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NON-REGULAR PSEUDO-DIFFERENTIAL OPERATORS ON THE WEIGHTED TRIEBEL-LIZORKIN SPACES

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Abstract. We consider certain non-regular pseudo-differential operators and study the question of their boundedness on the weighted Triebel-Lizorkin and Besov spaces.

1. Introduction. Let $\Psi \in C^\infty(\boldsymbol{R}^n)$ satisfy $\mathrm{supp}(\Psi) \subset \{1/2 \le |\xi| \le 2\}$, $|\Psi(\xi)| \ge c > 0$ for $3/5 \le |\xi| \le 5/3$ and

$$\sum_{j\in\boldsymbol{Z}} \Psi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \neq 0\,,$$

where $\boldsymbol{Z}$ denotes the set of all integers and $|\xi| = (\sum_{j=1}^n \xi_j^2)^{1/2}$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. Define $\Phi \in C_0^\infty(\boldsymbol{R}^n)$ by $\Phi(\xi) = 1 - \sum_{j\ge1} \Psi(2^{-j}\xi)$. We note that Φ is supported in $\{|\xi| \le 2\}$. In what follows, we also assume that Ψ is radial. We write $g_t(x) = t^{-n}g(x/t)$ for $t > 0$. Define $D_j(f) = f * (\hat{\Psi})_{2^{-j}}$ for $j \ge 1$ and $D_0(f) = f * \hat{\Phi}$, where $\hat{g}$ denotes the Fourier transform:

$$\hat{g}(\xi) = \int_{\boldsymbol{R}^n} g(x)e^{-2\pi i\langle x,\xi\rangle}dx\,, \quad \langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n\,;$$

we also write $\hat{g} = \mathcal{F}(g)$.

Let A_r, $1 \le r < \infty$, be the weight class of Muckenhoupt on $\boldsymbol{R}^n$. We recall that A_r, $1 < r < \infty$, is defined to be the class of all weight functions w on $\boldsymbol{R}^n$ satisfying

$$\sup_Q \left(|Q|^{-1}\int_Q w(x)dx\right)\left(|Q|^{-1}\int_Q w(x)^{-1/(r-1)}dx\right)^{r-1} < \infty\,,$$

where the supremum is taken over all cubes $Q \subset \boldsymbol{R}^n$ and $|Q|$ denotes the Lebesgue measure of Q. Also, for a weight function w, we say that $w \in A_1$ if there exists a constant C such that $\mathcal{M}(w)(x) \le Cw(x)$ for almost every x, where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator.

Let $w \in A_\infty$, where $A_\infty = \bigcup_{1\le r<\infty} A_r$. The weighted (inhomogeneous) Triebel-Lizorkin space $F_p^{s,q}(w)$, with $0 < p < \infty$, $0 < q \le \infty$ and $s \in \boldsymbol{R}$, is defined to be the space of all tempered distributions f on $\boldsymbol{R}^n$ satisfying

$$\|f\|_{F_p^{s,q}(w)} = \left\|\left(\sum_{j=0}^\infty \left|2^{sj}D_j(f)\right|^q\right)^{1/q}\right\|_{L^p(w)} < \infty\,,$$

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where $\|\cdot\|_{L^p(w)}$ denotes the weighted L^p norm: $\|g\|_{L^p(w)} = (\int |g(x)|^p w(x)dx)^{1/p}$. Also, the weighted (inhomogeneous) Besov space $B_p^{s,q}(w)$, with $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \boldsymbol{R}$, is defined to be the space of all tempered distributions f on $\boldsymbol{R}^n$ satisfying

$$\|f\|_{B_p^{s,q}(w)} = \left(\sum_{j=0}^{\infty} \left\|2^{sj} D_j(f)\right\|_{L^p(w)}^q\right)^{1/q} < \infty .$$

See [2], [4] and [13] for more details on these spaces.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, let $(\partial\xi)^\alpha$ denote a differential operator

$$(\partial/\partial\xi_1)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}$$

of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For a function F on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ and $x, \xi, h, \eta \in \boldsymbol{R}^n$, we write

$$(d_h F)(x,\xi) = F(x+h,\xi) - F(x,\xi) ,$$
$$(\delta_\eta F)(x,\xi) = F(x,\xi+\eta) - F(x,\xi) .$$

We also define $(\delta_\eta d_h F)(x,\xi) = (d_h \delta_\eta F)(x,\xi) = (\delta_\eta(d_h F))(x,\xi)$.

Let $r > 0$ and $0 \le \delta \le 1$. Suppose that r is not an integer and that

$$|(\partial\xi)^\alpha (\partial x)^\beta \sigma(x,\xi)| \le C(1+|\xi|)^{-|\alpha|+\delta|\beta|} \quad \text{for } |\alpha| \le N ,\ |\beta| < r ;$$
$$|(d_y(\partial\xi)^\alpha (\partial x)^\beta \sigma)(x,\xi)| \le C|y|^{r-[r]}(1+|\xi|)^{-|\alpha|+\delta r} \quad \text{for } |\alpha| \le N ,\ |\beta| = [r] ,$$

where N is an even integer greater than $3n/2 + 1$ ($[a]$ denotes the integer such that $a - 1 < [a] \le a$). Let T_σ be a pseudo-differential operator defined by

$$T_\sigma(f)(x) = \int_{\boldsymbol{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}d\xi , \quad f \in \mathcal{S}(\boldsymbol{R}^n) ,$$

where $\mathcal{S}(\boldsymbol{R}^n)$ denotes the Schwartz space. Then, Bourdaud [1] proved that T_σ is bounded on the Sobolev spaces $H_p^s(= F_p^{s,2})$ for $1 < p < \infty$, $s \in ((\delta-1)r, r)$ and that the range of s is optimal (see [5] for further developments). Also, the boundedness of T_σ on the Besov spaces was studied. Related results can be found in [3], [6], [7], [8], [10], [11], [12], [14] and [15]. In particular, in [10] a weighted L^p, $p \ge 2$, norm inequality for T_σ was proved under a minimal regularity condition for the symbol σ.

In [1], the case where r is an integer was also considered. In this note we confine ourselves to the case where r is not an integer and generalize results of [1] to the weighted (inhomogeneous) Triebel-Lizorkin spaces by using the idea of [10]. Moreover, by applying the results for the Triebel-Lizorkin spaces, we study the boundedness of T_σ on the weighted (inhomogeneous) Besov spaces. We refer to Sugimoto [12] for relevant results.

Let ω be a non-negative function on $[0,\infty) \times [0,\infty)$ satisfying the following:

(ω.1) there exist constants $C, M > 0$ such that

$$\omega(s,at) \le C(1+a)^M \omega(s,t) \quad \text{for all } a > 0 ;$$

(ω.2) there exists a constant $C > 0$ such that

$$\omega(s',t) \le C\omega(s,t) \quad \text{for } s/2 \le s' \le 2s ;$$

(ω.3) $0 < \omega(1, 1)$.

Let $0 \le \delta \le 1$. We assume that $r(> 0)$ is not an integer and the function ω satisfies

$$C(\omega) := \sup_{0\le j\le h, 0\le h} \omega(2^j, 2^{-h}) 2^{-j\delta(r-[r])} 2^{h(r-[r])} < \infty\,, \tag{1.1}$$

where $j, h \in \boldsymbol{Z}$. A prime example of ω satisfying (1.1), (ω.1)–(ω.3) is $\omega(s,t) = s^{\delta(r-[r])} t^{r-[r]}$ (see also Remark 3 below).

Let $\sigma(x, \xi)$ be a bounded function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$. For the rest of this note, we assume that $\sigma \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ for the sake of simplicity. We write $\sigma^{(\alpha)}(x, \xi) = (\partial\xi)^\alpha \sigma(x, \xi)$, $\sigma_{(\beta)}(x, \xi) = (\partial x)^\beta \sigma(x, \xi)$ and $\sigma^{(\alpha)}_{(\beta)}(x, \xi) = (\partial\xi)^\alpha (\partial x)^\beta \sigma(x, \xi)$. Let L be a non-negative integer. We consider the following conditions:

$$|\sigma^{(\alpha)}(x, \xi)| \le C_\alpha (1 + |\xi|)^{-|\alpha|} \quad \text{for } |\alpha| \le L\,; \tag{1.2}$$

$$|(d_y \sigma^{(\alpha)}_{(\beta)})(x, \xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{-|\alpha|+\delta|\beta|}\, \omega(1 + |\xi|, |y|) \quad \text{for } |\alpha| \le L, |\beta| = [r]\,. \tag{1.3}$$

Then we have the following

THEOREM 1. *Let $1 < q \le 2$. Suppose that $\sigma(x, \xi)$ satisfies* (1.2) *and* (1.3) *with $L = [n/q] + 1$. Let $s \in ((\delta - 1)r, r)$, $q \le p < \infty$ and $w \in A_{p/q}$. Then the operator T_σ is bounded on $F^{s,q}_p(w)$. The operator norm is bounded by a constant which is independent of σ if the constants C_α, $C_{\alpha,\beta}$ in* (1.2) *and* (1.3) *are fixed for $|\alpha| \le [n/q] + 1$, $|\beta| = [r]$.*

Let L be a non-negative integer and $0 < a, b \le 1$. We consider the following conditions:

$$|\sigma^{(\alpha)}(x, \xi)| \le C_\alpha (1 + |\xi|)^{-|\alpha|} \quad \text{for } |\alpha| \le L\,; \tag{1.4}$$

$$|(\delta_\eta \sigma^{(\alpha)})(x, \xi)| \le C_\alpha (1 + |\xi|)^{-|\alpha|-a} |\eta|^a \quad \text{for } |\eta| < |\xi|/2\,, \quad |\alpha| = L\,; \tag{1.5}$$

$$|(d_y \sigma^{(\alpha)}_{(\beta)})(x, \xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{-|\alpha|+\delta|\beta|}\, \omega(1 + |\xi|, |y|) \quad \text{for } |\alpha| \le L\,, \quad |\beta| = [r]\,; \tag{1.6}$$

$$|(\delta_\eta d_y \sigma^{(\alpha)}_{(\beta)})(x, \xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{-|\alpha|-b+\delta|\beta|} |\eta|^b\, \omega(1 + |\xi|, |y|) \quad \text{for } |\eta| < |\xi|/2\,, |\alpha| = L\,, \quad |\beta| = [r]\,. \tag{1.7}$$

Then, Theorem 1 follows from the following

THEOREM 2. *Let $1 < q \le 2$, $0 < a \le 1$ and $[n/q] + a > n/q$. Suppose that $\sigma(x, \xi)$ satisfies* (1.4)–(1.7) *with $L = [n/q]$, $a = b$. Let $s \in ((\delta - 1)r, r)$, $q \le p < \infty$ and $w \in A_{p/q}$. Then T_σ is bounded on $F^{s,q}_p(w)$. Moreover, the operator norm is bounded by a constant which is independent of σ if we fix the constants C_α, $C_{\alpha,\beta}$ in* (1.4)–(1.7) *for $|\alpha| \le [n/q]$, $|\beta| = [r]$.*

This is a consequence of a more general result (Theorem 3). Let ρ be a non-negative function such that $\rho^{-1} \in L^1(\boldsymbol{R}^n)$. Let $1 < q \le 2$. For an appropriate f, define

$$\|f\|_{A^q_\rho} = \left(\int_{\boldsymbol{R}^n} |\hat{f}(x)|^{q'} \rho(x)^{q'/q} dx \right)^{1/q'},$$

where q' is the exponent conjugate to q: $q' = q/(q-1)$. We consider the following conditions:

$$C_1 := \sup_{j\geq 1}\sup_{x\in \boldsymbol{R}^n} \|\sigma(x, 2^j\cdot)\Psi(\cdot)\|_{A_\rho^q} < \infty\,; \tag{1.8}$$

$$\sup_{x\in \boldsymbol{R}^n} \|(d_y\sigma_{(\beta)})(x, 2^j\cdot)\Psi(\cdot)\|_{A_\rho^q} \leq C_\beta 2^{j\delta[r]}\omega(2^j, |y|) \quad \text{for } |\beta| = [r]\,,\ j\geq 1\,; \tag{1.9}$$

$$C_2 := \sup_{x\in \boldsymbol{R}^n} \|\sigma(x, \cdot)\Phi(\cdot)\|_{A_\rho^q} < \infty\,; \tag{1.10}$$

$$\sup_{x\in \boldsymbol{R}^n} \|(d_y\sigma_{(\beta)})(x, \cdot)\Phi(\cdot)\|_{A_\rho^q} \leq C_\beta\omega(1, |y|) \quad \text{for } |\beta| = [r]\,. \tag{1.11}$$

Then we have the following:

THEOREM 3. *Suppose that the conditions* (1.8)–(1.11) *hold. Let* $s \in ((\delta-1)r, r)$, $q \leq p < \infty$ *and* $w \in A_{p/q}$. *We further assume that*

$$\sup_{t>0} \theta_t * f(x) \leq C\mathcal{M}(f)(x) \quad \textit{a.e.} \tag{1.12}$$

for all non-negative bounded functions f, *where* $\theta(x) = \rho(x)^{-1}$. *Then* T_σ *is bounded on* $F_p^{s,q}(w)$. *The operator norm is bounded by a constant which is independent of* σ *if the constants* C_1, C_2, C_β $(|\beta| = [r])$ *in* (1.8)–(1.11) *are fixed.*

REMARK 1. Let $\omega(s,t) = s^{\delta(r-[r])}t^{r-[r]}$, $r > 0$, $0 \leq \delta \leq 1$. By examples similar to those in [1], we can see the optimality of the range of s in Theorem 2 ($(\delta-1)r < s < r$).

Indeed, for $k \geq 10$, let

$$\sigma_k(x,\xi) = \sum_{j=10}^{k} 2^{-jr}\exp(2\pi i 2^j x_1) \quad (x = (x_1, x_2, \ldots, x_n))\,.$$

Then, $\sigma_k(x,\xi)$ uniformly satisfies (1.4)–(1.7) for all L and $a = b = 1$. We take Ψ in the definition of $F_p^{s,q}(w)$ such that $\Psi(\xi) = 1$ for $9/10 \leq |\xi| \leq 10/9$, $\Psi(2^{-m}\xi) = 0$ if $9/10 \leq 2^{-j}|\xi| \leq 10/9$ and $m \neq j$. Let $g \in \mathcal{S}(\boldsymbol{R}^n)$ satisfy $\operatorname{supp}(\hat{g}) \subset \{|\xi| \leq 10^{-9}\}$, $0 < \|g\|_{L^p(w)} < \infty$. Then we can see $D_h T_{\sigma_k}(g)(x) = 2^{-hr}\exp(2\pi i 2^h x_1)g(x)$, $h = 10, 11, \ldots, k$, and $D_h T_{\sigma_k}(g) = 0$ otherwise. Thus, if $s \geq r$,

$$\sup_{k\geq 10} \|T_{\sigma_k}(g)\|_{F_p^{s,q}(w)} = \sup_{k\geq 10}\left(\sum_{h=10}^{k} 2^{h(s-r)q}\right)^{1/q} \|g\|_{L^p(w)} = \infty\,.$$

Next, let

$$\sigma(x,\xi) = \sum_{j\geq 10} 2^{(\delta-1)rj}\exp(-2\pi i 2^j x_1)\psi(2^{-j}\xi)\,,$$

where $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ is supported in $\{7/8 \leq |\xi| \leq 8/7\}$ and $\psi(\xi) = 1$ for $9/10 \leq |\xi| \leq 10/9$. Then $\sigma(x,\xi)$ satisfies (1.4)–(1.7) for all L and $a = b = 1$. Put, for $m \geq 10$,

$$f_m(x) = \sum_{j=10}^{m} j^{-1}2^{j(1-\delta)r}\exp(2\pi i 2^j x_1)g(x)\,,$$

where g is as above. Then $T_\sigma(f_m)(x) = g(x)\sum_{j=10}^{m} j^{-1}$. On the other hand, $D_h(f_m)(x) = h^{-1}2^{h(1-\delta)r}\exp(2\pi i 2^h x_1)g(x)$ for $h = 10, 11, \dots, m$, $D_h(f_m)(x) = 0$ otherwise, where D_h is defined by Ψ specified above. Thus

$$\|f_m\|_{F_p^{s,q}(w)} = \left(\sum_{h=10}^{m} h^{-q}2^{h(s+(1-\delta)r)q}\right)^{1/q}\|g\|_{L^p(w)},$$

and hence the sequence $\{f_m\}$ is bounded in $F_p^{s,q}(w)$ if $s < (\delta-1)r$ or $s = (\delta-1)r$ and $1 < q \le \infty$, while $\{T_\sigma(f_m)\}$ is unbounded there.

REMARK 2. Let

$$\sigma_a(x,\xi) = \exp(-2\pi i\langle x,\xi\rangle - |x|^2)(1+|\xi|^2)^{-n/a}, \quad a > 0.$$

Suppose that $n/2$ is not an integer and put $\varepsilon_0 = n/2 - [n/2] = 1/2$. If M is a non-negative integer and if $0 \le M \le [n/2]-1$, then we can see that $\sigma_4(x,\xi)$ satisfies (1.4)–(1.7) with $L = M$, $a = b = 1$, $\varepsilon_0 \le r < 1$, $\omega(s,t) = s^{\delta r}t^r$, $\delta = (r-\varepsilon_0)/r$. Although $0 \in ((\delta-1)r, r)$, we can easily see that T_{σ_4} is not bounded on $L^2 = F_2^{0,2}$ (the unweighted Lebesgue space).

Also, we can see that $\sigma_4(x,\xi)$ satisfies (1.4)–(1.7) with $L = [n/2]$, $a = b$, $\omega(s,t) = s^{\delta r}t^r$, $0 < r < 1$, $\delta = (r-\varepsilon_0+a)/r$, where $\varepsilon_0 - r \le a \le \varepsilon_0$ if $0 < r < \varepsilon_0$ and $0 < a \le \varepsilon_0$ if $\varepsilon_0 \le r < 1$. If $L + a < n/2$, then $0 \in ((\delta-1)r, r)$; but as we mentioned above, T_{σ_4} is not bounded on L^2 (see Coifman-Meyer [3, p. 12] and Yabuta [14, Section 6]).

REMARK 3. Let $\varepsilon = r - [r]$, $\beta \ge \varepsilon$, $\gamma \ge 0$, $0 \le \delta \le 1$. For $s \ge 1$, $t \ge 0$, let

$$\omega_0(s,t) = \begin{cases} s^{\delta\varepsilon}t^{\varepsilon} & \text{if } st < 1, \\ s^{\delta\varepsilon}t^{\varepsilon}(st)^{\beta-\varepsilon} & \text{if } st \ge 1, \end{cases}$$

and for $0 \le s < 1$, $t \ge 0$, let $\omega_0(s,t) = s^\gamma t^\varepsilon$ $(t < 1)$, $\omega_0(s,t) = s^\gamma t^\beta$ $(t \ge 1)$. Then, ω_0 satisfies (1.1), $(\omega.1)$–$(\omega.3)$.

In Section 2, we recall results relevant to the proof of Theorem 3. In Section 3, we prove Theorem 3 by applying these results. Theorem 2 is proved in Section 4 by using Theorem 3. In Section 5, we state results on the boundedness of T_σ on $B_p^{s,q}(w)$ as applications of the results for $F_p^{s,q}(w)$.

2. Results for the proof of Theorem 3. Take radial functions $\psi, \varphi \in C_0^\infty(\boldsymbol{R}^n)$ such that $\mathrm{supp}(\psi) \subset \{1/4 < |\xi| < 4\}$, $\psi(\xi) = 1$ if $1/2 \le |\xi| \le 2$ and $\mathrm{supp}(\varphi) \subset \{|\xi| < 4\}$, $\varphi(\xi) = 1$ if $|\xi| \le 2$. Decompose

$$\begin{aligned}\sigma(x,\xi) &= \sigma(x,\xi)\Phi(\xi) + \sum_{j\ge1}\sigma(x,\xi)\Psi(2^{-j}\xi) \\ &= \sigma(x,\xi)\Phi(\xi)\varphi(\xi)^2 + \sum_{j\ge1}\sigma(x,\xi)\Psi(2^{-j}\xi)\psi(2^{-j}\xi)^2 \\ &= \left(\int_{\boldsymbol{R}^n} A_0(x,k)\exp(2\pi i\langle k,\xi\rangle)dk\right)\varphi(\xi)^2\end{aligned}$$

$$+\sum_{j\geq 1}\left(\int_{\boldsymbol{R}^n} A_j(x,k)\exp(2\pi i\langle 2^{-j}k,\xi\rangle)dk\right)\psi(2^{-j}\xi)^2\,,$$

where

$$A_j(x,k)=\int_{\boldsymbol{R}^n}\sigma(x,2^j\xi)\Psi(\xi)\exp(-2\pi i\langle k,\xi\rangle)d\xi\,,\quad j\geq 1\,,$$
$$A_0(x,k)=\int_{\boldsymbol{R}^n}\sigma(x,\xi)\Phi(\xi)\exp(-2\pi i\langle k,\xi\rangle)d\xi\,.$$

For $j\geq 1$, put

$$A_j^{(2)}(x,k)=\int_{\boldsymbol{R}^n}((\hat{\varphi})_{2^{-j+12}}*\sigma(\cdot,2^j\xi))(x)\Psi(\xi)\exp(-2\pi i\langle k,\xi\rangle)d\xi\,,$$

where $((\hat{\varphi})_{2^{-j+12}}*\sigma(\cdot,2^j\xi))(x)=\int(\hat{\varphi})_{2^{-j+12}}(y)\sigma(x-y,2^j\xi)dy$.

LEMMA 1. *Suppose that the conditions* (1.8) *and* (1.9) *hold. Put* $A_j^{(1)}(x,k)=A_j(x,k)-A_j^{(2)}(x,k)$, $j\geq 1$. *Then*

(2.1) $$\sup_{x\in\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}|A_j^{(1)}(x,k)|^{q'}\rho(k)^{q'/q}dk\leq C2^{-q'j[r]}2^{q'j\delta[r]}\omega(2^j,2^{-j})^{q'}\,,$$

(2.2) $$\sup_{j\geq 1}\sup_{x\in\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}|A_j^{(2)}(x,k)|^{q'}\rho(k)^{q'/q}dk<\infty\,.$$

Furthermore, the Fourier transform of $A_j^{(2)}(x,k)$ *with respect to the* x*-variable is supported in* $\{|\xi|\leq 2^{j-10}\}$ *for all* k.

PROOF. First we see that

$$\int|A_j^{(2)}(x,k)|^{q'}\rho(k)^{q'/q}dk\leq C\int|(\hat{\varphi})_{2^{-j+12}}(y)|\|\sigma(x+y,2^j\cdot)\Psi(\cdot)\|_{A_\rho^q}^{q'}dy$$
$$\leq C\sup_{x\in\boldsymbol{R}^n}\|\sigma(x,2^j\cdot)\Psi(\cdot)\|_{A_\rho^q}^{q'}\,.$$

Therefore by (1.8) we get (2.2). The support condition for the Fourier transform of $A_j^{(2)}$ is easily seen.

Next, we prove (2.1). Put

$$H(x,y,2^j\xi)=\sigma(x+y,2^j\xi)-\sum_{|\beta|\leq[r]}\frac{1}{\beta!}\,y^\beta\sigma_{(\beta)}(x,2^j\xi)\,,$$

where $\beta!=\beta_1!\cdots\beta_n!$, $\beta=(\beta_1,\dots,\beta_n)$, $y^\beta=y_1^{\beta_1}\cdots y_n^{\beta_n}$. Since

$$\int\hat{\varphi}(y)dy=1\,,\quad\int\hat{\varphi}(y)y^\alpha dy=0\quad\text{if }|\alpha|>0\,,$$

then we have

$$\int(\hat{\varphi})_{2^{-j+12}}(y)\sigma(x+y,2^j\xi)dy-\sigma(x,2^j\xi)=\int(\hat{\varphi})_{2^{-j+12}}(y)H(x,y,2^j\xi)dy\,,$$

and hence

$$\int |A_j^{(1)}(x,k)|^{q'}\rho(k)^{q'/q}dk \le C\int |(\hat\varphi)_{2^{-j+12}}(y)|\|H(x,y,2^j\cdot)\Psi(\cdot)\|_{A^q_\rho}^{q'}dy\,.$$

By Taylor's formula we see that

$$\|H(x,y,2^j\cdot)\Psi(\cdot)\|_{A^q_\rho} \le C\sum_{|\beta|=[r]}|y|^{[r]}\sup_{0\le t\le 1}\|\sigma_{(\beta)}(x+ty,2^j\cdot)\Psi(\cdot)-\sigma_{(\beta)}(x,2^j\cdot)\Psi(\cdot)\|_{A^q_\rho}\,.$$

Thus by (1.9)

$$\begin{aligned}\int |A_j^{(1)}(x,k)|^{q'}\rho(k)^{q'/q}dk &\le C\int |(\hat\varphi)_{2^{-j+12}}(y)||y|^{q'[r]}2^{q'j\delta[r]}\sup_{0\le t\le 1}\omega(2^j,t|y|)^{q'}dy\\ &\le C\int |\hat\varphi(y)||y|^{q'[r]}2^{-q'j[r]}2^{q'j\delta[r]}\sup_{0\le t\le 1}\omega(2^j,2^{-j+12}t|y|)^{q'}dy\\ &\le C2^{-q'j[r]}2^{q'j\delta[r]}\omega(2^j,2^{-j})^{q'}\int |\hat\varphi(y)||y|^{q'[r]}(1+|y|)^{q'M}dy\\ &\le C2^{-q'j[r]}2^{q'j\delta[r]}\omega(2^j,2^{-j})^{q'}\,,\end{aligned}$$

where we have used $(\omega.1)$. This proves (2.1), which completes the proof of Lemma 1.

LEMMA 2. *Suppose that the conditions* (1.8) *through* (1.11) *hold. Let* $A_0^{(1)}=A_0$, *and for* $j\ge 1$ *let* $A_j^{(1)}$ *be as in Lemma* 1. *Define* $A_{j,h}^{(1)}(x,k)=D_h(A_j^{(1)}(\cdot,k))(x)$ *for* $j,h\ge 0$. *Then*

$$\sup_{x\in\boldsymbol{R}^n}\left(\int |A_{j,h}^{(1)}(x,k)|^{q'}\rho(k)^{q'/q}dk\right)^{1/q'} \le C\min(2^{-j[r]}2^{j\delta[r]}\omega(2^j,2^{-j}),2^{-h[r]}2^{j\delta[r]}\omega(2^j,2^{-h}))\,.$$

PROOF. Let $j\ge 1$ and $h\ge 1$. We note that

$$A_{j,h}^{(1)}(x,k)=(\hat\Psi)_{2^{-h}}*A_j(\cdot,k)(x)-(\hat\varphi)_{2^{-j+12}}*(\hat\Psi)_{2^{-h}}*A_j(\cdot,k)(x)\,.$$

Since $\int\hat\Psi(y)y^\alpha dy=0$ for all α, we have

$$(\hat\Psi)_{2^{-h}}*\sigma(\cdot,2^j\xi)(x)=\int(\hat\Psi)_{2^{-h}}(y)H(x,y,2^j\xi)dy\,,$$

where H is as in the proof of Lemma 1. Thus, as in the proof of Lemma 1, we have

$$\int |D_h(A_j(\cdot,k))(x)|^{q'}\rho(k)^{q'/q}dk\le C2^{-q'h[r]}2^{q'j\delta[r]}\omega(2^j,2^{-h})^{q'} \tag{2.3}$$

and hence

$$\begin{aligned}&\int |(\hat\varphi)_{2^{-j+12}}*(\hat\Psi)_{2^{-h}}*A_j(\cdot,k)(x)|^{q'}\rho(k)^{q'/q}dk\\ &\quad\le C\int|(\hat\varphi)_{2^{-j+12}}(y)|\left[\int|(\hat\Psi)_{2^{-h}}*A_j(\cdot,k)(x-y)|^{q'}\rho(k)^{q'/q}dk\right]dy\\ &\quad\le C2^{-q'h[r]}2^{q'j\delta[r]}\omega(2^j,2^{-h})^{q'}.\end{aligned} \tag{2.4}$$

Also, by using Lemma 1, we get

$$\int |A^{(1)}_{j,h}(x,k)|^{q'}\rho(k)^{q'/q}dk \le C2^{-q'j[r]}2^{q'j\delta[r]}\omega(2^j, 2^{-j})^{q'} \quad \text{for } h \ge 0,\ j \ge 1\,. \tag{2.5}$$

Combining (2.3)–(2.5), we obtain the conclusion of Lemma 2 for $j \ge 1$. The proof for the case $j = 0$ can be done similarly by using (1.10) and (1.11). □

We can prove the following result by applying Hölder's inequality.

LEMMA 3. *Let $a > 1$, $1 \le r < \infty$, and let $\{x_k\}_{k=0}^{\infty}$ be a sequence of complex numbers such that $\sum_{k=0}^{\infty} |x_k|^r < \infty$. Then*

$$\sum_{j=0}^{\infty}\left|a^{-j}\sum_{k=0}^{j} a^k x_k\right|^r \le (a/(a-1))^r \sum_{k=0}^{\infty}|x_k|^r\,.$$

The following lemma generalizes a result stated in [1].

LEMMA 4. *For $j = 0, 1, 2, \ldots$, let f_j be a tempered distribution whose Fourier transform is supported in $\{|\xi| < c2^j\}$ for some constant $c > 0$ (note that f_j is a function by the support condition). We assume that $f_j = 0$ for all but a finite number of values of j. Let $s > 0$, $1 < p < \infty$, $1 < q < \infty$ and $w \in A_p$. Then we have*

$$\left\|\sum_{j=0}^{\infty} f_j\right\|_{F^{s,q}_p(w)} \le C\left\|\left(\sum_{j=0}^{\infty}2^{qjs}|f_j|^q\right)^{1/q}\right\|_{L^p(w)}.$$

PROOF. There exists a positive integer N such that

$$\sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty}\sum_{l=0}^{j+N} D_l(f_j) = \sum_{l=0}^{\infty}\sum_{j=\max(l-N,0)}^{\infty} D_l(f_j) = \sum_{l=0}^{\infty} D_l\Bigg(\sum_{j=\max(l-N,0)}^{\infty} f_j\Bigg)\,. \tag{2.6}$$

Now, by Hölder's inequality we have, for appropriate functions g_l,

$$\begin{aligned} \left|\sum_{l=0}^{\infty}\int D_l\Bigg(\sum_{j=\max(l-N,0)}^{\infty} f_j\Bigg)g_l dx\right| &= \left|\sum_{j=0}^{\infty}\int f_j\Bigg(\sum_{l=0}^{j+N} D_l(g_l)\Bigg)dx\right| \\ &\le \left(\int\left(\sum_{j\ge 0}2^{qjs}|f_j(x)|^q\right)^{p/q} w(x)dx\right)^{1/p} I^{1/p'}\,, \end{aligned} \tag{2.7}$$

where

$$I = \int\left(\sum_{j\ge 0}\left|2^{-js}\Bigg(\sum_{l=0}^{j+N} D_l(g_l)\Bigg)\right|^{q'}\right)^{p'/q'} w(x)^{-p'/p}dx.$$

By Lemma 3 with $a = 2^s$ and $r = q'$, we have

$$\sum_{j\ge 0}\left|2^{-js}\left(\sum_{l=0}^{j+N} D_l(g_l)\right)\right|^{q'} = \sum_{j\ge 0}\left|2^{-js}\left(\sum_{l=0}^{j+N} 2^{sl} D_l(2^{-sl}g_l)\right)\right|^{q'}$$
$$\le C\sum_{l=0}^{\infty} 2^{-q'sl}|D_l(g_l)|^{q'} .$$

Therefore

$$I \le C\int\left(\sum_{l=0}^{\infty} 2^{-q'sl}|D_l(g_l)|^{q'}\right)^{p'/q'} w(x)^{-p'/p}dx$$
$$\le C\int\left(\sum_{l=0}^{\infty} 2^{-q'sl}|g_l|^{q'}\right)^{p'/q'} w(x)^{-p'/p}dx ,$$

where the last inequality follows from a well-known vector valued inequality, since $w^{-p'/p} \in A_{p'}$ (see [9]). By a duality argument using this estimate in (2.7), we have

$$\int\left(\sum_{l=0}^{\infty} 2^{qsl}\left|D_l\left(\sum_{j=\max(l-N,0)}^{\infty} f_j\right)\right|^q\right)^{p/q} w(x)dx$$
$$\le C\int\left(\sum_{j\ge 0} 2^{qjs}|f_j(x)|^q\right)^{p/q} w(x)dx .$$

From this and (2.6) we can easily get the conclusion. □

3. Proof of Theorem 3. Put, for $j \ge 1$,

$$E_j(f)(x,k) = \int_{\boldsymbol{R}^n} \exp\left(2\pi i\langle 2^{-j}k,\xi\rangle\right)\psi(2^{-j}\xi)^2\hat{f}(\xi)\exp(2\pi i\langle x,\xi\rangle)d\xi$$
$$= (\tau_{-k}\mathcal{F}^{-1}(\psi))_{2^{-j}} * \Delta_j(f)(x) ,$$

where $\tau_k f(x) = f(x-k)$ and

$$\Delta_j(f)(x) = \int_{\boldsymbol{R}^n} \psi(2^{-j}\xi)\hat{f}(\xi)\exp(2\pi i\langle x,\xi\rangle)d\xi .$$

Also put $E_0(f)(x,k) = (\tau_{-k}\mathcal{F}^{-1}(\varphi)) * \Delta_0(f)(x)$, where

$$\Delta_0(f)(x) = \int_{\boldsymbol{R}^n} \varphi(\xi)\hat{f}(\xi)\exp(2\pi i\langle x,\xi\rangle)d\xi .$$

Then we can see that

$$T_\sigma(f)(x) = \sum_{j\ge 0}\int A_j(x,k)E_j(f)(x,k)dk , \quad f \in \mathcal{S}(\boldsymbol{R}^n) .$$

Decompose $A_j(x,k) = A_j^{(1)}(x,k) + A_j^{(2)}(x,k)$ for $j \geq 0$, where $A_j^{(1)}$ and $A_j^{(2)}$ are as in Lemmas 1 and 2 ($A_0^{(2)} \equiv 0$). Put

$$B_i(f)(x) = \sum_{j\geq 0} \int A_j^{(i)}(x,k)E_j(f)(x,k)dk\,, \quad i = 1, 2\,.$$

Then $T_\sigma(f) = B_1(f) + B_2(f)$. We note the following. For a positive integer N, let

$$S_N^{(i)}(f)(x) = \sum_{j=0}^{N} \int A_j^{(i)}(x,k)E_j(f)(x,k)dk\,, \quad i = 1, 2\,.$$

Then we can easily see that $|S_N^{(i)}(f)(x)| \leq C$ for some $C > 0$ independent of x and N. Also, we can see that $S_N^{(i)}(f)(x) \to B_i(f)(x)$ as $N \to \infty$ for all x.

We estimate $B_i(f)$, $i = 1, 2$, separately under the hypotheses of Theorem 3. We begin with the estimation of $B_2(f)$. First, observing that the Fourier transform of $\int A_j^{(2)}(x,k)E_j(f)(x,k)dk$, $j \geq 1$, is supported in an annulus of the form $\{c_1 2^j < |\xi| < c_2 2^j\}$ with $c_1, c_2 > 0$, by a vector valued inequality (see [9], [2]) we have, for $s \in \boldsymbol{R}$,

$$\|B_2(f)\|_{F_p^{s,q}(w)}^p \leq C \int \left(\sum_{j=1}^{\infty} 2^{qjs} \left| \int A_j^{(2)}(x,k)E_j(f)(x,k)dk \right|^q \right)^{p/q} w(x)dx\,.$$

By Hölder's inequality and Lemma 1 (2.2), the right hand side is bounded by, up to a constant factor,

$$\int \left(\sum_{j\geq 1} \int \rho(k)^{-1} 2^{qjs} |E_j(f)(x,k)|^q dk \right)^{p/q} w(x)dx\,.$$

Let g be a non-negative function on $\boldsymbol{R}^n$. Then, a direct computation yields

$$\begin{aligned}
&\int \sum_{j\geq 1} \int \rho(k)^{-1} 2^{qjs} \left|E_j(f)(x,k)\right|^q dk g(x)dx \\
&\quad \leq C \sum_{j\geq 1} \int \left(\int \rho(k)^{-1} \int 2^{jn} |\mathcal{F}^{-1}(\psi)(2^j(x-y)+k)| g(x)dxdk \right) 2^{qjs} |\Delta_j(f)(y)|^q dy \\
&\quad \leq C \sum_{j\geq 1} \int \left(\int \rho(k)^{-1} \mathcal{M}(g)(y - 2^{-j}k)dk \right) 2^{qjs} \left|\Delta_j(f)(y)\right|^q dy \\
&\quad \leq C \sum_{j\geq 1} \int \mathcal{M}\mathcal{M}(g)(y) 2^{qjs} \left|\Delta_j(f)(y)\right|^q dy\,,
\end{aligned}$$

where the last inequality follows from (1.12). Put

$$I = \int \left(\sum_{j\geq 1} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y)dy\,.$$

Now we assume that $p > q$. Then by Hölder's inequality

$$\begin{aligned}\sum_{j\ge 1}\int \mathcal{M}\mathcal{M}(g)(y)2^{qjs}\left|\Delta_j(f)(y)\right|^q dy \\ &\le C\left(\int \mathcal{M}\mathcal{M}(g)^{(p/q)'}(y)w(y)^{-q(p/q)'/p}\,dy\right)^{1/(p/q)'} I^{q/p} \\ &\le C\|g\|_{L^{(p/q)'}(w^{-q(p/q)'/p})} I^{q/p}\,,\end{aligned}$$

if $g \in L^{(p/q)'}(w^{-q(p/q)'/p})$ and $w \in A_{p/q}$. Therefore by the converse of Hölder's inequality we have

$$\int\left(\sum_{j\ge 1}\int \rho(k)^{-1}2^{qjs}\left|E_j(f)(x,k)\right|^q dk\right)^{p/q} w(x)dx \le CI\,. \tag{3.1}$$

The case $p = q$, $w \in A_1$ can be treated similarly and we also have this inequality. Thus

$$\|B_2(f)\|^p_{F_p^{s,q}(w)} \le C\int\left(\sum_{j\ge 1}2^{qjs}\left|\Delta_j(f)(y)\right|^q\right)^{p/q} w(y)dy\,. \tag{3.2}$$

Next, we estimate $B_1(f)$. For positive integers N, M, put

$$U_{N,M}(f)(x) = \sum_{j=0}^{N}\sum_{h=0}^{M}\int A^{(1)}_{j,h}(x,k)E_j(f)(x,k)dk\,,$$

where $A^{(1)}_{j,h}$ is as in Lemma 2. We estimate $U_{N,M}(f)$ on $F_p^{s,q}(w)$. The estimate will be uniform in N and M. Put $\tilde{A}^{(1)}_{j,h} = A^{(1)}_{j,h}$ if $0 \le h \le M$ and $\tilde{A}^{(1)}_{j,h} = 0$ if $h > M$; also $\tilde{E}_j(f) = E_j(f)$ if $0 \le j \le N$ and $\tilde{E}_j(f) = 0$ if $j > N$. Then

$$U_{N,M}(f)(x) = \sum_{j\ge 0}\sum_{h\ge 0}\int \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk = G_1(x) + G_2(x) + G_3(x)\,,$$

where

$$G_1(x) = \sum_{j=10}^{\infty}\sum_{h=0}^{j-10}\int \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk\,,$$

$$G_2(x) = \sum_{j=10}^{\infty}\sum_{h=j-9}^{\infty}\int \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk = \sum_{h=1}^{\infty}\sum_{j=10}^{h+9}\int \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk\,,$$

$$G_3(x) = \sum_{h=0}^{\infty}\sum_{j=0}^{9}\int \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk.$$

We estimate G_1, G_2 and G_3 separately.

Observing that the Fourier transform of $\int\sum_{0\le h\le j-10}\tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk$, $j \ge 10$, is supported in an annulus of the form $\{c_1 2^j < |\xi| < c_2 2^j\}$, $c_1, c_2 > 0$, via Hölder's inequality we have

$$\|G_1\|^p_{F^{s,q}_p(w)} \le C \int \Bigg(\sum_{j=10}^{\infty} 2^{qjs} \Bigg| \int \sum_{h=0}^{j-10} \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk \Bigg|^q \Bigg)^{p/q} w(x)dx$$

$$\le C \int \Bigg(\sum_{j=10}^{\infty} \Bigg(\int \Bigg| \sum_{h=0}^{j-10} \tilde{A}^{(1)}_{j,h}(x,k) \Bigg|^{q'} \rho(k)^{q'/q} dk \Bigg)^{q/q'} F_j(s,q,x) \Bigg)^{p/q} w(x)dx\,,$$

where

$$F_j(s,q,x) = \int \rho(k)^{-1} 2^{qjs} |\tilde{E}_j(f)(x,k)|^q dk\,.$$

We note that

$$\sum_{0\le h\le j-10} \tilde{A}^{(1)}_{j,h}(x,k) = g^{(j,M)} * A^{(1)}_j(\cdot,k)(x)\,, \quad j \ge 10$$

for some $g^{(j,M)} \in \mathcal{S}(\boldsymbol{R}^n)$ such that $\|g^{(j,M)}\|_{L^1} \le c$, where c is a constant independent of j and M. Therefore by Lemma 1 (2.1) we have

$$\int \Bigg| \sum_{h=0}^{j-10} \tilde{A}^{(1)}_{j,h}(x,k) \Bigg|^{q'} \rho(k)^{q'/q} dk \le C 2^{-q'j[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-j})^{q'}\,.$$

Thus for any $s \in \boldsymbol{R}$

$$\|G_1\|^p_{F^{s,q}_p(w)} \le CC(\omega)^p \int \Bigg(\sum_{j\ge 10} 2^{qjr(\delta-1)} F_j(s,q,x) \Bigg)^{p/q} w(x)dx$$

(3.3)

$$\le CC(\omega)^p \int \Bigg(\sum_{j\ge 10} 2^{qjs} 2^{qjr(\delta-1)} |\Delta_j(f)(y)|^q \Bigg)^{p/q} w(y)dy\,,$$

where the second inequality can be proved as above (see (3.1)).

Next we estimate G_2. Since the Fourier transform of $\int \sum_{j=10}^{h+9} \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk$, $h \ge 1$, is supported in $\{|\xi| < c2^h\}$, by Lemma 4 we have, for $0 < s < r$,

$$\|G_2\|^p_{F^{s,q}_p(w)} \le C \int \Bigg(\sum_{h=1}^{\infty} 2^{qhs} \Bigg| \int \sum_{j=10}^{h+9} \tilde{A}^{(1)}_{j,h}(x,k)\tilde{E}_j(f)(x,k)dk \Bigg|^q \Bigg)^{p/q} w(x)dx\,.$$

By Hölder's inequality, the right hand side is bounded by, up to a constant factor,

$$\int \Bigg(\sum_{h=1}^{\infty} 2^{h(s-r)} \sum_{j=10}^{h+9} 2^{(r-s)j} \Bigg| \int (2^{-rj} 2^{hr} \tilde{A}^{(1)}_{j,h}(x,k))(2^{sj}\tilde{E}_j(f)(x,k))dk \Bigg|^q \Bigg)^{p/q} w(x)dx\,.$$

By Hölder's inequality and Lemma 2, the inner integral is bounded by

$$\Bigg(\int |2^{-rj} 2^{hr} \tilde{A}^{(1)}_{j,h}(x,k)|^{q'} \rho(k)^{q'/q} dk \Bigg)^{1/q'} \Bigg(\int \Big| 2^{sj}\tilde{E}_j(f)(x,k) \Big|^q \rho(k)^{-1} dk \Bigg)^{1/q}$$

$$\le C 2^{-(r-\delta[r])j} 2^{h(r-[r])} \omega(2^j, 2^{-h}) \Bigg(\int |2^{sj}\tilde{E}_j(f)(x,k)|^q \rho(k)^{-1} dk \Bigg)^{1/q}.$$

Therefore, $\|G_2\|^p_{F^{s,q}_p(w)}$ is bounded by

$$CC(\omega)^p \int \Big(\sum_{h=1}^{\infty} 2^{h(s-r)} \sum_{j=10}^{h+9} 2^{(r-s)j} \int |2^{-(1-\delta)rj} 2^{sj} \tilde{E}_j(f)(x,k)|^q \rho(k)^{-1} dk \Big)^{p/q} w(x)dx$$
$$\le CC(\omega)^p \int \Big(\sum_{j=10}^{\infty} \int |2^{-(1-\delta)rj} 2^{sj} \tilde{E}_j(f)(x,k)|^q \rho(k)^{-1} dk \Big)^{p/q} w(x)dx\,.$$

Thus, we have (see (3.1))

$$\|G_2\|^p_{F^{s,q}_p(w)} \le CC(\omega)^p \int \Big(\sum_{j\ge 10} 2^{-q(1-\delta)rj} 2^{qjs} |\Delta_j(f)(y)|^q \Big)^{p/q} w(y)dy\,. \tag{3.4}$$

Finally, in the same way as in the case of G_2, if $0 < s < r$, we have

$$\begin{aligned} \|G_3\|^p_{F^{s,q}_p(w)} &\le C \int \Big(\sum_{h=0}^{\infty} 2^{qhs} \Big| \int \sum_{j=0}^{9} \tilde{A}^{(1)}_{j,h}(x,k) \tilde{E}_j(f)(x,k)\, dk \Big|^q \Big)^{p/q} w(x)dx \\ &\le C_\eta C(\omega)^p \int \Big(\sum_{j=0}^{9} 2^{-\eta j} 2^{qjs} |\Delta_j(f)(y)|^q \Big)^{p/q} w(y)dy \end{aligned} \tag{3.5}$$

for any $\eta \ge 0$.

By (3.3) through (3.5) we have

$$\big\| U_{N,M}(f) \big\|^p_{F^{s,q}_p(w)} \le C \int \Big(\sum_{j=0}^{\infty} 2^{qjs} |\Delta_j(f)(y)|^q \Big)^{p/q} w(y)dy\,, \tag{3.6}$$

where the constant C is independent of N and M. Fix j and put

$$T_M(f)(x) = \sum_{h=0}^{M} \int A^{(1)}_{j,h}(x,k) E_j(f)(x,k) dk\,.$$

Then we can see that $|T_M(f)(x)| \le C$ for some $C > 0$ independent of x and M and that $T_M(f)(x) \to \int A^{(1)}_j(x,k) E_j(f)(x,k)dk$ as $M \to \infty$ for all x. Therefore, letting $M \to \infty$ then $N \to \infty$ in (3.6), we have

$$\|B_1(f)\|^p_{F^{s,q}_p(w)} \le C \int \Big(\sum_{j=0}^{\infty} 2^{qjs} |\Delta_j(f)(y)|^q \Big)^{p/q} w(y)dy\,. \tag{3.7}$$

By the estimates (3.2) and (3.7), we can get the conclusion of Theorem 3, since

$$\int \Big(\sum_{j=0}^{\infty} 2^{qjs} |\Delta_j(f)(y)|^q \Big)^{p/q} w(y)dy \le C \|f\|^p_{F^{s,q}_p(w)}$$

for $s \in \boldsymbol{R}$, $0 < q < \infty$, $0 < p < \infty$ and $w \in A_\infty$ (see, e.g., [2]).

4. Proof of Theorem 2. Under the hypotheses of Theorem 2 we prove the validity of the conditions (1.8) through (1.11) with $\rho(k) = (1+|k|^q)^s$, $s = [n/q]+d$, where d is chosen so that $a > d$ and $[n/q]+d > n/q$.

Let $j \geq 1$. Then, integration by parts gives

$$A_j(x,k) = (2\pi i k_m)^{-[n/q]} \int_{\boldsymbol{R}^n} [(\partial/\partial\xi_m)^{[n/q]}(\sigma(x, 2^j\xi)\Psi(\xi))] \exp(-2\pi i\langle k,\xi\rangle) d\xi \,.$$

Let ψ be as in Section 2. Then by applying the Hausdorff-Young inequality we have, for $l \geq 0$,

$$\begin{aligned} &\int_{|k|\approx|k_m|,\, 2^l\leq|k|\leq 2^{l+1}} |A_j(x,k)|^{q'}(1+|k|^q)^{sq'/q} dk \\ &\leq C2^{q'sl}\int_{|k|\approx|k_m|} |\psi(2^{-l}k)A_j(x,k)|^{q'} dk \\ &\leq C2^{q'dl}\left(\int_{\boldsymbol{R}^n} |((\hat{\psi})_{2^{-l}} * F(x,\cdot))(\xi)|^q d\xi\right)^{q'/q}, \end{aligned} \tag{4.1}$$

where $F(x,\xi) = (\partial/\partial\xi_m)^{[n/q]}(\sigma(x,2^j\xi)\Psi(\xi))$. Then, by (1.4) and (1.5) with $L = [n/q]$ we have, for all $x, \xi, \eta \in \boldsymbol{R}^n$,

$$|F(x,\xi)| \leq C \quad \text{and} \quad |F(x,\xi+\eta) - F(x,\xi)| \leq C|\eta|^a \,. \tag{4.2}$$

By (4.2) we see that

$$\begin{aligned} |((\hat{\psi})_{2^{-l}} * F(x,\cdot))(\xi)| &= \left|\int [F(x,\xi+\eta) - F(x,\xi)](\hat{\psi})_{2^{-l}}(\eta)d\eta\right| \\ &\leq \left|\int_{|\eta|<|\xi|/2} [F(x,\xi+\eta) - F(x,\xi)](\hat{\psi})_{2^{-l}}(\eta)d\eta\right| \\ &\quad + \left|\int_{|\eta|\geq|\xi|/2} [F(x,\xi+\eta) - F(x,\xi)](\hat{\psi})_{2^{-l}}(\eta)d\eta\right| \\ &\leq C\chi_0(\xi)\int |\eta|^a |(\hat{\psi})_{2^{-l}}(\eta)| d\eta + C\min(2^{-al}, (2^l|\xi|)^{-2n}) \\ &\leq C2^{-al}(1+|\xi|)^{-2n}, \end{aligned}$$

where χ_0 is the characteristic function of the ball $\{|\xi| \leq 5\}$. Using this in (4.1), we have

$$\int_{|k|\approx|k_m|,\, |k|\geq 1} |A_j(x,k)|^{q'}(1+|k|^q)^{sq'/q} dk \leq \sum_{l\geq 0} C2^{q'dl}2^{-q'al} \leq C \,. \tag{4.3}$$

It is easier to get the estimate

$$\int_{|k|\leq 1} |A_j(x,k)|^{q'}(1+|k|^q)^{sq'/q} dk \leq C \,.$$

Using this and (4.3) for $m = 1, \ldots, n$, we see that the condition (1.8) holds.

Next we show that the condition (1.9) holds. Let $|\beta| = [r]$. Put

$$A(x, y, k, j, \beta) = \int_{\boldsymbol{R}^n} [\sigma_{(\beta)}(x + y, 2^j \xi) - \sigma_{(\beta)}(x, 2^j \xi)]\Psi(\xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi .$$

Then, by integration by parts

$$A(x, y, k, j, \beta) = (2\pi i k_m)^{-[n/q]} \int_{\boldsymbol{R}^n} G(x, y, \xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi ,$$

where $G(x, y, \xi) = (\partial/\partial \xi_m)^{[n/q]}[(\sigma_{(\beta)}(x + y, 2^j \xi) - \sigma_{(\beta)}(x, 2^j \xi))\Psi(\xi)]$. By the Hausdorff-Young inequality we have, as above, for $l \geq 0$

$$\begin{aligned} &\int_{|k| \approx |k_m|, 2^l \leq |k| \leq 2^{l+1}} |A(x, y, k, j, \beta)|^{q'} (1 + |k|^q)^{sq'/q} dk \\ &\quad \leq C 2^{q'dl} \left(\int_{\boldsymbol{R}^n} |((\hat{\psi})_{2^{-l}} * G(x, y, \cdot))(\xi)|^q d\xi \right)^{q'/q} . \end{aligned} \tag{4.4}$$

By (1.6), (1.7) with $L = [n/q]$ and $a = b$, we have, for all $x, y, \xi, \eta \in \boldsymbol{R}^n$,

$$\begin{aligned} &|G(x, y, \xi)| \leq C 2^{j\delta[r]} \omega(2^j, |y|) \quad \text{and} \\ &|G(x, y, \xi + \eta) - G(x, y, \xi)| \leq C |\eta|^a 2^{j\delta[r]} \omega(2^j, |y|) . \end{aligned} \tag{4.5}$$

Using (4.5) and arguing as in the proof for (1.8) above, we can see that

$$|(\hat{\psi}_{2^{-l}} * G(x, y, \cdot))(\xi)| \leq C 2^{-al} 2^{j\delta[r]} \omega(2^j, |y|)(1 + |\xi|)^{-2n} .$$

Using this in (4.4) and summing up in $l \geq 0$, we have

$$\int_{|k| \approx |k_m|, |k| \geq 1} |A(x, y, k, j, \beta)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq C 2^{q' j\delta[r]} \omega(2^j, |y|)^{q'} . \tag{4.6}$$

We also have

$$\int_{|k| \leq 1} |A(x, y, k, j, \beta)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq C 2^{q' j\delta[r]} \omega(2^j, |y|)^{q'} .$$

Using this and (4.6) for $m = 1, \dots, n$, we can get (1.9).

The validity of the conditions (1.10) and (1.11) can be proved similarly. Since $\rho(x) = (1 + |x|^q)^s$ satisfies (1.12), now Theorem 2 follows from Theorem 3.

5. Boundedness on the weighted Besov spaces. As applications of Theorems 1–3, we have the following:

THEOREM 4. *Let* $1 < t \leq 2$. *Suppose that* $\sigma(x, \xi)$ *satisfies* (1.2) *and* (1.3) *with* $L = [n/t] + 1$. *Then*

$$\|T_\sigma(f)\|_{B_p^{s,q}(w)} \leq C \|f\|_{B_p^{s,q}(w)} , \quad f \in \mathcal{S}(\boldsymbol{R}^n) ,$$

where $t \leq p < \infty$, $s \in ((\delta - 1)r, r)$, $0 < q \leq \infty$ *and* $w \in A_{p/t}$.

THEOREM 5. *Let* $1 < t \leq 2$, $0 < a \leq 1$ *and* $[n/t] + a > n/t$. *Suppose that* $\sigma(x, \xi)$ *satisfies* (1.4)–(1.7) *with* $L = [n/t]$, $a = b$. *Let* $t \leq p < \infty$, $s \in ((\delta - 1)r, r)$, $0 < q \leq \infty$ *and* $w \in A_{p/t}$. *Then, we have*

$$\|T_\sigma(f)\|_{B_p^{s,q}(w)} \leq C\|f\|_{B_p^{s,q}(w)}, \quad f \in \mathcal{S}(\boldsymbol{R}^n).$$

THEOREM 6. *Let* $1 < t \leq 2$. *Suppose that* (1.8)–(1.11) *hold with* t *in place of* q. *We further assume* (1.12). *Then*

$$\|T_\sigma(f)\|_{B_p^{s,q}(w)} \leq C\|f\|_{B_p^{s,q}(w)}, \quad f \in \mathcal{S}(\boldsymbol{R}^n),$$

where $t \leq p < \infty$, $s \in ((\delta - 1)r, r)$, $0 < q \leq \infty$ *and* $w \in A_{p/t}$.

Also, we have remarks similar to those in Theorems 1–3 for the dependence of the bounds on σ. We can derive Theorems 4–6 from Theorems 1–3, respectively, by applying interpolation arguments (see [2, Theorem 3.5]).

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