p,q}(\mathcal{S})$ for p, q satisfying $(**)$. Under these hypotheses, equation (1.1) belongs to the class of equations treated in references [6] and [21]. What follows is an annotated list of the specific results from these references which are used in this paper. The statements given below are not necessarily in their most general forms but are tailored to the applications which we will make here. Theorem D is from reference [21] and the remaining results are from reference [6].

In this section Ω will always denote a bounded domain in E^n and $Q = \Omega \times (0, T]$. The parabolic boundary of Q is the set $\{\Omega \times (t = 0)\} \cup \{\partial\Omega \times \times [0, T]\}$ When the nature of the basic domain is irrelevant we will denote it by $D = \Sigma \times (0, T]$, where Σ is either Ω or E^n. If W is one of the functions A_j, B_j, C, F_j or G, then $\| W \|$ means $\| W \|_{p,q}$ if the basic domain is Q or $\sup_{\mathcal{S}} \| W \|_{p,q}$ if the basic domain is $S = E^n \times (0, T]$. With this convention we define

$$k = \sum_{j=1}^{n} \| F_j \| + \| G \|.$$

All constant will be denoted by $\mathcal{C}$. The statement « $\mathcal{C}$ depends on the structure of (1.1) » means that $\mathcal{C}$ is determined by the quantities $\nu, M, n, \| A_j \|, \| B_j \|, \| C \|$ and θ, where θ is a positive constant which is determined by the values of p and q occuring in (H) and in the hypotheses on the F_j and G.

In reference [6] it is shown that every weak solution u of (1.1) in D has a representative which is continuous in D. We will therefore always assume that *u denotes the continuous representative of a given weak solution.* Hence there is no difficulty in talking about the value of u at any point of its domain of definition.

THEOREM A. (*Maximum Principle*) *Let u be a weak solution of equation* (1.1) *in Q. If $u \in C^0(\bar{Q})$ and $m_1 \le u \le m_2$ on the parabolic boundary of Q, then*

$$m_1 - \mathcal{C} k_1 \le u(x, t) \le m_2 + \mathcal{C} k_2$$

in $\bar{Q}$, *where* $\mathcal{C}$ *depends only on* T, Ω *and the structure of* (1.1), *while*

$$k_i = \left(\sum_{j=1}^{n} \| A_j \| + \| C \| \right) | m_i | + k$$

for $i = 1, 2$.

Let $(\bar{x}, \bar{t})$, be an arbitrary fixed point in the basic set D. We denote by $R(\varrho)$ the ball in E^n of radius $\varrho/2$ centered at $\bar{x}$ and define $Q(\varrho) = R(\varrho) \times (\bar{t} - \varrho^2, \bar{t}]$. The symbol $\| \cdot \|_{p,q,\varrho}$ will be used to denote the $L^{p,q}$ norm computed over the cylinder $Q(\varrho)$.

THEOREM B. (*Local Boundedness*). *Let* u *be a weak solution of equation* (1.1) *in* D. *Assume that* $\varrho \leq \varrho_0$ *and that* $Q(3\varrho) \subset D$. *Then in* $\bar{Q}(\varrho)$

$$| u(x, t) | \leq \mathcal{C} (\varrho^{-(n+2)/2} \| u \|_{2,2,3\varrho} + \varrho^{\theta} k)$$

where $\mathcal{C}$ *depends only on* ϱ_0 *and the structure of* (1.1).

To state the next result it is convenient to use the following notation. We write $x' = (x, t)$, $y' = (y, s)$, etc. to denote points in space-time and introduce a pseudo-distance according to the definition

$$| x' |^2 = \begin{cases} \max (x_i^2, -t/4) & \text{for } t \leq 0 \\ \infty & \text{for } t > 0. \end{cases}$$

Thus, for example, the set $|y' - x'| < \varrho$ for fixed x' is the cylinder $|x_i - y_i| < \varrho$, $t - 4\varrho^2 < s \leq t$.

THEOREM C. (*Interior Hölder Continuity*) *Let* u *be a weak solution of equation* (1.1) *in* Q *such that* $|u| \leq m$ *in* Q. *If* x', y' *are points of* Q *with* $s \leq t$ *then*

$$| u(y') - u(x') | \leq \mathcal{C} (m + k) \left(\frac{| y' - x' |}{R} \right)^{\alpha},$$

where $\mathcal{C}$ *and* α *are positive constants depending only on the structure of* (1.1), *and* R *is equal to either the pseudo-distance from* x' *to the parabolic boundary of* Q *or* 1, *whichever is smaller.*

In order that a solution of (1.1) in Q be continuous up to the parabolic boundary of Q it is clearly necessary that $\partial\Omega$ have some regularity properties. The following very weak condition was introduced by Ladyženskaja and Ural'ceva (cf. reference [15]). The boundary $\partial\Omega$ of Ω will be said to have property (A) if there exist constants a_0 and θ_0, $0 < a_0, \theta_0 < 1$, such that for any ball $R(\varrho)$ with center on $\partial\Omega$ and radius $\varrho/2 \leq a_0$ the inequality

$$\operatorname{meas}\{R(\varrho) \cap \Omega\} \leq (1-\theta_0) \operatorname{meas} R(\varrho)$$

holds.

THEOREM D. (*Continuity up to the Boundary*) *Let u be a weak solution of equation* (1.1) *in Q and suppose that $\partial\Omega$ has property* (A).

(i) *If u is continuous in $\bar{\Omega} \times (0, T]$ and bounded in Q then u has a modulus of continuity in $\bar{\Omega} \times [\delta, T]$ for any $\delta \in (0, T)$ which is completely determined by δ, the structure of* (1.1), *k, $\max_Q |u|$, the constants involved in property* (A), *and the modulus of continuity of u on $\partial\Omega \times [\delta/2, T]$.*

(ii) *If u is continuous in $\bar{Q}$ then the modulus of continuity of u in $\bar{Q}$ is completely determined by the structure of* (1.1), *k, $\max_Q |u|$, the constants involved in property* (A) *and the modulus of continuity of u on the parabolic boundary of Q. In particular, if u is Hölder continuous on the parabolic boundary of Q then it is also Hölder continuous in $\bar{Q}$.*

The remaining results concern non-negative solutions of (1.1). Since we apply them only in cases where $k = 0$ we will state them only for the equation

$$Lu = 0. \tag{1.2}$$

THEOREM E. (*Harnack Principle*) *Let u be a non-negative solution of equation* (1.2) *in S. Then for all points (x, t), (y, s) in S with $0 < s < t \leq T$ we have*

$$u(y, s) \leq u(x, t) \exp \mathcal{C}\left(\frac{|x-y|^2}{t-s} + \frac{t}{s}\right),$$

where $\mathcal{C}$ depends only on T and the structure of (1.2).

The next two theorems are concerned with the behavior of a non-negative solution in the neighborhood of $t = 0$.

THEOREM F. *Let u be a non-negative solution of equation* (1.2) *in S. Suppose that for some $\varkappa > 0$ we have*

$$\mathcal{M} = \inf_{0 < t \leq T} \int_{|x|^2 < \varkappa t} u(x, t)\, dx > 0.$$

Then

$$u(x, t) \geq \mathcal{C}_1 \mathcal{M} t^{-n/2} \exp(-\mathcal{C}_2 |x|^2/t).$$

Here $\mathcal{C}_1$ and $\mathcal{C}_2$ both depend on T and the structure of (1.2), *and $\mathcal{C}_1$ also depends on $\varkappa$.*

THEOREM G. *Let u be a non-negative solution of equation* (1.2) *in S. If*

$$\mathcal{N} = \sup_{0<t\leq T} \int_{E^n} u(x,t)\,dx < \infty$$

then

$$u(x,t) \leq \mathcal{C}\,\mathcal{N} t^{-n/2},$$

where $\mathcal{C}$ *depends only on the structure of* (1.2).

Finally, we have the somewhat more complicated versions of Theorems E and F which hold when the basic domain is bounded. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two point sets we will write $d(\mathcal{E}_1, \mathcal{E}_2)$ for the distance from $\mathcal{E}_1$ to $\mathcal{E}_2$.

THEOREM H. (*Harnack Principle*) *Let u be a non-negative solution of equation* (1.2) *in Q. Suppose* Ω' *is a convex subdomain of* Ω *such that* $\delta = d(\Omega', \partial\Omega) > 0$. *Then for all* x, y *in* Ω' *and all* s, t *satisfying* $0 < s < t \leq T$ *we have*

$$u(y,s) \leq u(x,t) \exp \mathcal{C}\left(\frac{|x-y|^2}{t-s} + \frac{t-s}{R} + 1\right),$$

where $\mathcal{C}$ *depends only on the structure of* (1.2), *and* $R = \min(1, s, \delta^2)$.

THEOREM I. *Let u be a non-negative solution of equation* (1.2) *in Q. Let* ξ *be a fixed point of* Ω *and suppose that for some* $\varkappa > 0$

$$\mathcal{M} = \inf \int_{|x-\xi|^2<\varkappa t} u(x,t)\,dx > 0 \quad for \quad 0 < t < \min\left\{T, \frac{1}{\varkappa} d^2(\xi, \partial\Omega)\right\}.$$

If Ω' *is a covex subdomain of* Ω *such that* $\xi \in \Omega'$ *and* $\delta = d(\Omega', \partial\Omega) > 0$ *then*

$$u(x,t) \geq \mathcal{C}_1\,\mathcal{M} t^{-n/2} \exp(-\mathcal{C}_2 |x-\xi|^2/t)$$

for all $x \in \Omega'$ *and* $0 < t < \min(T, 2\delta_1^2/\varkappa)$, *where* $\delta_1 = d(\xi, \partial\Omega')$. *Here* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *depend on* δ, T *and the structure of* (1.2). *and* $\mathcal{C}_1$ *also depends on* $\varkappa$.

Theorems B, C, E and H are local results. In particular, they are applicable only in subdomains where t is bounded away from zero. If, however, $u(x,t)$ has a limit in an appropriate sense as $t \to 0$ and if the limit satisfies certain additional conditions, then this restriction can be avoided. A special case of this remark will be used on several occasions in what follows and we conclude this section with a precise formulation of that special case.

Let u be a weak solution of equation (1.1) in D and let σ, τ be such that $0 < \sigma < \tau < T$. For each $\varepsilon > 0$ so small that $0 < \sigma - \varepsilon < \tau + \varepsilon < T$ there exists a smooth function $\eta_\varepsilon = \eta_\varepsilon(t)$ such that (i) $\eta_\varepsilon = 1$ for $\sigma \leq t \leq \tau$, (ii) $\eta_\varepsilon = 0$ for $t \leq \sigma - \varepsilon$ and $t \geq \tau + \varepsilon$, (iii) $0 \leq \eta_\varepsilon \leq 1$, and (iv) $0 \leq \eta'_\varepsilon \leq$ const.$/\varepsilon$ for $\sigma - \varepsilon \leq t \leq \sigma$ and const$/\varepsilon \leq \eta'_\varepsilon \leq 0$ for $\tau \leq t \leq \tau + \varepsilon$. If φ is a $C^1(D)$ function with compact support in Σ then clearly $\eta_\varepsilon \varphi \in C^1_0(D)$. Thus it follows from (2) that

$$\int_{\tau}^{\tau+\varepsilon} \eta'_\varepsilon \left(\int_{\Sigma} u\varphi \, dx \right) dt + \int_{\sigma-\varepsilon}^{\sigma} \eta'_\varepsilon \left(\int_{\Sigma} u\varphi \, dx \right) dt +$$

$$\iint_D \eta_\varepsilon \left(- u\varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + \dots - G\varphi \right) dx \, dt = 0.$$

Since $\int_\Sigma u\varphi \, dx$ is a continuous function of t on the interval $(0, T]$ we can let $\varepsilon \to 0$ to obtain [1]

$$(1.3) \qquad \int_\Sigma u\varphi \, dx \Big|_{t=\tau} + \iint_{\Sigma \times (\sigma, \tau)} \left(- u\varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + \dots - G\varphi \right) dx \, dt = \int_\Sigma u\varphi \, dx \Big|_{t=\sigma}$$

for any $\varphi \in C^1(\bar{D})$ with compact support in Σ, where $0 < \sigma < \tau < T$. By continuity, it follows that (1.3) also holds with $\tau = T$.

A weak solution u of equation (1.1) in D will be said to have *initial values* u_0 *on* $t=0$, where $u_0 \in L^2_{\text{loc}}(\Sigma)$, if $u \in L^\infty[0, T; L^2_{\text{loc}}(\Sigma)] \cap L^2[0, T; H^{1,2}_{\text{loc}}(\Sigma)]$ and

$$\lim_{t \to 0} \int_\Sigma u(x, t) \, \psi(x) \, dx = \int_\Sigma u_0(x) \, \psi(x) \, dx$$

for all $\psi \in C^1_0(\Sigma)$. If u has initial values u_0 on $t = 0$ and φ is a $C^1(\bar{D})$ func-

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[1] According to Lemma 2 (given in section 2), $u \in L^\infty[\delta, T; L^2_{\text{loc}}(\Sigma)] \cap L^2[\delta, T; H^{1,2}_{\text{loc}}(\Sigma)]$ implies that $u \in L^{2q'}[\delta, T; L^{2p'}_{\text{loc}}(\Sigma)]$ for all exponents p', q' whose Hölder conjugates p, q satisfy (**). It follows that $|\eta_\varepsilon(-u\varphi_t + \dots - G\varphi)|$ is dominated by an integrable function, and, using the dominated convergence theorem one obtains (1.3). A similar remark justifies the limit process which leads to (1.4).

tion with compact support in Σ, then since $u \in L^\infty[0, T; L^2_{\text{loc}}(\Sigma)]$

$$\int_\Sigma u(x, \sigma)\, \varphi(x, \sigma)\, dx = \int_\Sigma u(x, \sigma)\, \varphi(x, 0)\, dx +$$

$$+ \int_\Sigma u(x, \sigma)\, \{\varphi(x, \sigma) - \varphi(x, 0)\}\, dx \longrightarrow \int_\Sigma u_0(x)\, \varphi(x, 0)\, dx$$

as $\sigma \longrightarrow 0$. Therefore, letting $\sigma \longrightarrow 0$ in (1.3) we obtain the following result. If u is a weak solution of equation (1.1) in D with initial values u_0 on $t = 0$ then

$$\int_\Sigma u\varphi\, dx\,\Big|_{t=\tau} + \iint_{\Sigma \times (0, \tau)} (-u\varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + \ldots - G\varphi)\, dx\, dt = \tag{1.4}$$

$$= \int_\Sigma u_0(x)\, \varphi(x, 0)\, dx$$

for all $\tau \in [0, T]$ and for all $\varphi \in C^1(\bar{D})$ with compact support in Σ.

Let u be a weak solution (1.1) in D with initial values $u_0 = \text{constant}$ on $t = 0$. Extend the domain of definition of the F_j and G by setting them equal to zero in $\complement D$. Let $D^* = \Sigma \times (-\infty, T]$ and define the function

$$u^*(x, t) = \begin{cases} u(x, t) & \text{for } (x, t) \in \Sigma \times [0, T] \\ u_0 & \text{for } (x, t) \in \Sigma \times (-\infty, 0). \end{cases} \tag{1.5}$$

We assert that u^* is a weak solution of equation (1.1) in D^*, where L is defined for $t < 0$ according to the convention adopted in the Introduction and $F_j = G = 0$ for $t < 0$. It is clear that $u \in L^\infty[-\infty, T; L^2_{\text{loc}}(\Sigma)] \cap L^2[-\infty, T; H^{1,2}_{\text{loc}}(\Sigma)]$. Thus it remains to be shown that

$$\iint_D (-u^* \varphi_t + A_{ij} u^*_{x_i} \varphi_{x_j} + \ldots - G\varphi)\, dx\, dt + \tag{1.6}$$

$$+ \iint_{\Sigma \times (-\infty, 0)} (-u^* \varphi_t + u^*_{x_i} \varphi_{x_i})\, dx\, dt = 0$$

for arbitrary $\varphi \in C^1_0(D^*)$. For any $\sigma < 0$

$$\int_D u_0\, \varphi(x, \sigma)\, dx + \iint_{\Sigma \times (-\infty, \sigma)} (-u^* \varphi_t + u^*_{x_i} \varphi_{x_i})\, dx\, dt = 0.$$

Hence, letting $\sigma \to 0$,

$$\int_{\Sigma} u_0 \varphi(x, 0)\, dx + \iint_{\Sigma \times (-\infty, 0)} (- u^* \varphi_t + u^*_{x_i} \varphi_{x_i})\, dx\, dt = 0.$$

On the other hand, if we let $\tau \to T$ in (1.4) it follows that

$$\iint_D (- u^* \varphi_t + A_{ij} u^*_{x_i} \varphi_{x_j}) + \dots - G\varphi)\, dx\, dt = \int_{\Sigma} u_0 \varphi(x, 0)\, dx.$$

Combining the last two expression we obtain (1.6) and the assertion is proved.

Since the extension of equation (1.1) for $t < 0$ does not alter its structure, the results of reference [6] apply to u^* in D^*. In particular, the weak solution u^* defined by (1.5) has a representative which is continuous in D^*. Henceforth u^* will denote this continuous representative. Note that with this convertion $u^*(x, 0) = u_0$ for $x \in \Sigma$. To summarize these considerations we have the following result.

EXTENSION PRINCIPLE. *Let u be a weak solution of equation* (1.1) *in D with initial values $u_0 =$* constant *on $t = 0$. Then Theorem B holds for u^* in D^*. Moreover, if Theorem C, E or H holds for u in D, then the same Theorem holds for u^* in D^*.*

The operator adjoint to L is

$$\tilde{L}v \equiv - v_t - \{A_{ij} v_{x_j} - B_i v\}_{x_i} + A_i v_{x_i} - Cv,$$

and it is clear that if L satisfies (H) then the same is true for $\tilde{L}$. Thus all of the results given above also hold for the equation

$$\tilde{L}v = \{F_j(x, t)\}_{x_j} + G(x, t). \tag{1.7}$$

Indeed, any result which is stated for equation (1.1) can be reformulated to apply to equation (1.7) by replacing t with $-t$, and interchanging A_j and $-B_j$.

2. Energy Estimate.

In this section we consider a certain class of weak solutions of the equation

$$Lu = \{F_j(x, t)\}_{x_j} + G(x, t) \tag{2.1}$$

and derive estimates for a weighted $L^{2,\infty}$ norm of u and a weighted $L^{2,2}$ norm of u_x. These estimates will be used frequently in various contexts in the remainder of the paper and they will be given here in a form which is sufficiently general to cover all applications which occur.

Let Ω be a fixed bounded open domain in E^n and $Q = \Omega \times (0, T]$. We assume that L satisfies (H), $F_j \in L^{2,2}(Q)$, and $G \in L^{p,q}(Q)$ with p, q satisfying $(**)$. If u is a weak solution of (2.1) in Q and $u \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^{1,2}(\Omega)]$ then u is said to be a *global weak solution of* (2.1) *in* Q. For arbitrary $\tau \in (0, T]$ let $Q_\tau = \Omega \times (0, \tau]$. If u is a global weak solution of (2.1) in Q with initial values u_0 on $t = 0$ then it follows from (1.4) that

$$\int_\Omega u\varphi\, dx\,|_{t=\tau} + \iint_{Q_\tau} (-u\varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + \dots - G\varphi)\, dx\, dt = \int_\Omega u_0(x)\, \varphi(x, 0)\, dx \tag{2.2}$$

for all $\varphi \in C^1(\bar{Q}_\tau)$ with compact support in Ω. Note that if u is a weak solution of (2.1) in D then u is a global weak solution of (2.1) in $\Omega \times (\delta, T]$ with initial values $u(x, \delta)$ on $t = \delta$ for any bounded open domain Ω such that $\bar{\Omega} \subset \Sigma$ and any $\delta \in (0, T)$. In the proof of the main result of this section we will need two properties of global weak solutions. Before formulating the main result we derive these properties.

If u is a global weak solution of (2.1) in Q with initial values u_0 on $t = 0$ then

$$\| u(\cdot, t) \|_{L^2(\Omega)} \leq \| u \|_{2,\infty} \quad \text{for all } t \in [0, T] \tag{2.3}$$

and

$$\lim_{\tau \to t} \int_\Omega u(x, \tau)\, \psi(x)\, dx = \int_\Omega u(x, t)\, \psi(x)\, dx \tag{2.4}$$

for all $t \in [0, T]$ and $\psi \in L^2(\Omega)$, where $u(x, 0)$ is interpreted as $u_0(x)$. Let

$$E = \{t\,;\, 0 \leq t \leq T,\ \| u(\cdot, t) \|_{L^2(\Omega)} > \| u \|_{2,\infty}\}.$$

Since $u \in L^{2,\infty}(Q)$ we know that $|E| = 0$. Suppose that $E \neq \emptyset$. Then given $s \in E$ there exists a sequence of points $\{t_j\}$ in $\complement E$ such that $t_j \to s$. If this were not the case then E would contain a neighborhood of s in contradiction to $|E| = 0$. The weak compactness of bounded sets in $L^2(\Omega)$ and $\| u(\cdot, t_j) \|_{L^2(\Omega)} \leq \| u \|_{2,\infty}$ imply the existence of a function $U_s(x) \in L^2(\Omega)$ and a subsequence, which we again denote by $\{t_j\}$, such that $u(\cdot, t_j) \to U_s$ weakly in $L^2(\Omega)$. Moreover, $\| U_s \|_{L^2(\Omega)} \leq \| u \|_{2,\infty}$. Set $\tau = t_j$ in (2.2) and take

$\varphi = \varphi(x) \in C_0^1(\Omega)$. Then letting $t_j \to s$ we obtain

$$\int_\Omega U_s(x)\,\varphi(x)\,dx + \iint_{Q_s} (A_{ij} u_{x_i} \varphi_{x_j} + \ldots - G\varphi)\,dx\,dt = \int_\Omega u_0(x)\,\varphi(x)\,dx.$$

On the other hand, (2.2) also holds for $\tau = s$ and it follows that

$$\int_\Omega U_s(x)\,\varphi(x)\,dx = \int_\Omega u(x, s)\,\varphi(x)\,dx$$

for arbitrary $\varphi \in C_0^1(\Omega)$. Since $C_0^1(\Omega)$ is dense in $L^2(\Omega)$ this implies that $u(x, s) \equiv U_s(x)$ as functions in $L^2(\Omega)$ and, in particular, that $\| u(\cdot, s) \|_{L^2(\Omega)} = \| U_s \|_{L^2(\Omega)} \leq \| u \|_{2,\infty}$. Hence the assumption that $E \neq \emptyset$ leads to a contradiction and (2.3) holds. Now let σ and τ be any two points of $[0, T]$ with $\sigma < \tau$. It follows from (2.2) that

$$\int_\Omega u(x, \tau)\,\varphi(x)\,dx + \iint_{\Omega \times (\sigma, \tau)} (A_{ij} u_{x_i} \varphi_{x_j} + \ldots - G\varphi)\,dx\,dt = \int_\Omega u(x, \sigma)\,\varphi(x)\,dx$$

for any $\varphi \in C_0^1(\Omega)$. Thus

$$\lim_{\tau \to t} \int_\Omega u(x, \tau)\,\varphi(x)\,dx = \int_\Omega u(x, t)\,\varphi(x)\,dx$$

for all $t \in [0, T]$ and $\varphi \in C_0^1(\Omega)$. In view of (2.3) and the density of $C_0^1(\Omega)$ in $L^2(\Omega)$, (2.4) follows easily.

The main result of this section is the following lemma concerning global weak solutions of (2.1) in Q.

LEMMA 1. *Let u be a global weak solution of* (2.1) *in* $Q = \Omega \times (0, T]$ *with initial values* u_0 *and let* $\zeta = \zeta(x)$ *be a non-negative smooth function such that*

$$\zeta u \in L^2[0, T;\ H_0^{1,2}(\Omega)].$$

There exist positive constants α, β, C *such that for arbitrary* $\xi \in E^n$, $s \in [0, T)$ *and* $\mu > 0$

$$\| \zeta e^h u \|_{2,\infty}^2 + \| \zeta e^h u_x \|_{2,2}^2 \leq$$

$$C \Big\{ \int_\Omega \zeta^2 e^{2h} u^2\,dx \,\big|_{t=s} + \| \zeta e^h G \|_{2p/p+1,\, 2q/q+1}^2 + \sum_{j=1}^n \| \zeta e^h F_j \|_{2,2}^2 + \| e^h u \zeta_x \|_{2,2}^2 \Big\},$$

where the norms are taken over the set $\Omega \times (s, T')$ *with* $T' = \min(T, \mu + s)$, *and*

$$h = h(x, t) = -\frac{\alpha \,|\, x - \xi \,|^2}{2\mu - (t - s)} - \beta(t - s).$$

The constants α *and* β *depend only on the structure of* L, *while* $\mathcal{C}$ *depends only on the structure of* L, T, *and the exponents* p, q *for* G

By the structure of L we mean n and the quantities which occur in the hypotheses (H). In particular, α depends only on n, M and ν, while β depends only on n, M_0 and ν. The explicit forms of α, β and $\mathcal{C}$ are given in the proof of Lemma 1.

In the proof of Lemma 1 and elsewhere in the paper we will need the following interpolation lemma.

LEMMA 2. *If* $w \in L^\infty[0, T;\; L^2(\Omega)] \cap L^2[0, T;\; H_0^{1,2}(\Omega)]$ *then* $w \in L^{2p', 2q'}(Q)$ *for all values of* p' *and* q' *whose Hölder conjugates* p *and* q *satisfy*

$$\frac{n}{2p} + \frac{1}{q} \leq 1 \quad (< 1 \quad \text{if } n = 2)$$

Moreover

$$\| w \|^2_{2p', 2q'} \leq K T^\theta \{\| w \|^2_{2,\infty} + \| w_x \|^2_{2,2}\},$$

where $\theta = 1 - \dfrac{1}{q} - \dfrac{n}{2p}$, *and* K *is a positive constant which depends only on* n *for* $n \neq 2$ *and only on* p *for* $n = 2$.

This lemma is a slightly improved version of Lemma 3 of reference [6]. Specifically, in reference [6] the constant K depends on $|\,\Omega\,|$ in the case $n=2$, while here K is independent of $|\,\Omega\,|$ for all $n \geq 1$. We shall prove Lemma 1 only for the case $n = 2$ and refer the reader to [6] for the remaining cases.

PROOF OF LEMMA 2 FOR THE CASE $n = 2$. By Nirenberg's form of the Sobolev theorem [20]

$$\| w \|_{2p'}(t) \leq \widetilde{K} \,\| w_x \|_2^{1/p}(t)\, \| w \|_2^{1/p'}(t)$$

for $p > 1$ and almost all $t \in [0, T]$, where $\widetilde{K} = \{(p'+1)/2\sqrt{2}\}^{1/p}$. Hence

$$\| w \|^{2q'}_{2p', 2q'} \leq \widetilde{K}^{2q'} \| w \|^{2q'/p'}_{2,\infty} \int_0^T \left(\int_\Omega |\, w_x \,|^2 \, dx \right)^{q'/p} dt \leq$$

$$\leq \widetilde{K}^{2q'}\, T^{1-q'/p}\, \| w \|^{2q'/p'}_{2,\infty} \| w \|^{2q'/p}_{2,2}$$

provided that $1-\frac{1}{q}=\frac{1}{q'}\geq\frac{1}{p}$. Thus it follows from Young's inequality that

$$\| w \|^2_{2p', 2q'} \leq \widetilde{K}^2 T^\theta \| w \|^{2/p'}_{2,\infty} \| w_x \|^{2/p}_{2,2} \leq \widetilde{K}^2 T^\theta \{\| w \|^2_{2,\infty} + \| w_x \|^2_{2,2}\}$$

PROOF OF LEMMA 1. Without loss of generality we may set $s = 0$. To derive the required estimate it is necessary to take $\zeta^2 e^{2h} u$ as the test function in (2.2). It is obvious that this cannot be done directly since $\zeta^2 e^{2h} u$ is not a $C^1(\bar{Q}_\tau)$ function with compact support in Ω, and in particular, it does not have a derivate with respect to t even in the distribution sense. Thus the proof is divided into two parts. In the first part we show that the equation obtained by formally substituting $\varphi = \zeta^2 e^{2h} u$ in (2.2) is in fact valid. That is, if u is a global weak solution of (2.1) in Q then u satisfies

$$\begin{aligned}\frac{1}{2}\int_\Omega \zeta^2 e^{2h} u^2\, dx\,|_{t=\tau} + \iint_{Q_\tau}\zeta^2 e^{2h}\{A_{ij} u_{x_i} u_{x_j} + \\ + (A_j - B_j) u_{x_j} u + (-h_t - C) u^2 + F_j u_{x_j} - Gu\}\, dx\, dt + \\ + 2\iint_{Q_\tau}\zeta e^{2h} u (A_{ij} u_{x_i} + uA_j + F_j)(\zeta_{x_j} + \zeta h_{x_j})\, dx\, dt = \frac{1}{2}\int_\Omega \zeta^2 e^{2h} u_0^2\, dx\,|_{t=0}\end{aligned} \tag{2.5}$$

for all $\tau \in [0, T']$. To accomplish this we use a regularization technique as in reference [3]. The second part of the proof consists of the actual derivation of the required estimate from (2.5).

In view of the choice of ζ,

$$w(x,t) = \zeta(x)\, e^{h(x,t)}\, u(x,t) \in L^\infty[0,\tau; L^2(\Omega)] \cap L^2[0,\tau; H_0^{1,2}(\Omega)].$$

Thus, in particular, there exists a sequence $\{w^k\}$ of $C^1(\bar{Q}_\tau)$ functions with compact support in Ω such that

$$\| w - w^k \|_{2,2} + \| w_x - w_x^k \|_{2,2} \to 0 \text{ as } k \to \infty,$$

where the norms are computed over the set Q_τ for fixed $\tau \in (0, T']$. Let $K_l(t)$ be an *even* averaging kernel with support in $|t| < 1/l$, where l is a positive integer. Let w_l^k denote the convolution of w^k with K_l on $(0, \tau)$, that is,

$$w_l^k(x, t) = \int_0^\tau K_l(t - \eta)\, w^k(x, \eta)\, d\eta$$

and define w_l similarly. For each value of k and l it is clear that

$$\varphi_l^k(x, t) = \zeta(x)\, e^{h(x, t)}\, w_l^k(x, t)$$

is a $C^1(\overline{Q}_\tau)$ function with compact support in Ω. It is therefore an admissible test function in (2.2) and we may set $\varphi = \varphi_l^k$ to obtain

$$\int_\Omega \zeta\, e^h\, w_l^k\, u dx\,|_{t=\tau} + \iint_{Q_\tau} e^h \{-\zeta u\,(w_{lt}^k + w_l^k h_t) +$$

$$\zeta\,(A_{ij}\, u_{x_i} + A_j\, u + F_j)\, w_{lx_j}^k - \zeta\,(B_j u_{x_j} + Cu + G)\, w_l^k + \tag{2.6}$$

$$(A_{ij} u_{x_i} + A_j u + F_j)\,(\zeta h_{x_j} + \zeta_{x_j})\, w_l^k\}\, dx\, dt = \int u_0\, \zeta e^h\, w_l^k\, dx\,|_{t=0}\,.$$

We assert that if we hold l fixed and let $k \to \infty$ in (2.6) the result is that (2.6) is valid with k deleted. By Minkowski's and Schwarz's inequalities together with the standard properties of averaging kernels [20]

$$\| w_l^k - w_l \|_2(t) \leq \int_0^\tau K_l(t - \eta)\, \| w^k - w \|_2(\eta)\, d\eta \leq$$

$$\left(\int_{-\infty}^{\infty} K_l(t - \eta)\, d\eta\right)^{1/2} \left(\int_0^\tau K_l(t - \eta)\, \| w^k - w \|_2^2(\eta)\, d\eta\right)^{1/2} =$$

$$\left(\int_0^\tau K_l(t - \eta)\, \| w^k - w \|_2^2(\eta)\, d\eta\right)^{1/2},$$

whence

$$\| w_l^k - w_l \|_{2,2} \leq \| w^k - w \|_{2,2}\,.$$

Similarly, since $w_{lx} = w_{xl}$,

$$\| w^k_{lx} - w_{xl} \|_{2,2} \leq \| w^k_x - w_x \|_{2,2},$$

$$\| w^k_{lt} - w_{lt} \|_{2,2} \leq \left(\int\limits_{-\infty}^{\infty} | K_l'(t-\eta) | \, d\eta \right) \| w^k - w \|_{2,2} = C_l \| w^k - w \|_{2,2}$$

and

$$\| w^k_l - w_l \|_2 (t) \leq \left(\int\limits_{-\infty}^{\infty} K_l^2 (t-\eta) \, d\eta \right)^{1/2} \| w^k - w \|_{2,2} = C_l' \| w^k - w \|_{2,2}$$

for every $t \in [0, \tau]$. Thus

$$\| w^k_l - w_l \|_{2,\infty} \leq C_l' \| w^k - w \|_{2,2}$$

and, by Lemma 2,

$$\| w^k_l - w_l \|_{2p', 2q'} \leq K T^\theta \{ \| w^k_x - w_x \|_{2,2} + C_l' \| w^k - w \|_{2,2} \}$$

for all exponents p', q' such that $\frac{n}{2p} + \frac{1}{q} \leq 1$. Note that w itself also belongs to $L^{2p', 2q'}(Q_\tau)$ for the same range of exponenents. Using these facts together with the hypotheses satisfied by L, F_j, G and u_0 it is not difficult to check that each term in (2.6) tends as $k \to \infty$ to the corresponding term with k deleted. We omit further details.

Consider the integral

$$V = \iint\limits_{\Omega_\tau} \zeta \, e^h \, u w_{lt} \, dx \, dt = \iint\limits_{\Omega_\tau} w w_{lt} \, dx \, dt.$$

By Fubini's theorem and the definition of w_l this can be written

$$V = \int\limits_{\Omega} \left\{ \int\limits_0^\tau \int\limits_0^\tau K_l'(t-\eta) \, w(x,t) \, w(x,\eta) \, dt \, d\eta \right\} dx.$$

Thus, since $K_l(t)$ is an even function of t, it follows that $V = 0$. Using the translation continuity of the norm $\| \cdot \|_{p,q}$ one shows, in a standard fashion [20], that $\| w_{x_l} - w_x \|_{2,2} \to 0$ and $\| w_l - w \|_{2p', 2q'} \to 0$ as $l \to \infty$ for all appropriate p, q. Consider

$$\int_{\Omega} w(x,\tau)\, w_l(x,\tau)\, dx = \int_0^{1/l} K_l(\sigma) \left(\int_{\Omega} w(x,\tau)\, w(x,\tau-\sigma)\, dx \right) d\sigma$$

for $1/l < \tau$. Since $u(x,\tau) \in L^2(\Omega)$ and (2.4) holds we have

$$\lim_{\sigma \to 0} \int_{\Omega} w(x,\tau)\, w(x,\tau-\sigma)\, dx = \int_{\Omega} w^2(x,\tau)\, dx.$$

Thus given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_{\Omega} w(x,\tau) \{ w(x,\tau-\sigma) - w(x,\tau) \}\, dx \right| < \varepsilon$$

for all $|\sigma| < \delta$. In particular, if $1/l < \delta$ then

$$\left| \int_{\Omega} w(x,\tau)\, w_l(x,\tau)\, dx - \frac{1}{2} \int_{\Omega} w^2(x,\tau)\, dx \right| =$$

$$\int_0^{1/l} K_l(\sigma) \left[\int_{\Omega} w(x,\tau) \{ w(x,\tau-\sigma) - w(x,\tau) \}\, dx \right] d\sigma < \varepsilon,$$

that is,

$$\lim_{l \to \infty} \int_{\Omega} w(x,\tau)\, w_l(x,\tau)\, dx = \frac{1}{2} \int_{\Omega} w^2(x,\tau)\, dx.$$

Similarly,

$$\lim_{l \to \infty} \int_{\Omega} u_0(x)\, \zeta(x)\, e^{h(x,0)}\, w_l(x,0)\, dx = \frac{1}{2} \int_{\Omega} \zeta^2(x)\, e^{2h(x,0)}\, u_0^2(x)\, dx.$$

Using these observations it is not difficult to verify that if we first let $k \to \infty$ in (2.6) for fixed l and then let $l \to \infty$ in the resulting equation we obtain precisely the equation (2.5). This completes the first part of the proof of Lemma 1.

For the integrands of the second and third terms on the left in (2.5) we have the estimates

$$A_{ij} u_{x_i} u_{x_j} + \dots - Gu \geq \frac{3\nu}{4} |u_x|^2 + \{ - h_t - C - \frac{2}{\nu} \Sigma (A_j - B_j)^2 \} u^2 -$$

$$\frac{2}{\nu} \Sigma F_j^2 - | Gu$$

and

$$2 \mid \zeta e^{2h} u (A_{ij} u_{x_i} \zeta_{x_j} + \dots + \zeta F_j h_{x_j}) \mid \le \frac{\nu}{4} \zeta^2 e^{2h} \mid u_x \mid^2 +$$

$$\zeta^2 e^{2h} u^2 \left(\frac{8n^2 M^2}{\nu} \mid h_x \mid^2 + 2 \Sigma A_j^2 + 2 \mid h_x \mid^2 \right) +$$

$$e^{2h} u^2 \mid \zeta_x \mid^2 \left(\frac{8n^2 M^2}{\nu} + 2 \right) + 2 e^{2h} \zeta^2 \Sigma F_j^2 ,$$

where

$$\mid h_x \mid^2 = \frac{4\alpha^2 \mid x - \xi \mid^2}{(2\mu - t)^2} \text{ and } h_t = - \frac{1}{4\alpha} \mid h_x \mid^2 - \beta.$$

Let τ_1, τ_2 be such that $0 \le \tau_1 \le \tau_2 \le T'$. Write (2.6) for $\tau = \tau_1$ and $\tau = \tau_2$, form the difference between these expressions, and apply the estimates given above to obtain

$$\frac{1}{2} \int_\Omega \zeta^2 e^{2h} u^2 \, dx \Big|_{\tau_1}^{\tau_2} + \frac{\nu}{2} \iint \zeta^2 e^{2h} \mid u_x \mid^2 dx\, dt + \iint \mathcal{B}(x, t) \zeta^2 e^{2h} u^2 \, dx \, dt$$

$$(2.7) \qquad + \left(\frac{1}{4\alpha} - \mathcal{C}_1 \right) \iint \zeta^2 e^{2h} u^2 \mid h_x \mid^2 dx \, dt \le \mathcal{C}_1 \iint e^{2h} u^2 \mid \zeta_x \mid^2 dx \, dt$$

$$+ 2 \left(1 + \frac{1}{\nu} \right) \iint e^{2h} \zeta^2 \Sigma F_j^2 \, dx \, dt + \iint \zeta^2 e^{2h} \mid G u \mid dx \, dt.$$

Here $\mathcal{C}_1 = 2 \left(1 + \frac{4n^2 M^2}{\nu} \right)$,

$$\mathcal{B}(x, t) = \beta - C - \frac{2}{\nu} \Sigma (A_j - B_j)^2 - 2 \Sigma A_j^2 ,$$

and the double integrals are computed on the set $\Omega \times (\tau_1, \tau_2)$.

Set

$$\beta = M_0 \left(1 + 2n M_0 + \frac{4n M_0}{\nu} \right)$$

and define

$$\mathcal{F}(x, t) = \begin{cases} 2 \left(1 + \frac{2}{\nu} \right) \Sigma (A_j^2 + B_j^2) + \mid C \mid & \text{for } \mid x \mid < R_0 \\ 0 & \text{for } \mid x \mid \ge R_0 . \end{cases}$$

Since L satisfies (H) it is easily seen that

$$(2.8) \qquad \iint \mathcal{B}(x,t)\,\zeta^2 e^{2h} u^2\,dx\,dt \geq -\iint \mathcal{F}(x,t)\,\zeta^2 e^{2h} u^2\,dx\,dt.$$

The functions A_j^2, B_j^2, C and G each belong to some space $L^{p,q}$ with p and q satisfying (**). Let θ denote the minimum value of $1-\frac{n}{2p}-\frac{1}{q}$ for the $2n+2$ pairs (p,q) involved. Note that $0<\theta\leq 1$. Assume $\tau_1 \leq \tau_2 \leq \tau_1+\sigma$, where $\sigma\leq 1$. By Hölder's inequality and Lemma 2

$$\iint \mathcal{F}\zeta^2 e^{2h} u^2\,dx\,dt \leq K\,\mathcal{C}_2\,\sigma^\theta \left(\|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2\right),$$

where

$$\mathcal{C}_2 = 2\left(1+\frac{2}{\nu}\right)\Sigma\left(\|A_j\|^2+\|B_j\|^2\right)+\|C\|$$

with the norms computed on the set Q_0. Similarly, using the inequalities of Hölder and Young together with Lemma 2, we have

$$\iint \zeta^2 e^{2h} |Gu|\,dx\,dt \leq \|\zeta e^h u\|_{2p',2q'}\,\|\zeta e^h G\|_{2p/p+1,\,2q/q+1}$$

$$\leq \frac{1}{2}\|\zeta e^h G\|^2_{2p/p+1,\,2q/q+1} + \frac{1}{2}K\sigma^\theta\left(\|\zeta e^h u\|^2_{2,\infty}+\|(\zeta e^h u)_x\|^2_{2,2}\right).$$

Finally we note that

$$\|(\zeta e^h u)_x\|^2_{2,2} \leq 2\|\zeta e^h u_x\|^2_{2,2} + 4\|u e^h \zeta_x\|^2_{2,2} + 4\|\zeta e^h u h_x\|^2_{2,2}.$$

In view of these estimates,

$$(2.9) \qquad \iint \zeta^2 e^{2h}\left(\mathcal{F}u^2+|Gu|\right)dx\,dt \leq \frac{1}{2}\|\zeta e^h G\|^2_{2p/p+1,\,2q/q+1} +$$

$$+\,2K\left(\mathcal{C}_2+\frac{1}{2}\right)\sigma^\theta\left(\|\zeta e^h u\|^2_{2,\infty}+\|\zeta e^h u_x\|^2_{2,2}+2\|\zeta e^h u h_x\|^2_{2,2}+2\|e^h u\zeta_x\|^2_{2,2}\right).$$

Choose

$$\sigma = \min\left[1,\left\{\frac{\min(1,\nu)}{8K\left(\mathcal{C}_2+\frac{1}{2}\right)}\right\}^{1/\theta}\right]$$

and α such that

$$\frac{1}{4\alpha} = \mathcal{C}_1 + \frac{1}{2} \min(1, \nu).$$

Note that $2K\left(\mathcal{C}_2 + \frac{1}{2}\right)\sigma^\theta \leq \frac{1}{4} \min(1, \nu)$. Thus using (2.8) and (2.9) it follows from (2.7) that

$$(2.10) \quad \int_\Omega \zeta^2 e^{2h} u^2 \, dx \Big|_{\tau_1}^{\tau_2} + \frac{\nu}{2} \iint \zeta^2 e^{2h} |u_x|^2 \, dx \, dt \leq \mathcal{C}_3 \iint e^{2h} u^2 |\zeta_x|^2 \, dx \, dt +$$

$$\| \zeta e^h G \|^2_{2p/p+1,\, 2q/q+1} + 4\left(1 + \frac{1}{\nu}\right) \sum_j \| e^h \zeta F_j \|^2_{2,2} + \frac{\min(1, \nu)}{2} \| e^h u \zeta \|^2_{2,\infty},$$

where $\mathcal{C}_3 = 2\mathcal{C}_1 + \min(1, \nu)$.

Let

$$X(t) = \int_\Omega \zeta^2(x) \, e^{2h(x,t)} u^2(x, t) \, dx$$

and

$$J = \mathcal{C}_3 \| e^h u \zeta_x \|^2_{2,2} + \| \zeta e^h G \|^2_{2p/p+1,\, 2q/q+1} + 4\left(1 + \frac{1}{\nu}\right) \sum_j \| \zeta e^h F_j \|^2_{2,2},$$

where the norms are computed on the set $\Omega \times (0, T')$. Then (2.10) implies

$$(2.11) \quad X(\tau) + \frac{\nu}{2} \iint \zeta^2 e^{2h} |u_x|^2 \, dx \, dt \leq \frac{\min(1, \nu)}{2} \| \zeta e^h u \|^2_{2,\infty} + X(\tau_1) + J$$

for $\tau_1 \leq \tau \leq \tau_1 + \sigma$. Here the integral is taken over the set $\Omega \times (\tau_1, \tau)$ and the norm is computed over the set $\Omega \times (\tau_1, \tau_1 + \sigma)$. If we ignore the second term on the left in (2.11) it is easily seen that

$$X(\tau) \leq 2X(\tau_1) + 2J$$

for $\tau_1 \leq \tau \leq \tau_1 + \sigma$. If $(j - 1)\sigma \leq \tau \leq j\sigma$, it follows by iteration that

$$(2.12) \quad X(\tau) \leq 2^j X(0) + 2(2^j - 1) J \leq 2^j \{X(0) + 2J\}.$$

On the other hand, if we ignore the first term on the left in (2.11), set $\tau_1 = (j - 1)\sigma$ and apply (2.12) we obtain the estimate

$$(2.13) \quad \int_{(j-1)\sigma}^{j\sigma} \int_\Omega \zeta^2 e^{2h} |u_x|^2 \, dx \, dt \leq 2^j \left(1 + \frac{1}{\nu}\right) \{X(0) + 2J\}.$$

Suppose that $(l-1)\sigma < T' \leq l\sigma$ for some integer $l \geq 1$. Then (2.12) implies

$$\| \zeta e^h u \|^2_{2,\infty} \leq 2^{1+T'/\sigma} \{X(0) + 2J\},$$

and summing the estimates (2.13) on j from 1 to l yields

$$\| \zeta e^h u_x \|^2_{2,2} \leq 2^{2+T'/\sigma} \left(1 + \frac{1}{\nu}\right) \{X(0) + 2J\}.$$

Combining these two estimates, we obtain the assertion of Lemma 1.

[The conditions (H) which are imposed on L are sufficient but not necessary to guarantee the simultaneous applicability of the results enumerated in section 1 and of Lemma 1. To use theorems of section 1 in the strip S one needs, in addition to uniform parabolicity, only that each of the functions A_j^2, B_j^2 and C belong to some space $L^{p,q}(\mathcal{S})$. However, to obtain Lemma 1 it is necessary to have some further restriction on the behavior of the coefficients of L for large values of $|x|$. An examination of the proof of Lemma 1 reveals that the essential use of (H) is in obtaining a lower bound for the expression

$$\Theta = \zeta \{A_{ij} p_i + (A_j + B_j) u\} p_j - (h_t + C)\zeta u^2 + 2u (A_{ij} p_i + A_j u)(\zeta h_{x_j} + \zeta_{x_j}),$$

where $\zeta \geq 0$ and

$$h = -\frac{\alpha |x - \xi|^2}{2\mu - (t - s)} - \beta (t - s),$$

and that the proof can be carried through under less restrictive conditions on L. For example, we can assume that: (i) Each of the functions A_j^2, B_j^2 and C belong to some space $L^{p,q}(\mathcal{S})$ with p and q satisfying (**), and (II.1) holds. (ii) There exist positive constants a, b c, β and a function $\mathcal{F}(x, t)$ which is the sum of a finite number of non-negative functions each belonging to some space $L^{p,q}(S)$ with p, q satisfying (**) such that

$$\Theta \geq a\zeta |p|^2 + \left(\frac{1}{4\alpha} - b\right) \zeta u^2 |h_x|^2 - (c |\zeta_x|^2 + \zeta \mathcal{F}) u^2$$

holds for all $\alpha > 0$, $\mu > 0$, $\xi \in E^n$, $s \in [0, T)$ and $(x, t) \in E^n \times (s, T']$.]

In proving the existence of weak solutions we use Lemma 1 to obtain a sequence of functions $\{v^m\}$ such that $\| v^m \|^2_{2,\infty} + \| v^m_x \|^2_{2,2}$ is uniformly bounded. From such a sequence we then extract a subsequence which converges to the eventual weak solution by means of Lemma 2 and the following lemma.

LEMMA 3. *Let $\{v^m\}$ be a sequence of functions in $L^{2,2}(D)$ which converges weakly to a limit function v in $L^{2,2}(D)$. If $\| v^m \|_{2,\infty} \leq \mathcal{C}$ independent of m then $\| v \|_{2,\infty} \leq \mathcal{C}$.*

PROOF. According to the Banach-Saks theorem [17] there exist a subsequence $\{v^{m_i}\}$ such that the averages

$$\hat{v}^k = \frac{1}{k} \sum_{i=1}^{k} v^{m_i} \longrightarrow v$$

strongly in $L^{2,2}(D)$. Therefore, there exists a subsequence $\{\hat{v}^{k_i}\}$ of the averages which converge to v almost everywhere in D. Let $m(t)$ denote the n-dimensional measure of the set of $x \in \Sigma$ for which $\hat{v}^{k_i} \not\longrightarrow v(x, t)$ and E_0 denote the set of $t \in [0, T]$ for which $m(t) \neq 0$. The measure of E_0 is zero, for otherwise we would have a contradiction to $\hat{v}^{k_i} \longrightarrow v$ almost everywhere in D. Let E_m denote the set of $t \in [0, T]$ for which $\| v^m(\cdot, t) \|_{L^2(\Sigma)} > \| v^m \|_{2,\infty}$. Then for each $m \geq 1$ the measure of E_m is zero. Hence the set $E = \bigcup_{m=0}^{\infty} E_m$ also has measure zero. For any $t \in \complement E$ we have $\hat{v}^{k_i} \longrightarrow v$ and hence $(\hat{v}^{k_i})^2 \longrightarrow$ $\longrightarrow v^2$ almost everywhere in Σ. By Minkowski's inequality

$$\| \hat{v}^k(\cdot, t) \|_{L^2(\Sigma)} \leq \frac{1}{k} \sum_{i=1}^{k} \| v^{m_i}(\cdot, t) \|_{L^2(\Sigma)} \leq \mathcal{C}.$$

Thus, it follows from Fatou's lemma that $\| v(\cdot, t) \|_{L^2(\Sigma)} \leq \mathcal{C}$ for every $t \in \complement E$ Since the measure of E is zero this proves the Lemma.

2. The Boundary Value Problem.

Let Ω be a fixed bounded open domain in E^n and $Q = \Omega \times (0, T]$. Given functions $F_j \in L^{2,2}(Q)$, $G \in L^{p,q}(Q)$ with p, q satisfying (**), and $u_0 \in L^2(\Omega)$ we consider the boundary value problem

$$(3.1) \quad \begin{cases} Lu = \{F_j(x, t)\}_{x_j} + G(x, t) \text{ for } (x, t) \in Q \\ u(x, 0) = u_0(x) \text{ for } x \in \Omega, \; u(x, t) = 0 \text{ for } (x, t) \in \partial\Omega \times [0, T], \end{cases}$$

where L satisfies (H). A function $u = u(x, t)$ is said to be a *weak solution of problem* (3.1) if u is a global weak solution of equation (3.1) in Q with initial values u_0 on $t = 0$ and if $u \in L^2[0, T; H_0^{1,}(\Omega)]$.

By the convention adopted in the Introduction, L is defined for almost all (x, t). Extend the domains of definition of the F_j, G and u_0 by setting $F_j = G = 0$ in $\complement Q$ and $u_0 = 0$ in $\complement \Omega$. Let A_{ij}^m, $A_j^m, \ldots$, G^m denote integral averages of A_{ij}, $A_j, \ldots, G$ formed with an averaging kernel whose support lies in $|x|^2 + t^2 < m^{-2}$ for integers $m \geq 1$ and define the operators

$$L^m u \equiv u_t - (A_{ij}^m u_{x_i} + A_j^m u)_{x_j} - B_j^m u_{x_j} - C^m u.$$

By the standard properties of integral averages [20] it is easily verified that these operators have a uniform structure with respect to m which is completely determined by the structure of L. Let u_0^m denote the integral averge of u_0 formed with a kernel whose support lies in $|x| < m^{-1}$. If u_0 has compact support, so does u_0^m for m sufficiently large. Let $\{\Omega^m\}$ be a sequence of open domains with smooth boundaries such that $\bar{\Omega}^m \subset \Omega^{m+1} \subset \bar{\Omega}^{m+1} \subset \Omega$ for all $m \geq 0$ and $\lim_{m \to \infty} \Omega^m = \Omega$. If u_0 has compact support in Ω set $\zeta_m \equiv 1$, otherwise for each $m \geq 1$ let $\zeta_m = \zeta_m(x)$ denote a $C_0^\infty(\Omega^m)$ function such that $\zeta_m = 1$ on $\bar{\Omega}^{m-1}$ and $0 \leq \zeta_m \leq 1$.

Consider the sequence of boundary value problems

$$(3.2) \quad \begin{cases} L^m u = F_{j,x_j}^m + G^m \ \text{ for } (x,t) \in Q^m = \Omega^m \times (0, T] \\ u(x, 0) = \zeta_m(x)\, u_0^m(x) \ \text{ for } x \in \Omega^m,\ u(x,t) = 0 \ \text{ for } (x,t) \in \partial\Omega^m \times [0, T]. \end{cases}$$

Since the coefficients of L^m and the data are all C^∞ functions, and $\partial\Omega^m$ is smooth it follows from the classical existence theory [10] that for each $m \geq 1$ such that supp $(\zeta_m u_0^m) \subset \Omega^m$ the problem (3.2) has a unique classical solution $u^m = u^m(x, t)$ and $u^m \in C^1(\bar{Q}^m)$.

Theorem 1. *Suppose that L satisfies (H), $F_j \in L^{2,2}(Q)$, $G \in L^{p,q}(Q)$ with p and q satisfying (**), and $u_0 \in L^2(\Omega)$. Then there exists a unique weak solution u of the bondary value problem (3.1). Moreover, u is the weak limit in $L^2[0, T; H^{1,2}(\Omega)]$ of the sequence $\{u^m\}$ of classical solutions of problems (3.2) If each F_j belongs to some space $L^{p,q}(Q)$ with p and q satisfying (*) then u has the following additional properties.*

(i) *u is the uniform limit of $\{u^m\}$ in any compact subset of Q.*

(ii) *If $m_1 \leq u_0 \leq m_2$ almost everywhere on Ω then*

$$m_1 - \mathcal{C} k_1 \leq u(x, t) \leq m_2 + \mathcal{C} k_2$$

in Q, where $\mathcal{C}$ depends only on T, Ω and the structure of L, and the k_i are the same as in Theorem **A**.

(iii) *If $u_0 \in C_0^0(\Omega)$ and $\partial\Omega$ has property (A), then u is the uniform limit of $\{u^m\}$ and is continuous in $\bar{Q}$.*

PROOF. Since $u^m \in C^1(\bar{Q}^m)$ and $u^m = 0$ for $x \in \partial\Omega^m$ it follows that $u^m(\cdot, t) \in H_0^{1,2}(\Omega^m)$ for each $t \in [0, T]$, (cf. [20]). Thus, in particular, u^m is a weak solution of problem (3.2). By Lemma 1, with $\zeta = 1$, $s = 0$ $\mu = \infty$,

$$\|u^m\|^2_{2,\infty,Q^m} + \|u_x^m\|^2_{2,2,Q^m} \leq e^{\beta T} \mathcal{C} \left\{ \|\zeta_m u_0^m\|^2_{L^2(\Omega^m)} + \sum_{j=1}^n \|F_j^m\|^2_{2,2,Q^m} + \|G^m\|^2_{2p/p+1,\, 2q/q+1,\, Q^m} \right\},$$

where β and $\mathcal{C}$ are independent of m. It is easily verified that

$$\|\zeta_m u_0^m\|_{L^2(\Omega^m)} \leq \|u_0\|_{L^2(\Omega)}, \quad \|F_j^m\|_{2,2,Q^m} \leq \|F_j\|^2_{2,2,Q}$$

and

$$\|G^m\|_{2p/q+1,\, 2q/q+1,\, Q^m} \leq \|G\|_{2p/p+1,\, 2q/q+1,\, Q} \leq |\Omega|^{(p-1)/2p}T^{(q-1)/2q} \|G\|_{p,q,Q}.$$

Thus

$$\|u^m\|^2_{2,\infty,Q^m} + \|u_x^m\|^2_{2,2,Q^m} \leq e^{\beta T} \mathcal{C} \left\{ \|u_0\|^2_{L^2(Q)} + \sum_{j=1}^n \|F_j\|^2_{2,2,Q} + |\Omega|^{1/p'} T^{1/q'} \|G\|_{p,q,Q} \right\} = \mathcal{C}_1$$

and by Lemma 2,

$$\|u^m\|^2_{2p',\, 2q',\, Q^m} \leq K T^{\theta} \mathcal{C}_1 = \mathcal{C}_2$$

for all p', q' whose Hölder conjugates satisfy $(**)$. Extend the domain of definition of u^m by setting $u^m = 0$ in $\bar{Q} \setminus \bar{Q}^m$. It is clear that the extended function belongs to $L^\infty[0, T;\ L^2(\Omega)] \cap L^2[0, T;\ H_0^{1,2}(\Omega)]$, and that the last two estimates can be rewritten as

$$\|u^m\|^2_{2,\infty,Q} + \|u_x^m\|^2_{2,2,Q} \leq \mathcal{C}_1 \tag{3.3}$$

and

$$\|u^m\|^2_{2p',\, 2q',\, Q} \leq \mathcal{C}_2. \tag{3.4}$$

Let $\varphi \in C^1(\bar{Q})$ with compact support in Ω and suppose that φ vanishes in a neighborhood of $t = T$. If m is so large that $\{\text{supp}\,\varphi\} \cap Q \subset Q^m$ and $\text{supp}\,(\zeta_m u_0^m) \subset \Omega^m$ then

$$(3.5)\qquad \iint_Q (-u^m \varphi_t + A_{ij}^m u_{x_i}^m \varphi_{x_j} + A_j^m u^m \varphi_{x_j} + F_j^m \varphi_{x_j} - B_j^m u_{x_j}^m \varphi -$$

$$- C^m u^m \varphi - G^m \varphi)\, dx\, dt = \int_\Omega \zeta_m(x)\, u_0^m(x)\, \varphi(x, 0)\, dx.$$

In view of the weak compactness of bounded sets in $L^2[0, T; H_0^{1,2}(\Omega)]$ and $L^{2p', 2q'}(Q)$ it follows from (3.3) and (3.4) that there exists a subsequence of the u^m, again denoted by u^m, which converges weakly to an element u in $L^2[H_0^{1,2}]$ and in any finite collection of $L^{2p', 2q'}(Q)$ spaces. Moreover, it follows from (3.3) and Lemma 3 that $u \in L^{2,\infty}(Q)$. Since the integral averages of the coefficients and the data converge strongly in the appropriate spaces, the limit as $m \to \infty$ of the integral on the left in (3.5) exists and is equal to the corresponding integral with m deleted. Moreover, $\| \zeta_m u_0^m - u_0 \|_{L^2(\Omega)} \to 0$ as $m \to \infty$. Thus

$$(3.6)\qquad \iint_Q (-u \varphi_t + A_{ij} u_{x_i} \varphi_{x_j} + A_j u \varphi_{x_j} + F_j \varphi_{x_j} - B_j u_{x_j} \varphi - Cu\varphi - G\varphi)\,dx\,dt =$$

$$\int_\Omega u_0(x)\, \varphi(x, 0)\, dx$$

for all $\varphi \in C^1(\bar{Q})$ with compact support in Ω which vanish near $t = T$. In particulair, if $\varphi \in C_0^1(Q)$ then (3.6) holds with zero on the right hand side, and hence u is a global weak solution of equation (3.1) in Q. By an argument similar to the one employed in section 1 to derive (1.3) from (2), it follows from (3.6) that

$$\int_\Sigma u\varphi\, dx \,\Big|_{t=\tau} + \iint_{\Sigma \times (0,\tau)} (-u\varphi_t + A_{ij} u_{x_i} \varphi_{x_i} + \dots - G\varphi)\, dx\,dt = \int u_0(x)\, \varphi(x, 0)\, dx$$

for all $\tau \in [0, T]$ and for all $\varphi \in C^1(\bar{Q})$ with compact support in Ω. If we take $\varphi = \varphi(x) \in C_0^1(\Omega)$ then

$$\lim_{t \to 0} \int_\Omega u(x, t)\, \varphi(x)\, dx = \int_\Omega u_0(x)\, \varphi(x)\, dx.$$

Thus u is a weak solution of problem (3.1). The uniqueness is a trivial consequence of Lemma 1. Note that since u is unique the whole sequence $\{u^m\}$ converges to u.

If $F_j \in L^{p,q}(Q)$, with p, q satisfying (*), then the results of section 1 are applicable. Let K be a compact subset of Q and consider m so large that $K \subset Q^m$. By (3.4) and Theorem B, the sequence $\{u^m\}$ is uniformly bounded, and, by Theorem C, it is equicontinuous in K. Therefore, there is a subsequence which converges uniformly in K. Since any convergent subsequence from $\{u^m\}$ must converge to u it follows that $u^m \to u$ uniformly in K. To prove (ii) we first note that $m_1 \leq u_0 \leq m_2$ implies $m_1 \leq \zeta_m u_0^m \leq m_2$. Moreover, if m is sufficiently large $\| W^m \|_{Q^m} \leq \| W^m \|_Q \leq 2 \| W \|_Q$ for any $W \in L^{p,q}_{\text{loc}}(S)$. Thus, by Theorem A, for any $(x, t) \in Q$ and m so large that $(x, t) \in Q^m$

$$m_1 - \mathcal{C} k_1 \leq u^m(x, t) \leq m_2 + \mathcal{C} k_2, \tag{3.6}$$

where $\mathcal{C}$ depends only on T, Ω and the structure of equation (3.1). Assertion (ii) follows from (3.6) and (i).

If $\partial\Omega$ has property (A) then the sequence $\{\Omega^m\}$ can be chosen so that $\{\partial\Omega^m\}$ has property (A) uniformly whith respect to m. Moreover, the integral averages u_0^m have the same modulus of continuity as u_0. Thus, in view of (3.6) and Theorem D (ii), for m sufficiently large each u^m has a bound and a modulus of continuity in $\bar{Q}^m$ independent of m. If we extend the u^m by setting $u^m = 0$ in $\bar{Q} \setminus \bar{Q}^m$ the resulting sequence is uniformly bounded and equicontinuous in $\bar{Q}$. Thus the assertion (iii) follows from Arzela's theorem and the uniqueness of u.

In various special cases it is possible to simplify the construction of the approximating sequence $\{u^m\}$ in Theorem 1. For example, if $\partial\Omega$ is smooth and u_0 has compact support in Ω we can omit the approximation of Ω by smoothly bounded domains and take for $\{u^m\}$ the sequence of classical solutions of the boundary value problems

$$\begin{cases} L^m = F^m_{j, x_j} + G^m \text{ for } (x, t) \in Q \\ u(x, 0) = u_0^m(x) \text{ for } x \in \Omega, \ u(x, t) = 0 \text{ of } (x, t) \in \partial\Omega \times [0, T] \end{cases} \tag{3.7}$$

with m sufficiently large. If $\partial\Omega$ is smooth and u_0 is Hölder continuous on $\bar{\Omega}$ with $u_0 = 0$ on $\partial\Omega$ then $\{u^m\}$ can be taken to be the sequence of classical solutions of problems (3.7) with the initial values $u(x, 0) = u_0(x)$ instead of $u(x, 0) = u_0^m(x)$.

4. The Cauchy Problem.

In this section we consider the Cauchy problem

$$(4.1)\quad Lu = \{F_j(x,t)\}_{x_j} + G(x,t) \text{ for } (x,t) \in S,\; u(x,0) = u_0(x) \text{ for } x \in E^n,$$

where L satisfies (H), and the F_j, G, u_0 are given functions. The data will be required to satisfy certain conditions which will be specified later. A function $u = u(x,t)$ is said to be a *weak solution of problem* (4.1) if it is a weak solution of equation (4.1) in S and has initial values u_0 on $t = 0$. In particular, a weak solution of problem (4.1) is a global weak solution of equation (4.1) in Q with initial values u_0 on $t = 0$ for any bounded cylinder $Q \subset S$.

We are going to prove an existence and a uniqueness theorem for problem (4.1). Before doing so however let us recall some of the limitations on uniqueness and solvability of the Cauchy problem for the one dimensional equation of heat conduction

$$(4.2)\qquad u_{xx} = u_t .$$

Tihonov has shown that if u is a solution of (4.2) in S, then $u(x,0) = 0$ and

$$(4.3)\qquad u(x,t) = 0\,(e^{\lambda |x|^2})$$

in S for some $\lambda \geq 0$ imply $u \equiv 0$. On the other hand, there are examples which show that $u(x,0) = 0$ and $u = 0\,(e^{\lambda |x|^{2+\varepsilon}})$ do not imply $u \equiv 0$. Thus one cannot expect to have a unique solution of the Cauchy problem without excluding solutions which grow too rapidly as $|x| \to \infty$. The unique solution in the class of functions which satisfy (4.3) of equation (4.2) with the initial condition $u(x,0) = e^{\lambda x^2}$ is given by

$$u(x,t) = (1 - 4\lambda t)^{-1/2} \exp\,\{\lambda x^2/(1 - 4\lambda t)\}.$$

Clearly this solution is valid only for $t \in [0, 1/4\lambda)$. Thus if the data grow exponentially for large $|x|$ we cannot expect the solution of the Cauchy problem to exist for arbitrary values of t. Analogous restrictions enter into the statements of the results given below.

A function $u = u(x,t)$ defined and measurable on S will be said to belong to the class $\mathcal{E}^2(S)$ if there exists a number $\lambda \geq 0$ such that

$$\iint_S e^{-\lambda |x|^2} u^2(x,t)\, dx\, dt < \infty.$$

It is clear that if u satisfies (4.3) then $u \in \mathcal{E}^2(S)$.

THEOREM 2. *There is at most one solution of the Cauchy problem* (4.1) *in the class* $\mathcal{E}^2(S)$.

PROOF. If there were two solutions of the problem (4.1) in the class $\mathcal{E}^2(S)$, then their difference w would also belong to $\mathcal{E}^2(S)$ and be a weak solution of the problem

$$Lw = 0 \text{ for } (x, t) \in S, \; w(x, 0) = 0 \text{ for } x \in E^n .$$

We will show, using Lemma 1, that this implies $w \equiv 0$. Set $s = \xi = 0$. Then for $0 \leq t < 2\mu$ we have $h(x, t) \leq -\alpha |x|^2/2\mu$. If $\lambda = 0$ set $\mu = +\infty$, otherwise set $\mu = \alpha/\lambda$. Then $h(x, t) \leq -\lambda |x|^2/2$ and hence $e^h w \in L^{2,2}(S')$, where $S' = E^n \times (0, T')$ and $T' = \min(T, \mu)$. For $R \geq 1$ set $\zeta = \zeta_R(x)$, where $\zeta_R = 1$ for $|x| \leq R$, $\zeta_R = 0$ for $|x| \geq R + 1$, $0 \leq \zeta_R \leq 1$, and $|\zeta_{Rx}|$ is bounded independent of R. By Lemma 1

$$(4.4) \qquad \| \zeta_R e^h w \|_{2,\infty}^2 \leq \mathcal{C} \iint_{S'} e^{2h} w^2 |\zeta_{Rx}|^2 \, dx \, dt \leq \mathcal{C}_1 \int_0^{T'} \int_{|x| \geq R} e^{2h} w^2 \, dx \, dt,$$

where $\mathcal{C}_1$ is independent of R. Since $e^h w \in L^{2,2}(S')$ the integral on the right in (4.4) tends to zero as $R \to \infty$. Hence

$$\max_{[0, T']} \int_{|x| \leq \varrho} e^{2h(x,t)} w^2(x, t) \, dx = 0$$

for arbitrary $\varrho > 0$ and it follows that $w \equiv 0$ in S'. If $T' = T$ this completes the proof, otherwise the proof can be completed by a finite number of applications of the same argument on $E^n \times (T', 2T')$, $E^n \times (2T', 3T')$, etc.

THEOREM 3. *Let* $\gamma \geq 0$ *be fixed and assume* $T \leq \alpha/4\gamma$. *Suppose that* L *satisfies* (H), $e^{-\gamma |x|^2} F_j \in L^{2,2}(S)$, $e^{-\gamma |x|^2} G \in L^{p,q}(S)$ *with* p *and* q *satisfying* $(**)$ *and* $e^{-\gamma |x|^2} u_0 \in L^2(E^n)$. *Then there exists a weak solution* u *of the Cauchy problem* (4.1) *in* S. *Moreover*

$$\| e^h u \|_{2,\infty}^2 + \| e^h u_x \|_{2,2}^2 \leq \mathcal{C} \{ \| e^{-\gamma |x|^2} u_0 \|_{L^2(E^n)} + \\ + \sum_j \| e^{-\gamma |x|^2} F_j \|_{2,2}^2 + \| e^{-\gamma |x|^2} G \|_{p,q} \}$$

where $\mathcal{C}$ *is a positive constant which depends only on* T *and the structure of equation* (4.1), *and*

$$h(x, t) = -\frac{\alpha |x|^2}{2T - t} - \beta t.$$

In particular, u belongs to the class $\mathcal{E}^2(S)$ and hence is the unique solution of problem (4.1) in $\mathcal{E}^2(S)$.

PROOF. Let $Q_k = \Sigma_k \times (0, T]$, where $\Sigma_k = \{x ; |x| < k\}$ for integers $k \geq 1$, and consider the boundary value problems

$$(4.5)\qquad \begin{cases} Lu = F_{j,x_j} + G \text{ for } (x,t) \in Q_k \\ u(x,0) = u_0(x) \text{ for } x \in \Sigma_k,\ u(x,t) = 0 \text{ for } (x,t) \in \partial\Sigma_k \times [0, T]. \end{cases}$$

By Theorem 1, for each value of k there exists a unique weak solution u^k of problem (4.5). Since $u^k \in L^2[0, T; H_0^{1,2}(\Sigma_k)]$ it follows from Lemma 1 with $s = \xi = 0$, $\mu = T$ and $\zeta \equiv 1$ that

$$(4.6)\qquad \| e^h u^k \|_{2,\infty}^2 + \| e^h u_x^k \|_{2,2}^2 \leq \mathcal{C} \Big\{ \int_{\Sigma_k} e^{2h(x,0)} u_0^2(x)\, dx + \sum_j \| e^h F_j \|_{2,2}^2 + \| e^h G \|_{2p/p+1,\, 2q/q+1}^2 \Big\},$$

where $\mathcal{C}$ is independent of k and the norms are taken over the set Q_k. For $0 \leq t \leq T$ we have $h(x,t) + \gamma |x|^2 \leq -\nu |x|^2$, where $\nu = \alpha/4T$. Thus

$$\int_{\Sigma_k} e^{2h(x,0)} u_0^2\, dx \leq \| e^{-\gamma|x|^2} u_0 \|_{L^2(E^n)}^2,\ \| e^h F_j \|_{2,2} \leq \| e^{-\gamma|x|^2} F_j \|_{2,2,S}$$

and

$$\| e^h G \|_{2p/p+1,\, 2q/q+1} \leq T^{(q-1)/2q} \| e^{-\nu|x|^2} \|_{L^{2p/(p-1)}(E^n)} \| e^{-\gamma|x|^2} G \|_{p,q,S}.$$

Therefore (4.6) implies that

$$\| e^h u^k \|_{2,\infty}^2 + \| e^h u_x^k \|_{2,2}^2 \leq \mathcal{C}_1 \{ \| e^{-\gamma|x|^2} u_0 \|_{L^2(E^n)} + \sum_j \| e^{-\gamma|x|^2} F_j \|_{2,2,S}^2 + \| e^{-\gamma|x|^2} G \|_{p,q,S} \} \equiv \mathcal{C}_2,$$

where $\mathcal{C}_1$ depends only on T and the structure of the equation. If we extend the domain of definition of the u^k by setting $u^k = 0$ for $|x| > k$ and $0 \leq t \leq T$, then $u^k \in L^\infty[0, T; L^2_{\text{loc}}(E^n)] \cap L^2[0, T; H^{1,2}_{\text{loc}}(E)]$ and

$$(4.7)\qquad \| e^h u^k \|_{2,\infty,S}^2 + \| e^h u_x^k \|_{2,2,S}^2 \leq \mathcal{C}_2.$$

Furthermore

$$(4.8)\qquad \| e^h u^k \|_{2,2,S} \leq (T\mathcal{C}_2)^{1/2}.$$

In view of (4.7) and (4.8) there exists a subsequence of the vectors $(e^h u^k, e^h u^k_{x_1}, \dots, e^h u^k_{x_n})$, which we again index by k, and a vector $(e^h u, e^h u_1, \dots$ $\dots, e^h u_n)$ such that $e^h u^k \to e^h u$ and $e^h u^k_{x_i} \to e^h u_i$ weakly in $L^{2,2}(S)$. Let Q be any bounded cylinder in S and let φ be an arbitrary $L^{2,2}(Q)$ function. Let χ_Q denote the characteristic function of Q and set $\tilde{\varphi} = \varphi\chi_Q$. Then clearly $\varphi e^{-h} \in L^{2,2}(S)$. Thus

$$\iint_Q \varphi u^k \, dx\, dt = \iint_S (\tilde{\varphi} e^{-h})(u^k e^h)\, dx dt \to \iint_S (\tilde{\varphi} e^{-h})\,(u e^h)\, dx dt = \iint_Q \varphi u \, dx\, dt,$$

that is, $u^k \to u$ weakly in $L^{2,2}(Q)$ for any bounded $Q \subset S$. Similarly, $u^k_{x_i} \to u_i$ weakly in $L^{2,2}(Q)$ for any bounded $Q \subset S$. Since

$$\iint_S \varphi_{x_i} u^k \, dx\, dt = -\iint_S \varphi u^k_{x_i} \, dx\, dt$$

for all $\varphi \in C^1(\bar{S})$ with compact support in E^n, it follows that u_i is the distribution derivative of u with respect to x_i and we will write u_{x_i} instead of u_i. Note that for the limit function we have

$$\text{(4.9)} \qquad \| e^h u \|_{2,2,S} \leq (T\mathcal{C}_2)^{1/2}, \quad \| e^h u_x \|_{2,2,S} \leq \mathcal{C}_2^{1/2}$$

and $u \in L^2[0, T; H^{1,2}_{\text{loc}}(E^n)]$. Moreover, it follows from (4.7) and Lemma 3 that

$$\| e^h u \|_{2,\infty,S} \leq \mathcal{C}_2^{1/2}$$

and $u \in L^\infty[0, T; L^2_{\text{loc}}(E^n)]$.

Let Ω be an arbitrary bounded open set in E^n and let Ω' be another bounded open set such that $\bar{\Omega} \subset \Omega'$. Let $\zeta = \zeta(x)$ be a $C_0^\infty(E^n)$ function such that $\zeta = 1$ on $\bar{\Omega}$, $\zeta = 0$ on $\complement\,\Omega'$, and $0 \leq \zeta \leq 1$. By Lemma 2

$$\| \zeta u^k \|_{2p',2q'} \leq KT^\theta \{ \| \zeta u^k \|^2_{2\,\infty} + 2 \| \zeta u^k_x \|^2_{2,2} + 2 \| \zeta_x u^k \|^2_{2,2} \}$$

for all p' and q' whose Hölder coniugates satisfy (**). Thus

$$\| u^k \|_{2p',2q',Q} \leq KT^\theta \{ \| u^k \|^2_{2,\infty,Q'} + 2 \| u^k_x \|^2_{2,2,Q'} + 2 (\max |\zeta_x|)^2 \| u^k \|^2_{2,2,Q'} \},$$

where $Q = \Omega \times (0, T)$ and $Q' = \Omega' \times (0, T)$. Let $H = \min_{\bar{Q'}} e^h$. Then $H > 0$ and according to (4.7) and (4.8) we have $\| u^k \|_{2,\infty,Q'} \leq H^{-1} \mathcal{C}_2^{1/2}$, $\| u^k_x \|_{2,2,Q'} \leq$

$\leq H^{-1} \mathcal{C}_2^{1/2}$ and $\| u^k \|_{2,2,Q'} \leq H^{-1} (T \mathcal{C}_2)^{1/2}$. Hence

$$\| u^k \|_{2p',2q',Q} \leq \mathcal{C}_3$$

independent of k. Therefore, in any bounded cylinder Q, we can select a subsequence of the subsequence selected in the last paragraph which converges weakly to u in any finite collection of spaces $L^{2p',2q'}(Q)$ with p' and q' whose Hölder conjugates satisfy $(**)$.

Let φ be an arbitrary $C^1(\bar{S})$ function with compact support in E^n which vanishes near $t = T$. For all k so large that $(\operatorname{supp} \varphi) \cap S \subset Q_k$ we have

$$(4.10) \qquad \iint_S (- u^k \varphi_t + A_{ij} u^k_{x_i} \varphi_{x_j} + A_j u^k \varphi_{x_j} + \\ + F_j \varphi_{x_j} - B_j u^k_{x_j} \varphi - C u^k \varphi - G\varphi)\, dx\, dt = \int_{E^n} u_0 \varphi(x, 0)\, dx.$$

If we let $k \to \infty$ through the appropriate subsequence it follows from the considerations of the two previous paragraphs that (4.10) also holds with k deleted. Thus, as in the proof of Theorem 1, we conclude that u is a weak solution of the Cauchy problem (4.1). Since $h \geq -\alpha |x|^2/T - \beta T$ we have, in view of (4.9),

$$\iint_S e^{-2\alpha |x|^2/T} u^2(x, t)\, dx\, dt \leq T \mathcal{C}_2 e^{2\beta T}.$$

Hence u belongs to the class $\mathcal{E}^2(S)$ and is the unique solution of problem (4.1) in $\mathcal{E}^2(S)$. Note that any convergent subsequence of the u^k must converge to function $\mathcal{E}^2(S)$ and, therefore, to u. It follows that the whole sequence $u^k \to u$ weakly in the appropriate spaces.

Corollary 3.1 *Suppose that L satisfies (H), and the functions G and u_0 satisfy the hypotheses of Theorem 3. If for each j, the function $e^{-\gamma |x|^2} F_j$ belongs to some space $L^{p,q}(S)$ with p and q satisfying $(*)$ then the sequence $\{u^k\}$ of weak solutions of the boundary value problems* (4.6) *converge uniformly to u in any compact subset of S. Moreover*

$$\| e^h u \|^2_{2,\infty} + \| e^h u_x \|^2_{2,2} \leq \mathcal{C} \{ \| e^{-\gamma |x|^2} u_0 \|^2_{L^2(E^n)} + \sum_j \| e^{-\gamma |x|^2} F_j \|^2_{p,q} + \\ + \| e^{-\gamma |x|^2} G \|^2_{p,q} \}$$

where $\mathcal{C}$ depends only on T and the structure of equation (4.1), *and h is given in Theorem* 3.

PROOF. Observe that

$$\| e^h F_j \|_{2,2,Q_k} \leq T^{(q-2)/2q} \| e^{-\nu |x|^2} \|_{L^{2p/p-2}(E^n)} \| e^{-\gamma |x|^2} F_j \|_{p,q,S} .$$

Thus the proof of Theorem 4 can be carried through without essential change in the present case. In view of (4.8), the sequence $\{u^k\}$ is uniformly bounded in the $L^{2,2}$ norm in any bounded subset of S. Therefore, since the F_j and G are locally in the appropriate $L^{p,q}$ spaces, it follows from Theorems B and C that the sequence $\{u^k\}$ is uniformly bounded and equicontinuous in any compact subset of S. The uniform convergence of u^k to u then follows by Arzela's theorem and the uniqueness.

We now consides the special case of problem (4.1) in which $\gamma = 0$ and $u_0 \equiv 0$. In this case, using Theorem B and Lemma 1, we derive a pointwise bound for $|u|$ in terms of the F_j and G.

COROLLARY 3.2. *Let u be the weak solution in the class $\mathcal{E}^2(S)$ of the Cauchy problem*

$$(4.11) \qquad Lu = F_{j,x_j} + G \ \text{for} \ (x,t) \in S, \ u(x,0) = 0 \ \text{for} \ x \in E^n ,$$

where L satisfies (H) each $F_j \in L^{p,q}(S)$ for some p and q satisfying (*), *and $G \in L^{p,q}(S)$ with p and q satisfying* (**). *Then for all $(x,t) \in \bar{S}$*

$$(4.12) \qquad | u(x,t) | \leq \mathcal{C} \left(\sum_j \| F_j \|_{p,q,S} + \| G \|_{p,q,S} \right),$$

where $\mathcal{C}$ is a positive constant which depends on T and the structure of the equation. Moreover, u is uniformly Hölder continuous in $\bar{S}$.

PROOF. Let ξ be an arbitrary fixed point in E^n. By Lemma 1, with $\zeta \equiv 1$,

$$\| e^h u^k \|^2_{2,\infty,Q_k} \leq \mathcal{C} \{ \sum_j \| e^h F_j \|^2_{2,2,Q_k} + | e^h G \|^2_{2p/p+1,\,2q/q+1,\,Q_k} \}$$

where $\mathcal{C}$ is independent of k and

$$h(x,t) = -\frac{\alpha |x - \xi|^2}{2T - t} - \beta t.$$

Since $h(x,t) \leq -\alpha\,|x-\xi|^2/2T$ it is easily verified that

$$\| e^h \|_{r,s,S} \leq (2E^{2/n}/\alpha r)^{n/2r}\, T^{\frac{n}{2r}+\frac{1}{s}}$$

for all exponents $r, s \geq 1$, where

$$E = \int_{E^n} e^{-|z|^2}\, dz.$$

Thus, by Hölder's inequality,

$$\| e^h F_j \|_{2,2,Q_k} \leq \mathcal{C}^* \, T^{N-\frac{1}{2}+\theta} \| F_j \|_{p,q,S},$$

where $\theta = \dfrac{1}{2} - \dfrac{n}{2p} - \dfrac{1}{q} > 0$ and $N = \dfrac{n+2}{4}$, and

$$\| e^h G \|_{2p/p+1,\,2q/q+1,\,Q_k} \leq \mathcal{C}' \, T^{N-\frac{1}{2}+\theta'} \| G \|_{p,q,S},$$

where $\theta' = \dfrac{1}{2}\left(1 - \dfrac{n}{2p} - \dfrac{1}{q}\right) > 0$. Here $\mathcal{C}^*$ and $\mathcal{C}'$ depend only on the structure of the equation. It follows that

$$\| e^h u^k \|_{2,2,Q_k} \leq \mathcal{C}_1 T^N \{\sum_j \| F_j \|_{p,q,S} + \| G \|_{p,q,S}\},$$

where $\mathcal{C}_1$ depends only on T (or an upper bound for T) and the structure of the equation. Let $D_0 = \{|x-\xi| < 3\sqrt{T}/2\} \times (0, T]$. If k is so large that $D_0 \subset Q_k$ then

$$\| e^h u^k \|_{2,2,Q_k} \geq \| e^h u^k \|_{2,2,D_0} \geq e^{-\frac{9}{4}\alpha - \beta T} \| u^k \|_{2,2,D_0}$$

and hence

$$\| u^k \|_{2,2,D_0} \leq \mathcal{C}_2 T^N \{\sum_j \| F_j \|_{p,q,S} + \| G \|_{p,q,S}\} \equiv \mathcal{C}_3,$$

where $\mathcal{C}_2$ depends only on T and the structure. Since $u^k \to u$ weakly in $L^{2,2}(D_0)$ it follows that $\| u \|_{2,2,D_0} \leq \mathcal{C}_3$. Extend the domain of definition of u by setting $u = 0$ for $t < 0$ and let $D_1 = \{|x-\xi| < 3\sqrt{T}/2\} \times (-9T, T]$. Then clearly $\| u \|_{2,2,D_1} \leq \mathcal{C}_3$. By the Extension Principle and Theorem B, for $|x-\xi| \leq \sqrt{T}/2$ and $0 \leq t \leq T$ we have

$$|u(x,t)| \leq \mathcal{C}_4 \{T^{-N} \| u \|_{2,2,D_1} + T^{\theta/2} (\sum_j \| F_j \|_{p,q,D_1} + \| G \|_{p,q,D_1})\} \leq$$

$$+ \mathcal{C}_4 (\mathcal{C}_2 + T^{\theta/2}) \{\sum_j \| F_j \|_{p,q,S} + \| G \|_{p,q,S}\}.$$

Since ξ is arbitrary this proves the first assertion. The second assertion follows immediately from the Extension Principle, Theorem C and (4.12).

Consider the sequence of Cauchy problems

$$L^m u = F^m_{j,x_j} + G^m \text{ for } (x,t) \in S, \quad u(x,0) = 0 \text{ for } x \in E^n, \tag{4.13}$$

for integers $m \geq 1$, where

$$L^m u = u_t - \{A^m_{ij} u_{x_i} + A^m_j u\}_{x_j} - B^m_j u_{x_j} - C^m u,$$

and W^m denotes the integral average of W formed with a kernel whose support lies in $|x|^2 + t^2 < m^{-2}$. As we noted in section 3, the equations (4.13) have a uniform structure.

COROLLARY 3.3. *Under the hypotheses of Corollary* 3.2, *u is the uniform limit in any compact subset of $\bar{S}$ of the sequence $\{u^m\}$ of classical solutions of the Cauchy problems* (4.13).

PROOF. For each integer $m \geq 1$, the problem (4.13) has a unique classical solution u^m in the class $\mathcal{E}^2(S)$ (cf. [10]) and it is easily verified that u^m is also the weak solution of problem (4.13) in class $\mathcal{E}^2(S)$. By Corollary 3.1 and the fact that integral averaging on S does not increase norm

$$\| e^h u^m \|^2_{2,\infty} + \| e^h u^m \|^2_{2,2} \leq \mathcal{C} \{\Sigma \| F_j \|^2_{p,q} + \| G \|^2_{p,q}\},$$

where $\mathcal{C}$ depends only on T and the uniform structure. It follows, by the argument used to prove Theorem 3 and the strong convergence of integral averages, that the sequence $\{u^m\}$ converges to u weakly in the appropriate space. In view of Corollary 3.2, the sequence $\{u^m\}$ is uniformly bounded and equicontinuous in $\bar{S}$. The assertion then follows from Arzela's theorem and the uniqueness.

5. Non-negative Solutions of the Cauchy Problem.

The results obtained in the previous three sections are not restricted to non-negative solutions of linear parabolic equations. Beginning with this section, however, we shall use these general results together with the results quoted in section 1 to study the properties of various non-negative solutions. Specifically, in this section we will characterize the non-negative weak solutions of the Cauchy problem

$$Lu = 0 \quad \text{for } (x,t) \in S, \ u(x, 0) = u_0(x) \quad \text{for } x \in E^n, \tag{5.1}$$

where L satisfies (H) and u_0 is a given function in $L^2_{\text{loc}}(E^n)$. It is easily seen that if the problem (5.1) admits a non-negative solution then we must have $u_0 \geq 0$ almost everywhere in E^n. Our first result is a generalization of a theorem first proved by Widder [22] for the equation of heat conduction.

Lemma 4. *Suppose that* L *satisfies* (H). *If* u *is a non-negative weak solution of the Oauchy problem* (5.1) *with* $u_0 \equiv 0$ *then* $u \equiv 0$ *in* S.

Proof. Extend the domain of definiton of u by setting $u = 0$ for $t < 0$. The extended function is non-negative, locally bounded and continuous in $E^n \times (-\infty, T]$. By the Extension Principle and Theorem E

$$u(x, t) \leq u(0, T) \exp \mathcal{C} \left(\frac{|x|^2}{T - t} + \frac{T}{T + t} \right)$$

for all $(x,t) \in E^n \times (-T, T]$. Let $\varepsilon > 0$ be arbitrary and set $S_\varepsilon = E^n \times [0, T - \varepsilon]$. Then for $(x, t) \in S_\varepsilon$

$$u(x, t) \leq \{u(0, T) \exp \mathcal{C}\} \exp \mathcal{C} |x|^2/\varepsilon. \tag{5.2}$$

Clearly, if $u(0, T) = 0$ then $u \equiv 0$ in S_ε. On the other hand, if $u(0, T) > 0$ then (5.2) implies that $u \in \mathcal{E}^2(S_\varepsilon)$ and it follows, as in the proof of Theorem 2, that $u \equiv 0$ in S_ε. Finally, since ε is arbitrary and u is continuous we conclude that $u \equiv 0$ in S.

Since the difference between two non-negative functions does not necessarily have a constant sign, Lemma 4 does not imply uniqueness of a non-negative solution of problem (5.1). The uniqueness does follow from our next result which characterizes a non-negative solution of problem (5.1) as the limit of a certain well-defined sequence of functions associated with L and u_0.

Let $\Sigma_k = \{x ; |x| < k\}$ and $Q_k = \Sigma_k \times (0, T]$. For each integer $k \geq 3$, let $\zeta_k = \zeta_k(x)$ denote a $C_0^\infty(E^n)$ function such that $\zeta_k = 1$ for $|x| \leq k - 2$, $\zeta_k = 0$ for $|x| \geq k - 1$, $0 \leq \zeta_k \leq 1$, and $|\zeta_{kx}|$ is bounded independent of k. Consider the sequence of boundary value problems

$$(5.3) \qquad \begin{cases} Lu = 0 \quad \text{for } (x, t) \in Q_k \\ u(x, 0) = \zeta_k(x)\, u_0(x) \quad \text{for } x \in \Sigma_k, \ u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Sigma_k \times [0, T] \end{cases}$$

According to Theorem 1, for each k there exists a unique weak solution u^k of the boundary value problem (5.3). Extend the domain of definition of u^k by setting $u^k = 0$ for $|x| \geq k$ and $0 \leq t \leq T$.

THEOREM 4. *Suppose that L satisfies (H) and $u_0 \in L^2_{\text{loc}}(E^n)$. Let u be a non-negative weak solution of the Cauchy problem* (5 1). *Then at every point of S,*

$$\lim_{k \to \infty} u^k(x, t) = u(x, t),$$

where $\{u^k\}$ is the sequence of weak solutions of the boundary value problems (5.3). *Moreover, the convergence is uniform on every compact subset of S.*

PROOF. We first show that the sequence $\{u^k\}$ is non-negative and non-decreasing, that is,

$$(5.4) \qquad 0 \leq u^k(x, t) \leq u^{k+1}(x, t)$$

for all $(x, t) \in S$ and all $k \geq 3$. According to Theorem 1 (i) and the remark at the end of section 3, u^k is the limit at each point of Q_k of the sequence $u^{k,m}$ of classical solutions of the problems

$$\begin{cases} L^m u = 0 \quad \text{for } (x, t) \in Q_k \\ u(x, 0) = \{\zeta_k(x)\, u_0(x)\}^m \quad \text{for } x \in \Sigma_k, \ u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Sigma_k \times [0, T]. \end{cases}$$

Extend the domain of definition of $u^{k,m}$ by setting $u^{k,m} = 0$ for $|x| \geq k$ and $0 \leq t \leq T$. Clearly $u^{k,m} \geq 0$ and hence $u^k \geq 0$. Let $w^m(x, t) = u^{k+1,m}(x, t) - u^{k,m}(x, t)$. If $|x| \geq k$ and $0 \leq t \leq T$, then $u^{k,m} = 0$ and $w^m = u^{k+1,m} \geq 0$. On the other hand, $w^m \geq 0$ on the parabolic boundary of Q_k. Since $L^m w^m = 0$ in Q_k, it follows from the classical maximum principle that $w^m \geq 0$ in $\bar{Q}_k$. Therefore $u^{k+1,m} \geq u^{k,m}$ in $\bar{S}$ and we obtain (5.4) by letting $m \to \infty$.

We assert that the sequence $\{u^k\}$ is bounded above by u, that is,

$$(5.5) \qquad u^k(x, t) \leq u(x, t)$$

for all $(x, t) \in S$ and all $k \geq 3$. Since the inequality (5.5) is trivial for $|x| \geq k$ and $0 < t \leq T$, it suffices to prove it for points in Q_k. Let (x_0, t_0) be a fixed point in Q_k. For fixed k and integers l such that $t_0 > 1/l$ consider the sequence of boundary value problems

$$(5.6) \qquad \begin{cases} Lv = 0 \quad \text{for } (x, t) \in \Sigma_k \times (1/l, T] = Q_{kl} \\ v(x, 1/l) = \zeta_k(x) u(x, 1/l) \quad \text{for } x \in \Sigma_k, \\ v(x, t) = 0 \quad \text{for } (x, t) \in \partial\Sigma_k \times [1/l, T]. \end{cases}$$

Since $\partial\Sigma_k$ is smooth and $\zeta_k(x) u(x, 1/l) \in C_0^0(\Sigma_k)$, it follows from Theorem 1 (iii) and the remark at the end of section 3 that for each l the problem (5.6) has a unique weak solution $v^{k,l}$ and $v^{k,l} \in C^0(\bar{Q}_{kl})$. Set $w^{k,l} = u - v^{k,l}$. Then $w^{k,l} \in C^0(\bar{Q}_{kl})$, $w^{k,l} \geq 0$ on the parabolic boundary of Q_{kl}, and $w^{k,l}$ is a weak solution of $Lu = 0$ in Q_{kl}. Hence, by Theorem A, $w^{k,l} \geq 0$ in Q_{kl}. In particular,

$$(5.7) \qquad v^{k,l}(x_0, t_0) \leq u(x_0, t_0)$$

for $l > 1/t_0$. It remains to be shown that $v^{k,l} \to u^k$ as $l \to \infty$.

Define the function

$$\hat{v}^{k,l}(x, t) = \begin{cases} v^{k,l}(x, t) & \text{for } (x, t) \in \bar{Q}_{kl} \\ \zeta_k(x) u(x, t) & \text{for } (x, t) \in \bar{\Sigma}_k \times (0, 1/l]. \end{cases}$$

Clearly $\hat{v}^{k,l} \in C^0(Q_k)$. Moreover, it is easily verified that $\hat{v}^{k,l} \in L^2[0, T; H_0^{1,2}(\Sigma_k)]$ and satisfies

$$(5.8) \qquad \begin{aligned} &\iint\limits_{Q_k} (-\hat{v}^{k,l} \varphi_t + A_{ij} \hat{v}^{k,l}_{x_i} \varphi_{x_j} + A_j \hat{v}^{k,l} \varphi_{x_j} - B_j \hat{v}^{k,l}_{x_j} \varphi - C \hat{v}^{k,l} \varphi)\, dx\, dt \\ &+ \iint\limits_{\Sigma_k \times (0, 1/l)} \{-u A_{ij} \zeta_{kx_i} \varphi_{x_i} + u\varphi (A_j + B_j) \zeta_{kx_j} + \varphi A_{ij} u_{x_i} \zeta_{kx_j}\}\, dx\, dt \\ &\qquad = \int\limits_{\Sigma_k} \zeta_k(x) u_0(x) \varphi(x, 0)\, dx \end{aligned}$$

for all $\varphi \in C^1(\bar{Q}_k)$ with compact support in Σ_k which vanish near $t = T$. By Lemma 1, with $\zeta \equiv 1$, $s = 1/l$ and $\mu = +\infty$,

$$\| v^{k,l} \|^2_{2,\infty, Q_{kl}} + \| v^{k,l}_x \|^2_{2,2, Q_{kl}} \leq e^{\beta T} C \| \zeta_k u(x, 1/l) \|^2_{L^2(\Sigma_k)} \leq e^{\beta T} C \| u \|^2_{2,\infty, Q_k} \equiv C_1$$

independent of l. On the other hand, since u is a weak solution of the Cauchy problem, we have

$$\| \zeta_k u \|^2_{2, \infty, Q_k} + \| (\zeta_k u)_x \|^2_{2, 2, Q_k} = \mathcal{C}_2 < \infty.$$

Therefore

$$\| \hat{v}^{k, l} \|^2_{2, \infty, Q_k} + \| \hat{v}^{k, l}_x \|^2_{2, 2, Q_k} \leq \mathcal{C}_3$$

independent of l. In view of Lemma 2, there exists a subsequence of the $\hat{v}^{k, l}$ which converges weakly to a function $\hat{v}^k$ in $L^2[0, T; H_0^{1, 2}(\Sigma_k)]$ and $L^{2p', 2q'}(Q_k)$ for any finite collection of admissible exponents p and q. Moreover, according to Lemma 3, $\hat{v}^k \in L^{2, \infty}(Q_k)$. Note that the second integral on the left in (5.8) tends to zero as $l \to \infty$. Thus, letting $l \to \infty$ in (5.8), we conclude that $\hat{v}^k$ is the weak solution of problem (5.3). Therefore $\hat{v}^k = u^k$. Finally, by Theorems B and C, the sequence $\{\hat{v}^{k, l}\}$ is uniformly bounded and equicontinuous in any compact subset of Q_k. Hence, $\hat{v}^{k, l}(x_0, t_0) \to u^k(x_0, t_0)$ and (5.5) follows from (5.7).

According to (5.4) and (5.7), the sequence $\{u^k\}$ is non-decreasing and bounded above by u in S. Hence there exists a function $w = w(x, t)$ such that

$$0 \leq w(x, t) \leq u(x, t) \tag{5.9}$$

and $u^k(x, t) \to w(x, t)$ as $k \to \infty$ for all $(x, t) \in S$. In view of (5.5) and Theorem C, the sequence $\{u^k\}$ starting from a sufficiently large k is bounded and equicontinuous in any compact subset of S. Therefore the limit function w is continuous in S. We assert that w is a weak solution of the Cauchy problem (5.1).

Let Ω and Ω' be arbitrary bounded open domains in E^n such that $\bar{\Omega} \subset \Omega'$, $Q = \Omega \times (0, T]$ and $Q' = \Omega' \times (0, T]$. Let $\zeta = \zeta(x)$ be a $C_0^\infty(E^n)$ function such that $\zeta = 1$ on $\bar{\Omega}$, $\operatorname{supp} \zeta \subset \Omega'$ and $0 \leq \zeta \leq 1$. Take k so large that $\bar{\Omega}' \subset \Sigma_k$. By Lemma 1, with $s = 0$ and $\mu = +\infty$,

$$\| \zeta u^k \|^2_{2, \infty, Q'} + \| \zeta u^k_x \|^2_{2, 2, Q'} \leq e^{\beta T} \mathcal{C} \left\{ \int_{\Omega'} \zeta^2 \zeta_k^2 u_0^2 \, dx + \iint_{Q'} (u^k)^2 |\zeta_x|^2 \, dx \, dt \right\}.$$

Therefore, since $u^k \leq u$,

$$\| u^k \|^2_{2, \infty, Q} + \| u^k_x \|^2_{2, 2, Q} \leq e^{\beta T} \mathcal{C} \left\{ \int_{\Omega'} u_0^2 \, dx + (\max |\zeta_x|^2) \iint_{Q'} u^2 \, dx \, dt \right\} = \mathcal{C}_1$$

independent of k. Moreover, by Lemma 2,

$$\| u^k \|^2_{2p', 2q', Q} \leq \| \zeta u^k \|^2_{2p', 2q', Q'} \leq K T^\theta \{ \| \zeta u^k \|^2_{2, \infty, Q'} + \| (\zeta u^k)_x \|^2_{2, 2, Q'} \} \leq C_2$$

independent of k for any admissible exponents p and q. It follows that there exists a subsequence of the u^k which converges weakly to a function $\hat{u}$ in $L^2[0, T; H^{1,2}(\Omega)]$ and any finite collection of space $L^{2p', 2q'}(Q)$, where, by Lemma 3, $\hat{u} \in L^{2,\infty}(Q)$. However, since $u^k \to w$ pointwise in S we have $\hat{u} = w$. Thus, in particular, $w \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^{1,2}(\Omega)]$.

Let φ be an arbitrary $C^1(\bar{S})$ function with compact support in E^n and which vanishes near $t = T$. Choose Ω so that $\{\operatorname{supp} \varphi\} \cap S \subset Q$. For k so large that $Q \subset \Sigma_k$ we have

$$\iint_S (- u^k \varphi_t + A_{ij} u^k_{x_i} \varphi_{x_j} + A_j u^k \varphi_{x_j} - B_j u^k_{x_j} \varphi - C u^k \varphi) \, dx \, dt =$$

$$= \int_{E^n} \zeta_k(x) u_0(x) \varphi(x, 0) \, dx.$$

In view of the results of the last paragraph, if we let $k \to \infty$ we obtain

$$\iint_S (- w \varphi_t + A_{ij} w_{x_i} \varphi_{x_j} + A_j w \varphi_{x_j} - B_j w_{x_j} \varphi - C w \varphi) \, dx \, dt = \int_{E^n} u_0(x) \varphi(x, 0) \, dx ,$$

and it follows that w is a weak solution of the Cauchy problem (5.1).

The function $u - w$ is a weak solution of the Cauchy problem

$$Lv = 0 \quad \text{for } (x, t) \in S, \qquad v(x, 0) = 0 \quad \text{for } \quad x \in E^n ,$$

and, in view of (5.9), $u - w \geq 0$ in S. We therefore conclude from Lemma 4 that $w = u$ in S. Since u^k converges monotonically to u and since u is continuous in compact subsets of S, $u^k \to u$ uniformly on compact subsets of S by Dini's monotone convergence theorem.

An immediate consequence of Theorem 4 is the uniqueness of a non-negative solution of problem (5.1).

COROLLARY 4.1. *If L satisfies (H), $u_0 \in L^2_{\mathrm{loc}}(E^n)$ and $u_0 \geq 0$ almost everywhere in E^n, then the Cauchy problem* (5.1) *admits at most one non-negative weak solution.*

On the other hand, the proof of Theorem 4 can be reinterpreted to yield the following criterion for the existence of a non-negative weak solution of problems (5.1).

COROLLARY 4.2. *Suppose that L satisfies (H), $u_0 \in L^2_{\mathrm{loc}}(E^n)$ and $u_0 \geq 0$ almost everywhere in E^n. The Cauchy problem (5.1) possesses a non-negative weak solution if and only if there exists a function $U(x, t) \in L^2[0, T; L^2_{\mathrm{loc}}(E^n)]$ such that $u^k \leq U$ almost everywhere in S.*

PROOF. If there exists a non-negative weak solution u of problem (5.1) then $u \in L^2[0, T; L^2_{\mathrm{loc}}(E^n)]$ and, in view of (5.5), $u^k \leq u$ in S. Thus we may take $U = u$. If $u^k \leq U$ almost everywhere in S, then, since $\{u^k\}$ is a non-decreasing sequence of non-negative functions,

$$\lim_{k \to \infty} u^k = w$$

exists almost everywhere in S and $0 \leq w \leq U$. The proof that w is a solution of problem (5.1) is identical to the corresponding part of the proof of Theorem 4.

6. Existence of the Fundamental Solution and Green's Function.

Let $D = \Sigma \times (0, T]$ where Σ is either a bounded open domain contained in E^n or E^n itself, and consider the problem

$$(6.1) \qquad Lu = F_0(x, t) - \{F_j(x, t)\}_{x_j} \quad \text{for} \quad (x, t) \in D, \; u(x, 0) = 0 \quad \text{for} \quad x \in \Sigma,$$

where L satisfies (H), and the functions $F_0, F_1, \ldots, F_n$ all belong to a single space $L^{p,q}(D)$ with p, q satisfying $(*)$. If $\Sigma = E^n$ we take u to be the weak solution in $\mathcal{E}^2(S)$ of the Cauchy problem (6.1), while if Σ is bounded we take u to be the weak solution of the boundary value problem consisting of (6.1) and the condition $u = 0$ on $\partial\Sigma \times [0, T]$. In either case, according to Theorem 1 (ii) or Corollary 3.2, we have

$$(6.2) \qquad |u(x, t)| \leq C \sum_{j=0}^{n} \| F_j \|_{p, q, D}.$$

for all $(x, t) \in D$.

Let $\mathcal{L}^{p,q}$ denote the Banach space of vectors $W = (W_0, W_1, \dots, W_n)$ such that $W_j \in L^{p,q}(D)$ for $j = 0, 1, \dots, n$ with either of the equivalent norms

$$\| W \|_{p,q} = \sum_{j=0}^{n} \| W_j \|_{p,q,D} \text{ or } | W |_{p,q} = \max_j \| W_j \|_{p,q,D}.$$

According to (6.2), the value at a point in D of the solution u of the boundary value or Cauchy problem (6.1) is a bounded linear functional on $\mathcal{L}^{p,q}$, that is, $u(x,t) = \mathcal{G}_{x,t}(F)$, where $\mathcal{G}_{x,t} \in \{\mathcal{L}^{p,q}\}^*$. Moreover, $\| \mathcal{G}_{x,t} \| \leq \mathcal{C}$. The remainder of this section is devoted to deriving a more precise characterization of $\mathcal{G}_{x,t}$.

Denote by $\mathcal{N}^{p,q}$ the collection of vectors $W \in \mathcal{L}^{p,q}$ such that

$$\iint_D (W_0 \varphi + W_j \varphi_{x_j})\, dx\, dt = 0$$

for all $\varphi \in C_0^1(D)$. Let u be the weak solution of the boundary value or Cauchy problem (6.1) and let $\tilde{u}$ be the weak solution of the corresponding problem for

$$Lu = \tilde{F}_0 - \tilde{F}_{j,x_j},$$

where $\tilde{F} = (\tilde{F}_0, \dots, \tilde{F}_n) \in \mathcal{L}^{p,q}$ with the same p and q as $F = (F_0, \dots, F_n)$. Then

$$|\tilde{u}(x,t)| \leq \mathcal{C} \| \tilde{F} \|_{p,q}$$

in D, and $w = u - \tilde{u}$ is a solution of the corresponding problem for

$$Lu = (F_0 - \tilde{F}_0) - (F_j - \tilde{F}_j)_{x_j}.$$

In particular, w satisfies

$$\iint_D \{- w\varphi_t + A_{ij} w_{x_i} \varphi_{x_j} + A_j w \varphi_{x_j} - B_j w_{x_j} \varphi - C w \varphi - (F_0 - \tilde{F}_0)\varphi - (F_j - \tilde{F}_j)\varphi_{x_j}\}\, dx\, dt = 0$$

for all $\varphi \in C_0^1(D)$. Thus, if $F - \tilde{F} \in \mathcal{N}^{p,q}$ it follows from the uniqueness for the boundary value or Cauchy problem that $w \equiv 0$. Therefore $F - \tilde{F} \in \mathcal{N}^{p,q}$

implies that $u = \widetilde{u}$ and (6.2) can be replaced by

$$(6.3) \qquad |u(x,t)| \leq \inf \|\widetilde{F}\|_{p,q} \qquad (F - \widetilde{F} \in \mathcal{N}^{p,q}).$$

Assume now that p and q satisfy (*) and are finite. Consider

$$\mathcal{F}(\varphi) = \iint_D (F_0 \varphi + F_j \varphi_{x_j})\, dx\, dt,$$

where $F \in \mathcal{L}^{p,q}$. Since $\mathcal{F}$ is defined for all $\varphi \in L^{q'}[H_0^{1,p'}]$ we have $\mathcal{F} \in \{L^{q'}[H_0^{1,p'}]\}^* \equiv L^q[H^{-1,p}]$ (2). Let M denote the mapping from $L^{q'}[H_0^{1,p'}]$ to $\mathcal{L}^{p',q'}$ given by

$$M\varphi = (\varphi, \varphi_{x_1}, \dots, \varphi_{x_n}).$$

For the $L^{q'}[H^{1,p'}]$ norm of φ we take

$$\langle \varphi \rangle_{p',q'} = \max(\|\varphi\|_{p',q',D}, \|\varphi_{x_1}\|_{p',q',D}, \dots, \|\varphi_{x_n}\|_{p',q',D}).$$

Then, using the $|\cdot|_{p',q'}$ norm in $\mathcal{L}^{p',q'}$, M is an isometric isomorphism of $L^{q'}[H_0^{1,p'}]$ onto a linear subspace $\mathcal{M}^{p',q'} \subset \mathcal{L}^{p',q'}$. Define a linear functional $\mathcal{F}^*$ on $\mathcal{M}^{p',q'}$ by $\mathcal{F}^*(M\varphi) = \mathcal{F}(\varphi)$. Then

$$\|\mathcal{F}\| = \sup \frac{|\mathcal{F}(\varphi)|}{\langle \varphi \rangle_{p',q'}} = \sup \frac{|\mathcal{F}^*(M\varphi)|}{|M\varphi|_{p',q'}} = \|\mathcal{F}^*\|.$$

By the Hahn-Banach theorem there exists an element $\widetilde{\mathcal{F}} \in \{\mathcal{L}^{p',q'}\}^*$ such that $\widetilde{\mathcal{F}} = \mathcal{F}^*$ on $\mathcal{M}^{p',q'}$ and $\|\widetilde{\mathcal{F}}\| = \|\mathcal{F}^*\|$. Moreover, since $\{\mathcal{L}^{p',q'}\}^*$ is isometrically isomorphic to $\mathcal{L}^{p,q}$ it follows from a theorem of Bochner and Taylor [8] that there exists a unique $\psi \in \mathcal{L}^{p,q}$ such that $\|\widetilde{\mathcal{F}}\| = \|\psi\|_{p,q}$ and

$$\widetilde{\mathcal{F}}(W) = \iint_D \psi_j W_j\, dx\, dt$$

for all $W \in \mathcal{L}^{p',q'}$. If $\varphi \in L^{q'}[H_0^{1,p'}]$ then

$$\widetilde{\mathcal{F}}(M\varphi) = \iint_D (\psi_0 \varphi + \psi_j \varphi_{x_j})\, dx dt = \mathcal{F}^*(M\varphi) = \iint_D (F_0 \varphi + F_j \varphi_{x_j})\, dx\, dt.$$

Since $C_0^1(D) \subset L^{q'}[H_0^{1,p'}]$ it follows that $F - \psi \in \mathcal{N}^{p,q}$. On the other hand,

$$\|\psi\|_{p,q} = \|\widetilde{\mathcal{F}}\| = \|\mathcal{F}^*\| = \|\mathcal{F}\|.$$

(2) By definition $H^{1,p}(E^n) \equiv H_0^{1,p}(E^n)$.

Therefore, we conclude from (6.3) that

$$|u(x,t)| \leq \mathcal{C} \| \mathcal{F} \|,$$

where $\mathcal{F} \in L^q[H^{-1,p}]$, that is, $u(x,t) = \mathcal{G}_{x,t}(\mathcal{F})$, where $\mathcal{G}_{x,t} \in \{L^q[H^{-1,p}]\}^*$ and

$$\| \mathcal{G}_{x,t} \| \leq \mathcal{C}. \tag{6.4}$$

We now interpret the result derived above for the specific problems of interest. Let u denote the weak solution in $\mathcal{E}^2(S)$ of the Cauchy problem

$$Lu = F_0 - F_{j,x_j} \text{ for } (x,t) \in S, \; u(x,0) = 0 \text{ for } x \in E^n, \tag{6.5}$$

where L satisfies (H) and $F = (F_0, \ldots, F_n) \in \mathcal{L}^{p,q}$ for p and q satisfying $(*)$. In this case the constant $\mathcal{C}$ in (6.4) depends only on T, the structure of L, and the exponents p and q for F. Suppose first that both p and q are finite. Then since $\{L^q[H^{-1,p}]\}^* = \{L^{q'}[H_0^{1,p'}]\}^{**}$ and $L^{q'}[H_0^{1,p'}]$ is reflexive it follows from a theorem of Bochner and Taylor [8] that there exists a unique $\Gamma(x,t;\cdot,\cdot) \in L^{q'}[H_0^{1,p'}]$ such that $\langle \Gamma(x,t;\cdot,\cdot) \rangle_{p',q'} = \| \mathcal{G}_{x,t} \|$ and

$$u(x,t) = \iint_S \{\Gamma(x,t;\xi,\tau) F_0(\xi,\tau) + \Gamma_{\xi_j}(x,t;\xi,\tau) F_j(\xi,\tau)\} \, d\xi d\tau. \tag{6.6}$$

In view of (6.4), we have

$$\langle\langle \Gamma(x,t;\cdot,\cdot) \rangle\rangle_{p',q'} \equiv \| \Gamma(x,t;\cdot,\cdot) \|_{p',q',S} + \sum_{j=1}^{n} \| \Gamma_{\xi_j}(x,t;\cdot,\cdot) \|_{p',q',S} \leq \tag{6.7}$$

$$\leq (n+1)\mathcal{C}.$$

[It should also be shown that Γ is independent of the exponents p and q. However, this is an easy consequence of Theorem 2 and $\Gamma(x,t;\cdot,\cdot) \in$ $\in L^1[0,T; L^1_{\text{loc}}(E^n)]$, and we omit it.]

For integers $k \geq 1$ let $\Sigma_k = \{x ; |x| < k\}$ and $Q_k = \Sigma_k \times (0,T]$. For any pair of exponents p' and q' whose Hölder conjugates are finite and satisfy $(*)$ we have $\Gamma \in L^{q'}[0,T; H^{1,p'}(E^n)]$. Hence $\Gamma \in L^s[0,T; H^{1,r}(\Sigma_k)]$ for all exponents $r \in [1,p']$ and $s \in [1,q']$. Let $\mathcal{L}_k^{p,q}$ denote the subset $\mathcal{L}^{p,q}$ consisting of vectors which vanish for $|x| \geq k$ and $0 < t < T$. If $F \in \mathcal{L}_k^{\infty,q}$ with $2 < q \leq \infty$ then $F \in \mathcal{L}^{\bar{p},\bar{q}}$ for some finite exponents $\bar{p}$ and $\bar{q}$ which satisfy $(*)$. Let u be the weak solution in $\mathcal{E}^2(S)$ of the Cauchy problem (6.5) with data

$F=(F_0,\dots,F_n)\in\mathcal{L}_k^{\infty,q}$, where $2<q\le\infty$. Then the representation formula (6.6) holds for u and we have

$$u=\iint_{Q_k}(\Gamma F_0+\Gamma_{\xi_j}F_j)\,d\xi\,d\tau.$$

Since $\Gamma\in L^{q'}[0,T;H^{1,1}(\Sigma_k)]$ it follows from (6.2) that

$$\langle\Gamma(x,t;\,\cdot,\cdot)\rangle_{1,q',Q_k}\le\mathcal{C},$$

where $\mathcal{C}$ is independent of k. Thus, if we let $k\to\infty$, we obtain

$$\langle\Gamma(x,t;\,\cdot,\cdot)\rangle_{1,q',S}\le\mathcal{C}. \tag{6.8}$$

In particular then, (6.7) is also valid when $p=\infty$ and $2<q\le\infty$.

If $F\in\mathcal{L}^{\infty,q}$ let $F^k=\chi_k F$, where χ_k is the characteristic function of Q_k, and consider the Cauchy problem

$$Lu=F_0^k-F_{j,x_j}^k \text{ for } (x,t)\in S,\ u(x,0)=0 \text{ for } x\in E^n.$$

This problem has a unique weak solution u^k in $\mathcal{E}^2(S)$, where, according to Corollary 3.1,

$$\|e^h u^k\|^2_{2,\infty,S}+\|e^h u^k_x\|^2_{2,2,S}\le\mathcal{C}\|F^k\|^2_{\infty,q}\le\mathcal{C}\|F\|^2_{\infty,q}.$$

On the other hand, if φ is a $C^1(\bar S)$ function with compact support in E^n which vanishes near $t=T$ and if k is so large that $\{\sup\varphi\}\cap S\subset Q_k$ then

$$\iint_S(-u^k\varphi_t+A_{ij}u^k_{x_i}\varphi_{x_j}+A_j u^k\varphi_{x_j}-F_j\varphi_{x_j}-B_j u^k_{x_j}\varphi-Cu^k\varphi-F_0\varphi)\,dx\,dt=0.$$

By the arguments used to prove Theorem 3 and Corollary 3.1, u^k converges pointwise in S to the weak solution u in $\mathcal{E}^2(S)$ of the Cauchy problem (6.5) with data F. For each k, $F^k\in\mathcal{L}_k^{\infty,q}$ so that

$$u^k=\iint_{Q_k}(\Gamma F_0+\Gamma_{\xi_j}F_j)\,d\xi\,d\tau.$$

Therefore, in view of (6.8),

$$u=\lim_{k\to\infty}\iint_{Q_k}(\Gamma F_0+\Gamma_{\xi_j}F_j)\,d\xi d\tau=\iint_S(\Gamma F_0+\Gamma_{\xi_j}F_j)\,d\xi\,d\tau$$

and the representation formula (6.6) also holds when $p = \infty$ and $2 < q \leq \infty$.

Suppose that $F \in \mathcal{L}^{p,\infty}$ with $\max(n, 2) < p < \infty$. Then $F \in \mathcal{L}^{p,q}$ for all $q \geq 1$ and, in particular, for some finite q such that p and q satisfy (*). Thus if u is the weak solution in $\mathcal{E}^2(S)$ of the problem (6.5) with data F, then

$$u = \iint_S (\Gamma F_0 + \Gamma_{\xi_j} F_j)\, d\xi\, d\tau.$$

On the other hand, $\Gamma \in L^{q'}[0, T;\ H^{1,p'}(E^n)]$ for any pair of exponents p' and q' whose Hölder conjugates are finite and satisfy (*) implies that $\Gamma \in L^1[0, T;\ H^{1,p'}(E^n)]$. Therefore, it follows from (6.2) and the representation formula that

$$\langle \Gamma(x, t);\ \cdot, \cdot) \rangle_{p', 1, S} \leq \mathcal{C}$$

for $\max\ (n,\ 2) < p < \infty$.

Let u and $\tilde{u}$ be the weak solutions in $\mathcal{E}^2(S)$ of the Cauchy problem (6.6) with data F and $\tilde{F}$ respectively, and let $w = u - \tilde{u}$. If $F = \tilde{F}$ in $S_0 = E^n \times (0, t_0]$ for some $t_0 \in (0, T)$ then it follows from Theorem 2 that $w = 0$ in S_0. Thus the value of the solution of the problem (6.5) at a point $(x, t) \in S$ is independent of the values of F in $E^n \times (t, T]$. We therefore conclude from the representation formula (6.7) that $\Gamma(x, t;\ \xi, \tau) = 0$ for $\tau > t$. To summarize, we have proved the following theorem.

THEOREM 5. *Suppose that L satisfies (H). Then for each $(x, t) \in S$ there exists a unique function of (ξ, τ), $\Gamma(x, t;\ \xi, \tau)$, defined in S which has the following properties.*

(i) *For each pair of exponents p, q satisfyng (*),$\Gamma(x, t;\ \cdot, \cdot) \in L^{q'}[0, T;$ $H^{1,p'}(E^n)]$ and*

$$\langle\langle (\Gamma x, t;\ \cdot\ ,\ \cdot) \rangle\rangle_{p', q'} \leq \mathcal{C},$$

where $\mathcal{C}$ is a constant which depends only on T, the structure of L, aud the exponents p, q.

(ii) *If u is the weak solution in $\mathcal{E}^2(S)$ of the Cauchy problem (6.5), where the data $F = (F_0, \dots, F_n) \in \mathcal{L}^{p,q}$ with p, q satisfyng (*), then for all $(x,t) \in S$*

$$u(x, t) = \iint_S \{\Gamma(x, t;\ \xi, \tau)\, F_0(\xi, \tau) + \Gamma_{\xi_j}(x, t;\ \xi, \tau)\, F_j(\xi, \tau)\}\, d\xi\, d\tau.$$

(iii) $\Gamma(x, t;\ \xi, \tau) = 0$ *for* $\tau > t$.

The function $\Gamma(x, t;\ \xi, \tau)$ will be called the *weak fundamental solution* of the equation $Lu = 0$. Theorem 5 shows that it is analogous in many ways to the classical fundamental solution. The results which will be proved in the next four sections demonstrate that this analogy is quite complete.

Now let u denote the weak solution of the boundary value problem

$$(6.9)\qquad \begin{cases} Lu = F_0 - F_{j,x_j} \text{ for } (x,t) \in Q = \Omega \times (0, T] \\ u(x,t) = 0 \text{ for } (x,t) \in \{\Omega \times (t=0)\} \cup \{\partial\Omega \times [0, T]\}, \end{cases}$$

where $\Omega \subset E^n$ is bounded and $F = (F_0, \dots, F_n) \in \mathcal{L}^{p,q}$ with p, q satisfying (*). If p and q are finite, then by the same argnment which led to (6.6) and (6.7), there exists a unique $\gamma(x, t; \cdot, \cdot) \in L^{q'}[H_0^{1,p'}]$ such that

$$(6.10)\qquad u(x,t) = \iint_Q \{\gamma(x,t;\ \xi,\tau)\, F_0(\xi,\tau) + \gamma_{\xi_j}(x,t;\ \xi,\tau)\, F_j(\xi,\tau)\}\, d\xi d\tau$$

and

$$(6.11)\qquad \langle\langle\, \gamma(x,t;\ \cdot,\cdot)\rangle\rangle_{p',q'} \leq (n+1)\,\mathcal{C},$$

where $\mathcal{C}$ depends only on T, $|\Omega|$, the structure of L, and the exponents p, q. Moreover, the proof that the representation formula (6.10) and the estimate (6.11) remain valid when p or q is infinite is somewhat simpler in the present case since Ω is bounded. Finally, it is easily seen that $\gamma(x, t;\ \xi, \tau) = 0$ for $\tau > t$. The function $\gamma(x, t;\ \xi, \tau)$ will be called the *weak Green's function* for the equation $Lu = 0$ in Q.

Theorem 6. *Suppose that L satisfies (H) and that $Q = \Omega \times (0, T]$, where Ω is a bounded open domain in E^n. Then for each $(x, t) \in Q$ there exists a unique function of (ξ, τ), $\gamma(x, t;\ \xi, \tau)$, defined in Q which has the following properties.*

(i) *For each pair of exponents p, q satisfying (*), $\gamma(x, t; \cdot, \cdot) \in L^{q'}[0, T; H_0^{1,p'}(\Omega)]$ and*

$$\langle\langle\, \gamma(x,t;\ \cdot,\cdot)\,\rangle\rangle_{p',q'} \leq \mathcal{C},$$

where $\mathcal{C}$ is a constant which depends only on T, $|\Omega|$, the structure of L, and the exponents p, q.

(ii) *If u is the weak solution of the boundary value problem* (6.9), *where the data $F = (F_0, \dots, F_n) \in \mathcal{L}^{p,q}$ with p, q satisfying (*), then for all $(x, t) \in Q$*

$$u(x,t) = \iint_Q \{\gamma(x,t;\xi,\tau)\, F_0(\xi,\tau) + \gamma_{\xi_j}(x,t;\xi,\tau)\, F_j(\xi,\tau)\}\, d\xi\, d\tau.$$

(iii) $\gamma(x, t; \xi, \tau) = 0$ *for* $\tau > t$.

It is clear that everything which has been done in this section applies, with obvious modifications, to the adjoint operator $\tilde{L}$, and, in particular, to the adjoint Cauchy problem

$$\tilde{L}u = F_0 - F_{j, x_j} \text{ for } (x, t) \in S, u(x, T) = 0 \text{ for } x \in E^n,$$

and the adjoint boundary value problem

$$\begin{cases} \tilde{L}u = F_0 - F_{j, x_j} \text{ for } (x, t) \in Q \\ u(x, t) = 0 \text{ for } (x, t) \in \{\Omega \times (t = T)\} \cup \{\partial\Omega \times (0, T]\}. \end{cases}$$

The weak fundamental solution $\tilde{\Gamma}(x, t; \xi, \tau)$ of $\tilde{L}u = 0$ satisfies (i) and (the analogue of) (ii) of Theorem 5, while (iii) is replaced by $\tilde{\Gamma}(x, t; \xi, \tau) = 0$ for $\tau < t$. Similarly, the weak Green's function $\tilde{\gamma}(x, t; \xi, \tau)$ of $\tilde{L}u = 0$ in Q satisfies (i) and (ii) of Theorem 6 together $\tilde{\gamma}(x, t; \xi, \tau) = 0$ for $\tau < t$.

7. Classical Fundamental Solution and Green's Functions.

We consider the operators L and $\tilde{L}$ in strip S. If L satisfies (H.1) and if the coefficients of L are bounded and smooth, then L and $\tilde{L}$ possess classical fundamental solutions in S and Green's functions in smoothly bounded cylinders Q. In this section we enumerate various known properties of these functions and derive bounds for them. This information will be used in the next section to show that the weak fundamental solutions and Green's functions are limits of the corresponding classical functions and share many of their properties. Since the degree of smoothness beyond that required for the construction of classical solutions is irrelevant, we will make the usual quantitative assumption that L satisfies (H) and the temporary qualitative assumption that the coefficients of L are $C^\infty(\bar{S})$ functions.

Under the assumptions given above, it is well known that there exists a classical fundamental solution of $Lu = 0$, that is, a function $\Gamma(x, t; \xi, \tau)$ defined for $(x, t, \xi, \tau) \in S \times S$ except when both $x = \xi$ and $t = \tau$, which is continuous for $t > \tau$, and which has the following properties (cf. [10] and [12]).

(F.1) $\Gamma(x, t; \xi, \tau) \geq 0$ in $S \times S$ for $t > \tau$ and $\Gamma(x, t; \xi, \tau) = 0$ in $S \times S$ for $t < \tau$.

(F.2) $\Gamma(x, t; \xi, \tau) \leq K g_\varkappa (x - \xi, t - \tau)$ in $S \times S$ for $t > \tau$, where

$$g_\varkappa (x, t) = (4\pi \varkappa t)^{-n/2} \exp (-|x|^2/4\varkappa t)$$

is the fundamental solution of $\varkappa \Delta u = u_t$, and the positive constants $K, \varkappa$ depend on the bounds and moduli of continuity of the coefficients of L. In particular, $\Gamma(\cdot, \cdot; \xi, \tau)$ and $\Gamma(x, t; \cdot, \cdot)$ belong to $L^{p', q'}(S)$ for all p', q' whose Hölder conjugates satisfy $(**)$.

(F.3) For fixed $(\xi, \tau) \in E^n \times [0, T)$, Γ is continuously differentiable with respect to t, twice continuously differentiable with respect to x, and satisfies $L\Gamma = 0$ as function of (x, t) in $E^n \times (\tau, T]$. For fixed $(x, t) \in E^n \times (0, T]$, Γ is continuously differentiable with respect to τ, twice continuously differentiable with respect to ξ, and satisfies $\tilde{L}\Gamma = 0$ as a function of (ξ, τ) in $E^n \times [0, t)$.

(F.4) Suppose $e^{-\gamma|x|^2} u_0(x) \in L^2(E^n)$ for some $\gamma \geq 0$ and $G(x, t) \in C^\infty(\bar{S}) \cap L^{p, q}(S)$, where p and q satisfy $(**)$. Then

$$u(x, t) = \int_{E^n} \Gamma(x, t; \xi, \tau) u_0(\xi) \, d\xi + \int_\tau^t \int_{E^n} \Gamma(x, t; \xi, \eta) \, G(\xi, \eta) \, d\xi \, d\eta$$

is a classical solution of $Lu = G$ in $E^n \times (\tau, T^*]$ for $T^* = T^*(\gamma)$ sufficiently close to τ, and

$$v(\xi, \tau) = \int_{E^n} \Gamma(x, t; \xi, \tau) u_0(x) \, dx + \int_\tau^t \int_{E^n} \Gamma(x, \eta, \xi, \tau) \, G(x, \eta) \, dx \, d\eta$$

is a classical solution of $\tilde{L}v = G$ in $E^n \times [T', t)$ for $T' = T'(\gamma)$ sufficiently close to t. If γ is sufficiently small then $T' = 0$ and $T^* = T$. Moreover, if u_0 is continuous at $x = y$ then

$$\lim_{(x, t) \to (y, \tau+)} u(x, t) = \lim_{(\xi, \tau) \to (y, t-)} v(\xi, \tau) = u_0(y).$$

(F.5) For all $\eta \in (\tau, t)$

$$\Gamma(x, t; \xi, \tau) = \int_{E^n} \Gamma(x, t; \zeta, \eta) \, \Gamma(\zeta, \eta, \xi, \tau) \, d\zeta.$$

According to Theorem 5 there exists a weak fundamental solution of $Lu = 0$. We temporarily denote the weak fundamental solution by Γ' and show that we can identify it with the classical fundamental solution Γ. To this end, consider the Cauchy problem

$$Lu = G(x, t) \text{ for } (x, t) \in S, u(x, 0) = 0 \text{ for } x \in E^n, \tag{7.1}$$

where $G \in C^\infty(\bar{S}) \cap L^{p,p}(S)$ for some fixed finite $p > n + 2$. In view of (F.4),

$$u(x, t) = \int_0^t \int_{E^n} \Gamma(x, t; \xi, \tau) G(\xi, \tau) \, d\xi \, d\tau$$

is a classical solution of this problem, which, in view of (F.2), is bounded and therefore in class $\mathcal{E}^2(S)$. Moreover, it follows from the Schauder type theory for parabolic equations [10] that u is continuously differentiable with respect to t and twice continuously differentiable with respect to x in any compact subset of $\bar{S} = E^n \times [0, T]$. Thus u is also the weak solution in $\mathcal{E}^2(S)$ of problem (7.1) and, by Theorem 5, we have

$$u(x, t) = \int_0^t \int_{E^n} \Gamma'(x, t; \xi, \tau) G(\xi, \tau) \, d\xi \, d\tau$$

for all $(x, t) \in S$. Consequently

$$\int_0^t \int_{E^n} \{\Gamma(x, t; \xi, \tau) - \Gamma'(x, t; \xi, \tau)\} G(\xi, \tau) \, d\xi \, d\tau = 0 \tag{7.2}$$

for all $(x, t) \in S$. According to (F.2) and Theorem 5 (i), both Γ and Γ' belong to $L^{p',p'}(S)$ as functions of (ξ, τ). Since $C^\infty(\bar{S}) \cap L^{p,p}(S)$ is dense in $L^{p,p}(S)$, it follows from (7.2) that for every $(x, t) \in S$ we have $\Gamma(x, t; \xi, \tau) = \Gamma'(x, t; \xi, \tau)$ for almost every $(\xi, \tau) \in S$. Therefore it is unnecessary to distinguish between the weak and classical fundamental solutions of $Lu = 0$. Note that, in view of (F.3),

$$\Gamma(x, t; \xi, \tau) = \tilde{\Gamma}(\xi, \tau; x, t), \tag{7.3}$$

where Γ is the fundamental solution of $\tilde{L}u = 0$ in the variables (ξ, τ).

The upper bound for Γ given in (F.2) depends on the moduli of continuity of the coefficients of L and is therefore not applicable to the study of weak solutions of equations whose coefficients are not necessarily continuous. This bound does, however, hold for all parabolic equations with smooth coefficients regardless of whether the equation has divergence structure or not. As the next theorem shows, if we restrict our attention to divergence structure equations it is then possible to derive upper and lower bounds for Γ which have the same form as the upper bound in (F.2), but which are independent of the smoothness of the coefficients of L.

THEOREM 7. *Suppose that L satisfies (H) and the coefficients of L are smooth. Then there exist positive constants α_1, α_2 and $\mathcal{C}$ depending only on T and the structure of L such that*

$$\mathcal{C}^{-1} g_1(x-\xi, t-\tau) \le \Gamma(x, t; \xi, \tau) \le \mathcal{C} g_2(x-\xi, t-\tau)$$

for all $(x, t, \xi, \tau) \in S \times S$ with $t > \tau$, where $g_i(x, t)$ is the fundamental solution of $\alpha_i \Delta u = u_t$ for $i = 1, 2$.

PROOF OF THE LOWER BOUND. Since $\Gamma(x, t; \xi, \tau)$ is a non-negative weak solution of $Lu = 0$ for $(x, t) \in E^n \times (\tau, T)$ it follows from Theorem F that

$$\Gamma(x, t; \xi, \tau) \ge \mathcal{C}_1 \mathcal{M} (t-\tau)^{-n/2} \exp\{-\mathcal{C}_2 |x-\xi|^2/(t-\tau)\} \tag{7.4}$$

if for some $\varkappa > 0$

$$\mathcal{M} = \inf_{\tau < t < T} \int_{|x-\xi|^2 < \varkappa(t-\tau)} \Gamma(x, t; \xi, \tau)\, dx > 0.$$

Here $\mathcal{C}_1$ is a positive constant depending only on $\varkappa$, T and the structure of L, and $\mathcal{C}_2$ is a positive constant depending only on T and the structure of L. Hence in order to establish the lower bound in Theorem 7 it suffices to make an appropriate choice of $\varkappa$ and estimate $\mathcal{M}$ from below.

Let (ξ, τ) be an arbitrary fixed point in S and let t be an arbitrary fixed point in the open interval (τ, T). Consider the function

$$v(y, s) = \int_{|x-\xi|^2 < \varkappa(t-\tau)} \Gamma(x, t; y, s)\, dx$$

for $s < t$, where $\varkappa = 16/T$. In view of (F.1), (F.2) and (F.4), v is non-negative and bounded in $E^n \times [0, t]$, $\tilde{L}v = 0$ for $(y, s) \in E^n \times [0, t)$, and

$$\lim_{s \to t-} v(y, s) = 1$$

for all y such that $|y-\xi|^2 < \varkappa(t-\tau)$. It follows from the Schauder-type theory [10] that v is continuously differentiable with respect to y for (y, s)

in any compact subset of $\{|x-\xi|^2 < \varkappa(t-\tau)\} \times [0, t]$. Hence v is a weak solution of $\tilde{L}u = 0$ in $\{|x-\xi|^2 < \varkappa(t-\tau)\} \times [0, t)$ with initial values 1 on $s = t$. Extend the domain of definition of v by setting $v = 1$ for $s > t$. Then, by the Extension Principle and Theorem H (for $\tilde{L}u = 0$) with $\delta^2 = R = \varkappa(t-\tau)/16 \leq 1$, we have

$$\int_{|x-\xi|^2<\varkappa(t-\tau)} \Gamma(x, t; \xi, \tau)\, dx = v(\xi, \tau) \geq v(\xi, t) \exp\left\{-\mathcal{C}\left(\frac{16}{\varkappa}+1\right)\right\}$$

$$= \exp\{-\mathcal{C}(T+1)\},$$

where $\mathcal{C}$ depends only on T and the structure of L. Therefore for $\varkappa = 16/T$ we have $\mathcal{M} \geq \exp\{-\mathcal{C}(T+1)\}$ independent of (ξ, τ) in S and the assertion follows from (7.4).

To prove the validity of the upper bound in Theorem 7 we will need two lemmas concerning weak solutions of $Lu = 0$. For both of these lemmas the smoothness of the coefficients of L is totally irrelevant. The first result can be derived from Corollary 3.2 if the coefficients A_j and C belong to the appropriate $L^{p,q}(S)$ spaces. In any event, the proof is quite similar to that of Corollory 3.2.

LEMMA 5. *Suppose L satisfies (H) and let $S_\eta = E^n \times (\eta, T]$ for any $\eta \in [0, T)$ If u is the weak solution in $\mathcal{E}^2(S_\eta)$ of the Cauchy problem*

$$Lu = 0 \text{ for } (x, t) \in S_\eta;\; u(x, \eta) = 1 \text{ for } x \in E^n$$

then

$$0 \leq u(x, t) \leq \mathcal{C}$$

in S_η, where $\mathcal{C}$ is a constant which depends only on T and the structure of L.

PROOF. It suffices to prove the Lemma for $\eta = 0$. For integers $k \geq 1$, let $\Sigma_k = \{x; |x| < k\}$ and $Q_k = \Sigma_k \times (0, T]$. If u^k denotes the weak solution of the boundary value problem

$$\begin{cases} Lu = 0 \text{ for } (x, t) \in Q_k \\ u(x, 0) = 1 \text{ for } x \in \Sigma_k,\; u(x, t) = 0 \text{ for } (x, t) \in \partial\Sigma_k \times [0, T) \end{cases}$$

then, as we have seen in the proof of Theorem 3, $u^k \to u$ weakly in $L^{2,2}(Q)$ for any bounded cylinder $Q \subset S$. By lemma 1, with $\zeta \equiv 1$,

$$\| e^h u^k \|^2_{2,\infty,Q_k} \leq \mathcal{C} \int_{\Sigma_k} e^{2h(x,0)}\, dx,$$

where $\mathcal{C}$ depends only on T and the structure of L, and

$$h(x,t) = -\frac{\alpha\,|x-\xi|^2}{2T-t} - \beta t$$

for an arbitrary fixed $\xi \in E^n$. Since

$$\int_{\Sigma_k} e^{2h(x,0)}\,dx \leq \left(\frac{T}{\alpha}\right)^{n/2} \int_{E^n} e^{-|z|^2}\,dz$$

it follows that

$$\| e^h u^k \|_{2,2,Q_k} \leq \mathcal{C}_1\, T^N,$$

where $\mathcal{C}_1$ depends only on T and the structure of L, and $N = (n+2)/4$. Let $D_0 = \{|x-\xi| < 3\sqrt{T}/2\} \times (0, T]$. If k is so large that $D_0 \subset Q_k$ then

$$\| u^k \|_{2,2,D_0} \leq \mathcal{C}_1\, e^{\frac{9\alpha}{4}+\beta T}\, T^N = \mathcal{C}_2\, T^N$$

and hence $\| u \|_{2,2,D_0} \leq \mathcal{C}_2\, T^N$. Extend the domain of definition of u by setting $u = 1$ for $t < 0$ and let $D = \{|x-\xi| < 3\sqrt{T}/2\} \times (-9T, T]$. Clearly $\| u \|_{2,2,D} \leq \mathcal{C}_3\, T^N$, where $\mathcal{C}_3$ depends only on T and the structure. By the Extension Principle and Theorem B we have

$$| u(x,t) | \leq \mathcal{C}_4\, T^{-N} \| u \|_{2,2,D} \leq \mathcal{C}_3\, \mathcal{C}_4$$

for $|x-\xi| \leq \sqrt{T}/2$ and $0 \leq t \leq T$. Since ξ is arbitrary this proves the assertion.

If u is the solution of the Cauchy problem

$$(7.5) \qquad Lu = 0 \text{ for } (x,t) \in S_\eta,\ u(x,\eta) = u_0(x) \text{ for } x \in E^n$$

and if $u_0 = 0$ in some ball Σ centered at $x = y$, then it is plausable to expect that $u(y,t)$ will be small when $t - \eta$ is small in relation to the radius of Σ. The next lemma gives a precise formulation of this observation.

LEMMA 6. *Suppose that L satisfies (H) and u_0 is a $L^2(E^n)$ function such that $u_0(x) = 0$ for $|x-y| < \sigma$, where $y \in E^n$ and $\sigma > 0$ are fixed. If $u \in L^\infty(S_\eta)$ is the weak solution of the Cauchy problem* (7.5), *then for any $r \in (\eta, T]$ such that $0 < r - \eta \leq \sigma^2$*

$$| u(y,r) | \leq \mathcal{C}(r-\eta)^{-n/4} \| u_0 \|_{L^2(E^n)} \exp\{-\alpha\sigma^2/4(r-\eta)\},$$

where $\mathcal{C}$ is a constant which depends only on T and the structure of L.

PROOF. It suffices to prove the Lemma for $y = \eta = 0$. For $R \geq 1$, let $\zeta_R = \zeta_R(x)$ be a $C_0^\infty(E^n)$ function such that $\zeta_R = 1$ for $|x| \leq R$, $\zeta_R = 0$ for $|x| \geq R + 1$, $0 \leq \zeta_R \leq 1$ and $|\zeta_{Rx}|$ is bounded independent of R. By Lemma 1, with $\zeta = \zeta_R$, $s = \xi = 0$ and $\mu = r$

$$\| \zeta_R e^h u \|_{2,\infty}^2 \leq \mathcal{C} \left\{ \int_E \zeta_R^2 e^{2h(x,0)} u_0^2(x)\, dx + \| e^h u \zeta_{Rx} \|_{2,2}^2 \right\},$$

where the norms are computed over the set $\{|x| \leq R + 1\} \times (0, r)$ and $\mathcal{C}$ depends only on T and the structure of L. Since $u \in L^\infty(S)$ and $e^{2h} \in L^1(E^n \times (0, r))$ it follows that $\| e^h u \zeta_{Rx} \|_{2,2} \to 0$ as $R \to \infty$. Thus

$$\sup_{0<t<r} \left\{ \int_{|x|^2 < r/4} e^{2h(x,t)} u^2(x,t)\, dx \right\}^{1/2} \leq \tag{7.6}$$

$$\mathcal{C}^{1/2} \| e^{h(x,0)} u_0 \|_{L^2(E^n)} = \mathcal{C}_1 \left\{ \int_{|x| \geq \sigma} e^{2h(x,0)} u_0^2\, dx \right\}^{1/2}.$$

For $|x|^2 < r/4$ and $0 \leq t \leq r$,

$$-2h(x,t) = \frac{2\alpha |x|^2}{2r - t} + 2\beta t < \frac{\alpha}{2} + 2\beta T,$$

while for $|x|^2 \geq \sigma^2 \geq r$

$$-2h(x,0) = \frac{\alpha |x|^2}{r} \geq \frac{\alpha}{2} + \frac{\alpha\sigma^2}{2r}.$$

We therefore conclude from (7.6) that

$$\sup_{0<t<r} \left\{ \int_{|x|^2 < r/4} u^2(x,t)\, dx \right\}^{1/2} \leq e^{\beta T} \mathcal{C}_1 \| u_0 \|_{L^2(E^n)} e^{-\alpha\sigma^2/4r}.$$

Let $Q = \{|x|^2 < r/4\} \times (0, r]$. Then

$$\| u \|_{2,2,Q} \leq r^{1/2} \| u \|_{2,\infty,Q} \leq r^{1/2} e^{\beta T} \mathcal{C}_1 \| u_0 \|_{L^2(E^n)} e^{-\alpha\sigma^2/4r}$$

and it follows from Theorem B that

$$|u(0,r)| \leq \mathcal{C}_2 r^{-(n+2)/4} \| u \|_{2,2,Q} \leq r^{-n/4} e^{\beta T} \mathcal{C}_1 \mathcal{C}_2 \| u_0 \|_{L^2(E^n)} e^{-\alpha\sigma^2/4r},$$

where $\mathcal{C}_2$ depends only on T and the structure of L.

Note that results analogous to Lemmas 5 and 6 also hold for weak solutions of the adjoint equation. Let $\widetilde{S}_\eta = E^n \times [0, \eta)$ for $\eta \in (0, T]$. Then if v is the weak solntion in $\widetilde{S}_\eta$ of the Cauchy problem

$$\widetilde{L}v = 0 \quad \text{for} \quad (\xi, \tau) \in \widetilde{S}_\eta, \; v(\xi, \eta) = 1 \quad \text{for} \quad \xi \in E^n$$

we have $0 \leq v(\xi, \tau) \leq \mathcal{C}$ in $\widetilde{S}_\eta$. Moreover, if $v \in L^\infty(\widetilde{S}_\eta)$ is the weak solution of the Cauhy problem

$$\widetilde{L}v = 0 \quad \text{for} \quad (\xi, \tau) \in \widetilde{S}_\eta, \; v(\xi, \eta) = u_0(\xi) \quad \text{for} \quad \xi \in E^n$$

where u_0 satisfies the conditions of Lemma 6, then for any $r \in [0, \eta)$ such that $0 < \eta - r \leq \sigma^2$

$$|v(y, r)| \leq \mathcal{C}(\eta - r)^{-n/4} \| u_0 \|_{L^2(E^n)} \exp\{-\alpha\sigma^2/4(\eta - r)\}.$$

PROOF OF THE UPPER BOUND. We begin by establishing the preliminary estimate (3)

$$\Gamma(x, t; \xi, \tau) \leq \mathcal{C}(t - \tau)^{-n/2} \tag{7.7}$$

in $S \times S$ for $t > \tau$, where $\mathcal{C}$ depends only on T and the structure of L. Let $\tau \in [0, T)$ be fixed and consider the Cauchy problem

$$Lu = 0 \quad \text{for} \quad (x, t) \in S_\tau, \; u(x, \tau) = 1 \quad \text{for} \quad x \in E^n.$$

This problem has a unique classical solution w in $\mathcal{E}^2(S_\tau)$, and w is continuous and continuously differentiable in $\bar{S}_\tau$ (cf. [10]). Thus w is also the weak solution in $\mathcal{E}^2(S_\tau)$. In view of the properties of Γ, we have

$$w(x, t) = \int_{E^n} \Gamma(x, t; \xi, \tau)\, d\xi$$

and it follows from Lemma 5 that

$$0 \leq \int_{E^n} \Gamma(x, t; \xi, \tau)\, d\xi \leq \mathcal{C}_1 \tag{7.8}$$

(3) This estimate is due to Nash [16] in the special case in which L is given by (7).

in S_τ, where $\mathcal{C}_1$ depends only on T and the structure of L. Now hold $(x, t) \in E^n \times (0, T]$ fixed. Then, according to (F. 1) and (F. 3), Γ is a non-negative classical solution of $\tilde{L}u = 0$ as a function of (ξ, τ) in $E^n \times [0, t)$. It is therefore also a non-negative weak solution of $\tilde{L}u = 0$ for $(\xi, \tau) \in E^n \times [0, t)$. The assertion (7.7) then follows immediately from Theorem G (for $\tilde{L}u = 0$) and (7.8). Note that, by a similar argument, we can supplement (7.8) with

$$0 \leq \int_{E^n} \Gamma(x, t;\ \xi, \tau)\, dx \leq \mathcal{C}_1 \tag{7.8'}$$

for $(\xi, \tau) \in E^n \times [0, t)$.

Let $y \in E^n$, $\eta \in [0, t)$ and $\sigma > 0$ be fixed. For fixed r such that $0 < r - \eta \leq \sigma^2$ consider the function

$$u(x, t) = \int_{|y - \zeta| \geq \sigma} \Gamma(x, t;\ \zeta, \eta)\, \Gamma(y, r;\ \zeta, \eta)\, d\zeta$$

in S_η. We will use Lemma 6 to derive an estimate for $u(y, r)$. In view of (7.7) and (7.8),

$$0 \leq u(x, t) \leq \mathcal{C}\, (r - \eta)^{-n/2}$$

in $\bar{S}_\eta$, where $\mathcal{C}$ depends only on T and the structure of L. According to (F. 4), u is a classical solution of $Lu = 0$ in S_η, and for x such that $|x - y| \neq \sigma$

$$\lim_{t \to \eta+} u(x, t) = u_0(x),$$

where

$$u_0(x) = \begin{cases} 0 & \text{if } |y - x| < \sigma \\ \Gamma(y, r;\ x, \eta) & \text{if } |y - x| \geq \sigma. \end{cases} \tag{7.9}$$

Therefore u is a weak solution of $Lu = 0$ in S_η, and, by the bounded convergence theorem

$$\lim_{t \to \eta+} \int_{E^n} u(x, t)\, \psi(x)\, dx = \int_{E^n} u_0(x)\, \psi(x)\, dx$$

for any $\psi \in C_0^1(E^n)$. For any $\varepsilon \in (0, T - \eta)$, u is bounded, continuous and continuously differentiale in $\bar{S}_{\eta+\varepsilon}$. Thus it is the weak solution in $\mathcal{E}^2(S_{\eta+\varepsilon})$ of the Cauchy problem

$$Lv = 0 \quad \text{for} \quad (x, t) \in S_{\eta+\varepsilon}, \qquad v(x, \eta + \varepsilon) = u(x, \eta + \varepsilon) \ \text{for} \ x \in E^n .$$

By Theorem 3, with $\gamma > 0$ sufficiently small,

$$\| e^h u \|^2_{2,\infty,S_{\eta+\varepsilon}} + \| e^h u_x \|^2_{2,2,S_{\eta+\varepsilon}} \leq \mathcal{C}(r-\eta)^{-n/2} \| e^{-\gamma|x|^2} \|_{L^2(E^n)},$$

where $\mathcal{C}$ depends only on T and the structure of L. Letting $\varepsilon \to 0$ we conclude that $u \in L^\infty[\eta, T; L^2_{\text{loc}}(E^n)] \cap L^2[\eta, T; H^{1,2}_{\text{loc}}(E^n)]$. Therefore $u \in L^\infty(S_\eta)$ is the weak solution of the Cauchy problem (7.5) with u_0 given by (7.9).

Since, by (7.7) and (7.8), $u_0 \in L^2(E^n)$ we can apply Lemma 6 to u to obtain

$$0 \leq u(y, r) = \int_{|y-\zeta| \geq \sigma} \Gamma^2(y, r; \zeta, \eta)\, d\zeta \leq$$

$$\leq \mathcal{C}(r-\eta)^{-n/4} \exp\{-\alpha\sigma^2/4(r-\eta)\} \left\{ \int_{|y-x| \geq \sigma} \Gamma^2(y, r; x, \eta)\, dx \right\}^{1/2}.$$

Thus

$$\int_{|y-\zeta| \geq \sigma} \Gamma^2(y, r; \zeta, \eta)\, d\zeta \leq \mathcal{C}(r-\eta)^{-n/2} \exp\{-\alpha\sigma^2/2(r-\eta)\} \tag{7.10}$$

for $r \in (\eta, T]$ such that $0 < r - \eta \leq \sigma^2$, where $\mathcal{C}$ depends only on T and the structure of L. Similarly,

$$\int_{|y-\zeta| \geq \sigma} \Gamma^2(\zeta, \eta; y, r)\, d\zeta \leq \mathcal{C}(\eta - r)^{-n/2} \exp\{-\alpha\sigma^2/2(\eta-r)\} \tag{7.10'}$$

for $r \in [0, \eta)$ such that $0 < \eta - r \leq \sigma^2$.

Let (x, t) and (ξ, τ) be arbitrary fixed points of S with $t > \tau$. Set $\sigma = |x - \xi|/2$ and assume that $0 < t - \tau \leq \sigma^2$. By the Kolmogorov identity (F.5)

$$\Gamma(x, t; \xi, \tau) = \int_{E^n} \Gamma(x, t; \zeta, \eta)\, \Gamma(\zeta, \eta; \xi, \tau)\, d\zeta,$$

where $\eta = (t + \tau)/2$. Write

$$\Gamma(x, t; \xi, \tau) = \int_{|x-\zeta| \geq \sigma} \Gamma(x, t; \zeta, \eta)\, \Gamma(\zeta, \eta; \xi, \tau)\, d\zeta +$$

$$\int_{|x-\zeta| < \sigma} \Gamma(x, t; \zeta, \eta)\, \Gamma(\zeta, \eta; \xi, \tau)\, d\zeta = J_1 + J_2.$$

By the Schwarz inequality

$$J_1 \leq \left\{ \int\limits_{|x-\zeta| \geq \sigma} \Gamma^2(x,t;\zeta,\eta)\, d\zeta \right\}^{1/2} \left\{ \int\limits_{|x-\zeta| \geq \sigma} \Gamma^2(\zeta,\eta;\xi,\tau)\, d\zeta \right\}^{1/2}.$$

Using (7.10) to estimate the first factor on the right and (7.7), (7.8′) to estimate the second, we obtain

$$J_1 \leq \mathcal{C}(t-\tau)^{-n/2} \exp\{-\alpha\,|x-\xi|^2/8(t-\tau)\},$$

where $\mathcal{C}$ depends only on T and the structure of L. To estimate J_2 we note that $|x-\zeta| < \sigma = |x-\xi|/2$ implies $|\xi-\zeta| \geq |\xi-x| - |x-\zeta| \geq \sigma$. Thus

$$J_2 \leq \int\limits_{|\xi-\zeta| \geq \sigma} \Gamma(x,t;\zeta,\eta)\, \Gamma(\zeta,\eta;\xi,\tau)\, d\zeta \leq$$

$$\leq \left\{ \int\limits_{|\xi-\zeta| \geq \sigma} \Gamma^2(x,t;\zeta,\eta)\, d\zeta \right\}^{1/2} \left\{ \int\limits_{|\xi-\zeta| \geq \sigma} \Gamma^2(\zeta,\eta;\xi,\tau)\, d\zeta \right\}^{1/2}.$$

Using now (7.7), (7.8) to estimate the first factor on the right and (7.10′) to estimate the second, we find

$$J_2 \leq \mathcal{C}(t-\tau)^{-n/2} \exp\{-\alpha\,|x-\xi|^2/8(t-\tau)\}.$$

It follows that $\Gamma(x,t;\xi,\tau)$ has the required upper bound in case $0 < 4(t-\tau) \leq$ $\leq |x-\xi|^2$. If $|x-\xi|^2 < 4(t-\tau)$ then, by (7.7),

$$\Gamma(x,t;\xi,\tau) \leq \mathcal{C}(t-\tau)^{-n/2} \exp\{\alpha\,|x-\xi|^2/8(t-\tau)\} \exp\{-\alpha\,|x-\xi|^2/8(t-\tau)\} \leq$$

$$\leq \mathcal{C} e^{\alpha/2} (t-\tau)^{-n/2} \exp\{-\alpha\,|x-\xi|^2/8(t-\tau)\}.$$

This completes the proof of Theorem 7.

The following result is obtained from Theorem 7 by a simple computation.

Corollary 7.1. *For each pair of exponents p, q which satisfy* (**)

$$\|\Gamma(x,t;\cdot,\cdot)\|_{p',q'} \leq \mathcal{C} \quad \textit{and} \quad \|\Gamma(\cdot,\cdot;\xi,\tau)\|_{p',q'} \leq \mathcal{C},$$

where $\mathcal{C}$ depends only on T, the structure of L, and the exponents p, q.

Let Ω be a bounded open domain in E^n and $Q = \Omega \times (0, T]$. We assume that L satisfies (H), the coefficients of L are in $C^\infty(\bar{S})$, and $\partial\Omega$ is smooth. Then it is well known that there exists a classical Green's function for $Lu = 0$ in Q, that is, a function $\gamma(x, t; \xi, \tau)$ defined in $Q \times Q$ except when both $x = \xi$ and $t = \tau$, which is continuous for $t > \tau$ and which has the following properties (cf. [10] and [12]).

(G.1) $\gamma(x, t; \xi, \tau) \geq 0$ in $Q \times Q$ for $t > \tau$. $\gamma(x, t; \xi, \tau) = 0$ in $Q \times Q$ for $t < \tau$, and for $t > \tau$ with either x or ξ on $\partial\Omega$.

(G.2) $\gamma(x, t; \xi, \tau) \leq \Gamma(x, t; \xi, \tau)$ in $Q \times Q$ for $t > \tau$.

(G.3) For fixed $(\xi, \tau) \in \Omega \times [0, T)$, γ is continuously differentiable with respect to t, twice continuously differentiable with respect to x, and satisfies $L\gamma = 0$ as a function of (x, t) in $\Omega \times (\tau, T]$. For fixed $(x, t) \in \Omega \times (0, T]$, γ is continuously differentiable with respect to τ, twice continuously differentiable with respect to ξ, and satisfies $\tilde{L}\gamma = 0$ as a function of (ξ, τ) in $\Omega \times [0, t)$.

(G.4) Suppose that $u_0(x) \in L^2(\Omega)$ and $G(x, t) \in C^\infty(\bar{Q})$. Then

$$u(x, t) = \int_\Omega \gamma(x, t; \xi, \tau)\, u_0(\xi)\, d\xi + \int_\tau^t \int_\Omega \gamma(x, t; \xi, \eta)\, G(\xi, \eta)\, d\xi\, d\eta$$

is a classical solution of $Lu = G$ in $\Omega \times (\tau, T]$ and

$$v(\xi, \tau) = \int_\Omega \gamma(x, t; \xi, \tau)\, u_0(x)\, dx + \int_\tau^t \int_\Omega \gamma(x, \eta, \xi, \tau)\, G(x, \eta)\, dx\, d\eta$$

is a classical solution of $\tilde{L}v = G$ in $\Omega \times [0, t)$. Moreover, if u_0 is continuous at $y \in \Omega$ then

$$\lim_{(x,t)\to(y,\tau+)} u(x, t) = \lim_{(\xi,\tau)\to(y,t-)} v(\xi, \tau) = u_0(y).$$

According to Theorem 6, there also exists a weak Green's function for $Lu = 0$ in Q, but, by the argument we used for the fundamental solution, we can identify the weak and classical Green's functions. Thus, in particular,

$$\gamma(x, t; \xi, \tau) = \tilde{\gamma}(\xi, \tau; x, t), \tag{7.11}$$

where $\tilde{\gamma}$ is the Green's function for $\tilde{L}u = 0$ in Q in the variables (ξ, τ). In view of (G.2) and Theorem 7,

$$\gamma(x, t; \xi, \tau) \leq C(t - \tau)^{-n/2} \exp\{-\alpha\,|x - \xi|^2/8(t - \tau)\} \tag{7.12}$$

for all (x, t) and (ξ, τ) in Q with $t > \tau$, where $\mathcal{C}$ depends only on T and the structure of L. Note that the bound (7.12) is independent of the domain Q and the smoothness of the coefficients of L. There is also a lower bound of the same form for γ, but unlike (7.12) it is a local bound.

Theorem 8. *Suppose that L satisfies (H), the coefficients of L are smooth, and $\partial\Omega$ is smooth. Let Ω' be a convex subset of Ω such that the distance δ from Ω' to $\partial\Omega$ is positive. Then there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ depending only on δ, T and the structure of L such that*

$$\gamma(x, t\,; \xi, \tau) \geq \mathcal{C}_1 (t-\tau)^{-n/2} \exp\{-\mathcal{C}_2 |x-\xi|^2/(t-\tau)\}$$

holds for all $x, \xi \in \Omega'$ and either

$$\tau < t \leq \min\left\{T, \tau + \frac{T}{8} d^2(\xi, \partial\Omega')\right\}$$

for arbitrary $\tau \in [0, T)$ or

$$\max\left\{0, t - \frac{T}{8} d^2(x, \partial\Omega')\right\} \leq \tau < t$$

for arbitrary $t \in (0, T]$.

Proof. Let (ξ, τ) be a fixed point of $\Omega' \times [0, T)$. Since γ is a non-negative weak solution of $Lu = 0$ for $(x, t) \in \Omega \times (\tau, T]$, it follows from Theorem I that if for some $\varkappa > 0$

$$\mathcal{M} = \inf_{\tau < t < T^*} \int_{|x-\xi|^2 < \varkappa(t-\tau)} \gamma(x, t\,; \xi, \tau)\, dx > 0,$$

where $T^* = \min\left\{T, \tau + \frac{1}{\varkappa} d^2(\xi, \partial\Omega)\right\}$ then

$$\gamma(x, t\,; \xi, \tau) \geq \mathcal{C}_1' \mathcal{M} (t-\tau)^{-n/2} \exp\{-\mathcal{C}_2 |x-\xi|^2/(t-\tau)\}$$

for all $x \in \Omega'$ and $\tau < t < \min\left\{T, \tau + \frac{2}{\varkappa} d^2(\xi, \partial\Omega')\right\}$. Here $\mathcal{C}_1'$ depends only on δ, $\varkappa$, T and the structure of L. Hence in order to prove the first part of the Theorem it suffices to make a suitable choice of $\varkappa$ and estimate $\mathcal{M}$ from below. If we set $\varkappa = 16/T$ then exactly as in the proof of the lower bound in Theorem 7 we find $\mathcal{M} \geq \exp\{-\mathcal{C}(T+1)\}$ independent of (ξ, τ). The proof of the second part of the Theorem is completely analogous with the roles of L and $\widetilde{L}$ interchanged. We omit the details.

8. Weak Fundamental Solution and Green's Function.

In section 6 we showed that if L satisfies (H) then there exists a weak Green's function $\gamma(x, t;\ \xi, \tau)$ for $Lu = 0$ in any bounded cylinder Q. In particular, γ belongs to certain $L^{q'}[H_0^{1,p'}]$ spaces in Q and gives a representation formula for the solution of the boundary value problem (6.9). Here we will derive additional properties of γ which are analogous to (G. 1-4) for the classical Green's function. This derivation is based on the fact that γ can be approximated by classical Green's functions. We will also obtain similar results for the weak fundamental solution of $Lu = 0$.

Let Ω be a bounded open domain in E^n and $Q = \Omega \times (0, T]$. For $u_0 \in L^2(\Omega)$ and $G(x, t) \in L^{p,q}(Q)$ with p, q satisfyng $(**)$ we will show, among other things, that γ gives a representation for the solution of the boundary value problem

$$
(8.1) \qquad \begin{cases} Lu = G(x, t) \text{ for } (x, t) \in Q \\ u(x, 0) = u_0(x) \text{ for } x \in \Omega,\ u(x, t) = 0 \text{ for } (x, t) \in \partial\Omega \times [0, T] \end{cases}
$$

and the adjoint problem

$$
(8.2) \qquad \begin{cases} \widetilde{L}v = G(\xi, \tau) \text{ for } (\xi, \tau) \in Q \\ v(\xi, T) = u_0(\xi) \text{ for } \xi \in \Omega,\ v(\xi, \tau) = 0 \text{ for } (\xi, \tau) \in \partial\Omega \times [0, T]. \end{cases}
$$

In stating our result we use the following notation. If τ is an arbitrary point in $[0, T)$ we set $Q_\tau = \Omega \times (\tau, T]$, while if t is an arbitrary point in $(0, T]$ we set $\widetilde{Q}_t = \Omega \times [0, t)$.

THEOREM 9. *Suppose that L satisfies (H), $u_0 \in L^2(\Omega)$ and $G \in L^{p,q}(Q)$, where p, q satisfy $(**)$. Then the weak Green's function $\gamma(x, t;\ \xi, \tau)$ of $Lu = 0$ in Q has the following properties.*

(i) $\gamma(x, t;\ \xi, \tau) = \widetilde{\gamma}(\xi, \tau;\ x, t)$ *in $Q \times Q$ for $t > \tau$, where $\widetilde{\gamma}$ is the weak Green's function for $\widetilde{L}v = 0$ in Q.*

(ii) *If $\partial\Omega$ has property (A), then γ is continuous as a function of (x, t) in $\overline{\Omega} \times (\tau, T]$ for each $(\xi, \tau) \in \widetilde{Q}_T$, and as a function of (ξ, τ) in $\overline{\Omega} \times [0, t)$ for each $(x, t) \in Q_0$. Moreover, $\gamma = 0$ for either x or ξ in $\partial\Omega$ and $t > \tau$.*

(iii) *The bound* (7.12) *and the conclusion of Theorem* 8 *hold for γ.*

(iv) *For fixed $(\xi, \tau) \in \widetilde{Q}_T$, $\gamma(\cdot, \cdot; \xi, \tau) \in L^2[\tau + \delta, T;\ H_0^{1,2}(\Omega)]$ for arbitrary $\delta \in (0, T - \tau)$ with*

$$
\langle\langle \gamma(\cdot, \cdot\ ;\ \xi, \tau) \rangle\rangle_{2,2} \leq C\, \delta^{-n/4},
$$

where the norm is computed over the set $Q_{\tau+\delta}$, and $\mathcal{C}$ depends only on T and the structure of L. Moreover, γ is a weak solution of $Lu = 0$ for $(x, t) \in Q_\tau$. For fixed $(x, t) \in Q_0$, $\gamma(x, t; \cdot, \cdot) \in L^2[0, t-\delta; H_0^{1,2}(\Omega)]$ for arbitary $\delta \in (0, t)$ with

$$\langle\langle \gamma(x, t; \cdot, \cdot) \rangle\rangle_{2,2} \leq \mathcal{C}\, \delta^{-n/4}$$

where the norm is computed over the set $\tilde{Q}_{t-\delta}$, and $\mathcal{C}$ depends only on T and the structure of L. Morevoer, γ is a weak solution of $\tilde{L}v = 0$ for $(\xi, \tau) \in \tilde{Q}_t$.

(v) *For fixed $(\xi, \tau) \in \tilde{Q}_T$ let Σ denote an arbitrary open domain such that $\bar{\Sigma} \subset \Omega \setminus \{\xi\}$. Then γ is a global weak solution of $Lu = 0$ for $(x, t) \in \Sigma \times (\tau, T)$ with initial values zero on $t = \tau$. For fixed $(x, t) \in Q_0$ let Σ denote an arbitrary open domain such that $\bar{\Sigma} \subset \Omega \setminus \{x\}$. Then γ is a global weak solution of $\tilde{L}v = 0$ for $(\xi, \tau) \in \Sigma \times (0, t)$ with initial values zero on $\tau = t$.*

(vi) *The weak solution of the boundary value problem* (8.1) *is given by*

$$u(x, t) = \int_\Omega \gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi + \iint_Q \gamma(x, t; \xi, \tau)\, G(\xi, \tau)\, d\xi\, d\tau$$

and the weak solution of the boundary value problem (8.2) *is given by*

$$v(\xi, \tau) = \int_\Omega \gamma(x, T; \xi, \tau)\, u_0(x)\, dx + \iint_Q \gamma(x, t; \xi, \tau)\, G(x, t)\, dx dt.$$

If u_0 is continuous at a point y in Ω then

$$\lim_{(x,t)\to(y,0)} u(x, t) = \lim_{(\xi,\tau)\to(y,T)} v(\xi, \tau) = u_0(y).$$

Note that using the Extension Principle and Theorem C it follows from (i), (iii), (iv) and (v) that γ is continuous as a function of (x, t) in $Q \setminus \{(\xi, \tau)\}$ aud as a function of (ξ, τ) in $Q \setminus \{(x, t)\}$. Thus the behavior of γ in the interior of Q is independent of whether or not $\partial\Omega$ has property (A), and the assertion (ii) is actually only a statement about the boundary behavior of γ.

PROOF. As in Section 3, let

$$L^m u = u_t - (A_{ij}^m u_{x_i} + A_j^m u)_{x_j} - B_j^m u_{x_j} - C^m u,$$

where the superscript m on a function denotes an integral average formed with a kernel whose support lies in $|x|^2 + t^2 < m^{-2}$ for integers $m \geq 1$, and let $\{\Omega^m\}$ denote a sequence of smoothly bounded open domains such that $\bar{\Omega}^m \subset \Omega^{m+1} \subset \bar{\Omega}^{m+1} \subset \Omega$ for all m, and $\lim_{m \to \infty} \Omega^m = \Omega$. Recall that the operators L^m have a uniform structure which is, in turn, determined by the structure of L. Throughout this proof we will use $\mathcal{C}$ to denote any constant which depends only on T and the structure of L.

We assert that

$$(8.3) \qquad \gamma^m(x, t\,;\ \cdot,\cdot) \longrightarrow \gamma(x, t\,;\ \cdot,\cdot) \text{ and } \gamma^m(\cdot,\cdot\,;\ \xi, \tau) \longrightarrow \tilde{\gamma}(\xi, \tau\,;\ \cdot,\cdot)$$

weakly in $L^{p', q'}(Q)$ for any exponents p', q' whose Hölder conjugates satisfy

$$(8.4) \qquad 2 < p, q < \infty \text{ and } \frac{n}{2p} + \frac{1}{q} < \frac{1}{2},$$

where γ^m is the Green's function for $L^m u = 0$ in $Q^m = \Omega^m \times (0, T]$. Let F be an arbitrary $L^{p, q}(Q)$ function with p, q satisfyng (8.4) and consider the boundary value problem $Lu = F$ in Q, $u = 0$ on the parabolic boundary of Q. According to Theorem 6, the weak solution of this problem is given by

$$u(x, t) = \iint_Q \gamma(x, t\,;\ \xi, \tau)\, F(\xi, \tau)\, d\xi\, d\tau.$$

On the other hand, in view of Theorem 1 (i) and (G.4), for each $(x, t) \in Q$

$$\lim_{m \to \infty} \iint_Q \gamma^m(x, t\,;\ \xi, \tau)\, F^m(\xi, \tau)\, d\xi\, d\tau = \iint_Q \gamma(x, t\,;\ \xi, \tau)\, F(\xi, \tau)\, d\xi\, d\tau,$$

where we have set $\gamma^m = 0$ for $\xi \in \complement\, \Omega^m$. Since $F^m \longrightarrow F$ strongly in $L^{p, q}(Q)$ the first part of (8.3) follows immediately. The second part of (8.3) is proved by applying the same argument to the adjoint equation.

Let $(\xi, \tau) \in \tilde{Q}_T$ be fixed. For arbitrary $\delta \in (0, T - \tau)$ and m so large that $\xi \in \Omega^m$, $\gamma^m(x, t\,;\ \xi, \tau)$ is the classical solution of the boundary value problem

$$\begin{cases} L^m u = 0 \text{ for } (x, t) \in Q^m \\ u(x, \tau + \delta) = \gamma^m(x, \tau + \delta\,;\ \xi, \tau) \quad \text{for } x \in \Omega^m, \\ u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega^m \times [\tau + \delta, T]. \end{cases}$$

Since γ^m is continuously differentiable in $\bar{Q}^m_{\tau+\delta}$ and $\gamma^m = 0$ for $x \in \partial\Omega^m$, it is also the weak solution of this problem. By Lemma 1, with $\zeta \equiv 1$, $s = \tau + \delta$ and $\mu = +\infty$,

$$\| \gamma^m(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,\infty} + \| \gamma^m_x(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,2} \leq e^{\beta T}\, C\, \| \gamma^m(\cdot,\tau+\delta\,;\,\xi,\tau)\|^2_{L^2(\Omega)},$$

where the norms on the left are computed over the set $Q^m_{\tau+\delta}$. In view of (7.12) and Theorem 7

$$\| \gamma^m(\cdot,\tau+\delta\,;\,\xi,\tau)\|^2_{L^2(\Omega)} \leq C\, \delta^{-n/2}.$$

Thus we have

$$\| \gamma^m(\cdot,\cdot\,\xi,\tau)\|^2_{2,\infty} + \| \gamma^m_x(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,2} \leq C\, \delta^{-n/2} \tag{8.5}$$

where if we set $\gamma^m = 0$ for $x \in \complement\,\Omega^m$ we can regard the norms as being computed over the set $Q_{\tau+\delta}$. It follows from (8.5) and Lemma 2 that there exists a subsequence of the γ^m which converges weakly to a limit function in $L^2[\tau+\delta, T; H^{1,2}_0(\Omega)]$ and any finite collection of $L^{2p',2q'}(Q_{\tau+\delta})$ spaces for suitable exponents, and it follows from (8.3) that the limit function is $\tilde{\gamma}$. From this we conclude that $\tilde{\gamma}(\xi,\tau\,;x,t)$ is a weak solution of $Lu = 0$ for $(x,t) \in Q_\tau$ and $\langle\langle \tilde{\gamma}(\xi,\tau\,;\cdot,\cdot)\rangle\rangle_{2,2} \leq C\,\delta^{-n/4}$. If we hold $(x,t) \in Q_0$ fixed and apply the same argument to γ^m considered as a function of (ξ,τ), we find that $\gamma(x,t\,;\xi,\tau)$ is a weak solution of $\tilde{L}v = 0$ for $(\xi,\tau) \in \tilde{Q}_t$ and $\langle\langle \gamma(x,t\,;\cdot,\cdot)\rangle\rangle_{2,2} \leq C\,\delta^{-n/4}$.

On the other hand, for fixed $(\xi,\tau) \in \tilde{Q}_T$, it follows from (7.12), Theorem 7 and Theorem C that the sequence $\{\gamma^m\}$ is uniformly bounded and equicontinuous for (x,t) in any compact subset of Q_τ. Thus $\gamma^m \to \tilde{\gamma}$ uniformly in any compact subset of Q_τ. Similarly, for each $(x,t) \in Q_0$, $\gamma^m \to \gamma$ uniformly in any compact subset of $\tilde{Q}_t$. Thus, in particular, (i) and (iii) hold. Moreover, in view of the previous paragraph, (iv) also holds.

If $\partial\Omega$ has property (A) then the sequence $\{\Omega^m\}$ can be chosen so that $\{\partial\Omega^m\}$ has property (A) uniformly with respect to m. Let $(\xi,\tau) \in \tilde{Q}_T$ be fixed and set $\gamma^m = 0$ for $x \in \complement\,\Omega^m$, where we assume m so large that $\xi \in \Omega^m$. For arbitrary $\delta \in (0, T-\tau)$ the sequence $\{\gamma^m\}$ is uniformly bounded in $\bar{Q}_{\tau+\delta/2}$, and in view of Theorem D, equicontinuous in $\bar{Q}_{\tau+\delta}$. Therefore γ is continuous for $(x,t) \in \bar{\Omega} \times (\tau, T]$ and $\gamma = 0$ for $x \in \partial\Omega$. The proof of the continuity of γ as function of (ξ,τ) is similar and we omit it.

For fixed $(\xi,\tau) \in \tilde{Q}_T$, let Σ be an arbitrary open domain such that $\bar{\Sigma} \subset \Omega \setminus \{\xi\}$ and let $D_\tau = \Sigma \times (\tau, T]$. Then for m sufficiently large γ^m is a weak solution of $L^m u = 0$ in D_τ. If $|x-\xi| \geq \delta > 0$ and $t > \tau$ then

$$g_\varkappa(x-\xi, t-\tau) \leq 2^{n/2} g_{2\varkappa}(x-\xi, t-\tau) \exp\{-\delta^2/8\varkappa(t-\tau)\},$$

where $g_\varkappa$ denotes the fundamental solution of $\varkappa \Delta u = u_t$. Thus it follows from (7.12) that

$$\lim_{t\to\tau} \int_\Sigma \gamma^m(x, t; \xi, \tau)\, \psi(x)\, dx = 0$$

for all $\psi \in C_0^1(\Omega)$. Let $\zeta = \zeta(x)$ be a C^∞ function with compact support in $\Omega \setminus \{\xi\}$ such that $\zeta = 1$ on $\bar{\Sigma}$ and $0 \leq \zeta \leq 1$. By Lemma 1, with $s = \tau + \varepsilon$ for arbitrary $\varepsilon \in (0, T-\tau)$ and $\mu = +\infty$,

$$\|\zeta\gamma^m(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,\infty} + \|\zeta\gamma^m_x(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,2} \leq \tag{8.5}$$
$$e^{\beta T}\, C\{\|\zeta\gamma^m(\cdot, \tau+\varepsilon; \xi, \tau)\|_{L^2(\Sigma')} + \|\zeta_x\gamma^m(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,2}\},$$

where $\Sigma' = \operatorname{supp} \zeta$ and norms are computed over the set $\Sigma' \times (\tau+\varepsilon, T)$. If $|x-\xi| \geq \delta > 0$ and $t > \tau$ then $g^2_\varkappa(x-\xi, t-\tau) \leq g_x(\delta, t-\tau)\, g_\varkappa(x-\xi, t-\tau)$. Thus in view of (7.12) the first term on the right in (8.5) tends to zero as $\varepsilon \to 0$ and the second is bounded independent of m or ε. Therefore, letting $\varepsilon \to 0$, we obtain

$$\|\zeta\gamma^m(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,\infty} + \|\zeta\gamma^m_x(\cdot,\cdot\,;\,\xi,\tau)\|^2_{2,2} \leq \text{const.} \tag{8.6}$$

independent of m, where the norms are computed over the set $\Sigma' \times (\tau, T)$. In particular, each γ^m is a global weak solution of $L^m u = 0$ in D_τ with initial values zero on $t = \tau$. Moreover, we conclude from (8.6), Lemma 2 and the pointwise convergence of γ^m to γ that $\{\gamma^m\}$ converges weakly to γ in $L^2[\tau, T; H^{1,2}(\Sigma)]$ and in any finite collection of $L^{2p', 2q'}(D_\tau)$ spaces with suitable exponents. Hence it follows that γ is a global weak solution of $Lu = 0$ in D_τ with initial values zero on $t = \tau$. The corresponding result for the adjoint equation is proved analogously.

According to Theorem 1 (i) and (G.4), if u is the solution of the problem (8.1) then for each $(x, t) \in Q$ we have $u^m(x, t) \to u(x, t)$, where for m sufficiently large

$$u^m(x, t) = \int_\Omega \gamma^m(x, t; \xi, 0)\, \zeta^m(\xi)\, u_0^m(\xi)\, d\xi + \iint_Q \gamma^m(x, t; \xi, \tau)\, G^m(\xi, \tau)\, d\xi\, d\tau$$

and we have set $\gamma^m = 0$ for $\xi \subset \complement \Omega^m$. In view of (7.12) the sequence $\{\gamma^m(x, t; \cdot, 0)\}$ is uniformly bounded in $L^2(\Omega)$ for $t > 0$ and $\{\gamma^m(x, t, \cdot, \cdot)\}$ is uniformly bounded in $L^{p', q'}(Q)$. Hence if p and q are finite there exists a subsequence which converges weakly to $\gamma(x, t; \cdot, 0)$ in $L^2(\Omega)$ and to $\gamma(x, t; \cdot, \cdot)$ in $L^{p', q'}(Q)$. On the other hand $\zeta^m u_0^m \to u_0$ strongly in $L^2(\Omega)$ and $G^m \to G$ strongly in $L^{p, q}(Q)$. It follows that if $G \in L^{p, q}(Q)$ with finite exponents p, q satisfyng (* *) then

$$u(x, t) = \lim_{m \to \infty} u^m(x, t) = \int_\Omega \gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi + \iint_Q \gamma(x, t; \xi, \tau)\, G(\xi, \tau)\, d\xi, d\tau.$$

If $G \in L^{p, q}(Q)$ with p, q satisfyng (* *) and either p or q infinite, then, since Ω is bounded, we also have $G \in L^{\bar{p}, \bar{q}}(Q)$ for some finite exponents $\bar{p}, \bar{q}$ satisfyng (* *), and hence the representation formula is again valid. The proof of the representation formula for problem (8.2) is similar and we omit it.

Suppose that u_0 is continuous at a point $y \in \Omega$. The function

$$w(x, t) = u_0(y) \int_\Omega \gamma(x, t; \xi, 0)\, d\xi + \iint_Q \gamma(x, t; \xi, \tau)\, G(\xi, \tau)\, d\xi\, d\tau$$

is the weak solution of problem (8.1) with the initial condition $u(x, 0) = u_0(y)$ for $x \in \Omega$. Extend the domain of definition of w by setting $w(x, t) = u_0(y)$ for $t < 0$. By the Extension Principle and Theorem C, w is continuous in any compact subset of $\Omega \times (-T, T]$. Thus, in particular, $w(x, t) \to u_0(y)$ as $(x, t) \to (y, 0)$. Write the solution of problem (8.1) in the form

$$u(x, t) = w(x, t) + \int_\Omega \gamma(x, t; \xi, 0)\{u_0(\xi) - u_0(y)\}\, d\xi \equiv w(x, t) + J(x, t),$$

and write J as the sum of an integral J_1 taken over the set $\{\xi; |\xi - y| < \varrho\}$ and an integral J_2 taken over the set $\Omega_\varrho = \{\xi; \xi \in \Omega, |\xi - y| \geq \varrho\}$, where ϱ is smaller than the distance from y to $\partial\Omega$. Thus, in view of (7.12),

$$|J_1| \leq \mathcal{C} \max_{|\xi - y| < \varrho} |u_0(\xi) - u_0(y)|.$$

Moreover, if $|x - y| < \varrho/2$ then $|\xi - y| \geq \varrho$ implies $|x - \xi| > \varrho/2$ and hence

$$|J_2| \leq \mathcal{C}\, \| u_0(\cdot) - u_0(y) \|_{L^2(\Omega)}\, g_\varkappa(\varrho/2, t),$$

where $\varkappa$ depends only on T and the structure of L. Since u_0 is continuous at y, given $\varepsilon > 0$ we can fix a $\varrho = \varrho(\varepsilon) > 0$ such that $|J_1| < \varepsilon/2$. Then for

$|x-y| < \varrho/2$ there exists an $\eta>0$ depending on ε and $\varrho(\varepsilon)$ such that $0<t<\eta$ implies $|J_2| < \varepsilon/2$. Therefore $J \to 0$ and $u(x,t) \to u_0(y)$ as $(x,t) \to (y,0)$. The proof of the corresponding statement for $v(\xi,\tau)$ is similar and we omit it.

We are going to prove a theorem concerning the properties of the weak fundamental solution of $Lu=0$ which is analogous to Theorem 9. Before doing so, however, we pause to prove a lemma which establishes a useful relationship between the weak Green's function for certain domains and the weak fundamental solution. In stating this result we use the notation $\Sigma_k = \{x\,;\,|x|<k\}$ and $Q_k = \Sigma_k \times (0,T]$ for integers $k \geq 1$.

LEMMA 7. *Suppose that L satisfies (H). Then the sequence $\{\gamma^k(x,t;\xi,\tau)\}$ of weak Green's functions for $Lu=0$ in Q_k increases monotonically with k in $S \times S$ for $t>\tau$ and*

$$\lim_{k\to\infty} \gamma^k(x,t;\xi,\tau) = \Gamma(x,t;\xi,\tau),$$

where Γ is a representative of the fundamental solution which is continuous as a function of (x,t) in $E^n \times (\tau, T]$ for fixed (ξ,τ), and as a function of (ξ,τ) in $E^n \times [0,t)$ for fixed (x,t).

PROOF. We extend the domain of definition of $\gamma^k(x,t;\xi,\tau)$ by setting $\gamma^k=0$ if either x or ξ are in $\complement\bar{\Sigma}_k$. Let (x,t) and (ξ,τ) be fixed points with $0 \leq \tau < t \leq T$. We assert that

$$\gamma^{k+1}(x,t;\xi,\tau) \geq \gamma^k(x,t;\xi,\tau) \tag{8.7}$$

for all k. The assertion is trivial if for a given k, x or ξ are in $\complement\bar{\Sigma}_k$, and in view of Theorem 9 (ii), also if x or ξ are on $\partial\Sigma_k$. Hence we may assume that x and ξ are in Σ_k. Let $\varphi(y)$ be a non-negative $C_0^\infty(\Sigma_k)$ function such that $\{\xi\} \subset \operatorname{supp}\varphi$ and consider the boundary value problem

$$\begin{cases} Lu = 0 \text{ for } (y,s) \in \Sigma_k \times (\tau, T] \\ u(y,\tau) = \varphi(y) \text{ for } y \in \Sigma_k,\ u(y,s) = 0 \text{ for } (y,s) \in \partial\Sigma_k \times [\tau,T]. \end{cases}$$

By Theorem 1 (iii), this problem has a unique solution u^k for each k, and u^k is continuous in $\bar{\Sigma}_k \times [\tau, T]$. Since $u^{k+1} \geq 0 = u^k$ on $\partial\Sigma_k \times [\tau,T]$ and $u^{k+1}(y,\tau) = u^k(y,\tau)$ for $y \in \Sigma_k$, it follows from Theorem 1 (ii) that $u^{k+1}(x,t) \geq u^k(x,t)$. Thus, in view of Theorem 9 (vi)

$$\int_{\Sigma_{k+1}} \gamma^{k+1}(x,t;y,\tau)\,\varphi(y)\,dy \geq \int_{\Sigma_k} \gamma^k(x,t;y,\tau)\,\varphi(y)\,dy. \tag{8.8}$$

If we now replace φ in (8.8) by a sequence of $C_0^\infty(\Sigma_k)$ functions which approximates the Dirac distribution concentrated at $y = \xi$, it follows that (8.7) holds for x and ξ in Σ_k. According to Theorem 9 (iii) the sequence $\{\gamma^k\}$ is bounded above. Therefore, $\lim_{k\to\infty} \gamma^k$ exists in $S \times S$ for $t > \tau$. In view of Theorems 9 (iv) and C, the sequence $\{\gamma^k(x, t; \xi, \tau)\}$, starting from some sufficiently large k, is bounded and equicontinuous for (x, t) in any compact subset of $E^n \times (\tau, T]$ with (ξ, τ) fixed, and for (ξ, τ) in any compact subset of $E^n \times [0, t)$ with (x, t) fixed. Hence the limit function is continuous as a function of (x, t) and as a function of (ξ, τ) for $t > \tau$.

Let F be an arbitrary function in $L^{p,q}(S)$, where the exponents are finite and satisfy (8.4), and consider the Cauchy problem

$$Lu = F \text{ for } (x, t) \in S,\ u(x, 0) = 0 \text{ for } x \in E^n.$$

By Theorem 5 (ii) the weak solution in $\mathcal{E}^2(S)$ of this problem is given by

$$u(x, t) = \iint_S \Gamma(x, t; \xi, \tau) F(\xi, \tau)\, d\xi\, d\tau.$$

On the other hand, by Corollary 3.1 and Theorem 9 (vi),

$$\iint_S \Gamma(x, t; \xi, \tau) F(\xi, \tau)\, d\xi\, d\tau = \lim_{k\to\infty} \iint_S \gamma^k(x, t; \xi, \tau) F(\xi, \tau)\, d\xi\, d\tau$$

for each $(x, t) \in S$, where we have set $\gamma^k = 0$ for $\xi \in \complement \Sigma_k$. Thus $\gamma^k(x, t; \cdot, \cdot) \to \Gamma(x, t; \cdot, \cdot)$ weakly in $L^{p', q'}(S)$. Since the sequence $\{\gamma^k\}$ converges pointwise it follows that the pointwise limit is also Γ.

We will now prove the analogue of Theorem 9 for the fundamental solution. Suppose that for some $\gamma \geq 0$ we have $e^{-\gamma|x|^2} u_0(x) \in L^2(E^n)$ and $e^{-\gamma|x|^2} G(x, t) \in L^{p,q}(S)$ where p and q satisfy (**). We will show, among other things, that Γ gives a representation formula for the solution of the Cauchy problem

$$(8.9) \qquad Lu = G(x, t) \text{ for } (x, t) \in S,\ u(x, 0) = u_0(x) \text{ for } x \in E^n$$

and for the adjoint problem

$$(8.10) \qquad \widetilde{L}v = G(\xi, \tau) \text{ for } (\xi, \tau) \in S,\ v(\xi, T) = u_0(\xi) \text{ for } \xi \in E^n.$$

In stating the result we use the following notation. If τ is an arbitrary point in $[0, T)$ we set $S_\tau = E^n \times (\tau, T]$, while if t is an arbitrary point in $(0, T]$ we set $\tilde{S}_t = E^n \times [0, t)$.

THEOREM 10. *Suppose that L satisfies (H), and that for some $\gamma \geq 0$ we have $e^{-\gamma |x|^2} u_0 \in L^2(E^n)$ and $e^{-\gamma |x|^2} G \in L^{p,q}(Q)$, where p, q satisfy (**). The weak fundamental solution $\Gamma(x, t; \xi, \tau)$ of $Lu = 0$ has the following properties.*

(i) *$\Gamma(x, t; \xi, \tau) = \tilde{\Gamma}(\xi, \tau; x, t)$ in $S \times S$ for $t > \tau$, where $\tilde{\Gamma}$ is the fundamental solution of $\tilde{L}v = 0$.*

(ii) *The conclusion of Theorem 7 holds for Γ.*

(iii) *For all $\eta \in (\tau, t)$*

$$\Gamma(x, t; \xi, \tau) = \int_{E^n} \Gamma(x, t; \zeta, \eta) \Gamma(\zeta, \eta; \xi, \tau) \, d\zeta.$$

(iv) *For fixed $(\xi, \tau) \in \tilde{S}_T$, $\Gamma(\cdot, \cdot; \xi, \tau) \in L^2[\tau + \delta, T; H^{1,2}(E^n)]$ for arbitrary $\delta \in (0, T - \tau)$ with*

$$\langle\langle \Gamma(\cdot, \cdot; \xi, \tau) \rangle\rangle_{2,2} \leq \mathcal{C} \delta^{-n/4},$$

where the norm is computed over the set $S_{\tau+\delta}$ and $\mathcal{C}$ depends only on T and the structure of L. Moreover, Γ is a weak solution of $\Gamma u = 0$ for $(x, t) \in S_\tau$. For fixed $(x, t) \in S_0$, $\Gamma(x, t; \cdot, \cdot) \in L^2[0, t - \delta; H^{1,2}(E^n)]$ for arbitrary $\delta \in (0, t)$ with

$$\langle\langle \Gamma(x, t; \cdot, \cdot) \rangle\rangle_{2,2} \leq \mathcal{C} \delta^{-n/4},$$

where the norm is computed over the set $\tilde{S}_{t-\delta}$ and $\mathcal{C}$ depends only on T and the structure of L. Moreover Γ is a weak solution $\tilde{L}v = 0$ for $(\xi, \tau) \in \tilde{S}_t$.

(v) *For fixed $(\xi, \tau) \in \tilde{S}_T$ let Σ denote an arbitrary bounded open domain such that $\bar{\Sigma} \subset E^n \setminus \{\xi\}$. Then Γ is a global weak solution of $Lu = 0$ for $(x, t) \in \Sigma \times (\tau, T]$ with initial values zero on $t = \tau$. For fixed $(x, t) \in S_0$ let Σ denote an arbitrary bounded open domain such that $\bar{\Sigma} \subset E^n \setminus \{x\}$. Then Γ is a global solution of $\tilde{L}v = 0$ for $(\xi, \tau) \in \Sigma \times [0, t)$ with initial values zero on $\tau = t$.*

(vi) *Suppose that $T \leq 1/16\alpha_2\gamma$, where α_2 is the constant obtained in Theorem 7. Then the weak solution in $\mathcal{E}^2(S)$ of the Cauchy problem (8.9) is given by*

$$u(x, t) = \int_{E^n} \Gamma(x, t; \xi, 0) u_0(\xi) \, d\xi + \iint_S \Gamma(x, t; \xi, \tau) \, G(\xi, \tau) \, d\xi \, d\tau$$

and the weak solution in $\mathcal{E}^2(S)$ of the Cauchy problem (8.10) *is given by*

$$v(\xi, \tau) = \int_{E^n} \Gamma(x, T; \xi, \tau)\, u_0(x)\, dx + \iint_S \Gamma(x, t; \xi, \tau)\, G(x, t)\, dx\, dt.$$

If u_0 is continuous at a point $y \in E^n$ then

$$\lim_{(x,t) \to (y,0)} u(x, t) = \lim_{(\xi,\tau) \to (y,T)} v(\xi, \tau) = u_0(y).$$

Proof. Let F be an arbitrary $L^{p,q}(S)$ function for (p, q) satisfying (8.4) and consider the Cauchy problem

$$Lu = F \text{ for } (x, t) \in S,\ u(x, 0) = 0 \text{ for } x \in E^n.$$

According to Theorem 5 (ii) the weak solution in $\mathcal{E}^2(S)$ of this problem is given by

$$u(x, t) = \iint_S \Gamma(x, t; \xi, \tau)\, F(\xi, \tau)\, d\xi\, d\tau.$$

On the other hand, by Corollary 3.3 and Theorem 5 (ii)

$$\iint_S \Gamma(x, t; \xi, \tau)\, F(\xi, \tau)\, d\xi\, d\tau = \lim_{m \to \infty} \iint_S \Gamma^m(x, t; \xi, \tau)\, F^m(\xi, \tau)\, d\xi\, d\tau,$$

where Γ^m is the fundamental solution of $L^m u = 0$. Since $F^m \longrightarrow F$ strongly in $L^{p,q}(S)$ it follows that $\Gamma^m(x, t; \cdot, \cdot) \longrightarrow \Gamma(x, t; \cdot, \cdot)$ weakly in $L^{p',q'}(S)$ for each $(x, t) \in S_0$. Similarly, $\Gamma^m(\cdot, \cdot\, ; \xi, \tau) \longrightarrow \widetilde{\Gamma}(\xi, \tau; \cdot, \cdot)$ weakly in $L^{p',q'}(S)$ for each $(\xi, \tau) \in \widetilde{S}_T$.

Let $(\xi, \tau) \in \widetilde{S}_T$ be fixed. For arbitrary $\delta \in (0, T - \tau)$, $\Gamma^m(x, t; \xi, \tau)$ is the classical solution in $\mathcal{E}^2(S)$ of the Cauchy problem

$$L^m u = 0 \text{ for } (x, t) \in S_{\tau+\delta},\ u(x, \tau + \delta) = \Gamma^m(x, \tau + \delta; \xi, \tau) \text{ for } x \in E^n.$$

Since Γ^m is continuosly differentiable in $\bar{S}_{\tau+\delta}$ it is also the weak solution of this problem. For $R \geq 1$ let $\zeta_R = \zeta_R(x)$ be a $C_0^\infty(E^n)$ function such that $\zeta_R = 1$ for $|x| \leq R$, $\zeta_R = 0$ for $|x| \geq R + 1$, $0 \leq \zeta_R \leq 1$ and $|\zeta_{Rx}|$ bounded independent of R. By Lemma 1, with $\zeta = \zeta_R$, $\xi = 0$, $s = \tau + \delta$ and

$\mu = +\infty$,

$$\| \zeta_R \Gamma^m (\cdot, \cdot\, ; \xi, \tau) \|_{2,\infty}^2 + \| \zeta_R \Gamma_x^m (\cdot, \cdot\, ; \xi, \tau) \|_{2,2}^2 \leq$$

$$\leq \mathcal{C} \{ \| \zeta_R \Gamma^m (\cdot, \tau + \delta ; \xi, \tau) \|_{L^2\, E^n}^2 + \| \zeta_{Rx} \Gamma^m (\cdot, \cdot\, ; \xi, \tau) \|_{2,2}^2 \},$$

where the norms are computed over the set $S_{\tau+\delta}$. Here and elsewhere in this proof $\mathcal{C}$ stands for any constant which depends only on T and the structure of L. According to Theorem 7 we have $\| \Gamma^m \|_{L^2(E^n)}^2 \leq \mathcal{C}\, \delta^{-n/2}$ and $\|\Gamma^m\|_{2,2}^2 \leq \mathcal{C}\, \delta^{-n/2}$. Thus $\| \zeta_{Rx} \Gamma^m |_{2,2} \to 0$ as $R \to \infty$ and it follows that

$$(8.11) \qquad \| \Gamma^m (\cdot, \cdot ;\ \xi, \tau) \|_{2,\infty}^2 + \| \Gamma_x^m (\cdot, \cdot ;\ \xi, \tau) |_{2,2}^2 \leq \mathcal{C} \delta^{-n/2}.$$

The results of the last paragraph together with (8.11) and Lemma 2 imply that $\Gamma^m (\cdot, \cdot\, ; \xi, \tau) \to \tilde{\Gamma}(\xi, \tau\, ; \cdot, \cdot)$ weakly in $L^2[\tau + \delta, T\, ; H^{1,2}(E^n)]$ and any finite collection of $L^{2p', 2q'}(Q)$ spaces, where Q is any bounded cylinder in $S_{\tau+\delta}$. Thus, in particular, $\tilde{\Gamma}(\xi, \tau\, ; x, t)$ is a weak solution of $Lu = 0$ for $(x, t) \in S_\tau$ and $\langle\langle \tilde{\Gamma}(\xi, \tau\, ; \cdot, \cdot) \rangle\rangle_{2,2} \leq \mathcal{C}\delta^{-n/4}$. By a similar argument, we find that for fixed $(x, t) \in S_0$, $\Gamma(x, t\, ; \xi, \tau)$ is a weak solution of $\tilde{L}u = 0$ for $(\xi, \tau) \in \tilde{S}_t$ and $\langle\langle \Gamma(x, t\, ; \cdot, \cdot) \rangle\rangle_{2,2} \leq \mathcal{C}\delta^{-n/4}$.

In view of Theorem 7 and Theorem C, the sequence $\{\Gamma^m\}$ is uniformly bounded and equicontinuous in any compact subset of S_τ for fixed $(\xi, \tau) \in \tilde{S}_T$. Hence $\Gamma^m \to \tilde{\Gamma}$ uniformly for (x, t) in any compact subset of S_τ. Similarly, $\Gamma^m \to \Gamma$ uniformly for (ξ, τ) in any compact subset of $\tilde{S}_\tau$ for fixed $(x, t) \in S_0$. Therefore (i), (ii) and (iv) hold. The assertion (ii) follows directly from the corresponding result for Γ^m using the pointwise convergence of Γ^m to Γ, Theorem 7 and the dominated convergence theorem. The proof of (v) is almost identical to the proof of the Theorem 9 (v) and we omit the details.

According to Corollary 3.1 and Theorem 9 (vi), if u is the solution $\mathcal{E}^2(S)$ of problem (3.9) then for each $(x, t) \in S$ we have $u^k(x, t) \to u(x, t)$ where for k sufficiently large

$$u^k (x, t) = \int_\Omega \gamma^k (x, t\, ;\ \xi, 0)\, u_0 (\xi)\, d\xi + \iint_S \gamma^k (x, t\, ;\ \xi, \tau)\ G(\xi, \tau)\, d\xi\, d\tau.$$

Hence γ^k is the Green's function for $Lu = 0$ in $Q_k = \Sigma_k \times (0, T]$ and we have set $\gamma^k = 0$ for $\xi \subset \complement \Sigma_k$. By Lemma 7, $\gamma^k u_0 \to \Gamma u_0$ for almost all $\xi \in E^n$ and $\gamma^k G \to \Gamma G$ for almost all $(\xi, \tau) \in S$. In view of Theorem 9 (iii),

$\gamma^k(x, t; \xi, \tau) \le \mathcal{C} g_\varkappa(x-\xi, t-\tau)$, where $g_\varkappa$ is the fundamental solution of $\varkappa \Delta u = u_t$ and $\varkappa = \alpha_2$ is independent of k. It is easily verified that if $T \le 1/16\,\alpha_2\gamma$ then

$$(8.12) \qquad e^{\gamma|\xi|^2} g_\varkappa(x-\xi, t-\tau) \le 2^{n/2} e^{2|x|^2} g_{2\varkappa}(x-\xi, t-\tau).$$

For fixed $(x, t) \in S$, $g_{2\varkappa}(x-\xi, t) \in L^2(E^n)$ and $g_{2\varkappa}(x-\xi, t-\tau) \in L^{p', q'}(S)$. Hence $|\gamma^k u_0| \le \mathcal{C} g_\varkappa |u_0| \in L^1(E^n)$ and $|\gamma^k G| \le \mathcal{C} g_\varkappa |G| \in L^{1,1}(S)$. It follows from the dominated convergence theorem that

$$u(x, t) = \lim_{k \to \infty} u^k(x, t) = \int_\Omega \Gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi + \iint_S \Gamma(x, t; \xi, \tau)\, G(\xi, \tau)\, d\xi\, d\tau$$

for each $(x, t) \in S$. The proof of the corresponding statement for the adjoint equation is similar and we omit it.

The proof that $u(x, t) \to u_0(y)$ at points of continuity of u_0 is essentially the same as the proof of the corresponding part of Theorem 9. Here we must show that for $|x - y| < \varrho/2$ the integral

$$J_2 = \int_{|\xi - y| \ge \varrho} \Gamma(x, t; \xi, 0)\, \{u_0(\xi) - u_0(y)\}\, d\xi$$

can be made small by taking t sufficiently small. However, in view of (8.12) if $t < 1/32\,\varkappa\gamma$ then

$$|J_2| \le \mathcal{C} e^{2|x|^2} g_\varkappa(\varrho/2, . t)\, \| e^{-\gamma|\cdot|^2} \{u_0(\cdot) - u_0(y)\} \|_{L^2(E^n)} \int_{E^n} g_{2\varkappa}(x-\xi, t)\, d\xi \le$$

$$\le \mathcal{C} e^{2|x|^2} g_\varkappa(\varrho/2, t)\, \| e^{-\gamma|\cdot|^2} \{u_0(\cdot) - u_0(y)\} \|_{L^2(E^n)},$$

and the assertion follows. The proof that $v(\xi, \tau) \to u_0(y)$ is analogous.

We have shown that if u_0 is continuous at x then

$$\lim_{t \to 0} \int_{E^n} \Gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi = u_0(x).$$

In the next section we will also need the corresponding result when the integration is taken with respect to x.

LEMMA 8. *Suppose that L satisfies (H) and $e^{-\gamma|x|^2} u_0 \in L^2(E^n)$ for some $\gamma \geq 0$. Then, if u_0 is continuous at ξ,*

$$\lim_{t \to 0} \int_{E^n} \Gamma(x, t;\ \xi, 0)\, u_0(x)\, dx = u_0(\xi).$$

PROOF. In view of what we have shown in the proofs of Theorems 9 (vi) and 10 (vi) it suffices to prove that

$$\lim_{t \to 0} \int_{E^n} \Gamma(x, t;\ \xi, 0)\, dx = 1$$

for all $\xi \in E^n$. For $t \in (0, T]$ consider the family of Cauchy ploblems

$$\widetilde{L}v = 0 \quad \text{for } (y, s) \in \widetilde{S}_t,\ v(y, t) = 1 \quad \text{for } y \in E^n.$$

According to Theorem 3 and Lemma 5, for each $t \in (0, T]$ this problem has a unique weak solution $v(y, s;\ t)$ in the class $\mathcal{E}^2(\widetilde{S}_t)$ and

$$0 \leq v(y, s;\ t) \leq \mathcal{C}$$

in $\widetilde{S}_t$, where $\mathcal{C}$ depends on T and the structure of L. Extend the domain of definition of v by setting $v = 1$ for $s > t$. Then, by the Extension Principle and Theorem C, the extended function is continuous in $\bar{S}$ and for $(y, s) \in E^n \times [0, T/2]$ we have

$$v(y, s;\ t) - \mathcal{C}s^{\alpha/2} \leq v(y, 0;\ t) \leq v(y, s;\ t) + \mathcal{C}s^{\alpha/2},$$

where $\mathcal{C}$ depends only on T and the structure of L. In particular, if $t \in [0, T/2]$ then setting $s = t$ we obtain

$$1 - \mathcal{C}t^{\alpha/2} \leq v(y, 0;\ t) \leq 1 + \mathcal{C}t^{\alpha/2}.$$

Since, by Theorem 10 (vi),

$$v(y, 0;\ t) = \int_{E^n} \Gamma(x, t;\ y, 0)\, dx$$

the assertion follows.

9. Non-negative Weak Solutions.

In section 5 we characterized a non-negative weak solution of the Cauchy problem

$$Lu = 0 \quad \text{for } (x, t) \in S, \; u(x, 0) = u_0(x) \quad \text{for } x \in E^n \tag{9.1}$$

as the limit of a sequence of solutions of certain boundary value problems. Here we will re-examine this result in the light of the results which we have obtained concerning the weak Green's functions and fundamental solution for L. We show that if problem (9.1) admits a non-negative solution u then u is given by an integral involving Γ and u_0. Moreover, we will use this result to derive a representation formula for any non-negative weak solution of $Lu = 0$ in S.

THEOREM 11. *Suppose that L satisfies (H) and $u_0 \in L^2_{\mathrm{loc}}(E^n)$ is non-negative almost everywhere. If u is the non negative weak solution of problem* (9.1) *then*

$$u(x, t) = \int_{E^n} \Gamma(x, t\,;\, \xi, 0)\, u_0(\xi)\, d\xi \tag{9.2}$$

for all $(x, t) \in S$.

PROOF. According to Theorem 4 and Theorem 9 (vi)

$$u(x, t) = \lim_{k \to \infty} \int_{E^n} \gamma^k(x, t\,;\, \xi, 0)\, \zeta_k(\xi)\, u_0(\xi)\, d\xi,$$

where γ^k is the Green's function for $Lu = 0$ in $Q_k = \Sigma_k \times (0, T]$ and we have set $\gamma^k = 0$ for $\xi \in \complement \Sigma_k$. Recall that ζ_k is a $C_0^\infty(E^n)$ function such that $\zeta_k = 1$ for $|x| \leq k - 2$, $\zeta_k = 0$ for $|x| \geq k - 1$ and $0 \leq \zeta_k \leq 1$. In view of Lemma 7, the sequence $\{\gamma^k(x, t\,;\, \xi, 0)\, \zeta_k(\xi)\, u_0(\xi)\}$ is monotone increasing and converges to $\Gamma(x, t\,;\, \xi, 0)\, u_0(\xi)$ for almost all $\xi \in E^n$. Thus it follows from the monotone convergence theorem that u is given by (9.2).

Corresponding to Corollary 4.2 we have the following corollary to Theorem 11.

COROLLARY 11.1. *Suppose that L satisfies (H), $u_0 \in L^2_{\mathrm{loc}}(E^n)$, and $u_0 \geq 0$ almost everywhere in E^n. The Cauchy problem* (9.1) *possesses a non negative* *weak solution if and only if*

$$\int_{E^n} \Gamma(x, t;\ \xi, 0)\, u_0(\xi)\, d\xi \in L^2[0, T;\ L^2_{\mathrm{loc}}(E^n)]. \tag{9.3}$$

The non-negative solution u if it exists is given by (9.2).

PROOF. If there is a non-negative weak solution u of problem (9.1) then $u \in L^2[0, T; L^2_{\mathrm{loc}}(E^n)]$. By Theorem 11, u is given by (9.2) and hence (9.3) holds. On the other hand, by Lemma 7,

$$u^k(x, t) = \int_{\Sigma_k} \gamma^k(x, t;\ \xi, 0)\, \zeta_k\, u_0\, d\xi \leq \int_{E^n} \Gamma(x, t:\ \xi, 0)\, u_0\, d\xi.$$

Thus, it follows from Corollary 4.2, that if (9.3) is satisfied then the non-negative solution u of problem (9.1) exists and $u(x, t) = \lim_{k \to \infty} u^k(x, t)$. Therefore, as was shown in the proof of Theorem 11, u is given by (9.2).

THEOREM 12. *Suppose that L satisfies (H). If u is a non-negative weak solution of $Lu = 0$ in S, then there exists a unique non-negative Borel measure ϱ such that for all $(x, t) \in S$*

$$u(x, t) = \int_{E_n} \Gamma(x, t;\ \xi, 0)\, \varrho(d\xi). \tag{9.4}$$

The existence part of Theorem 12 was first proved by Widder [22] for classical solutions of the equation of heat conduction and subsequently extended for classical solutions of general second order parabolic equations with sufficiently smooth coefficients by Krzyżański [14]. Our existence proof is almost identical to Krzyżański's. Widder also gives a sufficient condition for a function defined by a formula such as (9.4) to be a solution of the equation of heat conduction. Our version of this result is given below in Corollary 12.1. The uniqueness proof which we give here is an adaptation of Frostman's proof that if the Newtonian potential due to a signed measure is almost everywhere zero then the measure is identically zero [11; pp 31-33].

PROOF. For any $s \in (0, T]$, we know that $u(x, s) \in L^2_{\mathrm{loc}}(E^n)$ and $u(x, t)$ is a non-negative weak solution of the Cauchy problem

$$Lv = 0 \ \text{ for } (x, t) \in E^n \times (s, T],\ \ v(x, s) = u(x, s) \quad \text{for } x \in E^n.$$

Thus, by Theorem 11,

$$u(x, t) = \int_{E_n} \Gamma(x, t; \xi, s)\, u(\xi, s)\, d\xi$$

for all $(x, t) \in E^n \times (s, T]$. If we restrict s to the interval $(0, T/2]$ then it follows from Theorem 10 (ii) that

$$\int_{E_n} e^{-\gamma|\xi|^2} u(\xi, s)\, d\xi \leq \mathcal{C}\, u(0, T), \tag{9.5}$$

where $\gamma = 1/2\alpha_1 T$ and $\mathcal{C}$ depends only on T and the structure of L. For each $s \in (0, T/2]$ define the Borel measure

$$\hat{\varrho}_s(E) = \int_E e^{-2\gamma|\xi|^2} u(\xi, s)\, d\xi.$$

In view of (9.5)

$$\hat{\varrho}_s(E) \leq \hat{\varrho}_s(E^n) \leq \mathcal{C}\, u(0, T) \tag{9.6}$$

for all Borel subsets E of E^n and

$$\hat{\varrho}_s(|x| \geq R) \leq \mathcal{C}\, e^{-\gamma R^2} u(0, T) \tag{9.7}$$

for all $s \in (0, T/2]$. According to the Frostman selection theorem [11; pp 11-13], (9.6) implies that there exists a sequence $s_j \to 0$ such that the corresponding measures $\hat{\varrho}_j = \hat{\varrho}_{s_j}$ converge to a Borel measure $\hat{\varrho}$. In particular,

$$\lim_{j \to \infty} \int_{E_n} f(\xi)\, \hat{\varrho}_j(d\xi) = \int_{E_n} f(\xi)\, \hat{\varrho}(d\xi)$$

for any $f \in C_0^0(E^n)$.

Hold $(x, t) \in S$ fixed and consider the functions

$$W_j(\xi) = \Gamma(x, t; \xi, s_j)\, e^{2\gamma|\xi|^2}$$

for $s_j < t$. By Theorem 10 (ii), if $0 < 8\alpha_2 (t - s)\gamma < 1$ we have

$$W_j(\xi) \leq \mathcal{C}\, \{4\pi\alpha_2 (t - s_j)\}^{-n/2} \exp\, [2\gamma\, |x|^2/\{1 - 8\alpha_2\, \gamma\, (t - s_j)\}].$$

Thus if $0 < s_j \leq t/2 < t \leq \tau = \alpha_1 T/8\alpha_2$

$$W_j(\xi) \leq C\, t^{-n/2} \exp \{2\,|x|^2/\alpha_1 T\}.$$

Moreover, it is clear that $W_j(\xi) \in C^0(E^n)$ for each j and $W_j(\xi) \to \Gamma(x, t; \xi, 0)\, e^{2\gamma|\xi|^2}$ uniformly for $|\xi| \leq R$. By Theorem 3 of reference [14], it follows from (9.7) and the properties of the sequence $\{W_j\}$ that

$$\lim_{j\to\infty} \int_{E_n} W_j(\xi)\, \hat{\varrho}_j(d\xi) = \int_{E_n} \Gamma(x, t;\ \xi, 0)\, e^{2\gamma|\xi|^2}\, \hat{\varrho}\,(d\xi).$$

Set

$$\varrho\,(E) = \int_{E} e^{2\gamma|\xi|^2}\, \hat{\varrho}\,(d\xi).$$

Then, since for sufficiently large j

$$\int_{E_n} W_j\,(\xi)\, \hat{\varrho}_j(d\xi) = u\,(x, t),$$

we have

$$u\,(x, t) = \int_{E^n} \Gamma(x, t;\ \xi, 0)\, \varrho\,(d\xi)$$

for $t \in (0, \tau]$. If $t > \tau$, then by Theorem 10 (iii), Theorem 11 and Fubini's theorem

$$u\,(x, t) = \int_{E^n} \Gamma\,(x, t;\ \zeta, \tau)\, u\,(\zeta, \tau)\, d\zeta$$

$$= \int_{E^n} \Gamma(x, t;\ \zeta, \tau) \left\{ \int_{E^n} \Gamma\,(\zeta, \tau;\ \xi, 0)\, \varrho\,(d\xi) \right\} d\zeta =$$

$$\int_{E^n} \left\{ \int_{E^n} \Gamma(x, t;\ \zeta, \tau)\, \Gamma(\zeta, \tau;\ \xi, 0)\, d\zeta \right\} \varrho\,(d\xi) = \int_{E^n} \Gamma\,(x, t;\ \xi, 0)\, \varrho\,(d\xi),$$

and the existence part of the Theorem is proved.

If u is a non-negative solution of $Lu = 0$ and ϱ is a Borel measure such that u is given by (9.4), then it follows from Theorem 10 (ii) that

$$\int_{E^n} e^{-\gamma|\xi|^2} \varrho(d\xi) \leq \mathcal{C}\, u(0, T), \tag{9.8}$$

where $\gamma = 1/4\alpha_1 T$, and $\mathcal{C}$ depends only on T and the structure of L.

Now suppose that there are two measures ϱ_1 and ϱ_2 each giving a representation of u in the form (9.4) and such that $\varrho_1 \not\equiv \varrho_2$. Set $\sigma = \varrho_1 - \varrho_2$. Then

$$0 = \int_{E^n} \Gamma(x, t;\ \xi, 0)\, \sigma(d\xi) = \int_{E^n} \Gamma(x, t;\ \xi, 0)\, e^{\gamma|\xi|^2} \lambda(d\xi) \tag{9.9}$$

for all $(x, t) \in S$, where

$$\lambda(E) = \int_E e^{-\gamma|\xi|^2} \sigma(d\xi).$$

By the Hahn-Jordan decomposition theorem there is a Borel set A such that $\lambda \geq 0$ on all Borel measurable subsets of A and $\lambda \leq 0$ on all Borel measurable subsets of $B = \complement A$. Set $\lambda^+(E) = \lambda(E \cap A)$ and $\lambda^-(E) = -\lambda(E \cap B)$. Then λ^+ and λ^- are mutually singular and $\lambda = \lambda^+ - \lambda^-$. Moreover,

$$\lambda^+(E) \leq \int_E e^{-\gamma|\xi|^2} \varrho_1(d\xi) \text{ and } \lambda^-(E) \leq \int_E e^{-\gamma|\xi|^2} \varrho_2(d\xi).$$

In particular, it follows from (9.8) that $\lambda^\pm(E^n) \leq \mathcal{C}\, u(0, T)$. Therefore the measures λ^+ and λ^- are regular [18; Theorem 2.18].

Since $\sigma \not\equiv 0$ we have either $\lambda^+(A)$ or $\lambda^-(B)$ positive. Suppose that $\lambda^+(A) = a > 0$. We will show that this leads to a contradiction. In view of the regularity of λ^+ and λ^-, there exists a compact set $K \subset A$ such that $\lambda(K) = \lambda^+(K) > 3a/4$ and a bounded open set $E \supset K$ such that $\lambda^-(E) = \lambda^-(E \setminus K) < a/4$. Let $\zeta = \zeta(x)$ be a continuous function in E^n with $\zeta = 1$ on K, $\zeta = 0$ on $\complement E$ and $0 \leq \zeta \leq 1$. Define

$$v(\xi, t) = \int_{E^n} \Gamma(x, t;\ \xi, 0)\, \zeta(x)\, e^{-\gamma|x|^2} dx.$$

By Lemma 8 we have

$$\lim_{t \to 0} v(\xi, t) = \zeta(\xi)\, e^{-\gamma|\xi|^2} \tag{9.10}$$

for all $\xi \in E^n$. On the other hand, if $0 < t \leq 1/16\alpha_2 \gamma$ and $\bar{E} \subset \{x;\ |v| \leq r\}$ then by Theorem 10 (ii)

$$(9.11) \qquad 0 \leq e^{\gamma |\xi|^2} v(\xi, t) \leq$$

$$\mathcal{C} (4\pi\alpha_2 t)^{-n/2} \int_{|x| \leq r} \exp \left\{ -\frac{|x-\xi|^2}{8\alpha_2 t} + \gamma |x|^2 \right\} dx \leq \mathcal{C}\, e^{\gamma r^2}$$

Thus, it follows from (9.8) and (9.11) that

$$(9.12) \qquad \int_{E^n} v(\xi, t)\, |\sigma|\, (d\xi) = \int_{E^n} e^{\gamma |\xi|^2} v(\xi, t)\, |\lambda|\, (d\xi) \leq \mathcal{C}\, e^{\gamma r^2}\, u(0, T).$$

Hence v is integrable with respect to σ for each $t \in (0, 1/16\alpha_2 \gamma)$ and

$$\int_{E^n} v(\xi, t)\, \sigma(d\xi) = \int_{E^n} e^{\gamma |\xi|^2} v(\xi, t)\, \lambda(d\xi).$$

Since $|\lambda|(E^n) < \infty$, we conclude from (9.10), (9.11) and the dominated convergence theorem that

$$\lim_{t \to 0} \int_{E^n} v(\xi, t)\, \sigma(d\xi) = \int_{E^n} \zeta(\xi)\, \lambda(d\xi) = \lambda(K) + \int_{E \setminus K} \zeta \lambda^+ (d\xi) - \int_{E\ K} \zeta \lambda^- (d\xi).$$

Therefore

$$(9.13) \qquad \lim_{t \to 0} \int_{E^n} v(\xi, t)\, \sigma(d\xi) > \frac{3a}{4} - \frac{a}{4} = \frac{a}{2} > 0.$$

For fixed $t > 0$, $\Gamma(x, t; \xi, 0)$ is the pointwise limit of a sequence of classical fundamental solutions $\Gamma^m(x, t;\ \xi, 0)$. The Γ^m are jointly continuous in (x, ξ). Hence the Γ^m are Borel measurable in $E^n \times E^n$ and the same is true for their limit Γ. Therefore, in view of (9.12), for each $t \in (0, 1/16\alpha_2 \gamma)$ we have

$$\int_{E^n} v(\xi, t)\, \sigma(d\xi) = \int_{E^n} \left\{ \int_{E^n} \Gamma(x, t;\ \xi, 0)\, \zeta(x)\, e^{-\gamma |x|^2}\, dx \right\} \sigma(d\xi) =$$

$$\int_{E^n} \left\{ \int_{E^n} \Gamma(x, t;\ \xi, 0)\, \sigma(d\xi) \right\} \zeta(x)\, e^{-\gamma |x|^2}\, dx = 0$$

and

$$\lim_{t \to 0} \int_{E^n} v(\xi, t)\, \sigma(d\xi) = 0$$

in contradiction to (9.13). Thus we cannot have $\lambda^+(A) > 0$. A similar argument shows that $\lambda^-(B) = 0$. Therefore $\lambda \equiv 0$ and the same is true for σ.

Corollary 12.1. *Suppose that L satisfies (H) and let*

$$u(x, t) = \int_{E^n} \Gamma(x, t;\ 0)\, \varrho\, (d\xi),$$

where ϱ is a non-negative Borel measure. Then u is a weak solution of $Lu = 0$ in S if and only if $u \in L^2[\delta, T;\ L^2_{loc}(E)]$ for every $\delta \in (0, T)$.

Proof. If u is a weak solution of $Lu = 0$ in S then it clearly has the required properties. To prove the converse, note that $u \in L^2[\delta, T;\ L^2_{loc}(E^n)]$ for every $\delta \in (0, T)$ implies there exists a sequence $\{t_j\}$ of points in $(0, T)$ such that $t_j \to 0$ and $u(\cdot, t_j) \in L^2_{loc}(E^n)$. Consider the Cauchy problem

$$\begin{cases} Lv = 0 \text{ for } (x, t) \in E^n \times (t_j, T], \\ v(x, t_j) = u(x, t_j) \text{ for } x \in E^n. \end{cases}$$

Since $u(\cdot, t_j) \in L^2_{loc}(E^n)$, it follows from Corollary 11.1 that this problem has a non-negative weak solution if and only if

$$\int_{E^n} \Gamma(x, t; \xi, t_j)\, u(\xi, t_j)\, d\xi \in L^2[t_j, T; L^2_{loc}(E^n)].$$

By Theorem 10 (iii) we have

$$\int_{E^n} \Gamma(x, t; \xi, t_j)\, u(\xi, t_j)\, d\xi = \int_{E^n} \Gamma(x, t; \xi, t_j) \left\{ \int_{E^n} \Gamma(\xi, t_j; \zeta, 0)\, \varrho\, (d\zeta) \right\} d\xi$$

$$= \int_{E^n} \left\{ \int_{E^n} \Gamma(x, t; \xi, t_j)\, \Gamma(\xi, t_j; \zeta, 0)\, d\xi \right\} \varrho\, (d\zeta) = u(x, t).$$

Thus, u is a weak solution in $E^n \times (t_j, T]$ for any j. Since $t_j \to 0$ we conclude that u is a weak solution in S.

We now consider a non-negative solution of $Lu = 0$ in S which has initial values on the hyperplane $t = 0$ except at an isolated point. Roughly speaking, the result is that such a solution is equal to a non-negative multiple of the fundamental solution with singularity at the point in question plus a non-singular solution. Thus, for non-negative solutions, any isolated singular point at which the singularity is weaker than that of the fundamental solution is a removable singular point.

COROLLARY 12.2. *Suppose that L satisfies (H), $u_0 \in L^1_{\text{loc}}(E^n)$ and $u_0 \geq 0$ almost everywhere in E^n. Let u be a non-negative weak solution of $Lu = 0$ in S such that*

$$\lim_{t \to 0} \int_{E^n} u(x, t)\, \psi(x)\, dx = \int_{E^n} u_0(x)\, \psi(x)\, dx \tag{9.14}$$

for every $\psi \in C^0_0(E^n \setminus \{0\})$. Then there exists a constant $\eta \geq 0$ such that

$$u(x, t) = \eta\, \Gamma(x, t; 0, 0) + \int_{E^n} \Gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi.$$

Note that if $u_0 \in L^2_{\text{loc}}(E^n)$ and

$$\hat{u}(x, t) = \int_{E^n} \Gamma(x, t; \xi, 0)\, u_0(\xi)\, d\xi \in L^2[0, T; L^2_{\text{loc}}(E^n)]$$

then, in view of Corollary 11.1,

$$u(x, t) = \eta\, \Gamma(x, t; 0, 0) + \hat{u}(x, t),$$

where $\hat{u}$ is the non-negative solution of the Cauchy problem

$$Lu = 0 \text{ for } (x, t) \in S,\; u(x, 0) = u_0(x) \text{ for } x \in E^n.$$

PROOF. It suffices to show that

$$\varrho(E) = \eta \delta_0(E) + \int_E u_0(x)\, dx$$

for every Borel set $E \subset E^n$, where δ_0 denotes the Dirac measure concentrated at $x = 0$. Let ψ be a non-negative $C^0_0(E^n \setminus \{0\})$ function. Then, by

Theorem 12,

$$\int_{E^n} u(x,t)\,\psi(x)\,dx = \int_{E^n}\Big\{\int_{E^n} \Gamma(x,t;\xi,0)\,\varrho(d\xi)\Big\}\,\psi(x)\,dx =$$
$$\int_{E^n}\Big\{\int_{E^n} \Gamma(x,t,\xi,0)\,\psi(x)\,dx\Big\}\,\varrho(d\xi).$$

In view of Lemma 8

$$\lim_{t\to 0}\int_{E^n} \Gamma(x,t;\xi,0)\,\psi(x)\,dx = \psi(\xi)$$

for all $\xi \in E^n$. Moreover, if $0 < t \leq 1/16\,\alpha_2\gamma$ and $\operatorname{supp}\psi \subset \{x; |x| \leq r\}$ then, by Theorem 10 (ii)

$$0 \leq e^{\gamma|\xi|^2}\int_{E^n} \Gamma(x,t;\xi,0)\,\psi(x)\,dx \leq \mathcal{C}'(\max\psi)\,e^{2\gamma r^2}.$$

Thus it follows from (9.14) and the dominated convergence theorem that

$$\int_{E^n} u_0(x)\,\psi(x)\,dx = \int_{E^n} \psi(\xi)\,\varrho(d\xi) \tag{9.15}$$

for any non-negative $\psi \in C_0^0(E^n \setminus \{0\})$.

In view of (9.8), for any compact set K we have $\varrho(K) < \infty$. Thus ϱ is regular [18; Theorem 2.18]. Moreover, since $u_0 \geq 0$ and $u_0 \in L^1_{\text{loc}}(E^n)$ the Borel measure

$$u_0(E) = \int_E u_0(x)\,dx$$

is also regular. Let E be any Borel set with compact closure in $E^n \setminus \{0\}$ Let K be any compact set contained in E, and for given $\varepsilon > 0$ let G be an open set with compact closure in $E^n \setminus \{0\}$ such that $E \subset G$ and $\varrho(G \setminus E) < \varepsilon$. If ψ is a $C^0(E^n)$ function such that $\psi = 1$ on K, $\psi = 0$ on $\complement G$ and $0 \leq \psi \leq 1$ then by (9.15), $u_0(K) < \varrho(E) + \varepsilon$. Taking the supremum over all $K \subset E$ and noting that ε is arbitrary, we obtain $u_0(E) \leq \varrho(E)$. On the other hand, if we choose G such that $u_0(G \setminus E) < \varepsilon$ and apply the same argument we find that $\varrho(E) \leq u_0(E)$. Therefore

$$\varrho(E) = \int_E u_0(x)\,dx \tag{9.16}$$

for all Borel sets E with compact closure in $E^n \setminus \{0\}$.

Let E be an arbitrary Borel set in $E^n \setminus \{0\}$. Then

$$E = \bigcup_{j=1}^{\infty} D_j,$$

where

$$D_j = E \cap [\{1/j + 1 < |x| \leq 1/j\} \cup \{j < |x| \leq (j+1)\}].$$

Each D_j is a Borel set with compact closure in $E^n \setminus \{0\}$ and the D_j are disjoint. It follows from (9.16) that

$$\varrho(E) = \sum_{j=1}^{\infty} \varrho(D_j) = \sum_{j=1}^{\infty} \int_{D_j} u_0 \, dx = \int_E u_0 \, dx.$$

Therefore, if ϱ has a singular part it must be concentrated on $\{0\}$ and ϱ has the form

$$\varrho(E) = \eta \delta_0(E) + \int_E u_0 \, dx$$

for some constant η. Suppose that $\eta < 0$. Then there exists an $r > 0$ such that for $E = \{x; |x| < r\}$ we have

$$\int_E u_0(x) \, dx < |\eta|/2.$$

Thus $\varrho(E) < \eta + |\eta|/2 = \eta/2 < 0$ which contradicts the non-negativity of ϱ.

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