PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 6, June 1998, Pages 1645–1653
S 0002-9939(98)04480-3

NON-NOETHERIAN REGULAR RINGS OF DIMENSION 2

JAMES J. ZHANG

(Communicated by Ken Goodearl)

ABSTRACT. We study connected, not necessarily noetherian, regular rings of global dimension 2.

0. INTRODUCTION

Throughout k is a field and A is a connected k-algebra; thus $A = \bigoplus_{n\geq 0} A_n$ and $A_0 = k$. The **augmentation ideal** is the unique maximal graded ideal $\mathfrak{m} := \bigoplus_{i>0} A_i$. We call $A/\mathfrak{m}$ the **trivial module**, and usually denote it by k. The (left and right) global dimension of A is equal to the projective dimension of k_A. The algebra A is **regular** if it has finite global dimension, d say, and

$$\operatorname{Ext}^i(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

In some papers this is called Artin-Schelter regular. In the definition we do not assume that A is finitely generated or even locally finite, but it is proved in [9, 3.1] that a regular ring is finitely generated. Also we do not require $\operatorname{GKdim} A < \infty$ in the definition.

Noetherian regular rings of global dimension no more than 2 are easy to classify. Noetherian regular rings of global dimension 3 are classified and studied in [1, 2, 3, 7, 8]. Here we classify regular rings of global dimension 2 which are not necessarily noetherian.

Theorem 0.1. *A connected ring is regular of global dimension 2 if and only if it is isomorphic to the algebra $k\langle x_1, \cdots, x_n\rangle/(b)$ satisfying the following conditions:*

1. *$n \geq 2$;*
2. *if the x_i's are labeled so that $1 \leq \deg x_1 \leq \cdots \leq \deg x_n$, then $\deg x_i + \deg x_{n-i}$ is a constant for all i;*
3. *there is a graded algebra automorphism σ of the free algebra $k\langle x_1, \cdots, x_n\rangle$ such that $b = \sum_{i=1}^{n} x_i \sigma(x_{n-i})$.*

Theorem 0.2. *Let A be an algebra in Theorem 0.1. Then the following hold.*

1. *A is noetherian if and only if A has finite GK-dimension, if and only if $n = 2$.*
2. *A is a domain. If $n > 2$, then it is not an Ore domain.*

Received by the editors December 3, 1996.
1991 *Mathematics Subject Classification.* Primary 16W50, 16E10, 16E70.
Key words and phrases. Connected algebra, global dimension, regular algebra.
This research was supported by an NSF Postdoctoral Fellowship.

3. *For every proper graded subspace $W \subset \bigoplus_{i=1}^n kx_i$, the subalgebra generated by W is free. As a consequence, if $n > 2$, A does not satisfy a polynomial identity.*
4. *If $n > 2$, A does not have non-trivial normal elements.*

We also study one-relator quadratic algebras. It is well-known that every one-relator quadratic algebra is Koszul and its global dimension is either infinite if the relation is $x^2 = 0$ or 2 otherwise. Here are some other properties.

Theorem 0.3. *Let V be a finite dimensional vector space and A a one-relator quadratic algebra $k\langle V\rangle/(b)$ for some $0 \neq b \in V \otimes V$. Then the following hold.*

1. *A is noetherian if and only if $\dim V = 1$ or $\dim V = 2$ and $b \neq xy$.*
2. *A is a domain if and only if $b \neq xy$ for some $x, y \in V$.*
3. *If A has a non-trivial normal element, then $\dim V = 1$ or $\dim V = 2$ and $b \neq xy$.*
4. *If $W \subset V$ is a subspace such that $b \notin W \otimes V$, then the subalgebra $k[W]$ is free.*

1. Regular rings of global dimension 2

Let $F = k\langle x_1, \cdots, x_n\rangle$ be a connected graded free algebra with augmentation ideal $\mathfrak{m}_F$. Let $\{y_1, \cdots, y_n\}$ be another minimal graded generating set of F, namely, $\{\bar{y}_1, \cdots, \bar{y}_n\}$ is a basis of the graded vector space $\mathfrak{m}_F/\mathfrak{m}_F^2$. Suppose that $b = \sum_{i=1}^r x_i y_i$ where $1 \leq r \leq n$ and where $\deg x_i + \deg y_i = e$ for all $1 \leq i \leq r$. The integer r is called the **rank** of b. Define A to be the connected one-relator algebra $F/(b)$. Since $b \in \mathfrak{m}_F^2$, A and F have the same minimal generating sets. We call F the **covering free algebra** of A. For any element f in F we will use the same letter f for the image in A. We will use $=_*$ for equality in the covering free algebra F and use ordinary $=$ for equality in A. These one-relator algebras form a special class of what W. Dicks studied in [5]. The **Hilbert series** of A is defined by

$$H_A(t) = \sum_i \dim A_i \, t^i.$$

Suppose that $H_A(t) = q(t)/p(t)$ for some relatively prime polynomials with integer coefficients and $p(0) = 1$ (we will see that this property holds for all algebras studied in this paper). If every root of $p(t)$ has absolute value 1, then A has finite Gelfand-Kirillov dimension; otherwise A has exponential growth [9, 2.2]. If $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a graded module, then the **l-th degree shift** of M is $M(l) := \bigoplus_{i\in\mathbb{Z}} M_{i+l}$.

Parts 1 and 2 of the following proposition are well-known. Since we will use these several times later, we include a proof here.

Proposition 1.1. *Let A be the connected algebra $k\langle x_1, \cdots, x_n\rangle/(b)$ defined above and $r = rank\ b$.*

1. *If $b = ax_1^2$ for some non-zero scalar a, then*
   (a) $\operatorname{gldim} A = \infty$,
   (b) $H_A(t) = (1 + t^{\deg x_1})[(1 + t^{\deg x_1})(1 - \sum_{i=1}^n t^{\deg x_i}) + t^{2\deg x_1}]^{-1}$.
2. *If $r > 1$, or if $r = 1$ and x_1, y_1 are linearly independent, then*
   (a) $\operatorname{gldim} A = 2$,
   (b) $H_A(t) = (1 - \sum_{i=1}^n t^{\deg x_i} + t^{\deg b})^{-1}$.
3. *A is regular if and only if $r = n > 1$.*

Proof. 1. Note that A has no non-resolvable ambiguities in the sense of [4] and all monomials not containing x_1^2 form a basis of A, whence the right annihilator $r(x_1) := \{a \in A \mid x_1 a = 0\}$ is equal to $x_1 A$. By using this property we see that the minimal projective resolution of k_A is

$$(1\text{-}1) \qquad \cdots \longrightarrow A(-3l) \longrightarrow A(-2l) \longrightarrow \bigoplus_i A(-l_i) \longrightarrow A \longrightarrow k \longrightarrow 0$$

where $l = \deg x_1$ and the boundary map from $A(-(p+1)l)$ to $A(-pl)$ is multiplication by x_1. Hence the statements follow from (1-1).

2. Let S be the idealizer $\mathbb{I}(bF) := \{f \in F \mid fb \in bF\}$. We claim that $S = k + bF$. Suppose that $fb =_* ba$ for some $f, a \in \mathfrak{m}_F - \{0\}$ and write f as $\sum_i x_i a_i$ in F. Then

$$(1\text{-}2) \qquad a_j b =_* y_j a \qquad \forall\, j = 1, \cdots, r;$$

$$(1\text{-}3) \qquad a_j b =_* 0 \qquad \forall\, j = r+1, \cdots, n.$$

Since F is a domain, $a_j \neq_* 0$ for $j \leq r$ and $a_j =_* 0$ for $j \geq r+1$. If $a_{j_0} \in k^\times$ for some $j_0 \leq r$, then (1-2) shows that $y_{j_0}(a_{j_0}^{-1} a) =_* b =_* \sum_{i=1}^r x_i y_i$. This cannot happen in the free algebra F when either $r \geq 2$ or $r = 1$ and $\{x_1, y_1\}$ are linearly independent. Thus we obtain a contradiction and hence $a_i \in \mathfrak{m}$ for all $j \leq r$. Since $\{y_1, \cdots, y_n\}$ is a minimal generating set of F, (1-2) implies that $a_j =_* y_j g_j$, so $g_j b =_* a$ by factoring out y_j from (1-2) for all $j \leq r$. Since F is a domain, $g_j =_* g_1$ and hence

$$f =_* \sum_{i=1}^n x_i a_i =_* \sum_{j=1}^r x_j a_j =_* \sum_{j=1}^r x_j y_j g_1 =_* b g_1.$$

Thus $S = k + bF$ and the **eigenring** (defined in [5, 2.1]) is $E := S/bF \cong k$. By [5, 5.3], $\operatorname{gldim} A = \operatorname{gldim} E + 2 = 2$ and by [5, 3.5] $H_A(t) = (1 - \sum_{i=1}^n t^{\deg x_i} + t^{\deg b})^{-1}$.

3. We only need to consider the algebras in part 2. The minimal projective resolution of k_A is of the form

$$(1\text{-}4) \qquad 0 \longrightarrow A(-e) \longrightarrow \bigoplus A(-l_i) \longrightarrow A \longrightarrow k \longrightarrow 0$$

where $e = \deg b$ and $l_i = \deg x_i$ for all i. The boundary map $\partial_1 : \bigoplus A(-l_i) \longrightarrow A$ sends $(a_1, \cdots, a_n)$ to $\sum_{i=1}^n x_i a_i$ and the boundary map $\partial_2 : A(-e) \longrightarrow \bigoplus A(-l_i)$ sends a to $(y_1 a, \cdots, y_r a, 0, \cdots, 0)$. By definition, A is regular if and only if the dual of (1-4), $\operatorname{Hom}((1-4), A)$:

$$(1\text{-}5) \qquad 0 \longleftarrow A(e) \longleftarrow \bigoplus A(l_i) \longleftarrow A \longleftarrow 0$$

is the minimal projective resolution of ${}_A k(e)$. The boundary map from $\bigoplus A(l_i)$ to $A(e)$ sends $(a_1, \cdots, a_n)$ to $\sum_{i=1}^r a_i y_i$. Hence A is regular if and only if $\{y_i\}_{i=1}^r$ is a minimal generating set of A, if and only if $r = n$. □

Proof of Theorem 0.1. Suppose that 1, 2, 3 hold. Let $y_i = \sigma(x_{n-i})$. Then A is regular of global dimension 2 by Proposition 1.1.3.

Conversely suppose that A is connected and regular of global dimension two. By [9, 3.1.1], we have a minimal free resolution of k_A

$$(1\text{-}6) \qquad 0 \longrightarrow A(-e) \longrightarrow \bigoplus_{i=1}^n A(-l_i) \longrightarrow A \longrightarrow k \longrightarrow 0$$

where n is finite. The Hilbert series is $H_A(t) = (1 - \sum_i t^{l_i} + t^e)^{-1}$. We may assume that $l_1 \le l_2 \le \cdots \le l_n$. If $n = 1$, we obtain an algebra with infinite global dimension (Proposition 1.1.1). Hence $n \ge 2$. By [9, 3.1.4],

$$t^e \left(1 - \sum_i t^{-l_i} + t^{-e}\right) = \left(1 - \sum_i t^{l_i} + t^e\right),$$

whence $l_i + l_{n-i} = e$ for all $i = 1, \cdots, n$. The boundary map from $\bigoplus_{i=1}^n A(-l_i)$ to A sends $(a_1, \cdots, a_n)$ to $\sum_i x_i a_i$ where $\{x_i\}$ is the minimal set of generators and $\deg x_i = l_i$, and the boundary map from $A(-e)$ to $\bigoplus_{i=1}^n A(-l_i)$ sends a to $(y_1 a, \cdots y_n a)$ for some $y_i \in \mathfrak{m}$. Consequently, A has one relation of degree e, of form $x_1 y_1 + x_2 y_2 + \cdots x_n y_n = 0$. Since A is regular, the dual of (1-6) (see (1-5)) is the projective resolution of ${}_A k(e)$, so $\{y_i\}_{i=1}^n$ is a minimal generating set of A. Now any homogeneous minimal generating set for A is also a minimal generated set for F. Thus $F = k\langle y_1, \cdots, y_n\rangle$ and $\sigma : x_i \longrightarrow y_{n-i}$ defines a graded algebra automorphism of F. Since the relation is homogeneous, we have $\deg y_i = e - l_i = l_{n-i}$ for all i. Therefore A is isomorphic to $F/(b)$ and 1, 2, 3 of Theorem 0.1 hold. □

By Theorem 0.1.2, if A is generated by three elements x_1, x_2, x_3 with $\deg x_1 \le \deg x_2 \le \deg x_3$ and $\deg x_1 + \deg x_3 \ne 2 \deg x_2$, then A is not regular of global dimension two. Noetherian regular rings of global dimension 2 are easily determined and that is the case when $n = 2$ in Theorem 0.1. The following Corollary is part 1 of Theorem 0.2.

Corollary 1.2. *Let A be a connected regular ring of global dimension 2, generated by n elements (see Theorem 0.1). Then the following statements are equivalent.*

1. $n = 2$.
2. *A is noetherian.*
3. $\mathrm{GKdim}(A) = 2$.
4. $\mathrm{GKdim}(A) < \infty$.

Proof. If A is in Theorem 0.1 and $n = 2$, then it is easy to see that A is noetherian of GK-dimension 2 [9, 3.5].

If A is in Theorem 0.1 (or even in Proposition 1.1) and $n > 2$, then it has exponential growth, and by [9, 1.2], it is neither left nor right noetherian. □

Next we prove other parts of Theorem 0.2. For simplicity, we use $\bar{f}$ for the image of f in $A/\mathfrak{m}^2$ for all $f \in A$.

Proposition 1.3. *Let A be regular of global dimension 2, generated by n elements. Then the following statements hold.*

1. *A is a domain.*
2. *Suppose that $n > 2$. Let x, f and g be homogeneous elements in $\mathfrak{m} - \{0\}$ and suppose that $x \notin \mathfrak{m}^2$. If $fg \in xA$ ($gf \in Ax$ respectively), then $f \in xA$ ($f \in Ax$ respectively).*

Remark 1.4. We do not have any example of a noetherian or non-noetherian regular ring which is not a domain.

Proof. If A is a regular ring of global dimension 2 and $n = 2$, then it is described in [9, 3.3], and in particular, it is a domain. In the rest of the proof we assume that $n > 2$.

Since A is $\mathbb{N}$-graded, A being a graded domain implies that A is a domain. Pick a minimal generating set $\{x_i \mid i = 1, \cdots, n\}$ such that $x = x_n$. Hence it suffices to show the following statement:

- Let f and g be homogeneous elements in $\mathfrak{m} - \{0\}$. Then
  (a) $fg \neq 0$,
  (b) $fg \in x_n A$ implies $f \in x_n A$.

We will prove ($\bullet$) by induction on $m := \deg f + \deg g$. Nothing needs to be proved when $m = 1$. Now suppose that $m > 1$ and assume that the statement ($\bullet$) holds for all cases when $\deg f + \deg g < m$.

Case 1: $m < e$.

Since $\bigoplus_{i<e} A_i = \bigoplus_{i<e} F_i$, it suffices to show that (a) and (b) hold in F. Hence (a) follows because F is a domain and (b) follows because x_n is in a minimal generating set of F.

Case 2: $m = e$.

(a) If $fg = 0$ in A, then $fg =_* lb$. Since $fg \neq_* 0$, $l \neq 0$. Write $f =_* \sum_i x_i a_i$; we have $a_i g =_* l y_i$ for all i. Passing to $\mathfrak{m}/\mathfrak{m}^2$, we have $\bar{y}_i \in k\bar{g}$ for all i. This contradicts the fact that $\{\bar{y}_i\}_{i=1}^n$ are linearly independent. Therefore $fg \neq 0$.

(b) If $fg \in x_n A$, then $fg =_* lb + x_n c$ for $l \in k$ and for some $c \in F$. By expanding f we have $a_i g =_* l y_i$ for $i < n$ and $a_n g =_* l y_n + c$. If $l \neq 0$, then this contradicts the fact $\{\bar{y}_i\}_{i<n}$ are linearly independent. Hence $l = 0$, whence $a_i =_* 0$ for all $i < n$. Therefore $f = x_n a_n \in x_n A$.

Case 3: $m > e$.

(a) If $fg = 0$, then, in F, we have

$$fg =_* \sum_{j=1}^{q} f_j b g_j + f_0 b + b g_{-1} \tag{1-7}$$

where $f_i, g_i \in \mathfrak{m}_F$. We always assume each term in the equation has degree m. Write $f_j =_* \sum_i x_i f_{ij}$ and $f =_* \sum_i x_i a_i$; we obtain

$$a_i g =_* \sum_j f_{ij} b g_j + f_{i0} b + y_i g_{-1} \ \ \forall\, i. \tag{1-8}$$

For every i, $a_i g = y_i g_{-1} \in y_i A$. If $g_{-1} = 0$, then by induction hypothesis (a), $a_i = 0$ for all i, so $f = 0$, a contradiction. Hence $g_{-1} \neq 0$, so $y_i g_{-1} \neq 0$ by induction hypothesis (a). This implies that $a_i \neq 0$ for all i. We claim that $a_i \in \mathfrak{m}$ ($*$). If not, let $a_{i_0} \in k^\times$. Thus $g = z_{i_0} g_{-1}$ where $z_{i_0} = y_{i_0} a_{i_0}^{-1} \in \mathfrak{m} - \mathfrak{m}^2$. For each i, $a_i z_{i_0} g_{-1} = y_i g_{-1}$. By induction hypothesis (a), $a_i z_{i_0} = y_i$ and hence $\bar{y}_i \in k\bar{z}_{i_0}$ for all i. This contradicts the fact that $\{\bar{y}_i\}_{i=1}^n$ are linearly independent. Thus we proved our claim ($*$). By induction hypothesis (b), $a_i = y_i w_i$. Factoring out a_i from $a_i g = y_i g_{-1}$ (by using induction hypothesis (a)), we obtain $w_i g = g_{-1}$ for all i. By induction hypothesis (a), $w_i = w_1$ and hence $f = \sum_i x_i y_i w_1 = b w_1 = 0$. This contradicts $f \neq 0$ in A, and hence (a) follows.

(b) If $0 \neq fg \in x_n A$, then $fg = x_n c$ for some $c \neq 0$ with $\deg c = m - \deg x_n > 0$. Similar to (1-7) we have

$$fg =_* \sum_{j=1}^{n} f_j b g_j + f_0 b + b g_{-1} + x_n c,$$

and similar to (1-8), we have

$$a_i g =_* \sum_j f_{ij} b g_j + f_{i0} b + y_i g_{-1} \ \ \forall\, i < n. \tag{1-9}$$

Hence $a_i g = y_i g_{-1} \in y_i A$ for all $i < n$. If $g_{-1} = 0$, then by induction hypothesis (a) $a_i = 0$ for all i and then $f = x_n a_n \in x_n A$. Now we suppose that $g_{-1} \neq 0$. Similar to $(*)$, we may assume $a_i \in \mathfrak{m}$ for all $i < n$ (in this case we use the fact that $\{\bar{y}_i\}_{i<n}$ are linearly independent). Since $y_i \in \mathfrak{m} - \mathfrak{m}^2$, by induction hypothesis (b), $a_i = y_i w_i$ for all $i < n$. Factoring out y_i from $a_i g = y_i g_{-1}$ we obtain that $w_i g = g_{-1}$ for all $i < n$. By induction hypothesis (a), $w_i = w_1$ for all $i < n$, and hence

$$\begin{aligned} f = \sum_i x_i a_i = x_n a_n + \sum_{i<n} x_i a_i = x_n a_n + \sum_{i<n} x_i y_i w_1 \\ = x_n a_n + (b - x_n y_n) w_1 = x_n (a_n - y_n w_1) \in x_n A. \end{aligned}$$

Thus (b) follows. □

Proposition 1.5. *Let A be a one-relator algebra $k\langle x_1, \cdots, x_n\rangle/(b)$ where $b = \sum_i x_i z_i$. Suppose that, for some s, z_s is a right regular element. Then $\sum_{i\neq s} x_i A = \bigoplus_{i\neq s} A_i$ and hence the subalgebra generated by $\{x_i \mid i \neq s\}$ is free.*

As a consequence, if A is a regular algebra of global dimension 2 and $\{x_i \mid 1 \leq i \leq n\}$ is a minimal generating set of A, then the subalgebra $k[x_i \mid i \neq s]$ is free for any s.

Proof. It suffices to show that $\sum_{i\neq s} x_i a_i = 0$ implies $a_i = 0$ in A for all $i \neq s$. Suppose that $\sum_{i\neq s} x_i a_i = 0$ for some $a_i \in A$. As the proof of Proposition 1.3, there are f_j and g_j with $f_j \in \mathfrak{m}_F$ such that

$$\sum_{i\neq s} x_i a_i =_* \sum_{j=0}^{q} f_j b g_j + b g_{-1} =_* \sum_{j=0}^{q} \sum_{i=1}^{n} x_i f_{ij} b g_j + \sum_{i=1}^{n} x_i z_i g_{-1},$$

and hence

$$(1 - \delta_{is}) a_i = z_i g_{-1} \quad \text{in} \quad A.$$

If $i = s$, then $z_s g_{-1} = 0$, so $g_{-1} = 0$ because z_s is right regular. If $i \neq s$, then $a_i = z_i g_{-1} = 0$.

If A is regular algebra, then A is a domain [Proposition 1.3] and hence z_s is a regular element. Therefore the statement follows. □

The proofs of parts 3 and 4 of the following are the same as ones of [11, 2.3] which are stated only for quadratic algebras.

Proposition 1.6. *Let A be a connected algebra and x a homogeneous element in $\mathfrak{m} - \{0\}$ satisfying the following two conditions:*

(a) *x is right regular and $xA \neq \mathfrak{m}$;*
(b) *if f and g are in $\mathfrak{m} - \{0\}$ and $fg \in xA$, then $f \in xA$.*

Then the following statements hold.

1. *A is a domain.*
2. *A is not an Ore domain and it does not satisfy a polynomial identity.*
3. *If a, y are in $A - \{0\}$, then $xa = ay$ implies that $x = y$ and $a = lx^n$ for some $n \geq 0$ and $l \in k^\times$.*
4. *A has no non-trivial normal elements.*

Proof. 1. Let f, g be non-zero elements in A. If $fg = 0$, then $f, g \notin k$ and $fg \in xA$. By (b) $f = xf'$ and $xf'g = 0$. By (a), $f'g = 0$. By induction on $\deg f'$, either $f' = 0$ or $g = 0$, a contradiction.

2. Pick $z \in \mathfrak{m} - xA$, by (b), $xA \cap zA = \{0\}$. Hence A is not Ore. Also the subalgebra $k[x, z]$ is free, hence A is not PI.

3. If $xa = ay$ and $a \notin k$, then $ay \in xA$, and by (b), $a = xa'$. Hence $xxa' = xa'y$, and by (a), $xa' = a'y$. The statement follows from induction on $\deg a$.

4. If A has a non-trivial normal element, then A has a homogeneous normal element of positive degree, say a. Hence $xa = ay$ for some $y \in \mathfrak{m}$. By part 3, $x = y$ and $a = lx^n$. Thus x^n is normal. By (a), there is $z \in \mathfrak{m} - xA$ and by (b) $xA \cap zA = \{0\}$. Therefore $zx^n \neq x^n c$ for all $c \in A$ and thus x^n is not normal, a contradiction. $\square$

Parts 2, 3, 4 of Theorem 0.2 follow easily from Propositions 1.3, 1.5 and 1.6.

2. One-relator quadratic algebras

In this section we will apply results from section 1 and results in [11] to one-relator quadratic algebras. From now on V is a vector space with a basis $\{x_i\}_{i=1}^n$ and A is a one-relator quadratic algebra $k\langle V\rangle/(b)$ where $b = \sum_{ij} l_{ij} x_i x_j$ for some $l_{ij} \in k$. It is well-known that A is Koszul and has global dimension 2 or infinity. The **rank** of the relation b is defined to be the rank of the matrix $(l_{ij}) \in M_n(k)$. This definition of rank coincides with the definition given in section 1 when we consider quadratic algebras. It is easy to see that the rank of b is independent of the choices of the basis of V. If $r = rank\ b$, then there are two bases $\{z_i\}$ and $\{y_i\}$ of V such that $b = \sum_{i \le r} z_i y_i$. Let ${}_bV$ (respectively V_b) denote the subspace generated by $\{z_i \mid i \le r\}$ (respectively $\{y_i \mid i \le r\}$). Then ${}_bV$ and V_b are only dependent on b. If $r = 1$, then $b = x_1^2$ or $b = x_1 x_2$ after changing a basis.

Proposition 2.1. *Let A be the algebra $k\langle V\rangle/(b)$ and suppose that $b = x_1 x_s$ where s is either 1 or 2. Then the following statements hold.*

1. *If $f, g \neq 0$ are in $\mathfrak{m}$, and $fg = 0$, then $f \in Ax_1$ and $g \in x_s A$.*
2. *$\sum_{i>1} x_i A = \bigoplus_{i>1} x_i A$.*
3. *The subalgebra generated by $\{x_i \mid i > 1\}$ is free.*
4. *If $j > 1$ and a, y are in $A - \{0\}$, then $x_j a = ay$ implies that $x_j = y$ and $a = l x_j^t$ for some $l \in k^\times$ and $t \ge 0$.*
5. *If $n > 1$, then A has no non-trivial normal elements.*

Proof. Since A has no non-resolvable ambiguities, all monomials not containing $x_1 x_s$ form a basis of A. In particular, x_i are right regular for all $i > 1$. Let I denote $(i_1, \cdots, i_t)$. For every element $a \in F$, write $a = \sum_I l_I x_{i_1} \cdots x_{i_t}$ where $(i_p, i_{p+1}) \neq (1, s)$ for all p. Define the support of a to be $Supp(a) := \{I \mid l_I \neq 0\}$. Part 1 and part 2 can be checked easily by expressing elements as sum of monomials, part 3 is a consequence of 2.

4. As in the proof of Proposition 1.6.3, it suffices to show $a = x_j a'$. Suppose that this is not true. Then $a = x_j a' + a_0$ where $a_0 = \sum_{i_1 \neq j} l_I x_{i_1} \cdots x_{i_t} \neq 0$. Since x_j is right regular, $x_j a = \sum_{I \in Supp(a)} l_I x_j x_{i_1} \cdots x_{i_t}$. Write $y = \sum_w h_w x_w$; then we have $ay = \sum_{(i_t, w) \neq (1,s)} l_I h_t x_{i_1} \cdots x_{i_t} x_w$. Comparing the monomials in $x_j a$ and ay, we obtain that, if $I = (i_1, \cdots, i_t) \in Supp(a_0)$, then $h_w x_{i_1} \cdots x_{i_t} x_w = 0$ for all w. Thus (i) $h_w = 0$ for all $w \neq s$ and $h_s \neq 0$ and (ii) $i_t = 1$. As a consequence

$ay = (x_j a' + a_0)h_s x_s = h_s x_j a' x_s$. Therefore

$$|Supp(ay)| = |Supp(a')| < |Supp(a)| = |Supp(x_j a)| = |Supp(ay)|,$$

a contradiction. Therefore part 4 follows.

5. If A has a non-trivial normal element, then A has a homogeneous normal element of positive degree, say a. Let $j > 1$. Then $x_j a = ay$ for some y of degree 1. By part 4, $x_j = y$ and $a = l x_j^t$ for some $l \neq 0$ and $t > 0$. Hence x_j^t is normal. But $x_j^t x_1 \neq z x_j^t$ for any z, so a contradiction. □

Now we are ready to prove Theorem 0.3.

Proof. 1. If $n = 1$, then A is $k[x]/(x^2)$, which is noetherian. If $n = r = 2$, then A is a noetherian regular algebra of global dimension two (Corollary 1.2). If $n = 2$ and $r = 1$, then A is either $k\langle x,y\rangle/(x^2)$ or $k\langle x,y\rangle/(xy)$; neither is noetherian. If $n > 2$, then by Proposition 1.1.1(b) and 2(b), A has exponential growth and hence it is not noetherian by [9, 1.2].

2 and 3. We discuss three cases.

Case 1: $r = rank\ b > 2$. Let Φ be the set of all subspaces of dimension no more than $rank\ b - 2$. Then by the proof of [11, 0.1(2)] we can apply [11, 2.2] to A and this Φ. Hence A is a domain and the following statement holds:

$(**)$ If f and g are homogeneous elements in $\mathfrak{m} - \{0\}$ and if $fg \in xA$ for some $x \in V - \{0\}$, then $f \in xA$.

By Proposition 1.6.4, A has no non-trivial normal elements.

Case 2: $r = 2$ and ${}_bV \neq V_b$. By [11, 2.2 and 2.3], A is a domain and the statement $(**)$ holds for $x \notin V_b$ (in this case Φ is the set of 1-dimensional subspaces not contained in V_b). By Proposition 1.6.4, A has no non-trivial normal elements.

Case 3: $r = 2$ and ${}_bV = V_b$. In this case $b = (l_{11}x_1 + l_{21}x_2)x_1 + (l_{12}x_1 + l_{22}x_2)x_2$ for some invertible matrix $(l_{ij})_{2\times 2}$. Hence $A = k\langle V\rangle/(b)$ is isomorphic to a twisted algebra B^σ where $B = k\langle V\rangle/(x_1x_2 - x_2x_1)$ and $\sigma : x_1 \to l_{12}x_1 + l_{22}x_2,\ x_2 \to -(l_{11}x_1 + l_{21}x_2),\ x_i \to x_i,\ \forall i > 2$. It is easy to see that the one-relator semi-group $G := \langle x_1, \cdots, x_n \mid x_1x_2 = x_2x_1\rangle$ is ordered and B is the semi-group algebra kG. Therefore B is a domain [6, Prop. A.II.1.4] and hence $A \cong B^\sigma$ is a domain [10, 5.2]. Now let $x \in V - {}_bV$ and $f, g \in \mathfrak{m} - \{0\}$. We may assume $x = x_n$ and $n > 2$. If $fg \in x_nA$, then $fg = x_nc$ and we have

$$\sum_i x_i a_i g =_* \sum_i x_i f_{ij} b g_i + \sum_{i=1}^{2} x_i y_i g_{-1} + x_n c$$

in $k\langle V\rangle$, where $f = \sum_i x_i a_i$. Comparing the coefficients in x_n, we have $a_n g =_* \sum_j f_{nj} b g_j + c$ and so $c = a_n g$ in A. Hence $fg = x_n c = x_n a_n g$. Since A is a domain, $f = x_n a_n \in x_n A$. Again we proved $(**)$ for $x \notin V_b$. By Proposition 1.6.4, A has no non-trivial normal elements if $V_b \neq V$.

Combining these cases and Proposition 2.1 we prove 2 and 3.

4. If $b \notin W \otimes V$, then there is a subspace $W' \supset W$ of dimension $n - 1$ such that $b \notin W' \otimes V$. So we may assume $\dim W = n - 1$. Hence this is a consequence of Proposition 1.5 and part 2 when $r \geq 2$. If $b = x_1^2$, this is Proposition 2.1.2. It remains to consider the case when $b = x_1x_2$. By changing a basis we may assume $b = x_1 y$, $W = \sum_{i>1} kx_i$ and $y \notin kx_1$. By Proposition 1.5, it suffices to show that y is a right regular element, which follows from Proposition 2.1.1. □

Acknowledgment

The author would like to thank S. P. Smith for several discussions on the subject.

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Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195

E-mail address: zhang@math.washington.edu