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Non-relativistic eigensolutions of molecular and heavy quarkonia interacting potentials via the Nikiforov Uvarov method.

Ekwevugbe Omugbe

Physics programme, Department of physics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria.

Email: omugbeekwevugbe@gmail.com

Abstract

In this paper, we obtained the analytical eigensolutions of the radial Schrodinger equation using the Nikiforov Uvarov method. Special cases of the modeled potential were discussed. We obtained the energy eigenvalues expressions for both quarkonia and diatomic molecular interacting systems. In particular, we computed the masses of charmonium and bottomonium mesons. The pure vibrational energy levels, inertia rotational constant and the first two centrifugal distortion constants of the Kratzer-Fues oscillator were obtained in closed form. The results obtained are in excellent agreement with the results in the literature.

Keywords. Schrodinger equation, Quarkonia spectrum, Nikiforov Uvarov method, Inertia rotational constant, Centrifugal distortion constants, diatomic molecules

1. Introduction

Quantum mechanical interacting potentials play vital role in the description of physical systems in applied physics and chemistry. In the non-relativistic limit, the potentials are solved with the Schrodinger equation so as to obtain the eigensolutions of the quantum system of interest. In particle and high energy physics, interacting potential such as the Cornell and its generalized forms [1-15] is used as a model to study the interaction of quarks. While in molecular physics and chemistry, several molecular potentials [16-26] have been used as a model to study the molecular structures and interactions by using analytical and computational techniques. Some of these potentials are exactly solvable for $l = 0$ (s-wave), while some are insoluble for $l \neq 0$ (that is for any arbitrary angular momentum quantum number). In such a case, the Schrodinger equation (SE) can be approximated by using analytical and numerical methods [27, 28]. Several analytical methods have been used to solve the SE with the potential of choice. In particular, quarkonia potential model such as the Cornell potential has been solved with different methods due to its non-trivial mathematical properties [29]. Ciftci and Hassan [1] used

the perturbation method within the framework of the asymptotic iterative method (AIM) to estimate the energy spectrum of quarkonia generated by the Cornell potential. Hall and Saad [2] applied the envelop method and AIM technique to compute the eigenvalues of the SE with the Cornell potential. Chung and co-workers [3] solved the SE numerically and obtained the energy eigenvalues for the s-wave heavy quarkonia. Vega and Flores [4], using the variation and supersymmetric quantum mechanics (SUSYQM) approach studied the properties of heavy quarkonia. Khoka et al. [5] applied the analytical exact iterative method (AEIM) to solve the N-dimensional SE with an extended Cornell potential. They obtained the energy and mass spectrum of heavy quarkonia. The Nikiforov Uvarov method has been used to obtain energy eigenvalues and mass spectrum of quarkonia systems [6-11]. Furthermore, the Laplace transformation approach [12], artificial neural network [13], algebraic [14], crank-Nicholson [15] methods have been applied to determine the properties of the quark interacting systems. On the other hand, molecular interacting potentials play a significant role in the description of the structure of diatomic molecules and are used to study the ro-vibrational states and energy spectra cum nuclear rotations and vibrations [30, 31].

In this paper, we will apply the Nikiforov Uvarov method to obtain the eigensolutions of the radial SE with an inversely quadratic plus Cornell potential shifted by a constant parameter. The proposed potential model can be applied to the description of both quarkonia and molecular interacting systems. We considered the potential of the form

$$V(r) = \frac{A}{r^2} + Br - \frac{C}{r} + D, \quad (1)$$

with A, B, C, D as arbitrary potential parameters. The potential in Eq. (1) reduces to the Cornell potential if we set the constants ($A = D = 0$)

$$V(r) = Br - \frac{C}{r}, \quad B, C > 0, \quad (2)$$

where C is a coupling constant and B is a linear confinement parameter. The potential for the non-relativistic regime can be used in the investigation of the masses and decay widths of charmonium states [2, 13]. The coulombic term arises from one gluon exchange between the quark and its anti-quark and dominates at short distances [32]. The linear term dominates at large distances and is supported by lattice Quantum Chromo-dynamics measurements [13, 33].

With $C = 0$ in Eq. (2), we will obtain the linear gravitational potential used as a model to investigate the dynamics of a quantum mechanical bouncing ball [27, 34-41].

$$V(r) = Br \quad (B = mg, r > 0). \quad (3)$$

The linear potential has practical implications in the study of quarkonia energy spectrum [34] and has also been used to investigate the condensation behaviour of a Bose-Einstein ideal gas [35]. Nesvizhevsky et al. [36], have shown experimentally that quantum effect can be observed macroscopically by using a very small mass such as a neutron bouncing on a perfectly reflecting mirror. Also, Jenke and co-workers [37], using the Gravity Resonance Spectroscopy (GRS) technique calculated the energy levels of ultra-cold neutrons bouncing on a smooth surface.

If we set $B = 0$ in Eq. (2) we will obtain the Coulomb potential function used in the description of the hydrogenic atom in quantum mechanics.

$$V(r) = -\frac{C}{r}. \quad (4)$$

Furthermore, when $A = D_e r_e^2$, $C = 2D_e r_e$, $B = 0$, and $D = D_e$ in Eq. (1), the potential reduces to the Kratzer-Fues molecular potential.

$$V(r) = \frac{Dr_e^2}{r^2} - \frac{2Dr_e}{r} + D_e, \quad (5)$$

where D_e is the dissociation energy between two atoms in a solid [17] and it corresponds to the vertical distance between the dissociation limit and the minimum point of the potential curve at the equilibrium internuclear distance $r = r_e$ [42]

The rotational-vibrational energy eigenvalue of diatomic molecules generated by the Kratzer-type potential has been intensively investigated in the literature [16-18, 21, 31, 42-50]

Finally if we let $A = \frac{V_0 a^2}{2}$, $C = V_0 a$, and $B = D = 0$ in Eq. (1), we will obtain the molecular Mie potential given as

$$V(r) = V_0 \left(\frac{a^2}{2r^2} - \frac{a}{r} \right). \quad (6)$$

It is worth mentioning that the modeled potential in this present work for the description of quarks and diatomic molecular interactions given in Eq. (1) differs slightly from the ones in the literature [1-15]. In the study of heavy quarkonia, some authors cited herein presented a generalized or extended Cornell potential containing the harmonic and also arharmonic term. While others solved the SE with the Cornell potential given in Eq. (2). The authors in [12] presented the special cases of the generalized Cornell potential which also differ from the ones obtained in this present work. Also we note that the potentials in Eqs. (2-6) have been solved individually in the literature. Therefore the objective of this present work is to obtain a generalized eigensolutions of the radial SE in such a way that the energy eigenvalues and wave functions of both quarkonia and molecular interacting systems can be elegantly retrieved. The shapes of the potentials in Eqs. (2-6) are given in Figures (1-2).

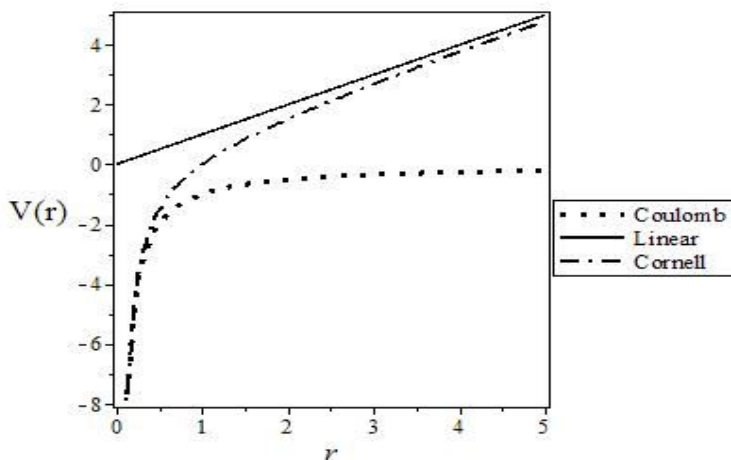


Fig 1. The variation of the potential energy as a function of radial distance. The solid, dashed-dot and dot lines represent the respective linear, Cornell and Coulomb potential functions.

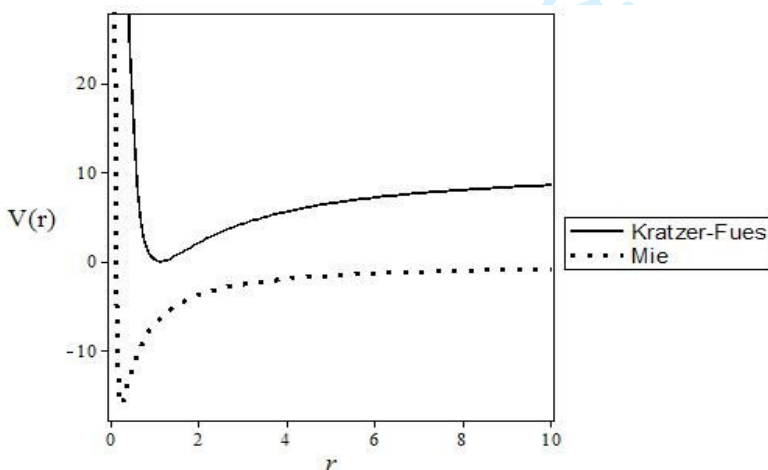


Fig 2. The variation of the molecular potential energy as a function of radial distance. The solid and dotted curves represent the respective Kratzer-Fues and Mie potential functions

The paper is organized as follows. In section two, we will present an overview of the Nikiforov Uvarov method. Section three contains the solution of the radial SE with the potential given in Eq. (1). We will find the energy eigenvalue and wave function solutions. While in section four, the applications of the obtained solutions to quarkonia and diatomic molecular interactions will be presented. The numerical results of the mass spectrum of charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) mesons including the pure vibrational energy levels, Inertia rotation constant and the first

two centrifugal distortion constants of CO and N₂ diatomic molecules will be obtained. Finally in section five, the paper is concluded.

2. Overview of the Nikiforov Uvarov method

The second order differential equations (ODE) whose solutions are the special orthogonal functions of hyper-geometric type can be solved by using the Nikiforov-Uvarov (NU) method [51]. The SE can be solved with this method, if it is transformed into a second order differential equation of the hyper geometric type with an appropriate coordinate transformation $r \rightarrow s$ [42]. The hyper geometric type equation is given [51] as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \quad (7)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomial of at most second degree and $\tilde{\tau}(s)$ is a first degree polynomial. The primed function $\psi(s)$ in Eq. (7) denotes derivative with respect to s .

Let the solution of Eq. (7) be

$$\psi(s) = \varphi(s)y(s). \quad (8)$$

If we substitute Eq. (8) into (7) we will obtain

$$y''(s) + \left(2\frac{\varphi'(s)}{\varphi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)}\right)y'(s) + \left(\frac{\varphi''(s)}{\varphi(s)} + \frac{\varphi'(s)}{\varphi(s)}\frac{\tilde{\tau}(s)}{\sigma(s)} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\right)y(s) = 0. \quad (9)$$

To write Eq. (9) in the most regular form [42] we will let

$$2\frac{\varphi'(s)}{\varphi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)} = \frac{\tau(s)}{\sigma(s)} \quad (10)$$

and

$$\frac{\varphi''(s)}{\varphi(s)} + \frac{\varphi'(s)}{\varphi(s)}\frac{\tilde{\tau}(s)}{\sigma(s)} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} = \frac{\bar{\sigma}(s)}{\sigma^2(s)}, \quad (11)$$

So that,

$$y''(s) + \left(\frac{\tau(s)}{\sigma(s)}\right)y'(s) + \left(\frac{\bar{\sigma}(s)}{\sigma^2(s)}\right)y(s) = 0, \quad (12)$$

where

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)}. \quad (13)$$

Substituting Eq. (13) into (10) implies that

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \quad (14)$$

The respective notations $\tau(s)$ and $\pi(s)$ in Eqs. (10) and (14) are polynomials of at most degree one. The first term of the coefficient of $y(s)$ in Eq. (9) can be written as

$$\frac{\varphi''(s)}{\varphi(s)} = \left(\frac{\varphi'(s)}{\varphi(s)}\right)' + \left(\frac{\varphi'(s)}{\varphi(s)}\right)^2 = \left(\frac{\pi(s)}{\sigma(s)}\right)' + \left(\frac{\pi(s)}{\sigma(s)}\right)^2 = \frac{(\sigma(s)\pi'(s) - \pi(s)\sigma'(s)) + \pi^2(s)}{\sigma^2(s)}. \quad (15)$$

If we substitute Eqs. (13) and (15) into (11) we will have

$$\frac{\sigma(s)\pi'(s) - \pi(s)\sigma'(s) + \pi^2(s) + \tilde{\tau}(s)\pi(s) + \tilde{\sigma}(s)}{\sigma^2(s)} = \frac{\bar{\sigma}(s)}{\sigma^2(s)}. \quad (16)$$

This implies that

$$\bar{\sigma}(s) = \sigma(s)\pi'(s) + \pi^2(s) + [\tilde{\tau}(s) - \sigma'(s)]\pi(s) + \tilde{\sigma}(s). \quad (17)$$

Suppose $\bar{\sigma}(s)$ is divisible by $\sigma(s)$ in Eq. (12) such that

$$\bar{\sigma}(s) = \alpha\sigma(s), \quad (18)$$

then we can write Eq. (12) as

$$\sigma(s)y''(s) + \tau(s)y'(s) + \alpha y(s) = 0, \quad (19)$$

where α is a constant.

By comparing Eq. (17) with (18), we found that

$$\pi^2(s) + [\tilde{\tau}(s) - \sigma'(s)]\pi(s) - \sigma(s)[\alpha - \pi'(s)] + \tilde{\sigma}(s) = 0. \quad (20)$$

Equation (20) is a quadratic equation in the variable $\pi(s)$.

If we set

$$k = \alpha - \pi'(s), \quad (21)$$

we will obtain the roots of the equation as

$$\pi(s) = \frac{(\sigma'(s) - \tilde{\tau}(s))}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 + k\sigma(s) - \tilde{\sigma}(s)}. \quad (22)$$

Since the polynomial $\pi(s)$ is of degree one, the expression under the square root has to be the square of a polynomial. We demand that the discriminant under the squared root of Eq. (22) be zero. This condition enables us to find the value of k .

To obtain the energy eigenvalues, we perform the differentiation of Eq. (19) with the substitution [42] $V_1(s) = y'(s)$

$$\begin{aligned} \sigma(s)y'''(s) + \sigma'(s)y''(s) + \tau(s)y'(s) + \tau'(s)y(s) + \alpha y'(s) \\ = \sigma(s)V_1''(s) + [\tau(s) + \sigma'(s)]V_1'(s) + [\alpha + \tau'(s)]V_1(s) = 0. \end{aligned} \quad (23)$$

Again we perform the second derivative of Eq. (19) with the substitution $V_2(s) = y''(s)$

$$\begin{aligned} \sigma'(s)V_2'(s) + \sigma(s)V_2''(s) + \sigma'(s)V_2'(s) + \sigma''(s)V_2(s) + \tau(s)V_2'(s) + \tau'(s)V_2(s) + \tau''(s)V_1(s) + \alpha V_2(s) \\ = \sigma(s)V_2''(s) + [\tau(s) + 2\sigma'(s)]V_2'(s) + [\alpha + 2\tau'(s) + \sigma''(s)]V_2(s) = 0. \end{aligned} \quad (24)$$

It is worth mentioning that higher derivative of the polynomial $\tau^n(s)$, would vanish for $n > 1$

It can be seen that the derivatives of the hyper-geometric-type equation in (19) also reproduces other hyper-geometric equations. The respective Eqs. (23 and 24) can be further simplified as

$$\sigma(s)V_1''(s) + \tau_1(s)V_1'(s) + \beta_1 V_1(s) = 0, \quad (25)$$

where

$$\tau_1(s) = \tau(s) + \sigma'(s), \quad (26)$$

$$\beta_1 = \alpha + \tau'(s), \quad (27)$$

and

$$\sigma(s)V_2''(s) + \tau_2(s)V_2'(s) + \beta_2 V_2(s) = 0, \quad (28)$$

where

$$\tau_2 = \tau(s) + 2\sigma'(s), \quad (29)$$

$$\beta_2 = \alpha + 2\tau'(s) + \sigma''(s). \quad (30)$$

Generally, if we set $V_n(s) = y^n(s)$, then we will obtain the equation

$$\sigma(s)V_n''(s) + \tau_n(s)V_n'(s) + \beta_n V_n(s) = 0, \quad (31)$$

where

$$\tau_n = \tau(s) + n\sigma'(s), \quad (32)$$

$$\beta_n = \alpha + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s). \quad (33)$$

When $\beta_n = 0$, then

$$\alpha_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \quad (n = 0, 1, 2). \quad (34)$$

To obtain the energy eigenvalues equation, we must establish the relation $\alpha_n = \alpha$ using Eqs. (21 and 34). The solution of Eq. (31) is a polynomial of the form $y(s) = y^n(s)$. The polynomial solutions $y^n(s)$ are given by the Rodrigues relation

$$y^n(s) = \frac{B_n}{\rho(s)d^n} [\sigma^n(s)\rho(s)], \quad (35)$$

where B_n is a normalization constant and $\rho(s)$ is a weight function which must satisfy the condition

$$[\sigma(s)\rho(s)]' = \tau(s)\rho(s). \quad (36)$$

3. Solution of the radial Schrodinger equation

The three-dimensional time-independent SE with a reduced mass μ and wave-function $\psi(r, \theta, \phi)$ is given as

$$-\frac{\hbar^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi(r, \theta, \phi) + V(r) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) \quad (37)$$

Using the method of separation of variables in Eq. (37) we can separate the equation into the radial part and the angular part by using the transformation $\psi(r, \theta, \phi) = \frac{R(r)Y(\theta, \phi)}{r}$. With the appropriate separation constant, we will obtain the radial SE as

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2} \right] R(r) = 0, \quad (38)$$

where the effective potential is given as

$$V_{eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}. \quad (39)$$

If we substitute the potential in Eq. (1) into (38) we will obtain

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E - \left(\frac{A}{r^2} + Br - \frac{C}{r} + D \right) - \frac{l(l+1)\hbar^2}{2\mu r^2} \right] R(r) = 0. \quad (40)$$

To transform the SE into a hyper-geometric type, we let $r = \frac{1}{s}$ and use the following calculus derivatives

$$R'(r) = R'(s) \times s'(r) = -s^2 R'(s). \quad (41)$$

$$R''(r) = (s'(r))^2 R''(s) + R'(s) \times s''(r) = s^4 R''(s) + 2s^3 R'(s). \quad (42)$$

Substituting Eq. (42) into (40) yields

$$R''(s) + \frac{2s}{s^2} R'(s) + \frac{2\mu}{s^4 \hbar^2} \left[E - \left(As^2 + \frac{B}{s} - Cs + D \right) - \frac{l(l+1)\hbar^2}{2\mu} s^2 \right] R(s) = 0. \quad (43)$$

To solve Eq. (43) we will use the approximation scheme used in the literature [6-8, 11]. We expand the term $\frac{B}{s}$ in power series form to the second order around r_0 ($\delta = \frac{1}{r_0}$) which is assumed to be the characteristic radius of a meson.

Next we let $y = s - \delta$ so that

$$\frac{B}{s} = \frac{B}{y + \delta} \approx B \left(\frac{3}{\delta} - \frac{3s}{\delta^2} + \frac{s^2}{\delta^3} \right). \quad (44)$$

This approximation allows us to solve Eq. (43) with the NU method. Equation (43) can be further simplified as

$$R''(s) + \frac{2s}{s^2} R'(s) + \frac{2\mu}{s^4 \hbar^2} \left[\left(E - \frac{3B}{\delta} - D \right) + \left(C + \frac{3B}{\delta^2} \right) s - \left(A + \frac{B}{\delta^3} + \frac{l(l+1)\hbar^2}{2\mu} \right) s^2 \right] R(s) = 0. \quad (45)$$

With

$$\frac{2\mu}{\hbar^2} \left(E - \frac{3B}{\delta} - D \right) = -T, \quad (46)$$

$$\frac{2\mu}{\hbar^2} \left(C + \frac{3B}{\delta^2} \right) = W, \quad (47)$$

$$\frac{2\mu}{\hbar^2} \left(A + \frac{B}{\delta^3} + \frac{l(l+1)\hbar^2}{2\mu} \right) = N, \quad (48)$$

we can write Eq. (45) as

$$R''(s) + \frac{2s}{s^2} R'(s) + \frac{-T + Ws - Ns^2}{s^4} R(s) = 0. \quad (49)$$

If we compare Eq. (49) with (7) we will have

$$\tilde{\tau}(s) = 2s, \quad \sigma(s) = s^2, \quad \tilde{\sigma}(s) = -T + Ws - Ns^2.$$

Using these polynomials in Eq. (22) implies that

$$\pi(s) = \pm \sqrt{k s^2 - (-T + Ws - Ns^2)}. \quad (50)$$

We will now obtain k by equating the discriminant of the quadratic equation under the squared root in Eq. (50) to zero

$$k = \frac{W^2}{4T} - N. \quad (51)$$

Substituting k into Eq. (50) yields

$$\pi(s) = \pm \sqrt{\frac{W^2}{4T} s^2 - Ws + T} = \pm \frac{(Ws - 2T)}{2\sqrt{T}}. \quad (52)$$

The NU method requires that for the polynomial $\pi(s)$, we choose the one for which $\tau(s)$ has a negative derivative. From Eq. (14) we have

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) = 2s \pm \frac{(Ws - 2T)}{\sqrt{T}}. \quad (53)$$

$$\tau'(s) = 2 - \frac{W}{\sqrt{T}} \quad (54)$$

The choice of $\pi(s)$ that will make $\tau'(s) < 0$ is $-\frac{(Ws - 2T)}{2\sqrt{T}}$.

To find the energy eigenvalues, we recall Eqs. (21 and 34)

$$\alpha = k + \pi'(s) = \frac{W^2}{4T} - N - \frac{W}{2\sqrt{T}} \quad (55)$$

$$\alpha_n = -n\tau'(s) - \frac{n(n-1)}{2} \sigma''(s) \quad (n = 0, 1, 2) \quad (56)$$

Substituting the notations $\tau'(s)$ and $\sigma''(s)$ into Eq. (56) yields

$$\alpha_n = -n \left(2 - \frac{W}{\sqrt{T}} \right) - n(n-1) = \frac{Wn}{\sqrt{T}} - n - n^2. \quad (57)$$

Now equating both sides of Eqs. (55) and (57) gives

$$\frac{W^2}{4T} - \frac{W}{\sqrt{T}} \left(n + \frac{1}{2} \right) + n + n^2 - N = 0. \quad (58)$$

Next, we let $x = \frac{1}{\sqrt{T}}$, so that Eq. (58) becomes

$$\frac{W^2 x^2}{4} - \left(n + \frac{1}{2} \right) W x + n + n^2 - N = 0. \quad (59)$$

Equation (59) is quadratic in the variable x . The roots are obtained as

$$x = \frac{1}{\sqrt{T}} = \frac{2 \left(n + \frac{1}{2} \right) W \pm 2 \sqrt{\left(n + \frac{1}{2} \right)^2 W^2 - W^2 (n + n^2 - N)}}{W^2}. \quad (60)$$

After some simplifications we obtained

$$T = \left[\frac{W}{(2n+1) \pm \sqrt{1+4N}} \right]^2. \quad (61)$$

On substituting the notations for W, T and N given in Eqs. (46-48) we obtained the energy eigenvalues generated by the potential in Eq. (1)

$$E_{nl} = -\frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2} \left(C + \frac{3B}{\delta^2} \right)}{(2n+1) \pm \sqrt{1 + \frac{8\mu}{\hbar^2} \left(A + \frac{B}{\delta^3} + \frac{l(l+1)\hbar^2}{2\mu} \right)}} \right]^2 + \frac{3B}{\delta} + D. \quad (62)$$

To obtain the wave function in Eq. (8), ($\psi(s) = \varphi(s)y(s)$) we use the relation in Eqs. (13 and 35), but we must first determine the weight function from (36). With the help of Eq. (13), we obtained

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)} = -\frac{(Ws-2T)}{s^2 2\sqrt{T}}. \quad (63)$$

Solving Eq. (63), yields

$$\varphi(s) = s^{-\frac{W}{2\sqrt{T}}} e^{-\frac{\sqrt{T}}{s}}. \quad (64)$$

Next we find the weight function from Eq. (36)

$$[s^2 \rho(s)]' = \left(2s - \frac{(Ws-2T)}{\sqrt{T}} \right) \rho(s)$$

$$\Rightarrow 2s \rho(s) + s^2 \rho'(s) = \left(2s - \frac{Ws}{\sqrt{T}} + 2\sqrt{T} \right) \rho(s). \quad (65)$$

Solving Eq. (65), we obtained the weight function as

$$\rho(s) = s^{-\frac{W}{\sqrt{T}}} e^{-\frac{2\sqrt{T}}{s}}. \quad (66)$$

Substituting Eq. (66) into (35) yields

$$y_{nl}(s) = (s)^{\frac{W}{\sqrt{T}}} e^{\frac{2\sqrt{T}}{s}} B_{nl} \frac{d^n}{ds^n} \left[(s)^{2n} s^{-\frac{W}{\sqrt{T}}} e^{-\frac{2\sqrt{T}}{s}} \right]. \quad (67)$$

Next, we substitute Eqs. (64 and 67) into (8) to get

$$R_{nl}(s) = (s)^{\frac{W}{2\sqrt{T}}} e^{\frac{\sqrt{T}}{s}} B_{nl} \frac{d^n}{ds^n} \left[(s)^{2n} s^{-\frac{W}{\sqrt{T}}} e^{-\frac{2\sqrt{T}}{s}} \right]. \quad (68)$$

If we change the variable from $s \rightarrow r$, we will obtain

$$R_{nl}(r) = (r)^{-\frac{W}{2\sqrt{T}} - 1} e^{\sqrt{T} r} B_{nl} \left(-r^2 \frac{d}{dr} \right)^n \left[(r)^{-2n + \frac{W}{\sqrt{T}}} e^{-2\sqrt{T} r} \right]. \quad (69)$$

4. Applications

4.1 Quarkonia interacting systems

The generalized energy eigenvalues equation obtained in Eq. (62), reduces to the Cornell potential if we set $A = D = 0$

$$E_{nl}^{Cornell} = -\frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2} \left(C + \frac{3B}{\delta^2} \right)}{(2n+1) \pm \sqrt{1 + \frac{8\mu}{\hbar^2} \left(\frac{B}{\delta^3} + \frac{l(l+1)\hbar^2}{2\mu} \right)}} \right]^2 + \frac{3B}{\delta}. \quad (70)$$

Also if we set $B = 0$ in Eq. (70) we will immediately retrieve the Coulomb's energy eigenvalue equation given as

$$E_{nl}^{CP} = -\frac{\mu Z^2 e^4}{2\hbar^2 n_p^2}, \quad n_p = 1, 2, 3 \quad (71)$$

where $C = Ze^2$, Z is the atomic number, e is the electronic charge and, n_p is a principal quantum number with notation $n_p = n + l + 1$

Next, we present the mass spectrum of charmonium and bottomonium for any arbitrary radial and angular momentum quantum numbers in Tables [1-2] using the meson mass relation [6, 9]

$$M_{nl} = m_q + m_{\bar{q}} + E_{nl}^{Cornell} \quad (72)$$

Where $m_q, m_{\bar{q}}$ are the respective quark and anti-quark masses

On substituting Eq.(70) into Eq.(72) we obtained

$$M_{nl} = m_q + m_{\bar{q}} - \frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2} \left(C + \frac{3B}{\delta^2} \right)}{(2n+1) \pm \sqrt{1 + \frac{8\mu}{\hbar^2} \left(\frac{B}{\delta^3} + \frac{l(l+1)\hbar^2}{2\mu} \right)}} \right]^2 + \frac{3B}{\delta} \quad (73)$$

In order to obtain the free parameters C , B and δ we fit Eq. (73) with experimental data given in Tables [1-2]. For the charmonium ($c\bar{c}$) meson, we solved simultaneously three algebraic equations with the numerical support of MAPLE package by substituting the experimental masses for the 1S, 2S, and 3S states into Eq. (73). For the bottomonium meson, we fixed B gotten from the charmonium fit into Eq. (73) and solve two algebraic equations for the parameters δ and C by substituting the experimental masses for the respective 1S and 2S states.

Table 1. Charmonium ($c\bar{c}$) mass spectra in GeV ($m_c = 1.209$ GeV, $B = 0.2$ GeV², $C = 1.234$, $\delta = 0.232$ GeV)

States	This work	Ref. [1]	Ref. [6]	Ref. [9]	Ref. [13]	Exp.[52]
1s	3.096839898	3.096	3.096	3.0969	3.098	3.097
2s	3.686005578	3.672	3.686	3.68697	3.688	3.686
3s	4.039049024	4.085	4.040	4.04143	4.029	4.039
4s	4.267153785	4.433	4.269	4.27086		
1p	3.256571621	3.521	3.255	3.25581	3.516	3.511
2p	3.778773823	3.951	3.779	3.77951	3.925	3.927
3p	4.097620019	4.310		4.09997	4.301	

Table 2. Bottomonium ($b\bar{b}$) mass spectra in GeV ($m_b = 4.823$ GeV, $B = 0.2$ GeV², $C = 1.553$, $\delta = 0.381$ GeV)

States	This work	Ref.[1]	Ref. [6]	Ref. [9]	Ref. [13]	Exp.[52]
1s	9.460424446	9.462	9.460	9.45851	9.460	9.460
2s	10.02337702	10.027	10.023	10.0218	10.026	10.023
3s	10.35386764	10.361	10.355	10.3539	10.354	10.355
4s	10.56426936	10.624	10.567	10.5661	10.572	10.579
1p	9.621553580	9.963	9.619	9.61781	9.891	9.899
2p	10.11488755	10.299	10.114	10.1127	10.258	10.260
3p	10.41073350	10.564		10.4106	10.518	10.512

4.2 Molecular interacting systems

Setting $A = D_e r_e^2$, $C = 2D_e r_e$, $B = 0$, and $D = D_e$, Eq. (1) reduces to the Kratzer-Fues potential and the energy eigenvalue equation in Eq. (62) becomes

$$E_{nl}^{KFP} = D_e - \frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2}(2D_e r_e)}{(2n+1) \pm \sqrt{1 + \frac{8\mu}{\hbar^2} \left(D_e r_e^2 + \frac{l(l+1)\hbar^2}{2\mu} \right)}} \right]^2. \quad (74)$$

If we substitute $A = \frac{V_0 a^2}{2}$, $C = V_0 a$, and $B = D = 0$ into Eq. (62), we will obtain the molecular Mie potential energy spectrum given as

$$E_{nl}^{MP} = -\frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2}(V_0 a)}{(2n+1) \pm \sqrt{1 + \frac{8\mu}{\hbar^2} \left(\frac{V_0 a^2}{2} + \frac{l(l+1)\hbar^2}{2\mu} \right)}} \right]^2. \quad (75)$$

To obtain the inertia rotational (B_{n0}) and the first two centrifugal distortion constants (D_{n0} , H_{n0}) of the Kratzer-Fues potential, we used the approach introduced in Ref. [47]. We used MAPLE software to expand Eq. (74) in power series in terms of $l(l+1)$ to the fourth order, and obtained

$$E_{nl}^{KFP} = E_{n0}^{KFP} + B_{n0}l(l+1) - D_{n0}l^2(l+1)^2 + H_{n0}l^3(l+1)^3 \dots O(l^4(l+1)^4), \quad (76)$$

where

$$E_{n0}^{KFP} = D_e - \frac{\hbar^2}{2\mu} \left[\frac{\frac{2\mu}{\hbar^2}(2D_e r_e)}{(2n+1) + \sqrt{1 + \frac{8\mu}{\hbar^2}(D_e r_e^2)}} \right]^2, \quad (77)$$

$$B_{n0} = \frac{32\mu D_e^2 r_e^2}{\left((2n+1) + \sqrt{\frac{(8\mu D_e r_e^2 + \hbar^2)}{\hbar^2}} \right)^3 \hbar \sqrt{(8\mu D_e r_e^2 + \hbar^2)}}, \quad (78)$$

$$D_{n0} = \frac{D_e^2 r_e^2 \left(32\mu D_e r_e^2 + 2\hbar^2 n \sqrt{8\mu \frac{D_e r_e^2}{\hbar^2} + 1} + \hbar^2 \sqrt{8\mu \frac{D_e r_e^2}{\hbar^2} + 1} + 4\hbar^2 \right)}{32\mu \left((2n+1) + \sqrt{8\mu \frac{D_e r_e^2}{\hbar^2} + 1} \right)^4 (8\mu D_e r_e^2 + \hbar^2)^2}, \quad (79)$$

$$H_{n0} = \frac{64D_e^2 r_e^2 \mu \hbar^2 [8r_e^2 \mu (7a + 10n + 5) + 4a \hbar^2 (n^2 + n + 2) + \hbar^2 (a^3 + 10n + 5)]}{(2n+1+a)^5 a^6 \hbar^6}, \quad (80)$$

where $a = \sqrt{\frac{(8\mu D_e r_e^2 + \hbar^2)}{\hbar^2}}$ in Eq. (80)

Using MAPLE package, we obtained the pure vibrational energy eigenvalues, inertia rotational constant and the first two centrifugal distortion constants for CO and N₂ diatomic molecules as shown in Tables (3-4). We used the conversions $\hbar c = 1973.29 \text{ eV}\text{\AA}$, $1 \text{ amu} = 931.494028 \text{ Mev}/c^2$ reported in Ref. [53]

Table3. Pure vibrational energy eigenvalues and inertia rotational constant of the Kratzer-Fues potential. The following spectroscopic parameters for CO ($D_e = 10.845073641 \text{ eV}$, $r_e = 1.128 \text{\AA}$, $\mu = 6.86058600 \text{ amu}$) and N₂ ($D_e = 11.938193820 \text{ eV}$, $r_e = 1.0940 \text{\AA}$, $\mu = 7.00335 \text{ amu}$) were taken from Ref.[53]

States	Inertia rotational constant B_{n0} (eV)		Energy eigenvalues E_{n0} (eV)	
n	CO	N ₂	CO	N ₂
0	0.0002376293307	0.0002476582831	0.050824940	0.005443680
5	0.0002216780654	0.0002314629222	0.539451730	0.578338160
10	0.0002071231165	0.0002166494376	0.995635920	1.068343360

15	0.0001938149822	0.0002030735317	1.422187272	1.527314640
20	0.0001816229639	0.0001906086169	1.821617824	1.957818388
25	0.0001704324662	0.0001791433309	2.196178952	2.362161092
30	0.0001601427268	0.0001685794428	2.547893176	2.742420276
35	0.0001506649083	0.0001588300807	2.878581510	3.100471276
40	0.0001419204832	0.0001498182292	3.189887104	3.438010380
45	0.0001338398643	0.0001414754484	3.483295665	3.756574956
50	0.0001263612399	0.0001337407804	3.760153226	4.057560983

Table 4. Centrifugal distortion constants in (eV) for the Kratzer-Fues potential. The following spectroscopic parameters for CO ($D_e = 10.845073641$ eV, $r_e = 1.1283$ Å, $\mu = 6.86058600$ amu) and N₂ ($D_e = 11.938193820$ eV, $r_e = 1.0940$ Å, $\mu = 7.00335$ amu) were taken from Ref. [53]

State	Centrifugal distortion constants			
n	$D_{n0}(\text{CO}) \times 10^{-9}$	$D_{n0}(\text{N}_2) \times 10^{-9}$	$H_{n0}(\text{CO}) \times 10^{-13}$	$H_{n0}(\text{N}_2) \times 10^{-13}$
0	5.2343481380000	5.1641635490000	1.15332750800000	1.0771375210000
20	3.7436924990000	3.7255746940000	0.78094998520000	0.7367124907000
40	2.7562209960000	2.7627818520000	0.5479856921000	0.5212465347000
60	2.0795367800000	2.0972876070000	0.3962468623000	0.3795346898000
80	1.6022592280000	1.6244309060000	0.9397202940000	0.2832300932000
100	1.2571608550000	1.2803490590000	0.2297880670000	0.2159099982000
200	0.4588356192000	0.4746830726000	0.0717250604500	0.0706230103900
400	0.1126483480000	0.1182933560000	0.0153622488800	0.0153489334900
800	0.0199507267500	0.0211742100100	0.0024395196750	0.0024593340250
2000	0.0015134282291	0.0001616964229	0.0001717725671	0.0001739404262

5. Conclusion

We have obtained the energy eigenvalue and wave function expressions of the radial SE with an inversely quadratic potential plus Cornell potential shifted by a constant parameter. The Pekeris-type approximation scheme within the framework of the NU approach was applied to obtain close form solutions. The eigenvalues and mass spectra of quarkonia interacting systems such as charmonium and bottomonium mesons were computed. Also molecular interacting systems were studied with the aid of the molecular Kratzer-Fues potential. The obtained energy spectra equations of the molecular Kratzer-Fues, Mie and Coulomb's potentials are exact but the spectrum generated by the Cornell potential is an approximation. Therefore it is worthwhile to state that the exact analytical equation of the energy spectra of quarkonia system described by the Cornell potential is unknown and has not been found yet [2]. Furthermore, we obtained the pure vibrational energy eigenvalues, inertia rotational constant (IRC) and the first two centrifugal distortion constants (CDC) using the method introduced in Ref. [47]. As the vibrational quantum number increases, both the IRC and CDC reduce monotonically and tend to zero at the dissociation limit. The results in this present work simulated with CO and N₂ diatomic molecules are in consonance with the work reported in [47].

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