# NON–QUADRATIC PERFORMANCE DESIGN FOR TAKAGI–SUGENO FUZZY SYSTEMS

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This paper improves controller synthesis of discrete Takagi-Sugeno fuzzy systems based on non-quadratic Lyapunov functions, making it possible to accomplish various kinds of control performance specifications such as decay rate conditions, requirements on control input and output and disturbance rejection. These extensions can be implemented via linear matrix inequalities, which are numerically solvable with commercially available software. The controller design is illustrated with an example.

**Keywords:** fuzzy control, Lyapunov functions, LMIs

## 1. Introduction

Fuzzy control systems have witnessed a strong growth of industrial applications in the recent years, mainly due to their reliability and satisfactory results in dealing with highly nonlinear behavior with a good compromise between accuracy and simplicity.

Since Takagi-Sugeno fuzzy systems (TSFSs) were described in (Takagi and Sugeno, 1985), they have been largely considered as one of the most suitable tools for modeling non-linear systems. Their structure facilitates stability analysis via common quadratic Lyapunov functions (Farinwata and Vachtsevanos, 1993; Tanaka and Sugeno, 1990), and controller synthesis via parallel distributed compensation (PDC), including many performance requirements like decay rate, input and output constraints, robustness and optimality (Tanaka and Sugeno, 1992; 1994; Tanaka *et al.*, 1998; Tanaka and Wang, 2001; Wang *et al.*, 1996). In addition to that, all these results can be stated as linear matrix inequalities (LMIs) that can be efficiently implemented and solved.

Nevertheless, when a large number of subsystems are involved, common Lyapunov functions are inadequate to establish stability or synthesize controllers, by virtue of their conservativeness. Several approaches have been developed to overcome these limitations. Piecewise quadratic Lyapunov functions were employed to enrich the set of possible Lyapunov functions used to prove stability (Bernal and Hušek, 2004a; 2004b; 2004c; Feng, 2004; Johansson *et al.*, 1999; Rantzer and Johansson, 2000). Controller synthesis under this approach recently appeared (Feng, 2003), but it is still limited to continuous-time TSFSs (Bernal and Hušek, 2005a; 2005b).

A more general approach based on non-quadratic Lyapunov functions was recently developed (Bernal and Hušek, 2005c; Guerra and Vermeiren, 2004; Tanaka *et al.*, 2003), not only to establish stability, but to synthesize controllers for a discrete TSFS. In contrast to the quadratic piecewise approach, the non-quadratic one can deal with non-linear premise variables, so the TSFS's approximation capabilities can be fully exploited. In this work, decay rate requirements, constraints on input and output and disturbance rejection are incorporated in one of the non-quadratic stabilizing controllers developed in (Guerra and Vermeiren, 2004). These extensions can be implemented via linear matrix inequalities (LMIs), which are numerically solvable with commercially available software.

This paper is organized as follows: Section 2 introduces dynamical fuzzy systems and the non-quadratic approach this work is based on. Section 3 develops extensions to the previous approach in order to include the underlying performance requirements. Section 4 exemplifies the results and, finally, Section 5 contains some concluding remarks.

## 2. Fuzzy Dynamic Model and the Non-Quadratic Approach

Consider the following discrete Takagi-Sugeno fuzzy system (Tanaka and Wang, 2001):

$\mathbf{R_i}$ : **If** $z_1(t)$ **is** $M_{i1}$ **and** $\cdots$ **and** $z_p(t)$ **is** $M_{ip}$ **then**

$$x(t+1) = A_i x(t) + B_i u(t)$$
$$y(t) = C_i x(t) \qquad i \in \{1,\dots,r\},$$

where $R_i$ denotes the $i$-th rule, $r$ is the number of rules, $M_{ij}$ is a fuzzy set, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the output vector, $A_i, B_i, C_i$ are matrices of suitable dimensions that represent the $i$-th local model of the fuzzy system, and $z(t) = [z_1(t) \dots z_p(t)]$ is the premise vector which depends on the state vector $x(t)$.

The previous rules can be compactly rewritten as follows:

$$x(t+1) = A_z x(t) + B_z u(t), \quad y(t) = C_z x(t), \quad (1)$$

where

$$A_z = \sum_{i=1}^{r} h_i\big(z(t)\big) A_i,$$
$$B_z = \sum_{i=1}^{r} h_i\big(z(t)\big) B_i,$$
$$C_z = \sum_{i=1}^{r} h_i\big(z(t)\big) C_i.$$

When disturbances are considered, the TSFS is modified as follows:

$$\begin{aligned} x(t+1) &= A_z x(t) + B_z u(t) + E_z v(t), \\ y(t) &= C_z x(t), \end{aligned} \quad (2)$$

where

$$E_z = \sum_{i=1}^{r} h_i(z(t)) E_i, \quad v(t) \in R.$$

The non-PDC control law

$$\begin{aligned} u(t) &= -\Big(\sum_{i=1}^{r} h_i\big(z(t)\big) F_i\Big)\Big(\sum_{j=1}^{r} h_j\big(z(t)\big) P_j\Big)^{-1} x(t) \\ &= -F_z P_z^{-1} x(t), \end{aligned} \quad (3)$$

with the Lyapunov function candidate

$$\begin{aligned} V(t) &= x^T(t)\Big(\sum_{i=1}^{r} h_i\big(z(t)\big) P_i\Big)^{-1} x(t) \\ &= x^T(t) P_z^{-1} x(t), \end{aligned} \quad (4)$$
$$P_i = P_i^T > 0,$$

is considered.

As in (Guerra and Vermeiren, 2004), in what follows, if $Y_z = Y_z(t)$, then $Y_{z+} = Y_z(t+1)$, $Y_z^{-1} = (Y_z)^{-1}$ and $Y_z^{-T} = (Y_z^{-1})^T$.

## 3. Control Performance Specifications

### 3.1. Decay Rate Specification

**Definition 1.** A discrete-time system is said to be *globally exponentially stable* if there exist positive constants $\alpha, 0 < \alpha < 1$ and $\beta > 0$, such that

$$\|x(t)\| \le \alpha^t \beta \|x(0)\|. \quad (5)$$

The number $\alpha$ is known as the decay rate.

**Lemma 1.** *If there exists a Lyapunov function such that*

$$\Delta V(x(t)) \le (\alpha^2 - 1) V(x(t)) \quad (6)$$

*for all trajectories of the fuzzy system (1), then the system is exponentially stable with a decay rate* $\alpha$.

*Proof.* Let $\lambda_i[M]$ denote the $i$-th eigenvalue of the matrix $M$. From (6) we have

$$V\big(x(t+1)\big) \le \alpha^2 V\big(x(t)\big) \le \alpha^{2t} V\big(x(0)\big). \quad (7)$$

Thus,

$$\begin{aligned} &\inf_z \min_i |\lambda_i\left[P_z^{-1}\right]| \|x(t)\|^2 \\ &\le \min_i |\lambda_i\left[P_z^{-1}\right]| \|x(t)\|^2 \\ &\le x^T(t) P_z^{-1} x(t) \\ &\le \alpha^{2t} x^T(0) P_{z_0}^{-1} x(0) \\ &\le \alpha^{2t} \max_i |\lambda_i\left[P_{z_0}^{-1}\right]| \|x(0)\|^2, \end{aligned}$$

where $P_{z_0} = \sum_{i=1}^{r} h_i(z(0)) P_i$, which is equivalent to (5) in the following form:

$$\|x(t)\| \le \alpha^t \left(\frac{\max_i |\lambda_i\left[P_{z_0}^{-1}\right]|}{\inf_z \min_i |\lambda_i\left[P_z^{-1}\right]|}\right)^{1/2} \|x(0)\|. \quad (8)$$

■

Defining $\Upsilon_{ij}^k$ as

$$\Upsilon_{ij}^k = \begin{bmatrix} \alpha^2 P_i & (*) \\ A_i P_j - B_i F_j & P_k \end{bmatrix}, \quad (9)$$
$$i,j,k \in \{1,\dots,r\}$$

where the asterisk denotes the transpose of the corresponding block below the main diagonal, the following theorem can be formulated:

**Theorem 1.** *Consider the discrete TSFS (1) and the non-PDC control law (3). With* $\Upsilon_{ij}^k$ *defined in (9), if there exist matrices* $P_i > 0$, $Q_i^k > 0$, $Q_{ij}^k = (Q_{ij}^k)^T$, $j > i$ *and matrices* $F_i$ *such that*

$$\Upsilon_{ii}^k > Q_i^k, \quad i, k \in \{1, \dots, r\}, \tag{10}$$

$$\Upsilon_{ij}^k + \Upsilon_{ji}^k > Q_{ij}^k, \ j > i, \quad i, j, k \in \{1, \dots, r\}, \tag{11}$$

$$\Psi^k = \begin{bmatrix} 2Q_1^k & (*) & \dots & (*) \\ Q_{12}^k & 2Q_2^k & \dots & \vdots \\ \vdots & \vdots & \ddots & (*) \\ Q_{1r}^k & \dots & Q_{(r-1)r}^k & 2Q_r^k \end{bmatrix} > 0,$$

$$k \in \{1, \dots, r\}, \tag{12}$$

*then the closed-loop TSFS is globally asymptotically stable with a decay rate* $\alpha, 0 < \alpha < 1$.

*Proof.* Consider the Lyapunov function candidate (4) for the system (1) under the control law (3). Since $\forall i,\ P_i > 0$ and $h_i(z(t)) \geq 0$ have a convex sum property, then $P_z > 0$ and $P_z^{-1} > 0$ (Guerra and Vermeiren, 2004). The variation of the Lyapunov function according to (6) can be rewritten as follows:

$$\begin{aligned} &\Delta V(x(t)) - (\alpha^2 - 1)V(x(t)) \\ &\quad = V(x(t+1)) - \alpha^2 V(x(t)) \\ &\quad = x^T(t+1)P_{z+}^{-1}x(t+1) - \alpha^2 x(t)^T P_z^{-1} x(t) \\ &\quad = x^T(t)((A_z - B_z F_z P_z^{-1})^T P_{z+}^{-1}(A_z - B_z F_z P_z^{-1}) \\ &\qquad -\alpha^2 P_z^{-1})x(t) \\ &\quad = x^T(t) R x(t) \leq 0. \end{aligned}$$

The previous inequality holds if $R < 0$. Premultiplying and postmultiplying $R$ by $P_z$ yields

$$(P_z A_z^T - F_z^T B_z^T)P_{z+}^{-1}(A_z P_z - B_z F_z) - \alpha^2 P_z < 0.$$

Now, taking the Schur complement of the previous expression gives

$$\begin{bmatrix} \alpha^2 P_z & (*) \\ A_z P_z - B_z F_z & P_{z+} \end{bmatrix} > 0$$

which is equivalent to the conditions (10) and (11) under the definitions (12) and (9) (Guerra and Vermeiren, 2004). ■

**Remark 1.** Note that the lowest upper bound on the decay rate can be found by solving the following generalized eigenvalue problem (GEVP): Minimize $\alpha$ subject to the LMIs (10)–(12) under the definition (9).

### 3.2. Constraints on the Input and Output

Without loss of generality, assume that for the Lyapunov function (4) the inequality

$$V(x(t)) \leq V(x(0)) \leq 1, \quad t \geq 0 \tag{13}$$

holds, which can be guaranteed by a proper choice of the initial conditions. Also, recall that $V(x(t)) \leq V(x(0)),\ t \geq 0$, holds for every Lyapunov function since it is, by definition, a positive monotonically decreasing function.

This condition can be expressed via LMIs, since

$$V(x(0)) \leq 1 \iff 1 - x^T(0)P_{z_0}^{-1}x(0) \geq 0$$

is equivalent to

$$\begin{bmatrix} 1 & (*) \\ x(0) & P_{z_0} \end{bmatrix} > 0$$

via the Schur complement, which is implied by

$$\begin{bmatrix} 1 & (*) \\ x(0) & P_i \end{bmatrix} > 0, \quad i \in \{1, \dots, r\}. \tag{14}$$

**Theorem 2.** *Consider the discrete TSFS (1) and the non-PDC control law (3). Assume that the initial condition* $x(0)$ *is known. The condition* $\|u(t)\| \leq \mu,\ \forall t \geq 0$ *holds if so do the LMIs (14) and*

$$\begin{bmatrix} P_i & (*) \\ F_j & \mu^2 I \end{bmatrix} > 0, \quad i, j \in \{1, \dots, r\}. \tag{15}$$

*Proof.* Recalling (13) and (14), it is clear that the condition $\|u(t)\| \leq \mu$ can be rewritten by means of (3) as follows:

$$\begin{aligned} &u^T(t)u(t) = x^T(t)P_z^{-T}F_z^T F_z P_z^{-1}x(t) \leq \mu^2 \\ &\quad \iff \frac{1}{\mu^2}x^T(t)P_z^{-T}F_z^T F_z P_z^{-1}x(t) \leq 1. \end{aligned}$$

Recalling (13), it is clear that the previous inequality holds if

$$\begin{aligned} &\frac{1}{\mu^2}x^T(t)P_z^{-T}F_z^T F_z P_z^{-1}x(t) \\ &\qquad \leq x^T(t)P_z^{-1}x(t) = V(x(t)). \end{aligned}$$

This condition is equivalent to

$$\begin{aligned} &\frac{1}{\mu^2}x^T(t)P_z^{-T}F_z^T F_z P_z^{-1}x(t) - x^T(t)P_z^{-1}x(t) \\ &\qquad = x^T(t)\left[\frac{1}{\mu^2}P_z^{-T}F_z^T F_z P_z^{-1} - P_z^{-1}\right]x(t) \leq 0, \end{aligned}$$

from which we get

$$\frac{1}{\mu^2}P_z^{-T}F_z^TF_zP_z^{-1} - P_z^{-1} \le 0.$$

Pre- and postmultiplying it by $P_z$ and rearranging some terms, we obtain

$$P_z - \frac{1}{\mu^2}F_z^TF_z \ge 0,$$

and, by the Schur complement,

$$\begin{bmatrix} P_z & (*) \\ F_z & \mu^2 I \end{bmatrix} > 0,$$

which is implied by LMIs (15). ■

**Theorem 3.** *Consider the discrete TSFS (1) and the non-PDC control law (3). Assume that the initial condition $x(0)$ is known. The condition $\|y(t)\| \le \lambda$, $\forall t \ge 0$ holds if so do the LMIs (14) and*

$$\begin{bmatrix} P_j & (*) \\ C_iP_j & \lambda^2 I \end{bmatrix} > 0, \quad i,j \in \{1,\dots,r\}. \tag{16}$$

*Proof.* As before, (13) implies (14). The condition $\|y(t)\| \le \lambda$ can be rewritten by means of (1) as follows:

$$y^T(t)y(t) = x^T(t)C_z^TC_zx(t) \le \lambda^2$$

$$\iff \frac{1}{\lambda^2}x^T(t)C_z^TC_zx(t) \le 1.$$

Recalling (13), it is clear that the previous inequality holds if

$$\frac{1}{\lambda^2}x^T(t)C_z^TC_zx(t) \le x^T(t)P_z^{-1}x(t) = V(x(t)),$$

which is equivalent to

$$\frac{1}{\lambda^2}x^T(t)C_z^TC_zx(t) - x^T(t)P_z^{-1}x(t)$$

$$= x^T(t)\left[\frac{1}{\lambda^2}C_z^TC_z - P_z^{-1}\right]x(t) \le 0,$$

which yields

$$\frac{1}{\lambda^2}C_z^TC_z - P_z^{-1} \le 0.$$

Pre- and postmultiplying this result by $P_z$ and rearranging some terms, we get

$$P_z - \frac{1}{\lambda^2}P_z^TC_z^TC_zP_z \ge 0,$$

and, by the Schur complement,

$$\begin{bmatrix} P_z & (*) \\ C_zP_z & \lambda^2 I \end{bmatrix} > 0,$$

which is implied by the LMIs (16). ■

### 3.3. Disturbance Rejection

Consider the TSFS (2), where $v(t)$ is the disturbance. In the sequel, disturbance rejection will be considered as minimizing $\gamma > 0$ subject to

$$\sup_{\|v(t)\|_2 \ne 0} \frac{\|y(t)\|_2}{\|v(t)\|_2} \le \gamma, \tag{17}$$

where $\|\cdot\|_2$ stands for the $\ell_2$ norm.

**Theorem 4.** *Consider the discrete TSFS (2) and the non-PDC control law (3). The condition (17) holds if so do the LMIs (10)–(12) and*

$$\begin{bmatrix} \gamma^2 I & (*) \\ E_i & P_j \end{bmatrix} > 0, \quad i,j \in \{1,\dots,r\}, \tag{18}$$

*under the definition*

$$\Upsilon_{ij}^k = \begin{bmatrix} P_i & (*) & (*) \\ A_iP_j - B_iF_j & P_k & 0 \\ C_iP_j & 0 & I \end{bmatrix}, \tag{19}$$

$$i,j,k \in \{1,\dots,r\}.$$

*Proof.* With no loss of generality, consider the Lyapunov function candidate (4) and $\gamma > 0$ such that, for all $t$,

$$\Delta V(x(t)) + y^T(t)y(t) - \gamma^2 v^T(t)v(t) \le 0. \tag{20}$$

This condition implies

$$\sum_{t=0}^{T_f}\left[\Delta V(x(t)) + y^T(t)y(t) - \gamma^2 v^T(t)v(t)\right] \le 0.$$

Assuming that $x(0) = 0$, we obtain

$$V(x(T_f)) + \sum_{t=0}^{T_f}\left[y^T(t)y(t) - \gamma^2 v^T(t)v(t)\right] \le 0.$$

Since $V(x(T_f)) \ge 0$, this implies (17).

The condition (20) can be transformed as follows:

$$\Delta V(x(t)) + y^T(t)y(t) - \gamma^2 v^T(t)v(t)$$

$$= V(x(t+1)) - V(x(t))$$

$$+ x^T(t)C_z^TC_zx(t) - \gamma^2 v^T(t)v(t)$$

$$= x^T(t+1)P_{z+}^{-1}x(t+1) - x^T(t)P_z^{-1}x(t)$$

$$+ x^T(t)C_z^TC_zx(t) - \gamma^2 v^T(t)v(t)$$

$$
\begin{aligned}
&= x^T(t)(A_z - B_zF_zP_z^{-1})^T P_{z+}^{-1}(A_z - B_zF_zP_z^{-1})x(t) \\
&\quad + v^T(t)E_{z+}^T P_{z+}^{-1} E_z v(t) - x^T(t)P_z^{-1}x(t) \\
&\quad + x^T(t)C_z^T C_z x(t) - \gamma^2 v^T(t)v(t) \\
&= x^T(t)\big[(A_z - B_zF_zP_z^{-1})^T P_{z+}^{-1}(A_z - B_zF_zP_z^{-1}) \\
&\quad - P_z^{-1} + C_z^T C_z\big]x(t) \\
&\quad + v^T(t)\left[E_z^T P_{z+}^{-1}E_z - \gamma^2 I\right] v(t) \le 0,
\end{aligned}
$$

which can be achieved if

$$
\begin{aligned}
&P_z^{-1} - C_z^T C_z \\
&\quad - (A_z - B_zF_zP_z^{-1})^T P_{z+}^{-1}(A_z - B_zF_zP_z^{-1}) > 0, \\
&\quad \gamma^2 I - E_z^T P_{z+}^{-1} E_z > 0,
\end{aligned}
$$

or, equivalently, by pre- and postmultiplying the first inequality by $P_z$, if

$$
\begin{aligned}
&P_z - P_z^T C_z^T C_z P_z \\
&\quad - (A_zP_z - B_zF_z)^T P_{z+}^{-1}(A_zP_z - B_zF_z) > 0, \\
&\quad \gamma^2 I - E_z^T P_{z+}^{-1} E_z > 0.
\end{aligned}
$$

Taking the Schur complements of the previous expressions gives

$$
\begin{bmatrix} \gamma^2 I(*) \\ E_zP_{z+} \end{bmatrix} > 0, \quad
\begin{bmatrix} P_z(*)(*) \\ A_zP_z - B_zF_zP_{z+}0 \\ C_zP_z0I \end{bmatrix} > 0,
$$

which is implied by the LMIs (10)–(12) and (18) under the definition (19) (Guerra and Vermeiren, 2004). ■

**Remark 2.** Recall that it is possible to find the lowest upper bound on $\gamma$ via a generalized eigenvalue problem (GEVP): Minimize $\gamma$ subject to the LMIs (10)–(12) and (18) under the definition (19).

**Remark 3.** Since the developed designs are specified in terms of LMIs, they can be combined without further adaptations. However, note that the more conditions are imposed on a certain plant, the more conservative the results can be.

## 4. Example

This section presents an example to illustrate the effect of decay rate design, constraints on the input and output and disturbance rejection. Every set of LMIs was solved via the MATLAB LMI toolbox.

Consider the following system (Guerra and Vermeiren, 2004):

$$\mathbf{R_1}: \quad \textbf{If } x_1 \textbf{ is } F_1^1(x_1(t)) \textbf{ then}$$

$$x(t+1) = \begin{bmatrix} -0.5 & 2 \\ -0.1 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 4.1 \\ 4.8 \end{bmatrix} u(t),$$

$$y(t) = [\ 0 \quad 1\ ]x(t),$$

$$\mathbf{R_2}: \quad \textbf{If } x_1 \textbf{ is } F_1^2(x_1(t)) \textbf{ then}$$

$$x(t+1) = \begin{bmatrix} -0.9 & 0.5 \\ -0.1 & -1.7 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 0.1 \end{bmatrix} u(t),$$

$$y(t) = [\ 0 \quad 1\ ]x(t), \tag{21}$$

with the membership functions $F_1^1(x_1(t)) = (x_1(t) + 1.3)/2.6$ and $F_1^2(x_1(t)) = 1 - F_1^1(x_1(t))$.

*Decay rate:* Employing Theorem 1 it is possible to find a controller for the system (21) to achieve a decay rate $\alpha = 0.77$, since the LMIs (10)–(12) were found feasible under the definition (9). The non-PDC control law (3) with the gains

$$
\begin{aligned}
P_1 &= \begin{bmatrix} 8687.5 & -252.3 \\ -252.3 & 34.4 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 4243.3 & -341.7 \\ -341.7 & 32.5 \end{bmatrix}, \\
F_1 &= [-265.3943\ 11.0817], \\
F_2 &= [-355.1544\ 13.4394]
\end{aligned}
$$

can be applied to stabilize the system with the given decay rate specification. When no decay rate is specified ($\alpha = 1$), the following gains are obtained:

$$
\begin{aligned}
P_1 &= \begin{bmatrix} 203.9358 & -12.0539 \\ -12.0539 & 6.9228 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 176.6025 & -15.0991 \\ -15.0991 & 1.8509 \end{bmatrix}, \\
F_1 &= [-7.2354\ \ 1.2357], \\
F_2 &= [-10.7477\ \ -0.7677].
\end{aligned}
$$

Figures 1 and 2 show the evolution of the states $x_1$ and $x_2$, respectively, from the initial condition $[0\ \ -3]^T$ with the previous decay rate specification (the solid line) and with no prescribed decay rate (the dashed line). The speed of the response can be increased by a decay rate specification at the expense of higher gains in the control law as well as higher transient responses.

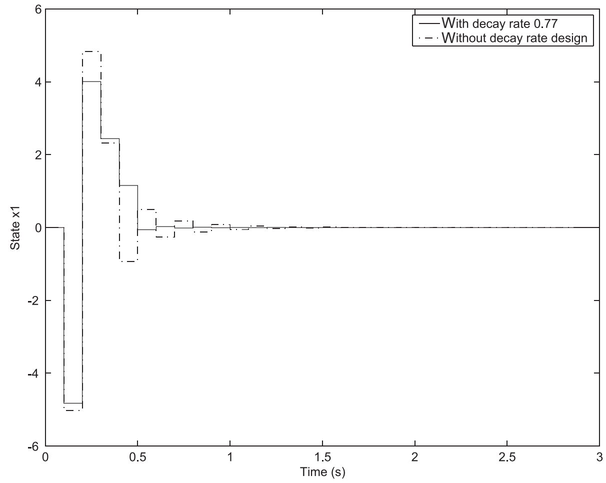


Fig. 1. Decay rate design: state $x_1$.

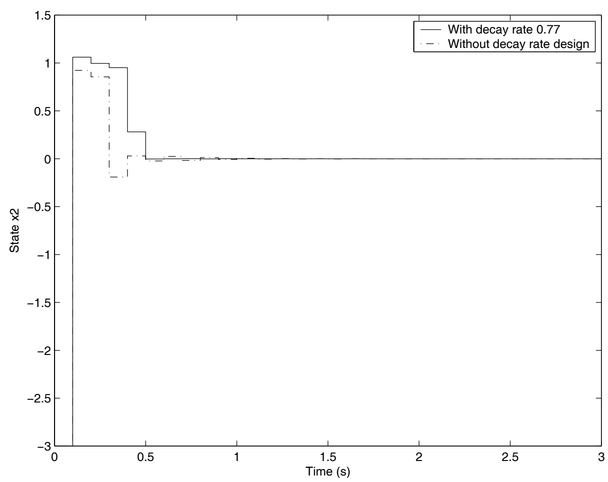


Fig. 2. Decay rate design: state $x_2$.

*Constraints on the input:* Consider again the system (21) and the condition $\|u(t)\| \leq 0.15,\ \forall t \geq 0$. With $\mu = 0.15$, the LMIs (10)–(12) and (14)–(15) were found feasible under the definition (9). The non-PDC control law (3) with the gains

$$P_1 = \begin{bmatrix} 7.9452 & -1.3043 \\ -1.3043 & 0.2538 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4.8357 & -0.5722 \\ -0.5722 & 0.0777 \end{bmatrix},$$

$$F_1 = [-0.4144\ \ 0.0731],$$

$$F_2 = [-0.0408\ \ -0.0101]$$

can be employed to stabilize the system and meet the required control input constraint. Figure 3 shows the control input signal (the dashed line) when no constraints on it are considered, whilst the solid line represents the control input signal under the underlying constraint.

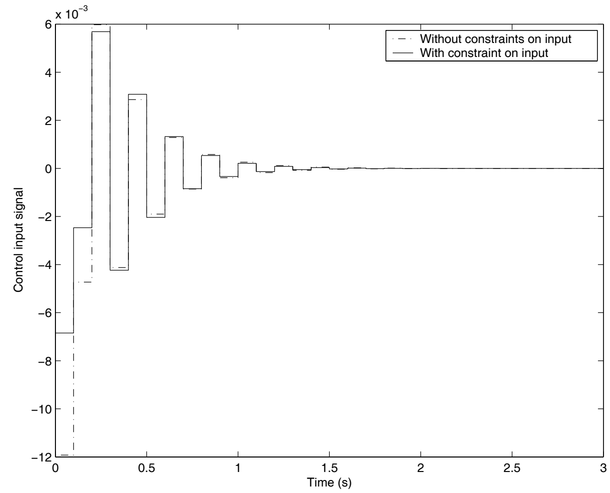


Fig. 3. Constraints on the input.

*Constraints on the output:* As for the system (21) subject to the condition $\|y(t)\| \leq 0.14,\ \forall t \geq 0$, the LMIs (10)–(12) and (14)–(16) are feasible under the definition (9), where $\lambda = 0.14$. The non-PDC control law (3) with the gains

$$P_1 = \begin{bmatrix} 21.9561 & 0.0307 \\ 0.0307 & 0.0195 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4.5025 & -0.2065 \\ -0.2065 & 0.0195 \end{bmatrix},$$

$$F_1 = [-0.5411\ \ 0.0018],$$

$$F_2 = [-0.7417\ \ -0.0052]$$

stabilizes the system under consideration and meets the underlying output constraint. Figure 4 shows the output signal (the dashed line) when no constraint on it was applied, whilst the solid line is the output signal under the output constraint scheme.

*Combining constraints on the input and output:* As was mentioned before, combinations of the previous designs are possible up to the feasibility of the LMIs. When the system (21) is considered, subject to the conditions $\|y(t)\| \leq 0.4,\ \|u(t)\| \leq 0.16,\ \forall t \geq 0$, then with $\mu = 0.15$ and $\lambda = 0.14$ the LMIs (10)–(12) and (14)–(16) were found feasible under the definition (9). The non-PDC control law (3) with the gains

$$P_1 = \begin{bmatrix} 5.4941 & -0.8212 \\ -0.8212 & 0.1658 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4.6619 & -0.5324 \\ -0.5324 & 0.0708 \end{bmatrix},$$

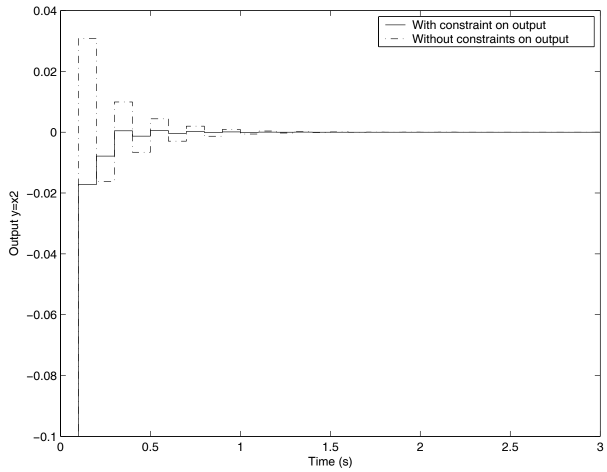


Fig. 4. Constraints on the output.

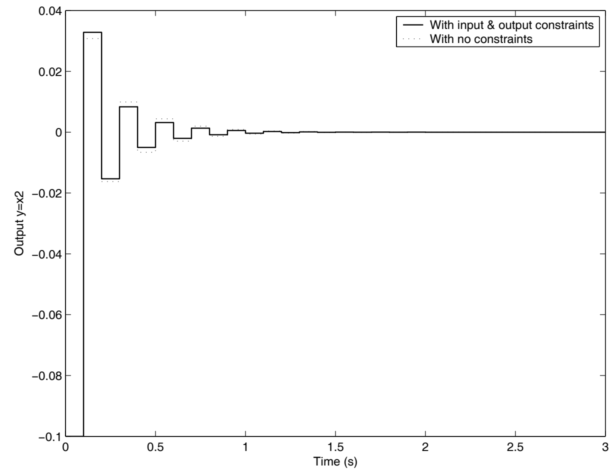


Fig. 5. Combining constraints on the input and output.

$$F_1 = [-0.2895 \;\; 0.0494],$$
$$F_2 = [-0.0709 \;\; -0.0076]$$

stabilizes the system under consideration and meets the underlying constraints. Figure 5 shows the output signal (the dashed line) when no constraint on it was applied, whilst the solid line is the output signal under the output and input constraint scheme. Figure 6 shows the input signal (the dashed line) when no constraint on it was applied and with the output signal under the output and input constraint scheme (the solid line).

*Disturbance rejection:* Consider the following modified version of the system (21):

$\mathbf{R_1}$ : **If** $x_1$ **is** $F_1^1(x_1(t))$ **then**

$$x(t+1) = \begin{bmatrix} -0.5 & 2 \\ -0.1 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 4.1 \\ 4.8 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t),$$

$$y(t) = [\; 0 \quad 1 \;] x(t),$$

$\mathbf{R}_2$ : **If** $x_1$ **is** $F_1^2(x_1(t))$ **then**

$$x(t+1) = \begin{bmatrix} -0.9 & 0.5 \\ -0.1 & -1.7 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t),$$

$$y(t) = [\; 0 \quad 1 \;] x(t), \qquad (22)$$

where $v(t)$ is a random disturbance with uniform distribution in the interval $[-0.01, 0.01]$.

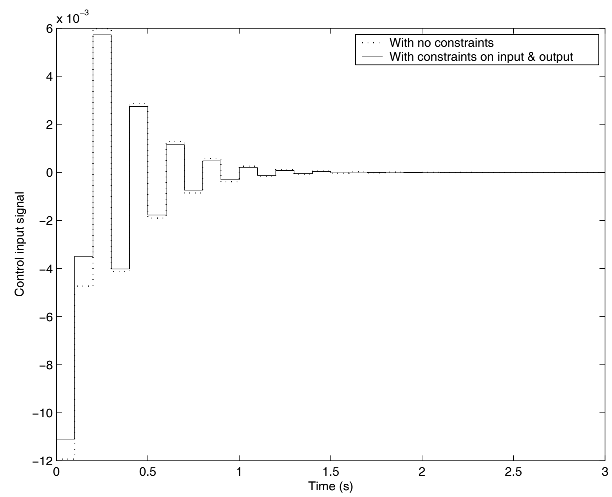


Fig. 6. Combining constraints on the input and output.

Applying Theorem 4 to the system (22) with $\gamma = 2.5$, a feasible solution to the LMIs (10)–(12) and (18) under the definition (19) was found, providing the control law (3) with the following gains:

$$P_1 = \begin{bmatrix} 379.6033 & 2.2591 \\ 2.2591 & 0.6821 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 93.5625 & -4.3447 \\ -4.3447 & 0.3697 \end{bmatrix},$$

$$F_1 = [-9.0012 \;\; 0.0251],$$

$$F_2 = [-14.2197 \;\; -0.0923].$$

In Figures 7 and 8, the evolution of the states under disturbance rejection is shown with solid lines, while the states under a simpler stabilizing controller are marked with dashed lines.

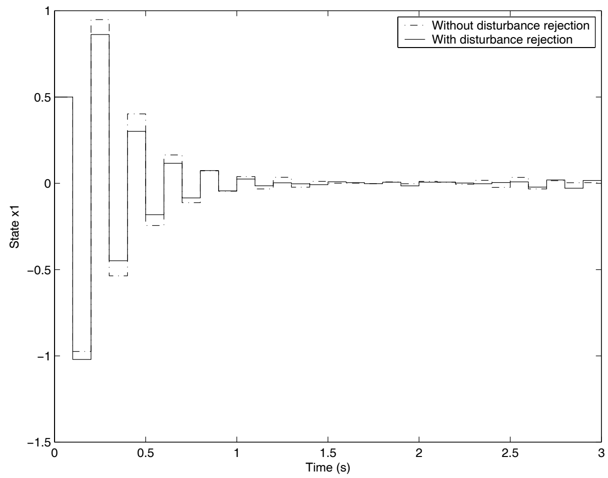


Fig. 7. Disturbance rejection: state $x_1$.

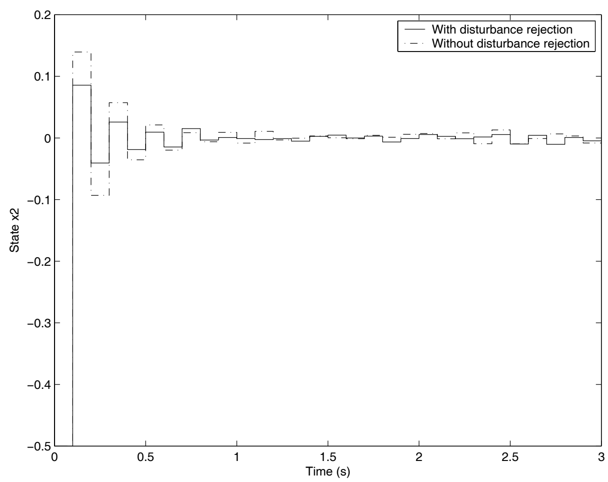


Fig. 8. Disturbance rejection: state $x_2$.

## 5. Conclusion

The paper develops some extensions for a non-quadratic fuzzy design, which permit us to specify the decay rate, meet constraints on the input and output and reject disturbances. The design employed uses a non-quadratic Lyapunov function with a non-PDC control law, which is proved to reduce conservativeness. Simulation examples are provided to illustrate the design procedure and performance.

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