

Non-Recurrent Behaviour in Quantum Dynamics

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Abstract. We study the motion of a quantum rotator under an external periodic perturbation. For the resonant case, i. e. when the frequency of driving pulses is rationally connected with the frequencies of the free rotator, the quasi-energy spectrum is known to be continuous. We prove that for a generic choice of the potential there is a non-empty set of non-resonant values of the external frequency such that the quasi-energy spectrum still has a continuous component.

I.

Up to now, quantum mechanics has offered but a poor surrogate for the variety of the forms in which chaotic behaviour can manifest in classical mechanics. Discreteness of the spectra of bounded systems and localization of states in unbounded systems with disordered potentials are major drawbacks that seem to constrain the search for chaotic quantum motion to the field of time-dependent Hamiltonians, and to time-periodic Hamiltonians in particular.

The so-called “ δ -kicked rotator” has been being used extensively as a model system in this line of research^[1-7]. It displays non-recurrent behaviour, associated with a continuous component in the quasi-energy spectrum, when the frequency of the driving pulses is rationally connected with the frequencies of the free rotator^[2]. However interesting in itself, this “quantum resonance” is quite another type of behaviour than chaotic; therefore, the latter can possibly take place only for “non-resonant” values of the driving frequency.

For such values, numerical evidence is that there are strong limitations to the occurrence – if any – of non-recurrent behaviour^[1-3]. These desultory numerical results, together with the recently discovered similarity of the rotator problem with the problem of localization^[6], seem to support the thesis that there are no possibilities for periodically driven quantum systems other than recurrence and resonance.

The result we present in this paper is that a “third way” actually exists. In fact, we prove that for a “generic” choice of the potential of the kicked rotator there are

non-resonant situations in which the quasi-energy spectrum still has a continuous component; also, there are reasons to conjecture that this component is actually a singular continuous one.

We give the proof in Sect. II. In the conclusive Sect. III we discuss somewhat the relevance of this result, as well as some possibilities of future development.

II.

We study the quantum properties of the δ -kicked rotator, described by the Hamiltonian:

$$\hat{H} = -\frac{1}{2} \frac{d^2}{d\theta^2} + \mu V(\theta) \sum_{n=-\infty}^{\infty} \delta(t - 4\pi n\tau),$$

where $V(\theta)$ is an *analytical*, real-valued, periodic function of θ , and we have assumed $\hbar = 1$. We denote by C the unit circle.

Mapping over one period involves free rotation and a kick:

$$\begin{aligned} (S\psi)(\theta) &= e^{-i\mu V(\theta)} \sum_{n=-\infty}^{\infty} c_n e^{-i(2\pi n^2 - n\theta)} \\ \psi(\theta) &= \sum c_n e^{in\theta}. \end{aligned} \tag{1}$$

From (1) we see that it suffices to consider values of τ within the interval $[0, 1]$.

Let us consider first the “resonant” case – in which $\tau = p/q$, p, q integers. To $\psi \in \mathcal{L}^2(C)$, we associate the q -vector

$$\left(\psi(\theta), \psi\left(\theta + \frac{2\pi}{q}\right), \dots, \psi\left(\theta + \frac{2\pi(q-1)}{q}\right) \right), \quad 0 \leq \theta < \frac{2\pi}{q}.$$

In this way $\mathcal{L}^2(C)$ is identified with $\mathcal{L}^2(I_q, C^q)$, where $I_q = \left(0, \frac{2\pi}{q}\right)$. As shown in ref. (2), in this case S allows for the integral decomposition

$$S = \int_0^{2\pi/q} d\theta S_\theta,$$

with S_θ a $q \times q$ unitary matrix: $S_\theta = W_\theta \Gamma$. W_θ is a diagonal matrix,

$$(W_\theta)_{nm} = \delta_{nm} e^{-i\mu V\left(\theta + \frac{2\pi n}{q}\right)} \quad (0 \leq n, m \leq q-1), \tag{1a}$$

and Γ is a circulant matrix independent of θ . We are not interested in further specification of Γ . For more details, see ref. (2).

S_θ has eigenvalues depending on θ , and, as shown in [2], at least one of them is not constant over I_q : this eigenvalue originates a continuous band in the spectrum of S .

We are interested in S having a purely continuous spectrum, i. e., no eigenvalues. Eigenvalues of S occur when S_θ has constant eigenvalues over sets of positive measure in I_q . Since S_θ is a matrix-valued analytic function of θ , its eigenvalues will be analytic functions of θ – except for branch points at values where level crossing occurs. Therefore, if an eigenvalue $\lambda(\theta)$ is constant over a set of positive measure, it will be constant all over I_q .

Then, consider the analytic, real-valued, periodic functions $V(\theta)$ in C , and let \mathcal{B}_α be the vector space of all such functions having an analytic continuation in the strip $C_\alpha = \{|\operatorname{Im} z| < \alpha\}$, continuous in \overline{C}_α . \mathcal{B}_α is a Banach space with the norm $\|\psi\|_\alpha = \max_{z \in \overline{C}_\alpha} |\psi(z)|$.

Theorem 1. *For any $\alpha > 0$, the set of $V \in \mathcal{B}_\alpha$ such that S , in resonance, has a purely continuous spectrum contains a set of the second Baire category in \mathcal{B}_α . In this sense we'll say that generic analytic V have this property.*

Proof. Suppose that λ is a constant, nondegenerate eigenvalue for a given $\tau = p/q$, and let $x = x(\theta)$ be a corresponding eigenvector (q -vector). Then $S_\theta x = \lambda x$, whence, differentiating,

$$\dot{S}_\theta x + S_\theta \dot{x} = \lambda \dot{x}, \text{ i.e. } (\lambda - S_\theta) \dot{x} = \dot{S}_\theta x = \dot{W}_\theta \Gamma x = M_\theta S_\theta x = \lambda M_\theta x,$$

where $(M_\theta)_{mn} = -i\mu V' \left(\theta + \frac{2\pi m}{q} \right) \delta_{mn}$. Therefore, we must have $(M_\theta x, x) = 0$ for all $\theta \in I_q$; in particular,

$$\sum_{j=0}^{q-1} V' \left(\frac{2\pi j}{q} \right) |x_j(0)|^2 = 0. \quad (2)$$

The matrix S_0 is fixed by the specification of the q numbers $V_j = V \left(\frac{2\pi j}{q} \right)$; therefore the left-handside of (2) defines a function of the $2q$ numbers: $V_j, V'_j = V' \left(\frac{2\pi j}{q} \right)$.

In a neighbourhood of any point (V_j, V'_j) corresponding to a matrix with no degenerate eigenvalues, this function is actually a continuous one. Therefore, the set of values (V_j, V'_j) such that

- (i) S_0 has no degenerate eigenvalues,
- (ii) Condition (2) is *not* satisfied by *any* eigenvector of S_0 ,

is open in C^{2q} , being the intersection of the open sets defined by (i) and (ii) separately; the set $X_{p,q,\alpha}$ of $V \in \mathcal{B}_\alpha$ such that (i) and (ii) are satisfied by (V_j, V'_j) is likewise open, because the map from \mathcal{B}_α into C^{2q} defined by $V(\theta) \rightsquigarrow (V_j, V'_j)$ is continuous, as a consequence of Cauchy's formula. This set is dense, also. In fact, given $V \in \mathcal{B}_\alpha$, we can first remove any degeneracy from the corresponding S_0 by an arbitrarily small variation of V . (This we prove in the Appendix.) If the \overline{S}_0 thus obtained satisfies (2) for one of its eigenvectors, then we may change $V(\theta)$ into $\overline{V} + \varepsilon \sin q\theta = \overline{V}(\theta)$. $\overline{V}(\theta)$ is arbitrarily close to $\overline{V}(\theta)$ in \mathcal{B}_α and its matrix S_0 has the same eigenvectors, but (2) is no longer satisfied.

Consider now the set $X_\alpha = \bigcap_{p,q} X_{p,q,\alpha}$. If $V \in X_\alpha$, then, for all p, q , S_0 will have no degenerate eigenvalues, and no eigenvector of S_0 will satisfy (2). Therefore, for any $\tau = p/q$, S_θ will have no constant eigenvalues. Thus, X_α is contained in the set of those $V \in \mathcal{B}_\alpha$ which have purely continuous spectra in resonance; on the other hand, X_α is a countable intersection of dense open sets in \mathcal{B}_α , i.e., it is a set of the second Baire category in \mathcal{B}_α .

So, for rational τ and generic $V(\theta)$, S_τ will have a purely continuous spectrum (from now on, the index of S will specify the period). This entails that, $\forall \psi \in L^2(C)$ [8],

$$\lim_{N \rightarrow \infty} \mathcal{R}_\tau(N, \psi) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{K=1}^N |(S_\tau^K \psi, \psi)|^2 = 0.$$

We shall restrict our attention to those $\psi \in L^2(\dot{C})$ which give a finite expectation value for the kinetic energy:

$$K(\psi) = \pi \sum_{-\infty}^{\infty} n^2 |c_n|^2 < \infty.$$

For $E > 0$, we define $\sigma_E = \{\psi \in L^2(C), \|\psi\| = 1, K(\psi) \leq E\}$.

Now we set out to prove our main result. We begin with some lemmas.

Lemma 1. *Let S_τ have purely continuous spectrum. Then $\mathcal{R}_\tau(N, \psi) \rightarrow 0$ as $N \rightarrow \infty$, uniformly with respect to $\psi \in \sigma_E$.*

Proof. For a fixed τ the sequence $\mathcal{R}_\tau(N, \psi)$ is equicontinuous in $\psi \in \sigma_E$; indeed:

$$|\mathcal{R}_\tau(N, \psi) - \mathcal{R}_\tau(N, \psi')| \leq \frac{2}{N} \sum_{K=1}^N \{ |(S_\tau^K \psi, \psi - \psi')| + |(S_\tau^K \psi - S_\tau^K \psi', \psi')| \} \leq 4 \|\psi - \psi'\|.$$

Moreover, σ_E is a compact set in $L^2(C)$.

Lemma 2. *Let ψ be entire analytic, $V \in \mathcal{B}_\alpha$, $F(\psi) = \sum n^{2r} |c_n|^2$; ($r > 0$). Then $\exists C$ such that, $\forall N$, $F(S^N \psi) \leq CN^{2r}$.*

Proof. $S^N \psi \in \mathcal{B}_\alpha$, $\forall N$. Therefore, by the Cauchy theorem the Fourier coefficients $c_n(N)$ of $S^N \psi$ and W_n of $e^{-i\mu V}$ satisfy:

$$q_\beta(N) = \sup_n e^{\beta|n|} |c_n(N)| < \infty, \quad \forall \beta \in [0, \alpha], \quad |W_n| \leq d e^{-\alpha|n|}.$$

Moreover, if $c'_n(N)$ are Fourier coefficients of $T\psi(N)$ [the “free” evolution of $\psi(N)$ over one period]

$$\begin{aligned} |c_n(N+1)| &= \left| \sum_K W_{n-K} c'_K(N) \right| \leq \sum_K |W_{n-K}| |c'_K(N)| \\ &= \sum_K |W_{n-K}| |c_K(N)| \leq d q_{\alpha/2}(N) \sum_K e^{-\alpha/2|K|} e^{-\alpha|n-K|} \\ &\leq \text{const } d q_{\alpha/2}(N) e^{-(\alpha/2)|n|}. \end{aligned}$$

(The const does not depend on n, N .)

Thus we may assume $q_{\alpha/2}(N+1) \leq \text{const } q_{\alpha/2}(N)$, and $|c_n(N)| \leq b^N e^{-\alpha/2|n|}$ for a suitable $b > 0$. This means that the pattern of the c'_n s cannot “spread” in time faster than linearly; it is then easy to conclude that $F(S^N \psi) < \text{const } N^{2r}$.

Lemma 3. *Let ψ be an entire function with $\|\psi\| = 1$. Then*

$$|\mathcal{R}_\tau(N, \psi) - \mathcal{R}_{\tau'}(N, \psi)| \leq |\tau - \tau'| \beta N^3$$

for a suitable $\beta > 0$ independent of τ and τ' .

Proof. Consider, first,

$$|(S_\tau^N \psi - S_{\tau'}^N \psi, \psi)| \leq \|(S_\tau^N - S_{\tau'}^N) \psi\| = Q_N.$$

Then,

$$Q_N \leq \|(S_\tau - S_{\tau'}) S_\tau^{N-1} \psi\| + \|S_\tau^{N-1} \psi - S_{\tau'}^{N-1} \psi\| = \|(S_\tau - S_{\tau'}) S_\tau^{N-1} \psi\| + Q_{N-1}.$$

Now, let $\phi_N = S_\tau^{N-1} \psi$:

$$\begin{aligned} \|(S_\tau - S_{\tau'}) \phi_N\| &= \|(T_\tau - T_{\tau'}) \phi_N\| = \left\| \sum_n c_n (N-1) (e^{-2\pi i \tau n^2} - e^{-2\pi i \tau' n^2}) u_n(\theta) \right\| \\ &\leq 2\pi |\tau - \tau'| \left(\sum_n n^4 |c_n (N-1)|^2 \right)^{1/2}. \end{aligned}$$

By Lemma 2, the term within brackets grows no faster than $\text{const } N^4$. Therefore, $Q_N \leq \gamma |\tau - \tau'| N^2 + Q_{N-1}$ for a suitable $\gamma > 0$. Since $Q_0 = 0$, we get $Q_N \leq |\tau - \tau'| \gamma N^3$. Now,

$$|\mathcal{R}_\tau(N, \psi) - \mathcal{R}_{\tau'}(N, \psi)| \leq \frac{2}{N} \sum_{K=1}^N Q_K \leq \frac{2}{N} \gamma |\tau - \tau'| \sum_{K=1}^N K^3 \leq 2\gamma |\tau - \tau'| N^3.$$

Theorem 2. *For generic V and for any given value of $E > 0$ there is a nonempty set of irrationals $\tau \in (0, 1)$ such that σ_E is contained in the continuous subspace of S_τ .*

Proof. Let V be generic in the sense of Theorem 1. For integer \bar{q} and for a given $\varepsilon > 0$, define $v_E(\bar{q}, \varepsilon)$ as the smallest integer such that

$$\forall \psi \in \sigma_E, \forall \tau = p/q, q \leq \bar{q}, p < q, \forall N \geq v_E(\bar{q}, \varepsilon), |\mathcal{R}_\tau(N, \psi)| < \varepsilon.$$

That such $v_E(\bar{q}, \varepsilon)$ exists is ensured by Lemma 1. By its very definition, $v_E(\bar{q}, \varepsilon)$ is non-decreasing for $\bar{q} \rightarrow \infty$ and for $\varepsilon \rightarrow 0$. Then let $\sigma(x), \eta(x)$ be real valued functions of $x > 0$, such that

$$\sigma(x) > 0, \eta(x) > 0, \quad \text{and} \quad \sigma \rightarrow 0, \eta \rightarrow 0 \quad \text{for} \quad x \rightarrow \infty.$$

Define

$$\phi_E(q) = \frac{\sigma(q)}{v_E^3(q, \eta(q))}.$$

As $q \rightarrow \infty$, $\phi_E(q) \rightarrow 0$ by the definition of σ, η, v .

Now consider the nonempty set $L_{\phi_E} \subset [0, 1]$ of irrationals $\bar{\tau}$ that can be obtained as the limit of a sequence $\tau_n = p_n/q_n$ of rationals in such a way that $|\bar{\tau} - \tau_n| < \phi_E(q_n)$. For any such $\bar{\tau}$, and for any entire $\psi \in \sigma_E$,

$$\mathcal{R}_{\bar{\tau}}(N, \psi) \leq |\mathcal{R}_{\bar{\tau}}(N, \psi) - \mathcal{R}_{\tau_n}(N, \psi)| + \mathcal{R}_{\tau_n}(N, \psi) \leq \beta |\bar{\tau} - \tau_n| N^3 + \mathcal{R}_{\tau_n}(N, \psi),$$

thanks to Lemma 3. Now consider $N_n = v_E(q_n, \eta(q_n))$. By the definition of v_E , $N_n \rightarrow \infty$ for $n \rightarrow \infty$. We have

$$\mathcal{R}_{\bar{\tau}}(N_n, \psi) \leq \beta |\bar{\tau} - \tau_n| v_E^3(q_n, \eta(q_n)) + \eta(q_n) \leq \beta \sigma(q_n) + \eta(q_n) \rightarrow 0 \quad \text{for} \quad n \rightarrow \infty.$$

On the other hand, we know that $\lim_{N \rightarrow \infty} \mathcal{R}_{\bar{\tau}}(N, \psi)$ exists $\forall \tau, \forall \psi$, its value being

$$\int_0^{2\pi} \mu(\{\theta\}) d\mu(\theta), \text{ where } d\mu \text{ is the spectral measure of } S_{\bar{\tau}} \text{ relative to } \psi.$$

It follows that $\lim_{N \rightarrow \infty} \mathcal{R}_{\tau}(N, \psi) = 0$; therefore, $\mu(\{\theta\}) = 0$ μ -almost everywhere, i.e., μ is a purely continuous measure. Therefore, all entire $\psi \in \sigma_E$ belong to the continuous subspace of S_{τ} . By a density argument, the same holds for $\forall \psi \in \sigma_E$.

III.

The result of Theorem 2 above is certainly far from complete. At least two points need to be clarified:

1) How large is the set L_{ϕ_E} and how does its size depend on the energy E and on the parameter μ ? If the quantum regime associated with values $\tau \in L_{\phi_E}$ is to correspond somehow to classical chaotic motion, then one expects the measure of L_{ϕ_E} to increase with μ , tending eventually to 1. To answer this question one needs more detailed information on the structure of the resonant spectra, in order to estimate $\nu_E(q, \varepsilon)$, and then ϕ_E .

2) What is the nature of the continuous nonresonant spectrum? On account of numerical results obtained for the standard rotator ($V = \cos \theta$) we expect the measure of the resonant spectrum to vanish in the limit $q \rightarrow \infty$, in that case, and in the generic case too. Then the nonresonant continuous spectrum would be singular-continuous. We conjecture that quantum chaos in periodically perturbed quantum systems is connected with the occurrence of this particular type of quasi-energy spectra.

Both 1) and 2) are the object of current investigations. Nevertheless, an objection of a more general nature might be raised against our result.

In this, as well as in other cases in which “generic” results are established, one may be doubtful about the extent to which properties which are “generic” in a technical sense are actually to be observed in a majority of cases.

However necessary, this remark affects but marginally the relevance of our result. As a matter of fact, we now positively know that in some periodically perturbed quantum systems, there is room for a type of behaviour that is neither recurrent nor resonant, and strongly suspect that it is connected with the occurrence of a singular continuous quasi-energy spectrum. We think that this fact gives some firmer ground to investigations on the possibility of quantum chaotic behaviour. Indeed, it raises some interesting questions: how is this type of quantum motion related to classical chaos? Is it liable to experimental detection?

Certainly, we will not be hasty in dismissing it as a mathematical curiosity just on account of its unusual mathematical nature: after all, classical chaos itself may be elusive to an exceedingly square attitude.

Appendix

We prove here that the family of unitary matrices $S_0 = S_0(V_0, V_1, \dots, V_{q-1})$ is generic in the sense that any degeneracy can be removed from any S_0 by an arbitrarily small variation of the V_i 's. The characteristic polynomial $P(\lambda, V_0, \dots, V_{q-1})$ of S_0 has multiple roots, if and only if the resultant $G(V_0, \dots, V_{q-1})$ of P and P'_λ vanishes^[9]. G is the determinant of a matrix whose elements are coefficients in P or in P'_λ : hence,

G is an entire function of V_0, \dots, V_{q-1} . There is at least one choice of V_0, \dots, V_{q-1} such that S_0 has no degenerate eigenvalues. This is $\bar{V}_j = \frac{2\pi j}{\mu q}$, ($j=0, \dots, q-1$). In fact, let $\alpha = \exp\left(\frac{2\pi i}{q}\right)$. The circulant matrix Γ has eigenvectors \underline{u}_j ($j=0, q-1$) with components $(\underline{u}_j)_n = \alpha^{jn}$. Therefore, the matrix elements of $S_0(\bar{V}_j)$ over the base \underline{u}_j are

$$(S_0)_{jK} = v_j \delta_{j+1,K} \quad (S_0)_{q-1,K} = v_{q-1} \delta_{0,K},$$

where v_j are the eigenvalues of Γ . It is then easy to find that the eigenvalues of S_0 are the q^{th} roots of $\prod_{j=0}^{q-1} v_j$; hence, they are distinct. Thus $G(\bar{V}_0, \dots, \bar{V}_{q-1}) \neq 0$.

Now let S_0 have degenerate eigenvalues for $V_j = \bar{V}_j$, so that $G(\bar{V}_0, \dots, \bar{V}_{q-1}) = 0$. Find a C^q -valued analytic function $t \rightarrow V_0, \dots, V_{q-1}$, such that $V_j(0) = \bar{V}_j$, $V_j(1) = \bar{V}_j$. $G(V_0(t), \dots, V_{q-1}(t))$ will be an analytic function of t , not identically 0; hence, $t=1$ must be an isolated zero, so that $G(V_0(t), \dots, V_{q-1}(t)) \neq 0$ for t arbitrarily near to 1.

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