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Research Article

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Non-occurrence of the Lavrentiev phenomenon for a class of convex nonautonomous Lagrangians

Abstract: We consider the classical functional of the Calculus of Variations of the form

$$I(u) = \int\limits_{\Omega} F(x, u(x), \nabla u(x)) \, dx,$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory convex function; the admissible functions u coincide with a prescribed Lipschitz function ϕ on $\partial\Omega$. We formulate some conditions under which a given function in $\phi + W_0^{1,p}(\Omega)$ with $I(u) < +\infty$ can be approximated in the $W^{1,p}$ norm and in energy by a sequence of smooth functions that *coincide* with ϕ on $\partial\Omega$. As a particular case we obtain that the Lavrentiev phenomenon does not occur when $F(x, u, \xi) = f(x, u) + h(x, \xi)$ is convex and $x \mapsto F(x, 0, 0)$ is sufficiently smooth.

Keywords: Lavrentieff, Lavrentiev, approximation, Lipschitz, regularity, convex, nonautonomous Lagrangian

MSC: Primary 49N99; Secondary 49N60

1 Introduction

In 1927, Lavrentiev [1] provided an example of the fact, later called the *Lavrentiev phenomenon*, that the infimum, over the set of absolutely continuous functions, of a one-dimensional functional of the calculus of variations may be strictly lower than the infimum of the same functional over the set of Lipschitz functions satisfying the same boundary conditions: the example was refined by Manià in [2] and, more recently, by Ball-Mizel in [3]. Finding the conditions that ensure the non-occurrence of the Lavrentiev phenomenon has some interest, if just for ensuring to catch the infimum of the functional via standard numerical methods. We point out that some authors refer to as the Lavrentiev phenomenon just the fact that the infima among the two aforementioned classes of functions differ without taking care the boundary datum. If one allows the boundary datum to vary, things change dramatically: in Lavrentiev's celebrated example itself the infima among Lipschitz/absolutely functions are the same if one allows one boundary datum to be just arbitrarily close to the initial one. Some recent results concerning the study of this kind of "local" Lavrentiev phenomenon have been recently obtained in [4, Theorem 4].

Alberti and Serra Cassano proved in [5] that, when the integration set is an interval in $\mathbb{R}$, the phenomenon does not occur for autonomous Lagrangians. For scalar problems, where the domain is multi-dimensional,

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few results appeared in the literature. Of course, the problem becomes much easier if one imposes growth conditions of the Lagrangian from above, since approximations are facilitated by Lebesgue's dominated convergence. A two-dimensional analogue of Manià's example was provided in [6]. There are examples in [7] of functionals of the form $F(x, \nabla u)$ depending on the independent variable x of the space and on the gradient ∇u of the admissible functions that exhibit the Lavrentiev phenomenon; there are also cases in which the Lavrentiev phenomenon does not occur (see [8]). No example is known to the authors for scalar problems when the Lagrangian is autonomous.

It was conjectured by Buttazzo-Belloni in [9] that the phenomenon should not occur when the Lagrangian $F(u, \nabla u)$ is autonomous and convex in both variables, a fact that they proved in the case of a (strongly) star-shaped domain under the hidden growth assumption that $F(u, 0)$ is summable and zero as a boundary datum. Other results that appeared aimed to prove the conjecture: we mention Ekeland-Temam who proved in [10] its validity for functionals of the gradient on a Lipschitz domain for a zero boundary datum; Bonfanti and Cellina in [6, 11] considered autonomous Lagrangians that are sum of a radial function of the gradient ∇u and a function of the variable u, under some smoothness assumptions on the boundary and on the boundary datum. A complete answer to the conjecture was given by Bousquet-Mariconda-Treu in [12], where they showed that whenever $F(u, \nabla u)$ is convex, given $u \in W_0^{1,1}(\Omega)$ with a Lipschitz boundary datum and finite energy (i.e., $F(u, \nabla u) \in L^1(\Omega)$), there is no *Lavrentiev gap* at u: there exists a sequence $(u_k)_k$ of Lipschitz functions that share the same boundary datum and converge to u both in $W^{1,1}$ and in energy, no matter if u is a minimizer.

We consider here a convex nonautonomous Lagrangian $F(x, u, \nabla u)$, and establish a sufficient condition under which no Lavrentiev gap occurs at any admissible function. As a byproduct it turns out that the Lavrentiev phenomenon does not occur if the Lagrangian is of the form

$$F(x, u, \nabla u) = f(x, u) + h(x, \nabla u),$$

with $f(\cdot, 0)$ of class $C^1(\overline{\Omega})$ and $h(\cdot, 0)$ of class $C^2(\overline{\Omega})$. The methods used here are mainly based on [12–14]: we show that we can approximate a function (both in $W^{1,1}$ and in energy) with a sequence of bounded functions that are Lipschitz in a neighbourhood of the boundary of the domain.

A partial motivation for studying these kind of functionals comes from minimization problems in the Heisenberg group where one wants to consider functionals that generalize those studied in [15] and in references therein.

We do not consider here the vectorial case for which, when the Lagrangian depends only on the gradient, there are both examples of the occurrence of the Lavrentiev phenomenon and cases where it does not occur [7].

2 Notation and assumptions

Notation

- The scalar product of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y \rangle$.
- The pointwise maximum (resp. minimum) of two functions u, v is denoted by $u \vee v$ (resp. $u \wedge v$), $u^+ = u \vee 0$ (resp $u^- = (-u) \vee 0$) is the positive (resp. negative) part of u.
- The convex subgradient of a function $g : \mathbb{R}^m \to \mathbb{R}$ at $\xi_0 \in \mathbb{R}^m$ is the set

$$\partial g(\xi_0) := \{v \in \mathbb{R}^m : g(\xi) - g(\xi_0) \geq \langle v, \xi - \xi_0 \rangle \quad \forall \xi \in \mathbb{R}^m\}.$$

- The partial convex subgradient of $F(x, s, \xi)$ with respect to x at (x_0, s_0, ξ_0) is the convex subgradient of $x \mapsto F(x, s_0, \xi_0)$ at $x = x_0$, it will denoted by $\partial_x F(x_0, s_0, \xi_0)$. Analogously we will denote by $\partial_s F(x_0, s_0, \xi_0)$ (resp. $\partial_\xi F(x_0, s_0, \xi_0)$) the partial convex subgradients of $F(x, s, \xi)$ with respect to s (resp. ξ) at (x_0, s_0, ξ_0). Also, the convex subgradient of $(s, \xi) \mapsto F(x, s, \xi)$ is denoted by $\partial_{s,\xi} F(x, s, \xi)$.

- For $E \subseteq \mathbb{R}^n$, $|E|$ is the n-dimensional Lebesgue measure of E.
- $\mathbf{1}_E$ is the indicator function of a set E.

2.1 Assumptions

- $\Omega \subset \mathbb{R}^n$ is an open and bounded set.
- $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $(x, s, \xi) \mapsto F(x, s, \xi)$ is a Carathéodory function, bounded below by $\langle \alpha(x), \xi \rangle + \beta(x)$ for some $\alpha \in L^1(\Omega; \mathbb{R}^n)$, $\beta \in L^1(\Omega)$.
- ϕ is a Lipschitz function on $\overline{\Omega}$.
- $p \geq 1$, and for $v \in \phi + W_0^{1,p}(\Omega)$ we define $I(v) := \int_\Omega F(x, v, \nabla v)\, dx$ (the "energy").

The following structure condition will be used, in alternative to the boundedness of the reference function u, in our main result.

2.2 Hypothesis (H)

Hypothesis (H).

1. There are positive sequences $(\tau_k)_k$ and $(\sigma_k)_k$ such that:

   - $\lim\limits_{k \to +\infty} \tau_k = +\infty$, $\lim\limits_{k \to +\infty} \sigma_k = +\infty$;
   - For each $k \in \mathbb{N}$, there are selections $(q_{\tau_k}(x), \zeta_{\tau_k}(x))$ of $\partial_{s,\xi} F(x, \tau_k, 0)$ and $(q_{\sigma_k}(x), \zeta_{\sigma_k}(x))$ of $\partial_{s,\xi} F(x, -\sigma_k, 0)$, and $C \geq 0$ satisfying

   $$\forall k \in \mathbb{N} \quad \zeta_{\tau_k}(x), \zeta_{\sigma_k}(x) \in W^{1,1}(\Omega),$$

   $$\operatorname{div} \zeta_{\tau_k}(x) \leq C, \quad \operatorname{div} \zeta_{\sigma_k}(x) \geq -C. \tag{2.1}$$

2. There is a measurable and bounded selection $q(x)$ of the subgradient $\partial_s F(x, 0, 0)$ of F at $(x, 0, 0)$ and, for all $k \in \mathbb{N}$,
   $$q_{\sigma_k}(x) \leq q(x) \leq q_{\tau_k}(x) \quad \text{a.e. } x \in \Omega. \tag{2.2}$$

Remark 2.1.

1. Condition (2.2) is satisfied if $s \mapsto F(x, s, \xi)$ is convex. Indeed the monotonicity of the subdifferential implies that, for a.e. $x \in \Omega$,

   $$(q_{\tau_k}(x) - q(x))(\tau_k - 0) \geq 0, \quad (q_{\sigma_k}(x) - q(x))(-\sigma_k - 0) \geq 0.$$

2. When F is of class $C^2(\overline{\Omega})$ and $(s, \xi) \mapsto F(x, s, \xi)$ is convex for a.e. x, Hypothesis (H) reduces to Condition 1, namely that there are increasing , divergent sequences $(\tau_k)_k$ and $(\sigma_k)_k$ such that

   $$\forall k \in \mathbb{N} \quad \operatorname{div} \nabla_\xi F(x, \tau_k, 0) \leq C, \quad \operatorname{div} \nabla_\xi F(x, -\sigma_k, 0) \geq -C$$

   for a suitable $C \geq 0$.

Here are some Lagrangians that satisfy Hypothesis (H).

Proposition 2.2 (Validity of Hypothesis (H))**.** Assume that the map $(s, \xi) \mapsto F(x, s, \xi)$ is convex for a.e. x and that

(i) Either $F(x, s, \xi) = F(s, \xi)$, i.e., F is *autonomous*, or
(ii) $F(x, s, \xi) = f(x, s) + h(x, \xi)$ for some Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \Omega \times \mathbb{R}^n \to \mathbb{R}$ with $x \mapsto \partial_x f(x, 0)$ bounded and $x \mapsto h(x, 0)$ of class $C^2(\overline{\Omega})$.

Then F fulfills Hypothesis (H).

Proof. Let $(\tau_k)_k, (\sigma_k)_k$ be arbitrary positive divergent sequences.

(i) Since $\zeta_{\tau_k}, \zeta_{\sigma_k}$ do not depend on x it turns out that their divergence is zero. Similarly, Point 2 of Hypothesis (H) is fulfilled since any $q \in \partial_s F(0, 0)$ does not depend on x and F is convex.

(ii) Assume now that $F(x, s, \xi) = f(x, s) + h(x, \xi)$. Then, for each k,

$$\nabla_\xi F(x, \tau_k, 0) = \nabla_\xi F(x, -\sigma_k, 0) = \nabla_\xi h(x, 0)$$

and $\operatorname{div} \nabla_\xi h(x, 0)$ is continuous, thus bounded on $\overline{\Omega}$. Moreover, any element of $\partial_s F(x, 0, 0)$ is an element of $\partial_x f(x, 0)$ and is thus bounded. Condition (2.2) follows from the convexity of $s \mapsto F(x, s, \xi)$, proving the validity of Hypothesis (H). □

3 Approximation lemmas

In this section, we establish two preliminary results that will be used in the proof of Theorem 4.2.

As a first step, we give a sufficient condition under which there is no Lavrentiev gap between $W_\phi^{1,p}(\Omega)$ and $W_\phi^{1,p}(\Omega) \cap L^\infty(\Omega)$. We have defined the space $W_\phi^{1,p}(\Omega)$ as the set of those functions $u \in W^{1,p}(\Omega)$ such that the extension of u by ϕ on $\mathbb{R}^n \setminus \Omega$ belongs to $W_{loc}^{1,p}(\mathbb{R}^n)$. We still denote by u this extension. In particular, $(u - \phi)$ belongs to $W^{1,p}(\mathbb{R}^n)$ and has compact support.

Lemma 3.1. Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy Hypothesis (H). Then for every u in $W_\phi^{1,p}(\Omega)$ such that $F(x, u, \nabla u) \in L^1(\Omega)$, there exists a sequence $(u_k)_k$ in $W_\phi^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $(u_k)_k$ converges to u in $W_\phi^{1,p}(\Omega)$ and

$$\lim_{k \to +\infty} I(u_k) = I(u). \tag{3.1}$$

Proof. Let $(\tau_k)_k$ and $(\sigma_k)_k$ satisfy the conditions formulated in Hypothesis (H). For k large enough such that both $\tau_k > |\phi|_{L^\infty(\Omega)}$ and $\sigma_k > |\phi|_{L^\infty(\Omega)}$, we define u_k by

$$u_k(x) = (u^+ \wedge \tau_k)(x) - (u^- \wedge \sigma_k)(x) = \begin{cases} u(x) & \text{if } -\sigma_k \le u(x) \le \tau_k, \\ \tau_k & \text{if } u(x) \ge \tau_k, \\ -\sigma_k & \text{if } u(x) \le -\sigma_k. \end{cases}$$

It is clear that $u_k \in W_\phi^{1,p}(\Omega) \cap L^\infty(\Omega)$ and that u_k converges to u in $W_\phi^{1,p}(\Omega)$. Moreover,

$$I(u_k) = \int_{\{-\sigma_k \le u \le \tau_k\}} F(x, u, \nabla u) \, dx + \int_{\{u \ge \tau_k\}} F(x, \tau_k, 0) \, dx + \int_{\{u \le -\sigma_k\}} F(x, -\sigma_k, 0) \, dx. \tag{3.2}$$

Let $q(x)$ and $(q_{\tau_k}(x), \zeta_{\tau_k}(x))$ be as in Hypothesis (H). We have

$$F(x, u, \nabla u) \ge F(x, \tau_k, 0) + q_{\tau_k}(x)(u - \tau_k) + \zeta_{\tau_k}(x) \cdot \nabla(u - \tau_k) \text{ a.e.}.$$

Moreover, from Point 2 of Hypothesis (H) we get

$$F(x, u, \nabla u) \ge F(x, \tau_k, 0) + q(x)(u - \tau_k) + \zeta_{\tau_k}(x) \cdot \nabla(u - k) \text{ a.e. on } \{u \ge \tau_k\}.$$

Since $(u - \tau_k)^+ \in W_0^{1,1}(\Omega)$, integration on $\{u \ge \tau_k\}$ then gives

$$\int_{\{u \ge \tau_k\}} F(x, u, \nabla u) \, dx \ge \int_{\{u \ge \tau_k\}} F(x, \tau_k, 0) - \|q\|_\infty u \, dx + \int_{\{u \ge \tau_k\}} \zeta_{\tau_k}(x) \cdot \nabla(u - \tau_k)^+ \, dx$$

$$\ge \int_{\{u \ge \tau_k\}} F(x, \tau_k, 0) - \|q\|_\infty u \, dx - \int_\Omega \operatorname{div} \zeta_{\tau_k}(x) (u - \tau_k)^+ \, dx.$$

Therefore, Hypothesis (H) yields

$$\int\limits_{\{u \geq \tau_k\}} F(x, u, \nabla u)\, dx \geq \int\limits_{\{u \geq \tau_k\}} F(x, \tau_k, 0) - (\|q\|_\infty + C)u\, dx. \tag{3.3}$$

Analogously we get

$$\int\limits_{\{u \leq -\sigma_k\}} F(x, u, \nabla u)\, dx \geq \int\limits_{\{u \leq -\sigma_k\}} F(x, -\sigma_k, 0) + (\|q\|_\infty + C)u\, dx. \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$I(u_k) \leq \int\limits_{\{-\sigma_k \leq u \leq \tau_k\}} F(x, u, \nabla u)\, dx + \int\limits_{\{u \geq \tau_k\}} F(x, u, \nabla u)\, dx + \int\limits_{\{u \leq -\sigma_k\}} F(x, u, \nabla u)\, dx + (\|q\|_\infty + C) \int\limits_{\{u \leq -\sigma_k\} \cup \{u \geq \tau_k\}} |u|\, dx$$

$$\leq I(u) + (\|q\|_\infty + C) \int\limits_{\{u \leq -\sigma_k\} \cup \{u \geq \tau_k\}} |u|\, dx. \tag{3.5}$$

Since $u \in L^1(\Omega)$, Lebesgue's Theorem implies that

$$\limsup_{k \to +\infty} I(u_k) \leq I(u).$$

By Fatou lemma,

$$\liminf_{k \to +\infty} I(u_k) \geq I(u)$$

and (3.1) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now prove that there is no Lavrentiev gap at $u \in W_\phi^{1,p}(\Omega)$ if u is Lipschitz continuous on a neighborhood of $\partial\Omega$.

Lemma 3.2. Assume that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is *convex* with respect to its three variables. Let u in $W_\phi^{1,p}(\Omega)$ be such that $F(x, u, \nabla u) \in L^1(\Omega)$. Assume, moreover, that either u is bounded or that F fulfils Hypothesis (H). If u is Lipschitz continuous on a neighborhood of $\partial\Omega$, then there exists a sequence $(u_k)_k$ in $\text{Lip}_\phi(\Omega)$ such that $(u_k)_k$ converges to u in $W_\phi^{1,p}(\Omega)$ and

$$\lim_{k \to +\infty} I(u_k) = I(u).$$

Moreover, if u is bounded in $L^\infty(\Omega)$, then the sequence $(u_k)_k$ may be taken to be bounded in $L^\infty(\Omega)$.

Proof. From Lemma 3.1 it is not restrictive to assume that u is bounded. We may consider u as extended by ϕ out of Ω. By assumption, there exists an open set $V \subset \mathbb{R}^n$ such that $\partial\Omega \subset V$ and u is Lipschitz continuous on $V \cap \Omega$. In particular u and ∇u are in $L^\infty(V \cap \Omega)$.

Let $\rho \in C_c^\infty(B_1, \mathbb{R}^+)$ be *even*, $\int_{\mathbb{R}^n} \rho\, dx = 1$ and for $k = 1, 2, \ldots, (\rho_k)_k$ be the sequence of mollifiers defined by $\rho_k(x) := k^n \rho(kx)$. Let also $\theta \in C_c^\infty(\Omega, [0, 1])$ be such that $\theta = 1$ on a neighborhood of $\Omega \setminus V$. We then define

$$u_k = \theta(u \star \rho_k) + (1 - \theta)u.$$

Notice first that $u_k \in \text{Lip}_\phi(\Omega)$. Indeed, if $\theta = 1$ then $u_k = u \star \rho_k$, otherwise $\{0 \leq \theta < 1\} \subset V \cap \Omega$ where ∇u is bounded. Clearly, $(u_k)_k$ converges to u in $W_\phi^{1,p}(\Omega)$. This implies

$$\liminf_{k \to +\infty} I(u_k) \geq I(u).$$

It remains to show that

$$\limsup_{k \to +\infty} I(u_k) \leq I(u). \tag{3.6}$$

For this purpose, we decompose $I(u_k) = \int_\Omega F(x, u_k, \nabla u_k)\, dx$ as the sum

$$I(u_k) = \int_{\{\theta=1\}} F(x, u_k, \nabla u_k)\, dx + \int_{\{0 \le \theta < 1\}} F(x, u_k, \nabla u_k)\, dx. \tag{3.7}$$

On the set $\{0 \le \theta < 1\} \subset V \cap \Omega$, $\nabla u \in L^\infty(V \cap \Omega)$ and

$$\nabla u_k = \theta(\nabla u \star \rho_k) + (1 - \theta)\nabla u + (\nabla\theta)(u \star \rho_k - u).$$

Let $\bar{k}$ be such that

$$\forall k \ge \bar{k} \qquad \{0 \le \theta < 1\} + B_{1/k} \subset V \cap \Omega.$$

Then, for $k \ge \bar{k}$ and $x \in \{0 \le \theta < 1\}$ we have

$$|u_k(x)| \le 2\|u\|_{L^\infty(V \cap \Omega)};$$

$$|\nabla u_k(x)| \le 2\|\nabla u\|_{L^\infty(V \cap \Omega)} + 2\|\nabla\theta\|_{L^\infty(V \cap \Omega)}\|u\|_{L^\infty(V \cap \Omega)}$$

which in turn means that under the above assumptions both u_k and ∇u_k are bounded by a constant that does not depend on k. Since $(u_k)_k$ converges to u in $W^{1,p}(\Omega)$ we may assume, by taking a subsequence, that $(u_k, \nabla u_k)_k$ converges a.e. to $(u, \nabla u)$. Now, since F is bounded on bounded sets, by Lebesgue's Theorem we have

$$\lim_{k \to +\infty} \int_{\{0 \le \theta < 1\}} F(x, u_k, \nabla u_k)\, dx = \int_{\{0 \le \theta < 1\}} F(x, u, \nabla u)\, dx. \tag{3.8}$$

On the set $\{\theta = 1\}$ we have

$$u_k = u \star \rho_k, \quad \nabla u_k = \nabla u \star \rho_k.$$

It remains to show that

$$\limsup_{k \to 0} \int_{\{\theta=1\}} F(x, u \star \rho_k, \nabla u \star \rho_k)\, dx \le \int_{\{\theta=1\}} F(x, u, \nabla u)\, dx; \tag{3.9}$$

afterwards, in view of (3.7) and (3.8), we get (3.6). Notice that, since ρ is even,

$$\rho_k \star x = x \quad \forall k \in \mathbb{N}.$$

By Jensen's inequality,

$$F(x, u \star \rho_k, \nabla u \star \rho_k) = F(x \star \rho_k, u \star \rho_k, \nabla u \star \rho_k) \le F(x, u, \nabla u) \star \rho_k.$$

Whence

$$\int_{\{\theta=1\}} F(x, u \star \rho_k, \nabla u \star \rho_k)\, dx \le \int_{\{\theta=1\}} F(x, u, \nabla u) \star \rho_k\, dx.$$

Since $F(x, u, \nabla u) \in L^1(\Omega)$, we get (3.9). $\qquad\square$

4 Main result

We consider here domains Ω that are *locally strongly star-shaped* in the sense of [12, Definition 2.9]. These include Lipschitz ones and allow even some cusps at some boundary points.

Definition 4.1. An open and bounded set Ω is called *locally strongly star-shaped* if for every $p \in \partial\Omega$, there exists an open set $H \subset \mathbb{R}^n$ such that $p \in H$ and $H \cap \Omega$ is strongly star-shaped, i.e., there is $z_H \in H \cap \Omega$ such that $z_H + \lambda(\Omega - z_H)$ is relatively compact in Ω for every $\lambda \in [0, 1[$.

Theorem 4.2 (Non-occurrence of the Lavrentiev gap). Assume that Ω is *locally strongly star-shaped* and that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is *convex*. Let $u \in W^{1,p}_\phi(\Omega)$ be such that $F(x, u, \nabla u) \in L^1(\Omega)$. Assume, moreover, that either u is *bounded* or that F satisfies the structure Hypothesis (H). Then the Lavrentiev gap for I does not occur at u, i.e. there exists a sequence $(u_k)_k$ in $\text{Lip}_\phi(\Omega)$ converging to u in $W^{1,p}(\Omega)$ and such that

$$\lim_{k \to +\infty} I(u_k) = I(u). \tag{4.1}$$

Moreover, if u is bounded in $L^\infty(\Omega)$, the sequence $(u_k)_k$ may be taken to be bounded in $L^\infty(\Omega)$.

Proof. In view of Lemma 3.2 it is enough to provide a sequence $(u_k)_k$ in $W^{1,p}_\phi(\Omega)$ satisfying the conditions of the claim with the exception that it is just Lipschitz continuous in a neighborhood of $\partial\Omega$ (instead of Lipschitz on Ω). We may consider u to be extended by ϕ out of Ω. Also, in view of Lemma 3.1, it is not restrictive to assume that u is bounded.

Without loss of generality, we can assume that $F \geq 0$. Indeed, since F is convex with respect to its variables, if $x_0 \in \Omega$ and $(\overline{a}, \overline{q}, \overline{\zeta}) \in \partial F(x_0, 0, 0)$ then

$$G(x, s, \xi) := F(x, s, \xi) - \overline{a} \cdot x - \overline{q}s - \overline{\zeta} \cdot \xi - F(x_0, 0, 0) \geq 0.$$

Moreover additive affine terms do not perturb our convergence results: if a sequence $(u_k)_k$ converges to u in $W^{1,p}(\Omega)$ then $(I(u_k))_k$ converges to $I(u)$ if and only if $\int_\Omega G(x, u_k, \nabla u_k)\, dx$ converges to $\int_\Omega G(x, u, \nabla u)\, dx$.

Consider first the case where Ω is strongly star-shaped with respect to the origin, i.e., for every $h \in [0, 1[$, $h\Omega$ is relatively compact in Ω. Given $\lambda, h \in]1/2, 1[$, set

$$u^\lambda_h := \phi(x) + \lambda h \, (u - \phi)\left(\frac{x}{h}\right).$$

Notice that u^λ_h converges to u in $W^{1,1}$ as $\lambda, h \to 1$ and that $u^\lambda_h = \phi$ on $\mathbb{R}^n \setminus h\Omega$. We then write

$$\left(x, u^\lambda_h, \nabla u^\lambda_h\right) = \left(x, \phi(x) + \lambda h \,(u - \phi)\left(\frac{x}{h}\right), \nabla\phi(x) + \lambda\nabla\,(u - \phi)\left(\frac{x}{h}\right)\right)$$

as a convex combination in λ, namely

$$\left(x, u^\lambda_h, \nabla u^\lambda_h\right) = \lambda\left(\frac{x}{h}, u\left(\frac{x}{h}\right), \lambda\nabla u\left(\frac{x}{h}\right)\right) + (1 - \lambda)\left(\frac{x\,(1 - \lambda/h)}{1 - \lambda}, \xi^\lambda_h(x), \frac{\nabla\phi(x) - \lambda\nabla\phi\left(\frac{x}{h}\right)}{1 - \lambda}\right), \tag{4.2}$$

where

$$\xi^\lambda_h(x) := u^\lambda_h - \lambda u\left(\frac{x}{h}\right) = \frac{1}{1 - \lambda}\left(\phi(x) - h\lambda\phi\left(\frac{x}{h}\right) + (h - 1)\lambda u\left(\frac{x}{h}\right)\right).$$

The convexity of F yields

$$I(u^\lambda_h) \leq \lambda \int_\Omega F\left(\frac{x}{h}, u\left(\frac{x}{h}\right), \nabla u\left(\frac{x}{h}\right)\right)\, dx + (1 - \lambda)\int_\Omega F\left(\frac{x\,(1 - \lambda/h)}{1 - \lambda}, \xi^\lambda_h(x), \frac{\nabla\phi(x) - \lambda\nabla\phi\left(\frac{x}{h}\right)}{1 - \lambda}\right)\, dx \tag{4.3}$$

Since $F \geq 0$, we get

$$\int_\Omega F\left(\frac{x}{h}, u\left(\frac{x}{h}\right), \nabla u\left(\frac{x}{h}\right)\right)\, dx = h^n \int_{h\Omega} F(x, u(x), \nabla u(x)\, dx \leq \int_\Omega F(x, u(x), \nabla u(x))\, dx.$$

Fix $\lambda \in]1/2, 1[$; we then study the second term of the right hand side of (4.3). Since $h \geq 1/2$, we have

$$\left\|\frac{x\,(1 - \lambda/h)}{1 - \lambda}\right\|_\infty \leq \frac{\sup\{|x| : x \in \Omega\}}{1 - \lambda}, \quad \|\xi^\lambda_h\|_\infty \leq \frac{1}{1 - \lambda}\,(2\|\phi\|_\infty + \|u\|_\infty)$$

and

$$\left\| \frac{\nabla \phi(x) - \lambda \nabla \phi \left(\frac{x}{h} \right)}{1 - \lambda} \right\|_\infty \le \frac{2 \|\nabla \phi\|_\infty}{1 - \lambda}.$$

Moreover

$$\lim_{h \to 1} \frac{x \left(1 - \lambda/h \right)}{1 - \lambda} = x, \quad \lim_{h \to 1} \xi_h^\lambda(x) = \phi(x) \quad \text{a.e.}$$

and

$$\lim_{h \to 1} \frac{\nabla \phi(x) - \lambda \nabla \phi \left(\frac{x}{h} \right)}{1 - \lambda} = \nabla \phi(x) \quad \forall x.$$

The function F being bounded on bounded sets, by means of the dominated convergence theorem we get

$$\lim_{h \to 1} \int_\Omega F \left(\frac{x \left(1 - \lambda/h \right)}{1 - \lambda}, \xi_h^\lambda(x), \frac{\nabla \phi(x) - \lambda \nabla \phi \left(\frac{x}{h} \right)}{1 - \lambda} \right) dx = \int_\Omega F(x, \phi, \nabla \phi) \, dx,$$

so that

$$\limsup_{h \to 1} I(u_h^\lambda) \le \lambda I(u) + (1 - \lambda) I(\phi).$$

The right-hand side term of the latter inequality tends to $I(u)$ as λ tends to 1. Hence, for every $i \in \mathbb{N}$, $i \ge 1$, there are sequences λ_i and $k_i \in \mathbb{N}$ with $k_i \ge i$ such that $u_{k_i}^{\lambda_i} \to u$ in $W^{1,p}(\Omega)$ as $i \to +\infty$, $u_{k_i}^{\lambda_i}$ are Lipschitz in a neighbourhood of $\partial \Omega$, and

$$I(u_{k_i}^{\lambda_i}) \le I(u) + \frac{1}{i} \quad \forall i \ge 1.$$

In particular we get

$$\limsup_{i \to +\infty} I(u_{k_i}^{\lambda_i}) \le I(u).$$

Also, Fatou's lemma gives

$$\liminf_{i \to +\infty} I(u_{k_i}^{\lambda_i}) \ge I(u),$$

and thus $\lim_{i \to +\infty} I(u_{k_i}^{\lambda_i}) = I(u)$, proving the claim.

The case of a general locally strongly star-shaped domain follows with the obvious changes as in the proof of [12, Theorem 4.1]. $\qquad \square$

Remark 4.3. Assume that $F(x, u, \nabla u) = f(x, u) + h(\nabla u)$ with f convex and superlinear. In Theorem 4.2 the (alternative) assumption that u is bounded is satisfied if, for instance, for every constant boundary datum $k \in \mathbb{Z}$, the minimizers of I among the functions that are equal to k in the boundary of Ω are bounded. Indeed, the fact that ϕ is bounded and the comparison principles of [16, 17] show that u is bounded too.

References

[1]   Mikhail Alekseevich Lavrentiev, *Sur quelques problèmes du calcul des variations*, Ann. Mat. Pura Appl. **4** (1927), 9–28.

[2]   Basilio Manià, *Sopra un esempio di Lavrentieff*, Boll. Unione Mat. Ital. **13** (1934), 147–153.

[3]   John MacLeod Ball and Victor J. Mizel, *One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation*, Arch. Rational Mech. Anal. **90** (1985), no. 4, 325–388.

[4]   Paolo Baroni, Maria Colombo, and Giuseppe Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, art. 62.

[5]   Giovanni Alberti and Francesco Serra Cassano, *Non-occurrence of gap for one-dimensional autonomous functionals*, Calculus of variations, homogenization and continuum mechanics (Marseille, 1993), vol. 18 of Ser. Adv. Math. Appl. Sci., pp. 1–17, World Sci. Publ., River Edge, NJ, 1994.

[6]   Giovanni Bonfanti and Arrigo Cellina, *On the non-occurrence of the Lavrentiev phenomenon*, Adv. Calc. Var. **6** (2013), no. 1, 93–121.

[7]   Jan Malý and William P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, vol. 51 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997.

[8]   Pierre Bousquet, Carlo Mariconda, and Giulia Treu, *Non-occurrence of the Lavrentiev gap for a class of nonconvex Lagrangians*, in preparation (2020).

[9]   Giuseppe Buttazzo and Marino Belloni, *A survey on old and recent results about the gap phenomenon in the calculus of variations*, Recent developments in well-posed variational problems, vol. 331 of Math. Appl., pp. 1–27, Kluwer Acad. Publ., Dordrecht, 1995.

[10] Ivar Ekeland and Roger Témam, *Convex analysis and variational problems*, vol. 28 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, english edition, 1999.

[11] Giovanni Bonfanti and Arrigo Cellina, *The nonoccurrence of the Lavrentiev phenomenon for a class of variational functionals*, SIAM J. Control Optim. **51** (2013), no. 2, 1639–1650.

[12] Pierre Bousquet, Carlo Mariconda, and Giulia Treu, *On the Lavrentiev phenomenon for multiple integral scalar variational problems*, J. Funct. Anal. **266** (2014), 5921–5954.

[13] Carlo Mariconda and Giulia Treu, *Non-occurrence of a gap between bounded and Sobolev functions for a class of nonconvex Lagrangians*, J. Convex Analysis **27** (2020), no. 4, to appear.

[14] Pierre Bousquet, Carlo Mariconda, and Giulia Treu, *A survey on the non occurence of the Lavrentiev gap for convex, autonomous multiple integral scalar variational problems*, Set-Valued Var. Anal. **23** (2015), no. 1, 55–68.

[15] Andrea Pinamonti, Francesco Serra Cassano, Giulia Treu, and Davide Vittone, *BV minimizers of the area functional in the Heisenberg group under the bounded slope condition*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **14** (2015), no. 3, 907–935.

[16] Carlo Mariconda and Giulia Treu, *A comparison principle and the Lipschitz continuity for minimizers*, J. Convex Anal. **12** (2005), no. 1, 197–212.

[17] Carlo Mariconda and Giulia Treu, *A Haar-Rado type theorem for minimizers in Sobolev spaces*, ESAIM Control Optim. Calc. Var. **17** (2011), 1133–1143.