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NON-PARAMETRIC UP-AND-DOWN EXPERIMENTATION¹

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1. Introduction. Let $Y(x)$ be a random variable such that $P(Y(x) = 1) = F(x)$ and $P(Y(x) = 0) = 1 - F(x)$ where $F(x)$ is a distribution function. It is sometimes of interest, as in sensitivity experiments, to estimate a given quantile of $F(x)$ with observations distributed like $Y(x)$ where the choice of x is under control. A procedure for estimating the median was suggested by Dixon and Mood [2]. The validity of their procedure depends on the assumption that $F(x)$ is normal. Robbins and Monro [6] suggested a general scheme which can be used for estimating any quantile and which imposes no parametric assumptions on $F(x)$. Their method does assume, however, that the range of possible experimental values of x is the real line. In practice, this will not be the case. Limitations on the precision of measuring instruments, or natural limitations such as when x is obtained by a counting procedure, will usually restrict the experimental range of x to a set of numbers of the form

$$a + hn \quad (-\infty < a < \infty, h > 0, n = 0, \pm 1, \dots).$$

In this note we suggest a non-parametric procedure for estimating any quantile of $F(x)$ on the basis of quantal response data when, experimentally, x is restricted to the form $a + hn$.

For convenience we assume $a = 0, h = 1$. Suppose we wish to estimate that value of $x = \theta$ such that $F(\theta - 0) \leq \alpha \leq F(\theta), \frac{1}{2} \leq \alpha < 1$. If $0 < \alpha \leq \frac{1}{2}$ or $a \neq 0$ or $h \neq 1$ the necessary modifications will be apparent. The experimental procedure is as follows: choose x_1 arbitrarily. Recursively, let

$$\begin{aligned} x_n &= x_{n-1} - 1, & \text{with probability } \frac{1}{2\alpha} \text{ if } y_{n-1} = 1, \\ (1) \quad &= x_{n-1} + 1, & \text{with probability } 1 - \frac{1}{2\alpha} \text{ if } y_{n-1} = 1, \\ &= x_{n-1} + 1, & \text{with probability 1 if } y_{n-1} = 0. \end{aligned}$$

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where y_k denotes the zero-or-one response at x_k . The estimate θ_n of θ based on n observations is the *most frequent* value of x , if unique, or the *arithmetic average of the most frequent levels*, if not unique.

We shall prove the following

THEOREM. *If $F(x)$ is strictly increasing for $\theta - 1 \leq x \leq \theta + 1$, then*

$$P(\max(|\limsup_{n \rightarrow \infty} \theta_n - \theta|, |\liminf_{n \rightarrow \infty} \theta_n - \theta|) < 1) = 1.$$

2. Two lemmas.

Let $\{X_n\} (n = 0, 1, \dots)$ be an irreducible Markov chain with recurrent non-null states and stationary transition probabilities $\{p_{ij}\}$ (see Feller [3] for definitions of terms) such that

$$(2) \quad p_{i,i+1} + p_{i,i-1} = 1 \quad (i = 0, \pm 1, \dots).$$

Let $v_i (i = 0, \pm 1, \dots)$ be the unique solution of the equations

$$(3) \quad \begin{cases} \sum_{i=-\infty}^{\infty} v_i p_{ij} = v_j & (j = 0, \pm 1, \dots), \\ v_i > 0, & \text{for all } i, \\ \sum_{i=-\infty}^{\infty} v_i = 1. \end{cases}$$

Since $\{X_n\}$ is irreducible and the states are recurrent non-null, the system (3) has such a unique solution. The v_i 's play the role of stationary absolute probabilities; i.e., if $P(X_0 = i) = v_j$, then $P(X_n = i) = v_i$ for every n .

LEMMA 1. *If for some $i = b$, $p_{b,b+1} \leq p_{b,b-1}$, $p_{b,b+1} > p_{b+1,b+2}$ and $p_{i,i+1}$ is non-increasing in i for $i \geq b + 1$, then $v_b > v_{b+1}$ and v_i is non-increasing in i for $i \geq b + 1$. Similarly, if for some $i = c$, $p_{c,c-1} \leq p_{c,c+1}$, $p_{c,c-1} > p_{c-1,c-2}$, and $p_{i,i+1}$ is non-decreasing in i for $i \leq c - 1$, then $v_c > v_{c-1}$ and v_i is non-decreasing in i for $i \leq c - 1$.*

Proof. Let $\pi_{ij} = P(X_n = j \text{ for some } n \geq 1, X_r \neq i \text{ or } j \text{ for } r < n | X_0 = i)$. From a result of Harris [5] we know that

$$(4) \quad \frac{v_{i+1}}{v_i} = \frac{\pi_{i,i+1}}{\pi_{i+1,i}}.$$

It is clear however that $\pi_{i,i+1} = p_{i,i+1}$ and $\pi_{i+1,i} = p_{i+1,i}$. Hence, from (4) and by the hypothesis

$$\frac{v_{b+1}}{v_b} = \frac{p_{b,b+1}}{p_{b+1,b}} = \frac{p_{b,b+1}}{1 - p_{b+1,b+2}} < \frac{p_{b,b+1}}{1 - p_{b,b+1}} \leq 1$$

and thus $v_{b+1} < v_b$. The remainder of the proof follows in the same manner.

Let $N_n(i)$ denote the number of r such that $X_r = i$ for $r \leq n$. For the truth of the following lemma we need not impose the condition (2).

LEMMA 2. *Let B be the set of states such that $v_{i'} = \max_i \{v_i\}$ for $i' \in B$. Then for every $i' \in B$.*

$$P\left(\lim_{n \rightarrow \infty} \frac{N_n(i')}{n} = v_{i'} > \lim_{n \rightarrow \infty} \max_{i \notin B} \left\{ \frac{N_n(i)}{n} \right\}\right) = 1.$$

Proof. Since $\sum_{i=-\infty}^{\infty} v_i = 1$, there exists a finite set A of states with $B \subset A$ such that $\sum_{i \notin A} v_i < v_{i'}$. From the strong law of large numbers for Markov chains [1], it follows that $P(\lim_{n \rightarrow \infty} (N_n(i)/n = v_i) = 1$ for every i and more generally $P(\lim_{n \rightarrow \infty} \sum_{i \notin A} (N_n(i)/n = \sum_{i \notin A} v_i) = 1$. Let ϵ be any number such that $0 < \epsilon < v_{i'} - \max(\max_{i \in A-B} \{v_i\}, \sum_{i \notin A} v_i)$ and let E_N denote the event that $(N_n(i')/n > v_{i'} - \epsilon$ for all $n > N$. By the previous remark and since $\{E_N\}$ is a monotone sequence, $\lim_{N \rightarrow \infty} P(E_N) = P(\lim_{N \rightarrow \infty} E_N) = 1$. Therefore there exists an N_1 such that $P(N_n(i')/n > v_{i'} - \epsilon$ for all $n > N_1) > 1 - \epsilon/3$. Similarly, since A is finite, there exists an N_2 such that $P(\max_{i \in A-B} \{N_n(i)/n\} < v_{i'} - \epsilon$ for all $n > N_2) > 1 - \epsilon/3$ and an N_3 such that $P(\sum_{i \notin A} N_n(i)/n < v_{i'} - \epsilon$ for all $n > N_3) > 1 - \epsilon/3$. Let $N_0 = \max(N_1, N_2, N_3)$. Then it follows that

$$P(N_n(i')/n > v_{i'} - \epsilon > \max(\max_{i \in A-B} \{N_n(i)/n\}, \sum_{i \notin A} N_n(i)/n \text{ for all } n > N_0) > 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$P(\lim_{n \rightarrow \infty} N_n(i')/n = v_{i'} > \limsup_{n \rightarrow \infty} \max_{i \notin B} \{N_n(i)/n\}) = 1.$$

The last assertion implies that $\lim_{n \rightarrow \infty} \max_i \{N_n(i)/n\}$ exists. By a similar argument applied to the finite set B_1 of states which have the second largest v_i 's it follows that $\limsup_{n \rightarrow \infty} \max_{i \notin B} \{N_n(i)/n\}$ can be replaced by $\lim_{n \rightarrow \infty} \max_{i \notin B} \{N_n(i)/n\}$. The lemma is proved.

3. Application of lemmas.

Let $\{X_n\}$ be the Markov chain defined by (1); i.e. let $X_n = i$ if $x_n = i$. The transition probabilities are of the form

$$p_{i,i+1} = 1 - \frac{F(i)}{2\alpha},$$

$$p_{i,i-1} = \frac{F(i)}{2\alpha}.$$

The chain is clearly irreducible and the states can be easily shown to be recurrent non-null using a theorem of Harris [5] or a modified version of a theorem of Foster [4]. The numbers $[\theta] + 1$ and $[\theta]$, where $[\theta]$ denotes the largest integer less than or equal to θ , can be taken as b and c of Lemma 1. From Lemma 1 and the condition of strict monotonicity of $F(x)$ for $\theta - 1 \leq x \leq \theta + 1$, it is clear that $[\theta]$ or $[\theta] + 1$ or both but no other states belong to B of Lemma 2. Thus, according to Lemma 2, the most frequent state, for n large enough, will be $[\theta] + 1$, $[\theta]$ or both with probability 1. In any case, the difference between θ and $[\theta] + 1$ or $[\theta]$ or the arithmetic average of the two is less than 1. The theorem is therefore proved.

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APPROXIMATE MOMENTS FOR THE SERIAL CORRELATION COEFFICIENT

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1. Introduction and summary. The first order Gaussian auto-regressive process (x_t) may be defined by the stochastic difference equation

$$(1) \quad x_t = \rho x_{t-1} + u_t,$$

where the u 's are NID(0, 1) and ρ is an unknown parameter. The choice of a statistic as an estimator for ρ depends on the initial conditions imposed on the difference equation (1). The so-called "circular" model is obtained by considering a sample of size N and then assuming that $x_{N+1} = x_1$. An appropriate estimator for ρ in this case is the circular serial correlation coefficient

$$(2) \quad r = \frac{\sum_{t=1}^N x_t x_{t+1}}{\sum_{t=1}^N x_t^2} \quad (x_{N+1} = x_1).$$

Leipnik [1] has derived an approximate density function

$$(3) \quad f(t) = \frac{\Gamma\left(\frac{N+2}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{1}{2}\right)} (1 - 2t\rho + \rho^2)^{-N/2} (1 - t^2)^{(N-1)/2}$$

for the estimator r . Leipnik also evaluated the first two moments of this distribution. In this paper a formula is obtained which gives $E(r^k)$ as a polynomial of degree k in ρ .

2. The general formula for $E(r^k)$. To calculate the moments of r we must evaluate the integral

$$(4) \quad E(r^k) = \int_{-1}^1 t^k f(t) dt.$$

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