NON-RELATIVISTIC QUANTUM MECHANICS

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Abstract. This chapter is a discussion of the philosophical and foundational issues that arise in non-relativistic quantum theory. After introducing the formalism of the theory, I consider: characterizations of the quantum formalism, empirical content, uncertainty, the measurement problem, and non-locality. In each case, the main point is to give the reader some introductory understanding of some of the major issues and recent ideas.

Keywords: quantum theory, quantum mechanics, measurement problem, uncertainty, nonlocality

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This article is an introduction to some of the most important philosophical and foundational issues that arise from or concern non-relativistic quantum theory. The chapter has six main sections. The first introduces the theory, including some of the important mathematical results required to formulate and address many of the philosophical and foundational issues. This section is the longest, and most important, for it will begin to give the careful reader the background needed to understand and evaluate much of the vast literature on non-relativistic quantum theory. And that literature is indeed vast—there is no way that it can even be summarized in a chapter of this length. Instead, in the five subsequent sections, I will consider some of the more important: foundational characterizations of the formalism of quantum theory, empirical content, quantum uncertainty, the measurement problem, and non-locality. There are many other issues one could discuss, and some recent movements that merit

consideration. Alas, we will not have time for them. A careful reading of the material here is a start, however, towards understanding these other issues.

Of these five issues, the first two are somewhat less discussed, especially in the Anglo-American philosophical literature. Those sections are therefore longer, relative to the final three, than some readers might expect. This fact is not meant to imply anything about the relative importance of the issues, but is an attempt to redress a relative lack of coverage in certain circles.

Much of the material presented here—especially from §4,§5, and §6—is largely my review of standard material that can be found in many places. I have therefore chosen not to provide extensive bibliographic information. Indeed, I have kept bibliographic references to a minimum. This article is thus *not* intended to be a compendium of work in the field, much less an extensive annotated bibliography. The reader is encouraged to seek additional resources to fill out the brief accounts given here. Such resources are numerous.

The final section is a brief mathematical appendix, reviewing essential definitions and results, mostly from the theory of Hilbert spaces and groups. It may serve one of two purposes, depending on the reader: a brief reminder of concepts learned elsewhere; or a prompt to learn the concepts elsewhere. It is unlikely that a reader who is completely unfamiliar with these concepts will absorb them just from what is said here. Reference is made to the relevant subsections of this appendix at the appropriate places in the text.

While at some points I have made some effort at rigor, for the most part, the discussion here is only partly rigorous, with the occasional attempt made to point towards what further would be required for complete rigor. The reader is, again, encouraged to consult the literature for mathematical details, and in any case is encouraged to bear in mind that much of the discussion here is *not* intended to be entirely mathematically rigorous, while, I hope, also not being misleading.

1. The Theory

This section is an introduction to the formalism of quantum theory. After a brief justification of this approach (§1.1), I will introduce the major elements of the formalism (§1.2), followed by a simple, but important, example (§1.3). I will then introduce the commonly used 'Dirac' notation (§1.4), and conclude by considering the role of transformations (groups) in the theory (§1.5), including dynamical transformations (equations of motion) and finally (§1.6) a brief preview of the philosophical issues to

come. More than subsequent sections, this section will rely heavily on the material from the mathematical appendix (§7), with references where appropriate.

1.1. **The Thought Behind Starting with Formalism.** Why begin an account of a physical theory with its formalism? Why not begin, instead, with its basic physical insights, or fundamental physical principles? One problem with such an approach here is that, in the case of quantum theory, there is not much significant agreement about what the basic physical insights, or fundamental physical principles, are. Some argue that the collapse postulate (to be discussed later) is at the heart of the theory. Others argue that it must be excised from the theory. Some argue that the theory is fundamentally indeterministic, while others argue that we can make sense of it only in terms of an underlying determinism. Some argue that the familiar notion of a 'particle' with a definite location is a casualty of the theory, while others argue that the theory makes sense only if one takes such a notion as fundamental.

Now, advocates of these different views tend also, it is true, to advocate different formulations of the theory, but they will not suggest that formulations other than their preferred one are *wrong*, only that they, perhaps, emphasize the wrong points. (Indeed, there is no disputing that the standard formalism—the one presented here—is empirically successful; advocates of different views will ultimately have to account for that success in their own terms.) Hence, while the choice of a single formalism at the start of our discussion might slant our point of view somewhat, it will, unlike the choice of basic physical insights or fundamental physical principles, not prejudge the central issues.

1.2. **The Standard Formalism.** I begin with a very brief sketch of a common understanding of the formalism, which I shall flesh out and generalize subsequently. (The reader is not expected to have a deep understanding of any aspect of the formalism merely as a result of reading this subsection.)

1.2.1. *Hilbert Space.* The formalism of quantum mechanics is normally understood in terms of the theory of Hilbert spaces (§7.1). A Hilbert space is a vector space (§7.1.1) with an inner product (§7.1.3) that is also complete with respect to the norm (§7.1.4) defined by this inner product. A standard example is the space, ℓ^2, of (modulus)-square-summable sequences of complex numbers. In this space, the inner product of two vectors, $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ is $\sum_{n=1}^{\infty} x_n^* y_n$. Another standard example is the space, $L^2(\mathbb{R}^N)$, of (modulus)-square-integrable, Lebesque-measurable, complex-valued functions on $\mathbb{R}^N$, where we identify two functions (i.e., they represent the same vector) if (and only if) they differ only on a set of Lebesque measure (§7.5.4) zero.

Here the inner product of two vectors, $f(x)$ and $g(x)$, is $\int f^*(x)g(x)dx$ (where $f(x)$ and $g(x)$ are arbitrary representatives from their respective equivalence classes).[1]

1.2.2. *Observables.* The 'observables' of the theory—the physical quantities, or properties, whose value or presence one can, in principle at least, measure, or 'observe'—are normally taken to be represented by the self-adjoint operators (§7.2.1, §7.2.3) on the Hilbert space. (The nature of the representation—that is, which operators represent which observables—can depend on the physical situation being described.) Via the spectral theorem (discussed below), one can identify each observable with a spectral family of projection operators, the observable being given, essentially, by a map from Borel sets (§7.5.5) of possible values of the observable to elements in the spectral family. This subsection reviews these ideas briefly.

1.2.2.1. *Positive Operator Valued Measures.* It is often useful to adopt a broader notion of an observable, as a 'positive-operator-valued measure' (POVM). In this approach, we begin with a set of 'possible values' for the observable, represented in the most general case as a locally compact topological space, S (§7.5.1). In most cases of interest to us, S is a subset of the real numbers, or things can be reworked so that it is.

A map, $E : \mathcal{B}(S) \to \mathcal{B}(\mathcal{H})$, from the Borel subsets of S to the bounded operators (§7.2.2) on some Hilbert space, $\mathcal{H}$, is a POVM just in case for any disjoint sequence of such subsets, $\Delta_n \subseteq S$,

$$E(\Delta_n) \text{ is a positive operator for all } n \tag{1.1}$$

$$E(S) = \mathbb{I}, \quad \text{the identity on } \mathcal{H} \tag{1.2}$$

$$E\left(\cup_n \Delta_n\right) = \sum_n E(\Delta_n). \tag{1.3}$$

In (1.1), an operator, E, is positive if $\langle v, Ev\rangle \geq 0$ for all $v \in \mathcal{H}$. The positive operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})^+$. The convergence intended in (1.3) is in the weak operator topology on $\mathcal{H}$ (§7.5.3). If, in addition, $E(\Delta_n \cap \Delta_m) = E(\Delta_n)E(\Delta_m)$ whenever $n \neq m$ then: everything in the image of E is a projection operator; E is then called a 'projection-valued measure' (PVM); and the family $\{E(\Delta_n)\}$ is a 'spectral family'. In this case, the $E(\Delta_n)$ are mutually orthogonal, meaning that $E(\Delta_m)E(\Delta_n) = 0$ (the zero operator) whenever $m \neq n$, and we write $E(\Delta_m) \perp E(\Delta_n)$.

[1]For those who have some familiarity with quantum theory: the space ℓ^2 is the space used in Heisenberg's 'matrix mechanics', while the space $L^2(\mathbb{R}^N)$ is the space used in Schrödinger's 'wave mechanics'. As Hilbert spaces, ℓ^2 and $L^2(\mathbb{R}^N)$ are isomorphic, meaning that the two theories are essentially the same.

We can recover a self-adjoint operator from any POV, E. If the cardinality of $S \subset \mathbb{R}$ is finite ($S = \{s_1, \ldots, s_N\}$), then the recovery is straightforward:[2]

$$A = \sum_{n=1}^{N} s_n E(s_n). \tag{1.4}$$

That is, the operator A is the weighted sum of the (mutually orthogonal) projections $E(s_n)$, the weights being the 'possible values' of the observable, i.e., elements of S. If S is countably infinite, then the situation is much the same, though one must worry about convergence. If S is uncountably infinite, then the sum becomes an integral, and matters become considerably more complicated. In any case, the resulting operator, A, is self-adjoint.

1.2.2.2. *Spectral Theorem.* The spectral theorem states the converse of the construction given by (1.4). Again, the finite case is simplest. There, every self-adjoint operator, F, can be written as

$$F = \sum_{n} s_n P_n, \tag{1.5}$$

where the s_n are real numbers and the P_n are mutually orthogonal projections. A spectral family therefore fixes a self-adjoint operator, and a self-adjoint operator fixes a spectral family. Hence the formalism of PVMs makes quick contact with a formalism (in terms of self-adjoint operators) that is perhaps more familiar to some readers, and certainly widely used in physics; thus that latter formalism can be seen as a special case of the more general formalism in terms of POVMs. The case of infinite-dimensional spaces is conceptually analogous, but mathematically trickier.

Notice that every vector inside the subspace corresponding to a projection P_n (henceforth, $\operatorname{ran} P_n$, the 'range' of P_n) is an eigenvector (§7.2.1) of F; the $\operatorname{ran} P_n$ are therefore often called 'eigenspaces' of F. When the eigenspaces of F are all one-dimensional, then F is called 'maximal'. The import of being maximal will become clear below.

Finally, notice that the spectral projections of F partially define an orthogonal, indeed orthonormal (§7.1.4), basis for the space. Within each P_n, choose a set of mutually orthogonal and normalized vectors, $\{e_{n,m}\}_{m=1}^{\dim(P_n)}$. Do the same for the kernel of F (the subspace that F maps to 0, denoted $\ker F$). The result (i.e., the union of all these sets) is an orthonormal basis (which, if F is maximal, is in fact fixed up to constant multipliers of the elements of the basis). Even when this basis

[2] In (1.4) $E(s_n)$ should strictly be written $E(\{s_n\})$, because $E()$ acts on Borel *sets*, but the notation is clumsy, and the meaning of $E(s_n)$ should be clear enough. I follow the same convention elsewhere.

is not uniquely fixed by F (because it is non-maximal), I will refer to such a basis as 'a basis determined by F'.

1.2.3. *States.*

1.2.3.1. *Probabilities.* The formalism in terms of POVMs (as well as the special case of PVMs) describes a probabilistic theory, inasmuch as it provides probabilities for (Borel sets of) values of observables, or (equivalently and sometimes more conveniently) expectation values for observables. I will take probabilities as fundamental; expectation values can then be generated from a probability measure over the possible values, f_n, of F in the usual way:

$$\mathrm{Exp}(F) = f_1 \Pr(f_1) + f_2 \Pr(f_2) + \cdots . \tag{1.6}$$

As we noticed above, rather than considering directly the possible values of an observable, we can also consider the corresponding (spectral) projectios, which can be taken, in a given physical situation, to represent those values.

A probability measure, p, defined on the projection operators should, minimally, be such that $p(P_1 + P_2) = p(P_1) + p(P_2)$ whenever $P_1 \perp P_2$. (Later I will motivate this condition. The basic idea is that it corresponds to the usual 'additivity axiom' of Kolmogorovian probability theory—see §7.5.6.) More specifically, and for now considering just the case of PVMs, we require a probability measure on the projections on a Hilbert space to be a map, p, from projections to the interval $[0, 1]$, where p is countably additive on sets of mutually orthogonal projections.

Precisely what one means by countable additivity for the operators that are in the image of a POVM (rather than a PVM) is a slightly subtle matter. In particular, in general the operators in the image of a POVM—normally they are called 'effects'—do not correspond to subspaces, and the notion of orthogonality does not apply. However, there is a natural generalization of the concept. Notice that for projections, $\{P_i\}$, in the image of a PVM, the condition that $\mathbb{I} - \sum_i P_i$ be a projection (or maybe the zero operator) is equivalent to the condition that the $\{P_i\}$ be mutually orthogonal.[3] The analogous condition in the case of positive operators is that, for effects $\{E_i\}$ in the image of a POVM, if $\mathbb{I} - \sum_i E_i$ is positive (or 0), then $\Pr(\sum_i E_i) = \sum_i \Pr(E_i)$.

1.2.3.2. *Statevectors and Wavefunctions.*

[3]Sketch of a proof: Write $(\mathbb{I} - \sum_i P_i)(\mathbb{I} - \sum_i P_i)$; expand; argue that if the $\{P_i\}$ are mutually orthogonal, then the result is $\mathbb{I} - \sum_i P_i$; argue (using the fact that projections are positive—this part is less trivial) that if the result is $\mathbb{I} - \sum_i P_i$, then the $\{P_i\}$ are mutually orthogonal; finally, argue that $\mathbb{I} - \sum_i P_i$ is self-adjoint.

1.2.3.2.a. Statevectors. Normalized vectors determine probability measures over the projections, via:

$$\text{probability of } P \text{ as given by } v := \Pr_v(P) := \langle v, Pv\rangle. \tag{1.7}$$

(One often sees the expression $|\langle \phi, v\rangle|^2$, where ϕ is a normalized vector from $\operatorname{ran} P$. The two expressions are equivalent.) Notice that the probabilities generated by the vectors v and $e^{i\phi}v$ (where ϕ is a real number) are the same. One says that 'overall phases do not affect probabilities'. The expectation value of a self-adjoint operator, F, given by the state v is

$$\text{expectation of } P \text{ as given by } v := \operatorname{Exp}_v(F) := \langle v, Fv\rangle. \tag{1.8}$$

(Note that the expectation value of a projection is also its probability.)

Note that if $v \in \operatorname{ran} P$ then $\Pr_v(P) = 1$. More generally, if P_n is an eigenspace of F corresponding to the eigenvalue f_n and $v \in \operatorname{ran} P_n$, then $\Pr_v(P_n) = 1$, i.e. the probability (in the state v) that F has the value f_n is 1. Such a state, v, is called an 'eigenstate' of F—it is a normalized vector inside the eigenspace, $\operatorname{ran} P_n$, of F. Notice that, in this case, writing F in terms of its spectral decomposition (recall 1.4) makes the determination of probabilities and expected values trivial. Indeed, even when dealing with general states, it is often convenient to write F in terms of its spectral decomposition, and the state in terms of a basis determined by F.

1.2.3.2.b. Superposition. It is a standard assumption of quantum theory that *every* vector in the Hilbert space for a system is a possible state for the system. This assumption is often expressed as the 'superposition principle', which asserts that (normalized) linear combinations of statevectors are again statevectors.

Given an observable, F, the superposition principle gives rise to (possible) states that are *not* eigenstates of F. Suppose, for simplicitly, that F is maximal, with eigenspaces and eigenvalues $\{P_n\}$ and $\{f_n\}$, and consider an orthonormal basis, $\{v_n\}$, determined by F (which, because F is maximal, just amounts to choosing one normalized vector from each $\operatorname{ran} P_n$). Now form the state vector

$$v = \sum_n k_n |v_n\rangle \tag{1.9}$$

where $\sum_n |k_n|^2 = 1$ and with at least *two* non-zero coefficients k_n. In this case, we say that v is a *superposition* of the v_n. (One sometimes here the word 'superposition' used in a way that suggests that some vectors are 'in superpositions' and others are not. *Relative to a given basis*, this distinction makes sense, but otherwise it does not. *Every* vector is a superposition for some choices of basis.) Notice that v is not an eigenstate of F, and assigns non-trivial probabilities to more than one possible value

of F. Of course, the superposition principle implies that v is nonetheless a possible state of a system.

1.2.3.2.c. Wavefunctions. Wavefunctions are just a specific way of representing statevectors. It is often convenient to take the Hilbert space for a quantum system to be the elements of $L^2(\mathbb{R}^3)$, in which case statevectors are (equivalence classes of) complex-valued functions on $\mathbb{R}^3$. The equation of motion that they standardly satisfy is a type of wave equation (e.g., the Schrödinger equation—see §1.5.2.3.a), and for this reason—as well as the fact that the equation was historically derived with wave phenomena in mind—these functions are called 'wavefunctions'. Linear combinations of waves may be conceived in terms of 'superposing' the waves—hence the term 'superposition'.

1.2.3.3. *Gleason's Theorem.* One can generate probability measures using non-negative trace-1 operators ('density operators'). The functional $\mathrm{Tr}[\cdot]$ is the 'trace functional', a map from the bounded operators on a Hilbert space to $\mathbb{R}$ defined by:

$$\mathrm{Tr}[F] = \sum_k \langle e_k, (F^*F)^{1/2} e_k \rangle \tag{1.10}$$

where $\{e_k\}$ is an orthonormal basis for $\mathcal{H}$. (Note that F^*F is self-adjoint and positive. It is in fact true that every positive operator, A, has a positive self-adjoint square root, B, defined by $B^2 = A$.) And if F itself is positive, then $F = \sqrt{F^2}$ and

$$\mathrm{Tr}[F] = \sum_k \langle e_k, F e_k \rangle. \tag{1.11}$$

The trace functional is provably independent of the choice of orthonormal basis, $\{e_i\}$. Moreover, a very useful property of the trace functional is that it is invariant under cyclic permutations of its arguments; for example,

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[BCA] = \mathrm{Tr}[CAB] \tag{1.12}$$

for any A, B, C.

Let W be any positive operator on a Hilbert space, $\mathcal{H}$, with $\mathrm{Tr}[W] = 1$. Let $E()$ be any POVM from some 'spectrum', S, of possible values to positive operators. Then $\mathrm{Tr}[WE(\cdot)]$ is a countably additive probability measure on (the σ-algebra of Borel sets of) possible values of the observable represented by the POVM E as follows:

$$\Pr(\Delta) = \mathrm{Tr}[WE(\Delta)]. \tag{1.13}$$

Countable additivity follows from (1.3) and the linearity of the trace functional. Normalization follows from (1.2) and the fact that W has unit trace.

When $E(\cdot)$ is a PVM, (1.13) defines a countably additive normalized measure on the projections on $\mathcal{H}$. Hence any density operator generates such a measure. The converse is (remarkably) true as well: every probability (i.e., countably additive, normalized) measure on the projections on a Hilbert space is generated as in (1.13) by some density operator. This theorem is due to Gleason (1957), and says, more precisely:

> **Theorem** (Gleason): Let $\mathcal{H}$ be a Hilbert space of dimension greater than 2. Then every countably additive normalized measure, $\Pr(\cdot)$, on the projections on (equivalently, closed subspaces of) $\mathcal{H}$ is generated by some trace-1 positive operator, W, on $\mathcal{H}$; for P a projection,
>
> $$\Pr(P) = \mathrm{Tr}[WP]. \tag{1.14}$$

The proof is non-trivial. Gleason's theorem is generalizable to the case of general POVMs. That is, the countably additive probability measures over effects are also given by the density operators. (Indeed, for POVMs, there is no restriction to the case $\dim(\mathcal{H}) > 2$. Again, the proofs are non-trivial. See Busch (2003).)

In this common understanding of quantum theory, then, the kinematics of a quantum system is, at its core, given by the POVMs on a Hilbert space together with a state, a density operator. In many cases of interest, one deals with PVMs, hence self-adjoint operators, rather than with POVMs.

Note, finally, that for any statevector, v, we can always represent v in terms of the density-operator formalism, by choosing as the state the projection, P_v, onto the subspace spanned by v. In this case, for any projection Q, $\mathrm{Tr}[P_v Q] = \langle v, Qv\rangle$. (To prove: take the trace in an orthonormal basis containing v.)

1.2.3.4. *Matrix Representation of States.* A vector—and in particular a statevector, ψ—can, of course, be written in terms of any orthonormal basis, $\{e_n\}$, and in this case, the coefficients c_n in the expansion $\psi = \sum_n c_n e_n$ may be considered as the 'coordinates of ψ in the e_n-basis'. It is, in fact, sometimes convenient (see, e.g., §1.3.3.2) to write the state as a column vector with these coordinates.

A similar construction is available for density operators. Again in the (orthonormal) basis $\{e_n\}$, consider a matrix whose elements are $\langle e_n, Fe_m\rangle$, for any operator, F, on a Hilbert space, $\mathcal{H}$. This map from operators on $\mathcal{H}$ to $N \times N$ matrices (where N could be infinite) is in fact an isomorphism from the (algebra of) operators on $\mathcal{H}$ to the (algebra of) $N \times N$ matrices

In particular, let W be a density operator on $\mathcal{H}$, and let $W_{nm} = \langle e_n, Fe_m\rangle$. Now let F be an observable whose eigenvectors are the e_n. Notice, in this case, that

$\langle e_n, Fe_m\rangle = \delta_{nm}$. One says that F is 'diagonal' in the basis $\{e_n\}$ (because all of the entries off of the diagonal are 0). If W is *also* diagonal in $\{e_n$, then the probabilities assigned by W to F behave completely classically, and in particular the classical 'sum rule' holds:

$$\Pr_W(f_n \text{ or } f_m) = \Pr_W(f_n) + \Pr_W(f_m) \tag{1.15}$$

(where $\Pr_W$ is the probability assigned by W via 1.14 and f_n is the eigenvalue of F corresponding to the eigenvector e_n). However, if W is *not* diagonal in $\{e_n\}$, then in general (1.15) fails. In this case, one speaks of 'interference' between the e_n (in the state W).

1.2.3.5. *Expectation Values.* It follows immediately that the expectation value of the observable (represented by the self-adjoint operator) F in the state (represented by the density operator) W is $\mathrm{Tr}[WF]$. To see why, write F in terms of its spectral resolution. The point is most easily seen when F has only a discrete spectrum, as in (1.5). Then by the linearity of the trace,

$$\mathrm{Tr}[WF] = \sum_n \mathrm{Tr}[WP_n]s_n. \tag{1.16}$$

(When F has a continuous spectrum, one must work with integrals whose definition must be treated carefully.) Notice that the expression $\mathrm{Tr}[WP_n]$ is the probability (in state W) that F takes the value s_n. Hence (1.16) is a weighted sum of the possible (spectral) values, s_n, for F, the weights given by the probabilities, $\mathrm{Tr}[WP_n]$, associated to those values in the state W. Note that the traces in (1.16) will in general be easiest to calculate in a basis determined by F.

1.2.3.6. *Quantum Probability Theory.* Classical probability theory standardly concerns measures over sigma-algebras of events (§7.5.5, §7.5.6). These sigma-algebras are defined in terms of the usual set-theoretic operations of complement and union. In quantum theory, we are dealing with a different structure. However it is sufficiently analogous to the structure considered in the classical setting that, mathematically at least, one can often easily carry over considerations from classical probability theory. Our 'sample space' is the set of all one-dimensional projections. Set-theoretic complement (E') becomes 'orthogonal complement' ($E^\perp$); set-theoretic union ($E \cup F$) becomes 'span' (the span of the subspaces E and F, written $E \vee F$); set-theoretic intersection ($E \cap F$) remains intersection (now written $E \wedge F$); and set-theoretic 'inclusion' ($E \subseteq F$) becomes subspace inclusion (often written $E \leq F$). Later, I will consider this structure in more detail—it is the 'lattice', $\mathcal{L}$, of subspaces of a Hilbert space (§7.4). For now, I simply note that it has the correct properties: (i) $\mathcal{H} \in \mathcal{L}$, (ii)

$E \in \mathcal{L}$ implies $E^{\perp} \in \mathcal{L}$; and (iii) for any countable sequence, $\{E_k\} \in \mathcal{L}$, $\vee_k E_k \in \mathcal{L}$. Analogous to classical probability theory, quantum probability theory is then the theory of normalized measures on such a structure. (Of course, if we are thinking in terms of POVMs rather than PVMs, then this story cannot be told, at least not in its present form. Instead, one considers the algebra of effects, and probability measures over it. However, I will not pursue the details here.)

1.2.3.7. *Lüder's Rule.* What about conditional probabilities? Although its interpretation can be highly contentious, and its application somewhat tricky, there is a standard expression for a conditional probability in quantum theory, called 'Lüder's Rule'. Indeed, one can derive it from elementary considerations.

Recall from basic probability theory that the conditional probability, $\Pr(A|B)$, of one event, A, given another, B, is defined by

$$\Pr(A|B) := \frac{\Pr(A \cap B)}{\Pr(B)}. \tag{1.17}$$

The thought behind this definition is that the probability of A (and B) *given* B is the probability that A and B occur jointly, 'renormalized' under the assumption that B occurred; i.e., it is the probabilty of A 'as if' B had probability 1. Indeed, (1.17) is the only probability measure that satisfies the condition that if $A \subseteq B$ then $\Pr(A|B) = \Pr(A)/\Pr(B)$. In other words, if A is contained in B, then $\Pr(A|B)$ is just a renormalization of the original probability measure to one that assigns probability 1 to B.

It turns out that this condition is already sufficient to determine the form of the conditional probability measure over the (lattice of) closed subspaces of (or projections on) a Hilbert space (Bub 1977). In other words, let $\Pr_W$ be the probability measure associated with the density operator, W, on $\mathcal{H}$. Let P be a subspace such that $\Pr_W(P) \neq 0$ (where, of course, $\Pr_W(P) = \mathrm{Tr}[WP]$). Then there is a unique probability measure, $\Pr_{W|P}$ (the 'probability in state W conditional on P'), over the closed subspaces of $\mathcal{H}$ such that

$$\Pr_{W|P}(Q) := \Pr_W(Q|P) = \frac{\Pr_W(Q)}{\Pr_W(P)} \tag{1.18}$$

for any $Q \leq P$. That measure is given by

$$\Pr_W(Q|P) = \frac{\mathrm{Tr}[PWPQ]}{\mathrm{Tr}[WP]}. \tag{1.19}$$

(1.19) is known as 'Lüder's Rule'. Note that for a statevector, $|v\rangle$, the same effect is achieved by projecting $|v\rangle$ onto P, normalizing the result, and using that new state

$(P|v\rangle/||Pv||)$ to calculate the probability of Q. Hence (using eq. 1.7)

$$\Pr_{|v\rangle}(Q|P) = \langle Pv|QPv\rangle/||Pv||^2. \tag{1.20}$$

1.2.3.8. *Mixed Versus Pure States.* Density operators that correspond to one-dimensional projections (equivalently, statevectors) are 'pure' states. These states assign probability 1 to that one-dimensional projection. Mixed states (i.e., states that are not pure) do not assign probabilty 1 to *any* one-dimensional projection. Moreover, mixed states are called 'mixed' because they can always be written as a linear combination of pure states. Indeed, by the spectral theorem, any mixed state, W, can be written as $W = \sum_n w_n P_n$ (and because W is a density operator, $0 \leq w_n \leq 1$ and $\sum_n w_n = 1$). If one or more of the P_n is not one-dimensional, we can always write it as a sum of mutually orthogonal one-dimensional projections—so we may assume, without loss of generality, that all of the P_n are one-dimensional.

The coefficients, or 'weights', w_n, must add to one (because $\mathrm{Tr}[W] = 1$), and in fact w_n is the probability assigned by W to P_n. Hence one can apparently think of W as representing, literally, a 'mixture' of systems in the pure states P_n, in the proportions w_n, so that w_n is the probability that a system chosen at random from the mixture will be found in the (pure) state P_n. We will explore (and qualify) this interpretation of mixed states below.

The converse is also true: any convex combination of pure states is again a state, in general mixed. Indeed, consider the operator

$$W = \sum_n w_n P_n \tag{1.21}$$

where the P_n are here one-dimensional but *not* necessarily mutually orthogonal (yet still, $\sum_n w_n = 1$).[4] This W has unit trace (because the trace functional is linear), and therefore it is a density operator. Note, however, that (1.21) is in general *not* its spectral decomposition.

1.2.3.9. *The Eigenstate-Eigenvalue Link.* According to a standard interpretation of quantum states, a system in the state W has a value for the observable F if and only if W assigns probability 1 to one of the possible values of F (and 0 to the others—in other words, 'trivial probabilities').[5] Notice, in particular, that this interpretation of states differs from the usual interpretation of classical probabilistic states. In the

[4]More generally, if some of the P_n are not one-dimensional, then we require that $\sum_n w_n \dim P_n = 1$, because in general, for a projection P, $\mathrm{Tr}[P] = \dim P$.

[5]For unbounded observables, such as position and momentum, one is naturally motivated to seek some other account. One possibility is to deny that they ever have definite values, but consider instead coarse-grained values, asserting, for example, that if the state W assigns probability 1 to some *region*, Δ, then the system is definitely confined to Δ, where this latter assertion is not meant

classical case, the probabilistic state is a measure over possible pure states, and one normally presumes that the system really is in one of those pure states.

This rule for assigning definite values has come to be called, following Fine (1973), the 'eigenstate-eigenvalue link'. Later (§5) we will consider in some detail the apparent consequences of this rule.

1.2.4. *Incompatibility.* An immediate consequence of this formalism is the fact that there are 'incompatible' physical quantities, at least in the minimal sense that if a state assigns probability 1 to some physical quantity (some projection, for example), then it necessarily assigns non-trivial probabilities (i.e. neither 0 nor 1) to others (and then, by the eigenstate-eigenvalue link, these other observables do not *have* values, in that state—recall §1.2.3.9). This fact follows directly from Gleason's theorem. (Note, however, that one can show in other, simpler, ways that there are no two-valued probability measures over the projections on a Hilbert space.)

Incompatibility is closely related to non-commutativity, and indeed the two terms are sometimes used interchangeably. Consider two projection operators, Q and Q'. To keep things simple, we will suppose throughout that Q and Q' are one-dimensional. Then if Q and Q' do not commute, i.e., $[Q, Q'] \neq 0$, there is no state that assigns probability 1 to Q and either 0 or 1 to Q'. To prove this claim, we will first show (next paragraph) that the only state assigning probability 1 to a one-dimensional projection, Q, is the state Q itself. (Notice that in the previous sentence, the first mention of Q is as the representative of some physical quantity, and the second is as a state.) We will then show (subsequent paragraph) that Q assigns non-trivial probabilities to any non-commuting Q'.

Let W be a state that assigns probability 1 to (one-dimensional) Q. Writing W in terms of its spectral decomposition, and taking the trace in a basis determined by W, we immediately find that

$$\mathrm{Tr}[WQ] = \sum_n w_n \langle e_n, Qe_n \rangle = 1 \tag{1.22}$$

where the weights w_n (from the spectral decomposition of W) sum to 1. Hence for some n, $Qe_n = e_n$, i.e., W is in fact pure, and equal to Q. Therefore, the only state assigning probability 1 to a one-dimensional projection, Q, is Q itself.[6]

to imply that there is some *point* in Δ that is the location of the system. There are, however, other approaches. See, for example, Halvorson (2001).

[6]This claim is also true in a more general form. Let the state W assign probability 1 to the projection Q (of any dimension). Then $(\mathrm{ran}\, Q)^\perp \subseteq \ker(W)$, with equality if Q is the smallest subspace to which W assigns probability 1.

Now suppose that (one-dimensional) $Q' \neq Q$ and $Q' \not\perp Q$, i.e., Q and Q' do not commute (for a discussion, see below). Then, by the same reasoning as above, replacing Q with Q' in (1.22), if $\mathrm{Tr}[WQ'] = 1$ then W must be pure and lie inside the subspace associated with Q'; i.e., $W = Q'$. But it cannot, because we assumed that $Q \neq Q'$. On the other hand, if we want $\mathrm{Tr}[WQ'] = 0$, then $\ker W \subseteq \mathrm{ran}\, Q'$. (The reasoning is essentially the same as above.) But again it cannot, because then $Q' \perp Q$, given our earlier conclusion that W is pure and lies in the subspace associated with Q, and we already assumed that $Q' \not\perp Q$.

This fact is also true in a more general form. Given two self-adjoint operators, F and G, if F and G do not share any eigenvectors then any state that assigns probabiltiy 1 to some value for F will necessarily assign non-trivial probabilities (neither 0 nor 1) to more than one of the possible values of G. I leave the proof (using essentially the same reasoning as above) to the reader.

Above I claimed that (one-dimensional) Q and Q' do not commute if $Q' \neq Q$ and $Q' \not\perp Q$. In fact, the following is true. For any subspaces, A and B, and the corresponding projections P_A and P_B, $[P_A, P_B] = 0$ if and only if

$$A = (A \wedge B) \vee (A \wedge B^\perp) \quad \text{and} \quad B = (B \wedge A) \vee (B \wedge A^\perp). \tag{1.23}$$

(Here we are not restricting to one-dimensional subspaces. Note, however, that (1.23) is implied by the disjunction '$A = B$ or $A \perp B$', and for one-dimensional subspaces, they are equivalent.) Here is the idea of the proof. Note that $A \wedge B$ and $A \wedge B^\perp$ are orthogonal. Hence, if (1.23) holds, we may write $P_A = P_Z + P_{A'}$ for some $Z \perp A'$. (Indeed, of course, $Z = A \wedge B$ and $A' = A \wedge B^\perp$.) Similarly, $P_B = P_Z + P_{B'}$, with $B' \perp Z$. Moreover, $A' \perp B'$. In other words, the conditions (1.23) imply that A and B 'are orthogonal apart from some shared part (Z)'. Then $[P_A, P_B] = [P_Z + P_{A'}, P_Z + P_{B'}] = [P_Z, P_Z] + [P_Z, P_{B'}] + [P_{A'}, P_Z] + [P_{A'}, P_{B'}] = 0$.

Going the other way, we will just sketch the idea. If P_A and P_B commute, then for any vector, v, $P_A P_B v = P_B P_A v$. First choose $v \in A$, so that $P_A P_B v = P_B v$. In general, if $P_A w = w$ (here $w = P_B v$), then either $w \in A$ or $w = 0$. Hence either (i) $P_B v = 0$, or (ii) $P_B v \in A$. If (i) is true for all $v \in A$, then $B \perp A$ and (1.23) clearly holds. If (ii) holds for all $v \in A$ then $B \leq A$ and again (1.23) clearly holds. Using the linearity of the operators involved, one can show that if (ii) holds for just *some* $v \in A$, then the $P_B v$ must form a subspace of A, and clearly this subspace is common to A and B; indeed it is $A \wedge B$. Similarly, one can show that choosing v from the subspace orthogonal to $A \wedge B$ gives rise to (i), so that indeed $A = (A \wedge B) \vee (A \wedge B^\perp)$. Repeating the argument for $v \in B$, we find that (1.23) holds.

The fact of incompatibility marks a significant departure from classical physics, where the structure of the space of states and observables allows for states that assign values to all observables with probability 1 (i.e., there are two-valued probability measures over the space of all 'properties' of the system). The probabilities of quantum theory appear, therefore, to be of a fundamentally different character from the probabilities of classical theory, which arise always because the state of the system is not maximally specific.[7]

1.2.5. *Canonical Commutation Relations.* An important and classic example of incompatibility involves the position and momentum observables. In fact, they obey the 'canonical commutation relations' (CCRs):

$$[P_i, Q_j] = -i\delta_{ij} \tag{1.24}$$

where i and j can be x, y, or z. (Henceforth, we will restrict our attention to one dimension, writing $[P, Q] = -i$. The generalization to three dimensions is straightforward.) Note that the constant on the right-hand side implicitly multiples the identity operator.

Any two observables that obey these commutation relations are typically called 'canonically conjugate'. These relations are central in quantum theory, and we will discuss them in detail in §4. For now, we simply notice them as a central example of incompatibility.

1.2.6. *Compound Systems.*

1.2.6.1. *Entangled States.* Compound systems are represented by tensor-product Hilbert spaces (§7.1.9), so that, for example, a system composed of two particles has a state that is a density operator on the tensor-product of the Hilbert spaces for the two particles individually. There is a fundamental and physically crucial distinction between two kinds of vector in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. A vector, v, in $\mathcal{H}$ is called 'factorizable' if it can be written as $x \otimes y$ for some $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Otherwise, v is called 'unfactorizable', or 'entangled'. An analogous definition applies to the operators (hence, the density operator states) on $\mathcal{H}$.

The existence of entangled states (whether represented as density operators or vectors) turns out to have numerous interesting consequences. It is connected with

[7]Here we are considering just cases where classical physics delivers genuine probability measures, and we ignore cases where classical physics is simply indeterminate. See Earman, Ch. 15, this volume.

'quantum-nonlocality', as well as the possibility of certain computational and information-theoretic (for example, cryptographic) feats that cannot be done with classical systems.[8] The existence of these states follows from the demand that the pure (vector) states for the compound system be closed under taking linear combinations. In other words, it follows from applying the superposition principle to compound systems as well as to simple systems.

1.2.6.2. *Bi-orthogonal Decomposition.* An important result about vectors in tensor-product spaces is the 'bi-orthogonal decomposition theorem' (Schrödinger 1935b), which states that, given a vector, v, in a Hilbert space, $\mathcal{H}$, and a factorization of $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, there exist orthonormal bases $\{e_n\}$ of $\mathcal{H}_1$ and $\{f_m\}$ of $\mathcal{H}_2$ such that

$$v = \sum_n c_n (e_n \otimes f_n). \tag{1.25}$$

If the $|c_n| \neq |c'_n|$ for all $n \neq n'$, then the bases are unique (up to a phase $e^{i\theta}$ on each element of the basis). Note that, in general, for *arbitrary* bases $\{x_n\}$ and $\{y_m\}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$, v is expressed in general in terms of a double sum:

$$v = \sum_{n,m} c_{nm} (x_n \otimes y_m) \tag{1.26}$$

and compare this expression with (1.25).

1.2.6.3. *Reduced States.*

1.2.6.3.a. Partial Trace and the Reduced Density Operator. Suppose we are given the state of a compound system, and wish to derive from it a state for one of the components. If the compound state is factorizable, then the procedure is straightforward. (The state $W = W_1 \otimes W_2$ fixes the component states to be W_1 and W_2 respectively.) But what about when it is entangled? Here we face a problem. If the state is entangled, then there is no obvious sense in which it can be 'divided' into a 'part' corresponding to one system, and a 'part' corresponding to the other.

The usual solution to this problem is to take the state of the component systems to be given by a *partial trace*. For any tensor-product Hilbert space, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the 'partial trace over $\mathcal{H}_1$' is a map, $\mathrm{tr}^{(1)}[\cdot]$, from the operators on $\mathcal{H}$ to operators on $\mathcal{H}_2$. It is the unique such map satisfying the condition that, for any density operator W on $\mathcal{H}$ and any observable F_2 on $\mathcal{H}_2$, the operator $\mathrm{tr}^{(1)}[W]$ generates the same expectation value for F_2 as W does for $\mathbb{I}_1 \otimes F_2$ (Jauch 1968, §11-8). The idea is that $\mathrm{tr}^{(1)}[\cdot]$ 'traces out' system 1, extracting just that part of the compound state that applies to system

[8] See Bub, Ch. 6, this volume.

2. Unless W is a 'product state' (i.e., $W = W_1 \otimes W_2$), the reduced states derived from W are necessarily mixed states.

1.2.6.3.b. Proper Versus Improper Mixtures. In §1.2.3.8 I introduced the idea that a mixed state can be understood as a literal mixture of systems each in some pure state. Certainly when we are describing the state of a system chosen at random from an ensemble that was produced by literally 'mixing' systems in various pure states, it is quite proper to interpret the mixed state in this way. However, we now see that mixed states can arise in another way, namely, as the state of one component of a compound system that is in a non-factorizable compound state. In these cases, it is far from clear that the state (of the component) should be understood as above. Indeed, there need not even *be* an ensemble of which this component is a part. Hence mixtures that arise from taking the partial trace of the state of a compound system are normally called 'improper mixtures', while those that arise from a mixing of individual systems in pure states are normally called 'proper mixtures' (a terminology introduced by dEspagnat §1971). Whether the probabilities generated by improper mixtures can reasonably be understood as 'ignorance about the true pure state' (as they can for proper mixtures) is a matter for interpretative investigation.

1.2.6.4. *Correlations.* Compound systems that are in a non-factorizable state will exhibit correlations between the measured values of observables on the two (or more) components. Consider, for example, the statevector $v = c_1 f_1 \otimes g_1 + c_2 f_2 \otimes g_2$ (where c_1 and c_2 are non-zero coefficients), and suppose that the f_n and the g_n are eigenvectors of the observables F and G respectively. In this state, there is a correlation between the value of F on system 1 and G on system 2. Indeed, let P_{f_n} and P_{g_n} be the projections onto the subspaces spanned by f_n and g_n respectively, and let P_v be the projection onto the subspace spanned by v. Then, applying Lüder's Rule (1.19), we find

$$\Pr_{P_v}(\mathbb{I}_1 \otimes P_{g_{n'}} | P_{f_n} \otimes \mathbb{I}_2) = \frac{\mathrm{Tr}[(P_{f_n} \otimes \mathbb{I}_2) P_v (P_{f_n} \otimes \mathbb{I}_2)(\mathbb{I}_1 \otimes P_{g_{n'}})]}{\mathrm{Tr}[P_v (\mathbb{I}_1 \otimes P_{f_n})]} \tag{1.27}$$

where $\mathbb{I}_k$ is the identity on $\mathcal{H}_k$. Taking the trace in a basis that includes the $f_n \otimes g_{n'}$ reveals that this conditional probability is 0 when $n \neq n'$ and 1 when $n = n'$. In other words, the values of F (on system 1) and G (on system 2) are perfectly correlated.[9]

[9] Authors will sometimes say that two observables are 'perfectly anti-correlated' if the two observables have the same spectrum and the value of one is always minus the value of the other. They will also occasionally reserve the term 'perfect correlation' for a similar case, where the value of one is always equal to the value of the other. Our use of the term 'perfect correlation'—according to which two observables are perfectly correlated in a state just in case the conditional probabilities for values of one, given a value of the other, are always 0 or 1—covers both cases.

Consideration of other observables would reveal additional correlations (not always perfect correlations). We will see an example later.

1.2.7. *Structure of the Space of States.* We noted above (§1.2.3.8) that every convex combination of pure states is again a state. Of course, a convex combination of mixed states is (by the spectral theorem) also a convex combination of pure states, so that in fact the set of states forms a convex set (§7.1.10), a point that I shall discuss in detail later (§2.2.1). Here we note the fundamental point that the convex set of states in quantum theory is not a simplex.

This point marks a departure from classical physics, where every mixed state is uniquely decomposable in terms of pure states. One thus naturally takes the mixed state as a measure of ignorance over the pure states that appear in its decomposition. No correspondingly straightforward interpretation of mixed states in quantum theory is available, in part because the mixed states are multiply decomposable into a convex combination of pure states.

1.3. **Simple Example: A Spin-$\frac{1}{2}$ Particle.** An understanding of the formalism, and the issues to which it gives rise, is much aided by some experience with actual calculations, however simple. In that spirit, let us consider the example of a spin-$\frac{1}{2}$ particle. The example is well-worn, but deservedly so. While there *are* some important foundational and philosophical issues concerning quantum theory that cannot be illustrated or investigated in the context of spin-$\frac{1}{2}$ particles, many such issues can be investigated in this context.

1.3.1. *Introduction of Spin into Quantum Theory.* Spin was introduced in 1924 in the course of an attempt to understand the spectrum of electromagnetic radiation emitted by certain metals. In the course of that explanation, electrons were supposed to have some "two-valued quantum degree of freedom".[10] This degree of freedom was soon associated with a rotation of the electron. Because the electron is a charged body, its rotation creates a magnetic field—the electron acts as a magnet whose north and south poles lie on the axis of rotation. This magnetic property was just what was needed to explain the phenomena.

So far, the story sounds good. However, it was seen almost immediately that the rotation cannot be literal. Nonetheless, the theory of 'spin' was developed in the context of the new quantum theory; the name stuck, and we continue to refer to this magnetic property of electrons (and as current theory tells us, other particles) as 'spin'.

[10]See Massimi (2004, chs. 2,4) for discussion.

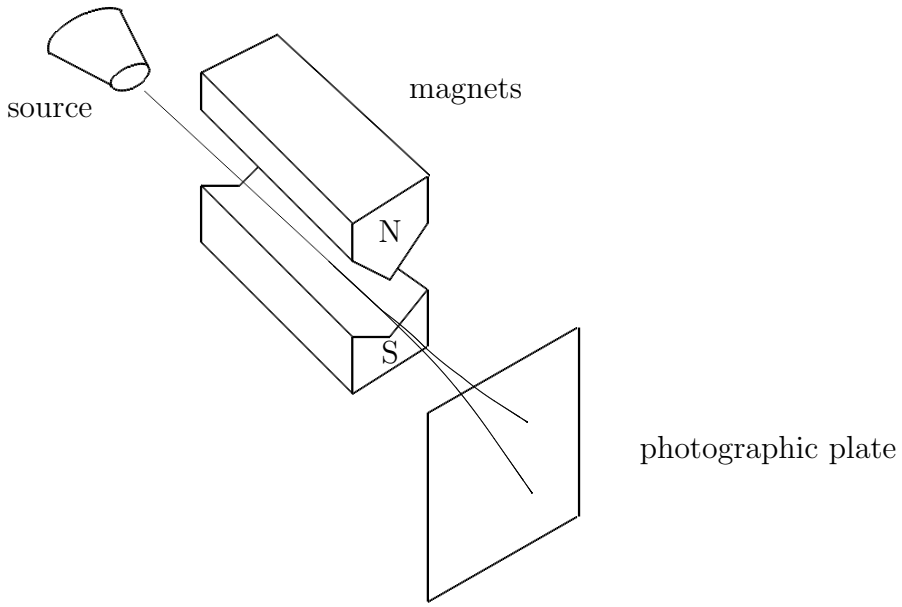

FIGURE 1. An experiment involving Stern-Gerlach magnets

1.3.2. *Quantization of Spin.* It turns out that spin is 'quantized', a fact already anticipated in Pauli's characterization of the property as a 'two-valued degree of freedom'. This fact is, classically, unexpected. To see why, consider a standard method for measuring the spin of a particle. (The method does not, in fact, work for electrons, but it illustrates the point well enough, and does work for electrically neutral particles with spin.) The relevant device is a 'Stern-Gerlach' device, a pair of magnets shaped and arranged to create an inhomogeneous magnetic field, that is, a magnetic field that is stronger in one direction (say, the north) than in the other. (See figure 1.)

Imagine a simple bar magnet passing between the Stern-Gerlach magnets. If the north pole points straight up so that it is close to the top magnet, then the top magnet pushes the north pole (of the bar magnet) down more than the bottom magnet pushes the south pole up, and the net result will be that the bar magnet is deflected downward. If the bar magnet enters the Stern-Gerlach magnets with the south pole facing up, then the result is the opposite: overall upward deflection. If, on the other hand, the bar magnet enters the Stern-Gerlach magnets horizontally, then it will pass straight through with no overall deflection in its path. Finally, if the bar magnet passes through neither vertically nor horizontally, then the result will be deflection, up or down, that is somewhere between the extreme cases. (The trajectories of the magnet in the two extreme cases are illustrated in figure 1.)

(One's physical intuition might be that in the intermediate cases, the axis of rotation of the electron would snap into alignment with the magnetic field so that the intermediate cases would quickly collapse into one of the extreme cases. However, such is not the case, alas. Instead, the spinning electron would classically be expected to act as a gyroscope, thereby maintaining its original inclination with respect to the magnetic field.)

Now, imagine putting an ensemble of particles with 'spin' through the magnets. Rather than the classically expected result (namely, a distribution of different amounts of deflection, from 'maximum down' to 'maximum up'), one finds only two results: 'maximum down' and 'maximum up'—these results are illustrated in figure 1.

This result holds no matter how the Stern-Gerlach magnets are oriented. That is: notice that we could reorient the magnets so that the axis passing between them is pointing in any direction in space. Passing an electron through the device, we would, again, find that it goes either 'up' or 'down' (relative to this new direction in space). Hence we can measure the spin of a particle in any direction, and we refer to the observables thus measured as 'spin-u', where u refers to some specified direction in space. These facts about spin make it clear, moreover, that classical 'spin' is at best a metaphor for whatever property particles with 'spin' have. (In any case, in non-relativistic quantum theory, the electron is normally treated as a point particle, so that its spin could not be coordinated with any spatial rotation. As it is commonly said, spin has no 'classical analogue'.)

1.3.3. *Quantum Formalism for Spin.* Let us see, now, how the observables for and states of a particle with spin are represented in the formalism that I sketched above. I will consider just the degrees of freedom related to spin, ignoring, for example, spatial degrees of freedom.

1.3.3.1. *Hilbert Space and Observables.* The Hilbert space for a single spin-$\frac{1}{2}$ particle is $\mathbb{C}^2$, the space of complex column-vectors with 2 components (§7.3). The 'observables' of the system correspond to 'spin' in various directions (every direction in space), and each will have just two possible values, which we may call 'up' (represented by the number $+1/2$) and 'down' ($-1/2$).[11] The spin observables in the x, y, and z directions

[11] The particles that I have been discussing—those that have just two degrees of freedom ('up' or 'down') for any given direction of spin, are called 'spin-$\frac{1}{2}$ particles', in part because their angular momentum about any given axis is either $+\hbar/2$ ('up') or $-\hbar/2$ ('down'), where $\hbar$ is a unit of angular momentum equal, in familiar units, to 1.054×10^{-34}kg m^2/s. (As is often done, I have adopted units in which $\hbar = 1$.) There are, in addition, deeper group-theoretic reasons for calling these particles 'spin-$\frac{1}{2}$' particles, but we cannot go into that point here. (There are also particles with higher spin, which means, operationally, that they have more than two degrees of freedom for each direction of spin.)

are defined in terms of the Pauli matrices by $S_x = (1/2)\sigma_x$, and similarly for S_y and S_z. (See §7.3.1).

1.3.3.2. *States.* The pure states can be represented by norm-1 vectors, or by projections onto the space spanned by them. Consider, for example, the statevectors

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.28}$$

The vector ψ, for example, corresponds to the (pure) density operator (one-dimensional projection operator)

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.29}$$

The vectors ψ and χ are an eigenvectors of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.30}$$

with eigenvalues $+1$ and -1 respectively.

Note that the expectation value of S_z in the state W is

$$\begin{aligned} \mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right] &= \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= (1 \;\; 0)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0 \;\; 1)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \tfrac{1}{2} + 0 = \tfrac{1}{2}. \end{aligned} \tag{1.31}$$

(Recall our earlier comments about calculating traces in an appropriately chosen basis.) Of course, in general a system's having an expectation value equal to some value, r, is not sufficient to imply that the system has the value r. (Indeed, r might not even be in the spectrum of possible values.) In this case, however, we may also note that the probability associated with the appropriate projection operator is 1. So, first, note that the spectral decomposition of S_z is:

$$\begin{aligned} S_z &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = (+\tfrac{1}{2})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-\tfrac{1}{2})\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &:= (+\tfrac{1}{2})\, P_{z_+} + (-\tfrac{1}{2})\, P_{z_-}\ . \end{aligned} \tag{1.32}$$

Hence the projection associated with the value $+\frac{1}{2}$ for S_z is P_{z_+} and the probability for the value $+\frac{1}{2}$ (for S_z) in the state W is

$$\mathrm{Tr}[W P_{z_+}] = \mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = 1. \tag{1.33}$$

(We leave the details of the calculation to the reader. Notice that taking the trace of a matrix amounts to just adding the numbers along the diagonal. The reader might

wish to prove this fact.) As I noted above in a more general context, this expression is, equivalently, the expectation value of P_{z_+} in the state W. Hence, in particular, if one agrees that 'value r for observable F has probability 1 in state W' implies 'a system in state W has value r for F' then we may conclude, from (1.33), that a system in the state W has the value $+1/2$ for S_z. (We will discuss such interpretive principles in more detail later.)

1.3.4. *Incompatibility.* Finally, notice that in this state, W, the expectation value of spin in the x and y directions is 0. For example,

$$\mathrm{Tr}[WS_x] = \mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}\right] = 0. \tag{1.34}$$

This fact suggests (indeed, in this two-dimensional case, implies) that the probabilities for $S_x = +\frac{1}{2}$ and $S_x = -\frac{1}{2}$ in the state W are $\frac{1}{2}$, as we can also verify by a direct calculation. First, note that the spectral resolution of S_x is:

$$\begin{aligned} S_x &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \left(\tfrac{1}{2}\right)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \left(-\tfrac{1}{2}\right)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &:= \left(+\tfrac{1}{2}\right) P_{x_+} + \left(-\tfrac{1}{2}\right) P_{x_-} \ . \end{aligned} \tag{1.35}$$

As the reader may verify, $\mathrm{Tr}[WP_{x_+}] = \mathrm{Tr}[WP_{x_-}] = \frac{1}{2}$.

We have thus verified, in this particular case, a claim made previously made abstractly, namely, that a state that is dispersion-free (i.e., generates probabilties of just 0 or 1 for all possible values) for one observable, will necessarily not be dispersion-free for some other observables. Indeed, I said earlier that non-commuting observables that do not share eigenvectors are always incompatible, in the sense that any state that is dispersion-free on one of them is necessarily *not* dispersion-free on the other. Now notice that S_x, S_y, and S_z are mutually non-commuting, and indeed share no eigenvectors. (In this two dimensional case, non-commuting maximal observables cannot share any eigenvectors.) Hence a state that is dispersion-free for one will necessarily generate non-trivial probabilities for the others.

Indeed, consider any direction, u, in space specified relative to the z-axis by the polar angles θ and ϕ, i.e., in Cartesian coordinates, $u = (x, y, x) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. (See Figure 2.) Then the associated spin observable is represented by the matrix

$$S_u = \frac{1}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}. \tag{1.36}$$

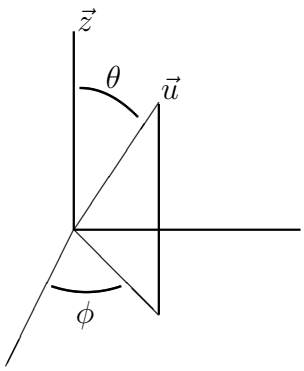

FIGURE 2. Polar angles.

(One reasonable and quick justification of this expression is to note that $S_u = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta$.) The only pairs of such operators that commute are anti-parallel; i.e., they correspond to spin in anti-parallel directions (and such operators are just multiples of one another by a factor of -1).

(One should keep in mind, however, that Gleason's theorem does not hold for our 2-dimensional space. Hence the density operators do not define all states, in this case. Indeed, Bell (1964) shows how to define a dispersion-free measure over the projections on $\mathbb{C}^2$ in terms of an additional 'hidden' parameter. Moreover, the quantum-mechanical states are obtainable by averaging over the possible values of the hidden parameters with an appropriate probability distribution over them.)

1.3.5. *The Bloch Sphere.* The Hilbert space $\mathbb{C}^2$ is used to represent any two-level quantum system, and such systems are of great interest in quantum theory, all the more so in recent years, as increasing interest in quantum information and quantum computation has focused attention even more on such systems (because they are the quantum analog of a classical 'bit'—see Bub, Ch. 6, this volume). A careful study of the pure states on $\mathbb{C}^2$ is often aided by the representation of those states in terms of the Bloch sphere. Note that any pure state on $\mathbb{C}^2$ can be represented by a vector of the form $v = \cos(\theta/2)\psi + e^{i\phi}\sin(\theta/2)\chi$ (using the notation of equations 1.28).[12] Hence, again referring to figure 2, we can represent each distinct pure state as a unique point on the surface of a unit sphere (in $\mathbb{R}^3$), normally called the 'Bloch sphere'. The 'north pole' of the sphere corresponds to the state ψ and the 'south pole' to the state χ.

[12]The claim is not that every vector can be written in this form, but that every pure state can be represented in this form. Recall that an overall phase factor does not affect the probabilities generated by a vector. Hence we may assume, without loss of generality, that the coefficient of ψ is real.

In fact, however, the 'Bloch sphere' is a ball. The interior points correspond to mixed states, as follows. Every density operator, W, on $\mathbb{C}^2$ can be written as

$$W = \frac{\mathbb{I} + \vec{r} \cdot \vec{\sigma}}{2} \tag{1.37}$$

for $\vec{\sigma}$ the 'vector' of Pauli matrices (§7.3.1) and $\vec{r}$ a vector from $\mathbb{R}^3$ with $||\vec{r}|| \leq 1$. The components of $\vec{r}$ determine a point inside the Bloch sphere representing the corresponding density operator. (Note, in particular, that $\vec{r} = (0, 0, 1)$ corresponds to the pure state given by $\theta = 0$, as it should.)

1.4. **Dirac Notation.** We will return to the example of a spin-$\frac{1}{2}$ particle later to illustrate a number of issues in quantum theory. When I do so—and, indeed, throughout the remainder of this essay—it will be helpful to have at hand a useful notation, the so-called 'Dirac bra-ket' notation, used commonly by both physicists and philosophers.

1.4.1. *Bras and Kets.* In the bra-ket notation, vectors are denoted by (and sometimes called) 'kets', $|v\rangle$. In the discussion above, for example, the column vector ψ in (1.28) might be denoted $|z_+\rangle$. Elements of the dual space (the 'row vectors' in our discussion above–see §7.1.8) are denoted by 'bras', $\langle v|$. In our example above, there is a natural 1-1 map from the kets (column vectors) to the bras (row vectors):

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a^* \quad b^*). \tag{1.38}$$

The bras thus define (continuous) linear functionals in the obvious way. Letting

$$|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad |w\rangle = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{1.39}$$

the linear functional (bra) $\langle v|$ acting on the vector (ket) $|w\rangle$ is

$$(a^* \quad b^*) \begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d \tag{1.40}$$

and is written, in the Dirac notation, as (the 'bra-ket') $\langle v|w\rangle$. (The reader might wish to check that the functional thus defined is indeed linear.) Of course, as it must be, $\langle v|w\rangle$ is also the inner product of $|v\rangle$ with $|w\rangle$, given (1.38). (In this notation, we continue to write $||v||$ for the norm of a vector, instead of $|||v\rangle||$.)

In the general case, i.e. where $\mathcal{H}$ is any (complex) Hilbert space (countable-dimensional at most), we take the elements of $\mathcal{H}$ to be kets, and the elements of the dual space $\mathcal{H}^*$ to be bras. Inner products may now be written $\langle v|w\rangle$, which denotes *both* the linear functional $|v\rangle$ acting on the vector $|w\rangle$ *and* the inner product of the vectors $|v\rangle$ and $|w\rangle$.

1.4.2. *Operators.* The operator, F, acting on the vector $|v\rangle$ is written $F|v\rangle$. The expectation value of the observable F in the state $|v\rangle$ is written $\langle v|F|v\rangle$, which is notationally (and numerically) equivalent to $\langle v|Fv\rangle$, the latter to be read as the inner product of $|v\rangle$ with the vector $F|v\rangle$. The expression $\langle w|F|v\rangle$ is defined similarly.

Corresponding to what is sometimes called the 'vector direct product'

$$\begin{pmatrix} a \\ b \end{pmatrix} (c \;\; d) = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}, \tag{1.41}$$

we can define $|v\rangle\langle w|$ to be the operator on $\mathcal{H}$ defined by

$$\Big(|v\rangle\langle w|\Big)|x\rangle = \langle w|x\rangle\,|v\rangle. \tag{1.42}$$

Notice that simple symbol-manipulation would generate the same result.

1.4.3. *Using the Dirac Notation.* As I just hinted, the Dirac notation is enormously useful, once its true meaning is understood, and dangerous otherwise. It's power—and danger—lies in the fact that it allows one more or less to ignore various distinctions, such as the distinction between a vector and a linear functional (element of a dual space). It also can be very helpful for 'coordinate-free' calculations. For example, we can discuss the theory of spin-$\frac{1}{2}$ particles without bothering with Pauli matrices and so on. Consider the basis $\{|z_+\rangle, |z_-\rangle\}$ for $\mathbb{C}^2$, where $|z_+\rangle$ is the state that assigns probability 1 to the value $+\frac{1}{2}$ for S_z and so on—note that we do not need to worry about how to represent this state as a column of complex numbers. It is sufficient to carry out calculations to note that for a direction in space, u, specified by the angles θ and ϕ relative to the z-axis:

$$|u_+\rangle = \cos\left(\tfrac{\theta}{2}\right) e^{-i\frac{\phi}{2}}|z_+\rangle + \sin\left(\tfrac{\theta}{2}\right) e^{i\frac{\phi}{2}}|z_-\rangle \tag{1.43}$$

$$|u_-\rangle = -\sin\left(\tfrac{\theta}{2}\right) e^{-i\frac{\phi}{2}}|z_+\rangle + \cos\left(\tfrac{\theta}{2}\right) e^{i\frac{\phi}{2}}|z_-\rangle. \tag{1.44}$$

The spin observables are then represented by

$$S_u = \frac{1}{2}|u_+\rangle\langle u_+| - \frac{1}{2}|u_-\rangle\langle u_-|. \tag{1.45}$$

Note, for example, that $\langle z_+|u_+\rangle = \cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}}$ and $\langle z_-|u_+\rangle = \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}}$, facts that are immediately read off of (1.43). Hence, for example, the probability that a system in the state $W = |z_+\rangle\langle z_+|$ has the value $+\frac{1}{2}$ for the observable S_u can be quickly

calculated as

$$\mathrm{Tr}\big[|z_+\rangle\langle z_+|\big(|u_+\rangle\langle u_+|\big)\big] \tag{1.46}$$
$$= \langle z_+|\big(|u_+\rangle\langle u_+|\big)|z_+\rangle \tag{1.47}$$
$$= \langle z_+|u_+\rangle\langle u_+|z_+\rangle \tag{1.48}$$
$$= |\langle z_+|u_+\rangle|^2 \tag{1.49}$$
$$= \cos\left(\tfrac{\theta}{2}\right)^2. \tag{1.50}$$

(To get from the first to the second line, calculate the trace using the basis $\{|z_+\rangle, |z_-\rangle\}$.) The genuis of Dirac's notation is that one can, as illustrated here, simply 'do the symbolically natural thing' and get the correct answer. For example, the third line follows from the second by 'erasing the parentheses and joining the bars'. Conceptually, we allowed the operator $|u_+\rangle\langle u_+|$ to act on $|z_+\rangle$, obtaining the vector $\langle u_+|z_+\rangle|u_+\rangle$, then took the inner product of this vector with $|z_+\rangle$ (or, applied the linear functional $\langle z_+|$ to $\langle u_+|z_+\rangle|u_+\rangle$). The convenience of the notation can also, however, lead one to forget conceptually important distinctions.

Keep in mind, moreover, that the convenience of not having to worry about explicit (e.g., matrix) representations of vectors and observables can also lead one to write down some rather silly, or at least physically opaque, states. One frequently, for example, sees written down 'states' such as $|\text{cat dead}\rangle$ or $|\text{Sarah sees the pointer}\rangle$. The Dirac notation naturally tempts one to write down such expressions, but we are so far from knowing whether such 'states' correspond to some pure vector state, and if so, what their properties are, that such expressions are best left to cartoons.

1.5. **Transformations.** We have now seen how to represent observables, and how to calculate expectation values (and probabilities). While such matters are indeed at the heart of the theory, there are other aspects of the formalism that are important for philosophical and foundational discussions. In particular, this subsection discusses transformations, both of the states of physical systems and of the observables associated with those systems. Along the way, I will have occasion to mention some theorems that are fundamental for the foundations of quantum mechanics.

1.5.1. *Groups and Their Representations.*

1.5.1.1. *Motivation.* Galileo observed that the laws of motion do not depend on the constant velocity of the 'lab' (frame of reference) in which they are applied. (For example, in the hull of a ship moving with constant velocity — more precisely, moving inertially — "jumping with your feet together, you pass equal spaces in every direction", as Galileo writes, just as you would back on shore.) Neither do they depend on

one's location, nor on the time at which they are applied, nor on the direction in which one is facing. In other words, the laws are invariant under certain transformations, namely, boosts (changes in velocity), spatial translations, temporal translations, and rotations. These sorts of transformation are represented, mathematically, by groups, and in the case of the 'Galilean transformations' that I just mentioned, the group is normally called the 'Galilean group'.[13] Hence group theory (§7.6) is the natural context in which to study, among other things, the 'invariances' of quantum theory.

The motivation here is that the properties of a group are exactly the properties normally thought to apply to 'invariance transformations'. In particular, if α and β are transformations that each individually leave the laws unchanged, then the composition of α followed by β is also such a transformation. Similarly, if α is a such transformation, then there is the transformation that 'undoes' what α did, that is the inverse of α. Notice, for example, that the composition of two Galilean transformations is another one, and that each transformation has an inverse.[14]

Groups show up in other contexts as well. Suppose, for example, that we are interested (as we soon will be) in the dynamics of a closed physical system. One way to think about the time-evolution of the state of a system is as a transformation on the set of states. The set of all such time-evolutions, then, plausibly should form a group. The identity represents 'no change' (or the degenerate case of evolution over no time). The product represents one period of evolution followed by another. And the inverse represents 'reversed' evolution, or evolution backwards in time. (If a given theory is not time-reversible, then we would be dealing with a semi-group rather than a group.)

Now, often one specifies a group abstractly, that is, by specifying the products and inverses in the group without representing it as a group of transformations on some set (such as the set of physical states of a system). The most trivial example is the group Z_2, which contains two elements, x and y. The multiplication rule is: $xy = x$, $yx = x$, $xx = y$, and $yy = y$. The identity is (clearly) y, while x and y are their own inverses. Notice that we specified this group without referring to any specific mathematical objects—the symbols 'x' and 'y' are just names for the two

[13]More precisely, the Galilean group is $(\mathcal{R} \ltimes \mathcal{V}) \ltimes (\mathcal{A} \times \mathcal{T})$, where $\times$ is the direct product, $\ltimes$ is the semi-direct product, and $\mathcal{T}$, $\mathcal{A}$, $\mathcal{V}$, and $\mathcal{R}$ are the (sub-)groups of temporal translations, spatial translations, boosts, and rotations, respectively (§7.6.2). If the Galilean group is defined, first and foremost, as the set of affine (parallel-line-preserving) maps from E, the Euclidean 4-dimensional manifold of events (space-time), to itself that preserve simultaneity of events and the distance between simultaneous events, then it turns out that the subgroups mentioned above are not all normal, as implied by the use of semi-direct products where one might expect direct products.

[14]See Brading and Castellani, Ch. 13, this volume, for more nuanced discussion.

elements of this group and by themselves have no further mathematical content. But we could also 'represent' the group Z_2 as, for example, the group of maps from any two-element set to itself, with y being the identity map, and x being the map that swaps the elements (maps each to the other). (Another representation of Z_2 takes x to be complex conjugation, *, and y to be **.)

1.5.1.2. *Wigner's Theorem.* Thinking of groups as 'collections of symmetry transformations', the very idea that these transformations are 'symmetries' suggests that they should not change the relationships amongst states. In particular, a symmetry transformation on the space of states should be such that a system in state $|\psi\rangle$ generates the same probabilities for observables both before and after the transformation (at least for observables that are supposed to be invariant under this symmetry, or have been 'transformed along' with $|\psi\rangle$, in the sense that their eigenvectors are also transformed). How might such transformations be represented?

Notice that a unitary operator (§7.2.6) fits the bill very nicely. Indeed, we *define* a unitary operator as, in part, one that preserves inner products. There is an important near-converse to this fact, due to Wigner (1931, p. 251).

> **Theorem** (Wigner): Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and let $T : \mathcal{H} \to \mathcal{H}$ be a 1-1 (but not necessarily linear) map satisfying $\langle Tw|Tv\rangle = \langle w|v\rangle$ for any $|w\rangle, |v\rangle \in \mathcal{H}$. Then
>
> $$T|v\rangle = \varphi(v)U|v\rangle \tag{1.51}$$
>
> where U is either unitary or anti-unitary and $\varphi()$ is a 'phase function', a complex-valued function on $\mathcal{H}$ whose values have modulus 1.

(Any anti-unitary operator, T, can be written as $T = UK$, where K is the 'complex conjugation' operator. Hence the anti-unitary transformations are just the unitary ones, followed by complex conjugation. Time-reversal, for example, is often associated with complex conjugation.)

One normally rules out the anti-unitary case on various grounds related to the 'unphysical' nature of such transformations; in particular, they are not continuously connected to the identity. In order to make this notion precise, one would need to introduce a topology on the group. In the typical cases of interest, the group is continuously parametrized (§7.6.4) by some set of real indices so that the group in fact forms a manifold (§7.5.2); i.e., it is a Lie group (§7.6.5). In these cases, a topology is already given. The significance of being continuously connected to the identity is just that in this case, one has the picture of the group transformations being built up from transformations that are 'infinitesimal', i.e., 'as close as you like to doing nothing

at all to the system' (the identity transformation). Of course, if we are talking just about symmetries, there is no reason to suppose that being continuously connected to the identity is a necessary condition—consider, just to mention the most obvious examples, time-reversal, or spatial reflection. On the other hand, if the symmetries in question are supposed to correspond, ultimately, to actual physical processes (such as dynamical evolution of a closed system), then continuous connectedness to the identity begins to look more compelling.

Hence, in general, symmetries in quantum theory are represented in terms of these maps, T, with U unitary or anti-unitary, and often under the assumption (or hope) that U is unitary.

1.5.1.3. *Projective Representations.* In the expression (1.51) one not only (normally) sets aside the case where U is anti-unitary, but also (normally) seeks maps, T, such that $\varphi(v)$ is identically 1. In this case, the representation of the symmetry group is just given in terms of a group of unitary operators. Such representations are particularly nice because much is known about unitary operators. (See §1.5.1.4 for an important example.) But one is not always so fortunate as to be able to find this sort of representation, often called a 'unitary' or 'ordinary' representation (§7.6.8). Sometimes one must live with the phase function's being non-trivial. In this case, the representation is called 'projective'.

The reason is as follows. Let $\mathcal{H}$ be a Hilbert space, and consider the set, $P\mathcal{H}$, of equivalence classes of vectors from $\mathcal{H}$, where two vectors are equivalent if and only if they lie in the same one-dimensional subspace. $P\mathcal{H}$ is a *projective Hilbert space*, whose structure is given by the 'angles' between the rays of $\mathcal{H}$ (the modulus of the inner product of normalized representatives from the rays). When the phase function in (1.51) is non-trivial, the resulting transformation still generates an automorphism of $P\mathcal{H}$. (Moreover, we have already observed that the pure states in quantum theory can, for the purposes of calculating probabilities, be just as well represented by one-dimensional projections as by state-vectors. Hence it should come as no surprise that projective representations of a group can still preserve all probabilities.) Hence, while ordinary representations tend to be easier to handle, there is nothing terribly inconvenient or problematic about projective representations, and one is sometimes forced to use them.

1.5.1.4. *Stone's Theorem.* Unitary representations are particularly nice, because they can be 'generated' by self-adjoint operators. Note, first, that given any self-adjoint operator, F, the operator e^{iF} is unitary. Moreover, the family of operators $e^{i\alpha F}$ with α

a real parameter forms a continuously parametrized group of unitary operators, where $e^{i\alpha F}e^{i\alpha' F} = e^{i(\alpha+\alpha')F}$. (Note that $\lim_{\alpha\to 0} e^{i\alpha F} = I$, i.e., this group is continuously connected to the identity.) Now suppose that we are interested in representing a continuously parametrized group, $\mathcal{G}$, as a family of unitary operators on a Hilbert space. Because of the nice behavior of the $e^{i\alpha F}$, one would very much like to find an F that generates a representation of $\mathcal{G}$. We are in luck:

> **Theorem** (Stone 1932): Let U_α be a (weakly) continuous unitary representation of $\mathcal{G}$ on a Hilbert space, $\mathcal{H}$.[15] Then there exists a self-adjoint operator, F, on $\mathcal{H}$ such that $U_a = e^{-iFa}$.

Because so many of the groups of interest in non-relativistic (and indeed relativistic) quantum mechanics have the requisite properties, Stone's theorem is of fundamental importance for the theory. We shall see some examples of its use later.

1.5.2. *Dynamics.*

1.5.2.1. *Some Initial Assumptions about Dynamical Evolution.* Dynamical evolution of the state of a system is just a kind of transformation on the space of states. I begin with a few simplifying assumptions, which will turn out to be sufficient to determine the form of dynamical evolution.

Keeping in mind that density operators are mixed states, and hence linear combinations of pure states, let us assume that the evolution of a density operator is induced by the evolution of the pure states of which is it composed. The idea here is that a density operator can represent simple physical mixing, and in that case at least, it should evolve as described. Suppose, for example, that we have a mixture, in proportions r and $1-r$ (with $0 < r < 1$), of two different types of system, the first in the pure state P_1 and the second in the pure state P_2. The corresponding density operator is $rP_1 + (1-r)P_2$. If the systems evolve without interacting with one another (for example, they might be physically isolated from one another), then one would expect that if the systems in P_n evolve to the state P'_n, the mixture evolves to $rP'_1 + (1-r)P'_2$, or so I shall assume. In that case, we can focus our attention on the pure states, and hence on the (normalized) vectors of a Hilbert space.

Notice that this argument certainly does not apply to density operators that arise from partial tracing (i.e., 'reduced density operators'). Indeed, in general, such operators will not evolve in the way described. But rather than determining their

[15] The map $\alpha \mapsto U_\alpha$ is *weakly continuous* if and only if $\langle w|U_\alpha|v\rangle$ is a continuous function of α for every $|v\rangle, |w\rangle \in \mathcal{H}$. Indeed, Stone's theorem holds under weaker conditions when $\mathcal{H}$ has a countable basis (i.e., it is 'separable'), in which case the functions $\langle w|U_\alpha|v\rangle$ need only be Lebesque-measurable. See Riesz and Sz.-Nagy (1955, §137).

evolution directly, one may derive it from evolution of the state of the compound system of which it is a component—and if that compound system is in an improper mixture, then repeat the procedure.

We are therefore primarily interested in the dynamics of an isolated physical system (though certainly the system would in general experience internal interactions), represented by a pure state. The question then becomes: which transformations on a Hilbert space for this system are possible dynamical evolutions of the state of the system?

Symmetry is again a helpful tool. Let $T : \mathcal{H} \to \mathcal{H}$ be a map from the Hilbert space $\mathcal{H}$ to itself representing the time-evolution (over some given stretch of time) of a closed system. Because the system is closed, it seems reasonable to suppose that this T should be a symmetry, in the sense that we have already noticed: $|\langle v|w\rangle|^2 = |\langle Tv|Tw\rangle|^2$ for all $|v\rangle, |w\rangle \in \mathcal{H}$. (Of course, the ultimate justification for this supposition is empirical success.) Hence (1.51) applies. We will assume, further, that the phase function $\varphi(v)$ is identically 1, i.e., that time-evolution is given by an ordinary representation of some group of evolution operators. Finally, recall that the anti-unitary operators are not continuously connected to the identity, meaning that, in this case, if they were used to represent time-evolution, there would be no way to represent evolution over infinitesimal times. Under the assumption that time is continuous, we are led to suppose that time-evolution is given by some group of unitary operators.

Notice, now, that our original argument for the conclusion that the evolution of mixed states should follow from the evolution of pure states, essentially by the assumption of linearity, is encapsulated in this result. As a transformation on $\mathcal{H}$, a unitary operator U also generates a transformation on the operators on $\mathcal{H}$, and assuming that dynamical evolution ought not change the relationship (that is, the relations definable purely in terms of the structure of the Hilbert space) between pure states and the mixtures that they compose, we must use U to generate the transformation on mixed states by $W \mapsto U^{-1}WU$. Indeed, this expression is the most general form of the standard dynamics of a closed system in quantum mechanics.

To see where it comes from, consider the density operator, $W(t) = \sum_n w_n(t)P_n(t)$. Letting $\{|\psi_{n,i}(t)\rangle\}$ be an orthonormal basis determined by $W(t)$ (where the index n ranges over the spectral projections $P_n(t)$ and the index i ranges over the dimension of $P_n(t)$), we may write

$$W = \sum_{n,i} w_{n,i}(t)|\psi_{n,i}(t)\rangle\langle\psi_{n,i}(t)|. \tag{1.52}$$

According to our earlier assumption, the evolution of W will be given in terms of the evolution of the $|\psi_{n,i}(t)\rangle$, which means in particular that the coefficients $w_{n,i}(t)$ will be time-independent. (Keep in mind, here, that we presume W to be the state of a closed (isolated) system. Without that assumption, nothing said here would be at all plausible.) Hence, if U_t is the evolution operator for the system, we may simply apply it to the summands:

$$\begin{aligned} W(t) &= \sum_{n,i} w_{n,i}\Big(U_t|\psi_{n,i}(0)\rangle\Big)\Big(\langle\psi_{n,i}(0)|U_t^*\Big) && (1.53)\\ &= U_t\left(\sum_{n,i} w_{n,i}|\psi_{n,i}(0)\rangle\langle\psi_{n,i}(0)|\right)U_t^* && (1.54)\\ &= U_t W(0) U_t^*, && (1.55)\end{aligned}$$

where we have used the linearity of the U_t. Finally, recall that for any unitary operator, U, $U^* = U^{-1}$.

Finally, it is worth noticing that for any given unitary map, U, on $\mathcal{H}$, considered to be a 'symmetry', the map $F \mapsto U^{-1}FU$ for all operators F is the 'correct' corresponding symmetry of operators, at least in the sense that for any $|v\rangle \in \mathcal{H}$ and any operator, F, on $\mathcal{H}$, $U(F|v\rangle) = (UFU^{-1})U|v\rangle$. That is, one can either 'apply the operator F to the vector $|v\rangle$, then transform according to U', or 'transform according to U then apply the transformed operator to the transformed vector', and in both cases the result is the same.

1.5.2.2. *The Hamiltonian.* We can say, then, that the evolutions on $\mathcal{H}$ are generated by unitary operators. But which ones? For example, which operator represents the evolution of a free particle? Which operator represents the evolution of a particle under the influence of some given potential energy? Some progress on that question is made via Stone's Theorem.

Above I claimed that dynamical evolution has all the properties of a group. In particular, let $U_{1,2}$ represent the evolution of a system from time t_1 to time t_2, and similarly for $U_{2,3}$. Then it would seem that evolving from t_1 to t_2, and thence to t_3, is the same as evolving from t_1 to t_3; in other words, $U_{1,3} = U_{2,3}U_{1,2}$. (Keep in mind that we are thinking of these $U_{m,n}$ as operating on some space of states—hence the ordering.)

A slightly stronger, but still quite compelling, assumption is that of 'time homogeneity'. Imagine a system evolving under the influence of some time-independent constraints (for example, time-independent potential energies). Then, if $t_3 - t_2 = t_2 - t_1$, the evolution operator $U_{1,2}$ should in fact be the same as $U_{2,3}$. (Keep in mind

that these operators transform the entire space; we are not assuming that a given single system will 'do the same thing' from t_2 to t_3 as it did from t_1 to t_2, but rather that two different isolated systems in the same state at two different times will do the same thing for the next equal stretch of time.) In this case, evolution operators require just a single parameter indicating the length of the time interval concerned, and we then have the relation: $U_t U_{t'} = U_{t+t'}$. We assume, as well, that U_0 is the identity, meaning that 'nothing happens instantaneously'.

Notice the similarity to an additive group. Indeed, a set of operators, U_t, obeying this rule forms a semi-group. But there is, as well, reasonable motivation for the existence of inverses in this case, namely, that they correspond to reverse evolution (evolution backwards in time). Writing the time-reversal of U_t as U_{-t} then we require that $U_t U_{-t} = U_{t-t} = \mathbb{I}$. In that case, the U_t form a one-parameter group.

Finally, we will add an assumption of continuity in time. In particular, we will assume the 'weak continuity' of the group U_t (see footnote 15). We can then apply Stone's Theorem to learn that, for *any* group representing time-evolution of a quantum system, there is some self-adjoint operator, H, such that $U_t = e^{-iHt}$.

How do we know which operator H to choose for a given system? Alas, the usual answer to this question is via 'quantization' of the Hamiltonian of an analogous classical system. For such a system, the Hamiltonian is usually the classical total energy. I say 'Alas', because although at a practical level quantization is *normally* straightforward, there is, as yet, no completely satisfactory foundational account of the nature of the 'analogy' between a given quantum system and its classical 'analogue'.[16] The most frequent examples are, however, straightforward. For example, the classical kinetic energy of a particle moving in one dimension is $p^2/2m$ (where p is the classical momentum and m is the mass of the particle), and the quantum-theoretic ('quantized') Hamiltonian is $P^2/2m$ (where P is the momentum operator). Hence the time evolution of a free particle in quantum theory is given by $|\psi(t)\rangle = e^{-iP^2t/2m}|\psi(0)\rangle$.

1.5.2.3. *Equations of Motion.* In this final section about the formalism of quantum theory, I briefly introduce the standard equations of motion in non-relativistic quantum mechanics. Much of the actual practice of quantum mechanics consists in solving these equations, either exactly (in rare cases where analytic solutions are obtainable), or approximately (most of the time, either using standard techniques from perturbation theory, or using numerical approximations).

[16] See Landsman, Ch. 5, §4, this volume.

1.5.2.3.a. The Schrödinger Equation. Consider the group of time-evolutions $U_t = e^{-iHt}$. These U_t uniquely solve the differential equation

$$\frac{\partial U_t}{\partial t} = -iHU_t. \tag{1.56}$$

But $\frac{\partial U_t}{\partial t}|\psi(0)\rangle$ is just the time-derivative of $|\psi(t)\rangle$ at time t, so that, allowing the operators on both sides of (1.56) to act on $|\psi(0)\rangle$:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -iH|\psi(t)\rangle. \tag{1.57}$$

Equation (1.57) is the *Schrödinger equation* for a time-independent Hamiltonian. A similar equation holds for 'bra' vectors:

$$\frac{\partial}{\partial t}\langle\psi(t)| = \langle\psi(t)|(iH). \tag{1.58}$$

(Recall our earlier comments about the simultaneous usefulness and seductiveness of the Dirac notation. The reader is invited, here, to consider what the terms in this equation *really* represent.) Of course, the evolution of mixed states is still given by (1.53).

If H depends on time, then we can still consider each infinitesimal evolution (from t to $t + dt$) to be given by unitary operators $e^{-iH(t)}$. In general, it is non-trivial to build up finite-time evolution operators from these infinitesimal ones. But when the Hamiltonian is time-independent, then of course we may define $U_t = e^{-iHt}$. If, moreover, the system is in a 'stationary state' at time $t = 0$—that is if it is in an energy eigenstate, $|\psi_E(0)\rangle$, of fixed energy, E, i.e., $H|\psi_E(0)\rangle = E|\psi_E(0)\rangle$—then the evolution takes the simple form $|\psi_E(t)\rangle = e^{-iHt}|\psi_E(0)\rangle = e^{-iEt}|\psi_E(0)\rangle$. That is, the system remains in the same one-dimensional subspace; only the phase, e^{-iEt}, changes with time.

Of course, (1.57) tells us how expectation values change as well. A straightforward application shows that for any operator, F, (1.57) implies:

$$\frac{d}{dt}\langle F\rangle = -i\langle[F, H]\rangle + \left\langle \frac{\partial F}{\partial t} \right\rangle, \tag{1.59}$$

where $\langle\cdot\rangle$ is the expectation value of the operator on the inside in some (here unspecified) state and H is the Hamiltonian.

Such is the standard account of 'Schrödinger evolution'. One should note, however, that in practice the system is often *not* isolated, so that it is necessary to add potentials to the Hamiltonian that represent the influence of some external system, σ_{ext}, on the system of interest, σ. The problem here is that it is often impractical to attempt to model the entire compound system (σ_{ext} *and* σ) whereas one has at least a fighting chance to model the influence of σ_{ext} on σ as an external potential

applied to σ. A typical example involves nano-electronics, where, for example, one might be interested in studying a potential difference applied across a molecule. In principle, one would include the electrodes at either end of the molecule. In practice, just modeling the molecule is already very difficult, and including the electrodes in the system is completely unfeasible. Instead, one simply encodes their effect on the molecule into the potential term in the Hamiltonian. In general, doing so results in *non-unitary* evolution (because the system is not closed).

1.5.2.3.b. The Schrödinger and Heisenberg Pictures. We have been thinking about dynamics in terms of the evolution of states. One may equivalently consider that the state is constant in time, but evolve the observables instead. Indeed, suppose that the state of the system is $|\psi(t)\rangle$, evolving under the unitary evolution U_t. Then the expectation value of the observable F at time t is $(\langle\psi(0)|U^*)\,F\,(U|\psi(0)\rangle)$. So we may just as well let the state be constant in time, i.e. $|\psi(t)\rangle = |\psi(0)\rangle$, while supposing that the *observables* change according to $F(t) = U_t^* F(0) U_t$. Clearly, the expression for the expectation value will be the same in either case. Hence the two pictures are empirically equivalent.

The first picture (where states evolve in time and observables are constant) is normally called the 'Schrödinger picture', while the second picture (where states are constant and observables evolve in time) is normally called the 'Heisenberg picture'.[17]

1.5.2.3.c. The Heisenberg Equation. In the Heisenberg picture, how do the observables change in time? We will (temporarily) index 'Schrödinger' observables with an 'S' and the corresponding 'Heisenberg' observables with an 'H'. We will assume, as well, that the Schrödinger observables do not depend explicitly on time (as is normally the case in basic applications). At any finite time, then, we would have $F^{\mathrm{H}}(t) = U_t^* F^{\mathrm{S}} U_t$. (See §1.5.2.3.b.) Then

$$\begin{aligned}
\frac{dF^{\mathrm{H}}}{dt} &= \frac{\partial U_t^*}{\partial t} F^{\mathrm{S}} U_t + U_t^* F^{\mathrm{S}} \frac{\partial U}{\partial t} && (1.60)\\
&= i U_t^* H^{\mathrm{S}} U_t U_t^* F^{\mathrm{S}} U_t - i U_t^* F^{\mathrm{S}} U_t U_t^* H^{\mathrm{S}} U && (1.61)\\
&= i[U_t^* H^{\mathrm{S}} U_t, F^{\mathrm{H}}] && (1.62)\\
&= i[H, F^{\mathrm{H}}] && (1.63)
\end{aligned}$$

[17] There is a third picture, the 'interaction picture' (sometimes called the 'Dirac picture'), that combines the first two. In this picture, both the states and the observables evolve in time. The evolution of a system due to the free part of its Hamiltonian is encoded into the evolution of the state, and the evolution of a system due to 'the rest' of the Hamiltonian (the 'interaction' part of the Hamiltonian) in encoded in the evolution of the observables.

where we have first used (1.56) and inserted $U_t U_t^*$ into each term, and then we used the fact that H^{S} commutes with U_t, hence $U_t^* H^{\mathrm{S}} U_t = H^{\mathrm{S}} = H^{\mathrm{H}}$. (We have therefore also dropped the index from the Hamiltonian in (1.63).) This equation is normally called the 'Heisenberg equation'.[18]

There is an important similarity between the Heisenberg equation of motion and the equation of motion for a classical observable (function on phase space) $f(x, p)$, written in terms of the Poisson bracket,

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \tag{1.64}$$

The commutator and the Poisson bracket have similar algebraic properties, and moreover, the classical equation of motion is given by $\frac{d}{dt} f = \{H, f\}$, where H is the classical Hamiltonian. Hence one way of thinking about the relationship between classical and quantum theory is in terms of an algebraic analogy between commutators and Poisson brackets.[19]

1.6. **Preview of Philosophical Issues.** Philosophy of non-relativistic quantum theory has traditionally been largely concerned with four issues (which is not to suggest, of course, that there are not other issues that have been discussed).

The first issue (discussed in §2) is in some ways the most fundamental, but also, in some circles, the least discussed, namely, the origins or justification of the quantum-theoretic formalism. How do we find ourselves using Hilbert spaces and self-adjoint operators on them rather than phase spaces and functions on them? Given that the other issues facing the philosopher of quantum theory are mirrored in certain features of the formalism, answering (or at least addressing) this question could prove very useful. There are some reasonably illuminating attempts to explain why quantum theory uses the formalism it does. The next section is devoted to a discussion of a few such attempts.

The second issue (discussed in §3) concerns empirical content. A close reading of, for example, some of the founders of quantum theory (especially Niels Bohr) reveals a deep concern on their part with the question how the quantum formalism gets its

[18] There is a similar equation for the evolution of the density operator in the Schrödinger picture: $\frac{d}{dt} W = -i[H, W]$. Its derivation is similar to the one given above. Notice the change of sign. There is no contradiction here. This equation describes the time-evolution of density operators (states) in the Schrödinger picture. Equation (1.63) describes the time-evolution of operators (observables) in the Heisenberg picture.

[19] Dirac proposed to *define* quantization in terms of this analogy. It turns out that the analogy cannot be carried through rigorously, at least not in its original form. For further discussion of quantization and this algebraic analogy, see Landsman, Ch. 5, this volume, especially §4. For further discussion of Poisson manifolds in classical mechanics (hence, the classical side of the analogy), see Butterfield, Ch. 1, this volume, especially §5.

empirical meaning. I will sketch a proposal, motivated by Bohr, for how to understand the connection between formalism and observation.

The issue of empirical content is closely connected with the issue of incompatibility, and more precisely, the uncertainty relations (discussed in §4). Why is it not possible to determine precise values for all observables? Does it follow from a deeper claim that one observable's having a definite value can preclude another, incompatible, observable from doing so? Some of the founders of quantum theory (again, especially Bohr) were concerned to explain how it is that the conditions required for the empirical well-definedness of some pieces of the formalism somehow preclude the simultaneous satisfaction of the conditions required for the empirical well-definedness of other pieces of the formalism. Others (especially Einstein) were concerned, instead, to 'beat' the uncertainty principle, initially by arguing that one *can* determine precise values for incompatible observables, and later by arguing for the weaker claim (in the famous Einstein-Podolsky-Rosen paper) that they must *have* precise values (simultaneously), even if we cannot determine (for example, measure) them.

The fourth issue (discussed in §5), again related to the previous one, arises from Gleason's Theorem: if there are no dispersion-free (sometimes in this context called 'two-valued') states in quantum theory, then apparently some observables can sometimes be in 'indeterminate' states. Even worse, it is easy to describe physically plausible scenarios in which observables that one normally believes to be definite will *not* be assigned a definite value by the state assigned via quantum theory, under a standard interpretation of the relationship between states and values for observables. Famously, Schrödinger described such a scenario, in which a cat was somehow 'neither alive nor dead'. The obvious way to avoid this problem—by interpreting the probabilities generated by the quantum state epistemically (i.e., as expressions of ignorance about the actual values, as classical probabilities often are) seems to run into serious philosophical, indeed logical, difficulties. Hence other solutions are proposed, in the form of sometimes quite exotic interpretations of the formalism. While many of these solutions can be seen to avoid the basic problem, each has its serious pitfalls as well—and therefore, detractors. No generally accepted response to the problem seems to be on offer.

The final major issue (discussed in §6) is the non-locality of the theory, as implied by the use of tensor-product spaces to represent compound systems. As we noticed, such spaces allow for so-called 'entangled' (non-factorizable) states that imply strong (even perfect) correlations between remote systems. Again, the obvious ways to make sense of these correlations, in terms of a common causal history or processes that propagate

purely locally in space-time, turn out not to work. Such is the upshot of a series of theorems, the first and most famous of which is due to Bell (1964). The result is an apparent conflict with the tendency among space-time theorists of the past century or more to adopt some form of a principle of 'locality'. While various attempts have been made to resolve the apparent conflict, or to argue that it is not a problem, no very satisfying resolution seems to be on offer.

2. Whence the Kinematical Formalism?

Our goal in this section is to review some attempts to say why we use the formalism that we do in quantum theory. There are numerous attempts to 'derive' the Hilbert-space formalism from physically 'intuitive' axioms. The attitude that we shall adopt here is that many of these attempts are worthwhile—they can contribute to our understanding of quantum theory—even if none of them succeeds in deriving the formalism from axioms whose physical import (much less truth) is always clear. It therefore makes sense to survey more than one such route to the formalism, rather than relying on just one of them. Here we will consider a few important representatives of the genre.

The first (§2.1) begins from the notion of a physical proposition, and argues that these propositions are properly represented by the subspaces of a Hilbert space. The second two routes (§2.2) begin from the notion of a physical state, and argue that these states are properly represented as probability measures over the subspaces of a Hilbert space. The final route (§2.3) is similar to these latter two, but while they attempt to characterize the state spaces in a somewhat abstract way (in terms of C^*-algebras), and then 'represent' them as states on a Hilbert space, this final route constructs the Bloch sphere (§1.3.5) directly, and builds (the state spaces of) higher-dimensional Hilbert spaces from it. (However, having said that, we will not in fact examine the proofs enough to notice these differences in any detail.)

Our intention is not to claim that any of these routes has, once and for all, illuminated the true reasons behind the use of Hilbert spaces in quantum theory. Rather, in each case, certain interesting or important aspects of the use of Hilbert spaces in quantum theory can be illuminated by following through some of the arguments that take us (ideally) from relatively uncontroversial and physically clear principles to the Hilbert space formalism. Our emphasis here will be on those aspects of each route that seem to be potentially helpful for understanding quantum theory, skipping over what appear to be merely technical conditions. In each case, our discussion will be necessarily brief (relative to the full story); proofs and technicalities will be omitted.

Finally, many concepts (such as the lattice of propositions, and the convex space of states) that are important for the foundations of quantum theory are discussed here. Readers who are not necessarily looking to learn about routes to the Hilbert space formalism will still learn something independently important from reading at least some parts of this section.

2.1. **From Propositions to Hilbert Space.** In this section, we trace a route to Hilbert space that begins with the logic of physical propositions. The goal, here, is to identify quantum theory (i.e., the Hilbert space formalism) as, in a sense, the unique theory that satisfies certain logical constraints.[20]

2.1.1. *The Lattice of Propositions.*

2.1.1.1. *Physical Meaning of Propositions.* We begin by taking as fundamental the set of propositions about a physical system at a time. The idea is that such propositions will take the form 'the system has the property P at time t'. (Hence, equivalently, one may take properties as fundamental. For each property, there is a corresponding proposition stating that the system possesses the property, at a given time. Here we shall speak solely in terms of these propositions.)

2.1.1.2. *Definition.* It is standard in the algebraic approach to logic to assume that the set of all propositions (syntactically, sentences) forms a lattice (§7.4). The partial order in the lattice corresponds to implication: $P \leq Q$ means that P implies Q. The supremum (join) on the lattice corresponds to disjunction, and the infimum (meet) corresponds to conjunction. These identifications are far from arbitrary. Consider, for example, disjunction. The join of two elements, P and Q, in the lattice, L, is the logically weakest proposition implied by both P and Q, which is, arguably, just what one means by 'disjunction'. Similar considerations apply to conjunction. Finally, assuming (again, as is standard) that the lattice is an ortholattice, the orthocomplement in the lattice corresponds to negation.

2.1.1.3. *Motivation and Interpretation of the Lattice Operations.* Apart from the direct appeal to algebraic logic, such lattices have sometimes been motivated by an operational description of measurements. One speaks, for example, of yes-no experiments, to test for the truth of some proposition. The partial order on the set of such tests is derived from the idea that one test, Q, might be passed every time some other

[20]There are many versions of this program, which got its start with Birkhoff and von Neumann (1936). The discussion here is largely in the spirit of Piron's (1976) work. A recent monograph covering much of the territory is that of Dalla Chiara, Giuntini, and Greechie (2004). A more operational approach, somewhat different from these others, is that of Ludwig (1983).

test, P, is, and in this case we write $P \leq Q$. However, I shall not pursue this approach in any detail here, but instead take it for granted that there is some way to understand these logical operations and relations in physical terms. (I do not mean to suggest that any particular approach—especially the various operational approaches—are entirely satisfactory. Indeed, one could argue that a firm understanding of the physical significance of the lattice-theoretic connectives is the Achilles heel of the route to Hilbert space via propositions.[21]

2.1.1.4. *Additional Constraints on the Lattice of Propostions.* In addition to the structure of the lattice of propositions that we have already established, we will need to assume a number of additional properties. First, we assume that the lattice has a bottom and a top (§7.4.5), 0 and 1, corresponding respectively to the logically false proposition and the logically true proposition. (We then require of the orthocomplement that $P \vee P^{\perp} = 1$ for all P. It follows that $0^{\perp} = 1$.)

Second, we assume that the lattice of propositions is complete and atomic (§7.4.5). This assumption amounts to the idea that there are some fundamental propositions—maximally specific propositions—and that one can interpret the truth of any weaker proposition as a consequence of the truth of some fundamental proposition. (It does not follow that one *must* interpret them in this way.) Another approach to justifying this assumption refers to the convex structure of the set of states. As we will see below (§2.2.1.3.c), there is good reason to suppose that this set has extremal points, corresponding to pure states, i.e., states of maximal information. That characterization suggests that each pure state should assign probability 1 to some maximal (logically strongest) proposition, which would be an atom in the lattice of propositions.

Next we must assume that the lattice is irreducible (§7.4.4). The assumption of irreducibility is far from trivial, but does have something approaching a physical interpretation, due to the following theorem from the theory of lattices:

> An ortholattice is irreducible if and only if its center (§7.4.3) is trivial, i.e., $\{0, 1\}$.

Now, a proposition's being in the center means, logically, that it is compatible with all other propositions (§1.2.4). Therefore, one way to understand the irreducibility of a lattice is as a consequence of the assumption that its center is trivial, i.e., that *every* proposition (except 0 and 1) is incompatible with at least some other proposition.

[21] I do not mean to suggest, of course, that this issue has not been addressed by advocates of the quantum-logic program. Jauch (1968), for example, proposes an understanding of the conjunction of non-commuting projections (which, apparently, cannot be understood simply in terms of the proposition 'both are measured') in terms of a limit of repeated measurements.

Note that this assumption is in a sense a 'maximal' violation of the classical law $p = (p \wedge q) \vee (p \wedge q^{\perp})$, for it asserts that the law fails for *every* p (which is not to say that there are not *some* p, q for which the law holds, but that for every p, some q exists for which the law fails—recall, also, the discussion surrounding the conditions 1.23).

Our final assumption is that L satisfies the covering property (§7.4.5). The motivation here is, alas, less clear, although some arguments have been given.[22]

2.1.2. *Piron's Program.* Piron's program was to characterize exactly those lattices that are (isomorphic to) the lattice of subspaces of a Hilbert space. He got as far as the following theorem (Piron 1964):

> If L is a complete, atomic, irreducible, orthomodular lattice that satisfies the covering law and has at least 4 orthogonal atoms, then it is (isomorphic) to the lattice of subspaces of an inner product space, V.

In fact, Piron was able also to say something about the field over which V is defined, but not much. In the end, Piron's theorem is suggestive, but far from the desired end, which is to characterize the lattice of subspaces of a Hilbert space as the unique structure satisfying certain logical constraints.

Piron's program made a huge step forward with the proof of the following theorem, due to Solér (1995):

> **Theorem** Solér: If the lattice L in the statement of Piron's Theorem contains an infinite orthonormal sequence, then the vector space in question is a Hilbert space over the reals, complex numbers, or quaternions.

This result is an important contribution to Piron's program, although there is an obvious limitation to its applicability: it does not cover the case of finite-dimensional Hilbert spaces (for example, those describing the spin of a particle, as in §1.3).

2.2. **From States to Hilbert Space.**

2.2.1. *An Approach in Terms of Convex Spaces of States.* We will now consider two approaches to Hilbert space that begin with the structure of the space of states. The first begins with the observation that the states form a convex set (§2.2.1.1). Our procedure will be to embed an arbitrary convex set into a vector space, V (§2.2.1.2), and argue (in the subsequent sections) for progressively adding more structure to V,

[22] For example: Piron (1964), and Cohen and Svetlichny (1987).

until it has enough structure to support a theorem to the effect that the convex set of states can in fact be represented as density operators on a Hilbert space.

2.2.1.1. *Convex Spaces of States.* Earlier we noticed that given any two probability measures, p and p', they can be combined to form a third, $q = rp+(1-r)p'$, where $0 < r < 1$. It is easily checked that q thus defined satisfies Kolmogorov's axioms (§7.5.6), if p and p' do. Indeed, any convex combination (§7.1.10) of probability measures yields another probability measure, called a 'mixture' of the measures appearing in the convex combination.

The general idea, then, will be to take a convex space of states as fundamental. At this point, we make no commitment about the space of states other than that it is convex. We will then impose extra conditions on this space, eventually forcing our space of states to be, in fact, the space of states on a Hilbert space. In other words, we will have characterized the space of states (density operators) on a Hilbert space.[23]

2.2.1.2. *Embedding in a Vector Space.* So consider, to start, an arbitrary convex set, S. It is mathematically natural and convenient to embed S into a real vector space, V, in part because convex combinations are just a special type of real linear combination, and the latter is naturally defined in the context of a (real) vector space.

If S is generated by its extreme points, then our immediate task is easy. (An extreme point of a convex set is a point that is not itself a convex combination of other points in the set. S is generated by its extreme points just in case every element of S can be expressed as a convex combination of extreme points.) In that case, we can define the embedding by letting V be the free vector space generated by the extreme points of S. (Intuitively, V is then all formal real linear combinations of the extreme points of S.) However, this approach clearly requires that we assume that S has extreme points; and while it does in quantum theory as standardly understood, it is worth seeing that one need not assume so, but can instead derive this fact from other considerations. (In the context of a physical theory, where S is a set of states, the extreme points are just the pure states—recall, also, §2.1.1.4.) So let V be a real vector space having S as a subset (i.e., S is embedded in V—given the real linear structure of S, it should be clear that it is always possible to find a vector space that contains a subset that is isomorphic to S as a convex set; our problem here is not to show that there is such a V, but to construct it from S, as above).

[23]The discussion in this section is largely an expansion and explanation of the program outlined by Haag (1992, §VII.2). The program has since taken on a new form, in the hands of Alfsen and Shultz (2003), as briefly mentioned below.

2.2.1.3. *Sufficient Conditions for the Existence of Extreme Points.*

2.2.1.3.a. Observables. We will take an arbitrary observable, f, of our theory to be a function from states to expectation values. That is, $f(v)$ is the expectation value of f in the state v. Indeed, after embedding the states in a vector space, V, we will consider the observables to be (possibly a subset of the) real-valued linear functionals on V. It is clear that observables, understood as maps from states to expectation values, should be linear functionals, because for any state, $v = \sum_n w_n v_n$ (written here as a convex combination of states v_n), we must have $f(v) = f(\sum_n w_n v_n) = \sum_n w_n f(v_n)$. Otherwise, $f(v)$ would in general be numerically different from the expectation value of f on a system randomly chosen from the mixture $\sum_n w_n v_n$).

Let O be the set of observables. We need not (yet) make any commitments about the contents of O except to require that it separate V; that is, for any non-zero $v \in V$, there is an $f \in O$ such that $f(v) \neq 0$ (equivalently, if $v_1 \neq v_2$ then there exists $f \in O$ such that $f(v_1) \neq f(v_2)$).[24] In S, this condition amounts to requiring that O be rich enough to make probabilistic distinctions amongst elements of S. To extend to all of V, assume now that O separates S, and note that if O does not also separate all of V, then in fact V is 'unnecessarily large' to embed S. For consider the subspace of all v such that $f(v) = 0$ for all $f \in O$. Then consider the quotient space V/W. There is a homomorphism from V to V/W that is an isomorphism from S to its image in V/W as convex sets. So we can just as well work with V/W, and doing so in fact eliminates structure from V that was not needed in order to embed S.

2.2.1.3.b. Topology on the Embedding Space. Therefore, we assume that O separates V. We now introduce a topology (§7.5) for V. Here, the guiding idea is that 'infinitesimally small' changes in the state should result in 'infinitesimally small' changes in expectation values.[25] We therefore introduce the coarsest topology on V that makes all of O continuous. Call it the O-topology.[26] In this topology, it makes a certain

[24]This condition can always be met—i.e., one can always find some linear functional that does the job—for any normed vector space, by the Hahn-Banach theorem. Our space V will in fact be normed.

[25]This assumption, while apparently natural, is certainly not compelling. See §2.3.2 for discussion of this idea in the context of Hardy's approach.

[26]A basis for this topology is given by all sets ('open balls'), B, constructible as follows. Choose $v \in V$, $f_1, f_2, \ldots, f_n \in O$, and $\epsilon > 0$. B is then the set of all $w \in V$ such that $|f_i(w) - f_i(v)| < \epsilon$ for $i = 1 \ldots n$.

amount of sense to require that S be compact (§7.5.1).[27] For example, that assumption guarantees that (the expectation values of) observables are bounded, because the continuous image of a compact set is compact.

2.2.1.3.c. The Existence of Extreme Points. Given these assumptions about S and its embedding into a vector space, V, and the associated observables, O, we can apply the following theorem from functional analysis:

> **Theorem** (Krein-Millman; see Rudin 1973, p. 70) Let S be a convex subset of V, O a separating set of linear functionals on V, and let S be compact in the O-topology. Then S has extremal points, and it is the smallest closed convex set containing all of those points.

One says that S is the 'closure of the convex hull of its extreme points'. In other words, S has extreme points, and is 'generated' by them in the sense that S is the closure of all convex combinations of extreme points. Hence every element of S can be written as a convex combination of extreme points (pure states), or is the limit of a sequence of such states.[28]

2.2.1.4. *Further Properties of the Embedding.* Recall the prescription (§2.2.1.2) for constructing the embedding space, V, as the free vector space over the extreme points of S. Now that we have established that S indeed has extreme points, we can, almost, follow this prescription. The only addendum is that we wish V to be closed under the O-topology. Hence, V is constructed as the closure of the free vector space over the extreme points of S. (By 'closure', here, we mean: ensure that the closure of every open set in V is also in V.) In this case, the pure states in S form a basis for V (because, in fact, the free vector space over the pure states is in essence 'all formal linear combinations' of pure states).

Let us now insist that O contain a linear functional, I, that assigns the value 1 to every pure state (extreme point) in S. This I is unique (because the pure states form a basis, and I is continuous by assumption and so is extended uniquely from the pure states to all of V). Now, let V^+ be the positive convex cone (§7.1.10) generated by the pure states. The states in V are then identified as the elements of V^+ that take the value 1 for I.

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[27] In the specific context in which we will soon work, compactness amounts to the assumption that the identity—the linear functional that has value 1 everywhere—is an observable.

[28] Careful! Those familiar with quantum theory might be tempted to suppose that V is, or is isomorphic to, the Hilbert space containing the statevectors for a system. It is neither. As we will see (once we have defined a norm on V), it is a vector space in which the density operators form the unit sphere.

Indeed, I generates a norm on V in a natural way. For any $v \in V^+$, define the norm of v by $||v|| = I(v)$. Now, any $v \in V$ can be decomposed as $v_1 - v_2$ for some $v_1, v_2 \in V^+$ (or is the limit of a sequence that can be so decomposed).[29] Hence we can define $||v||$ (now for any $v \in V$) as the infimum (over all such decompositions) of $||v_1|| + ||v_2||$, and require the norm to be continuous. V is thus a real Banach space (§7.1.7), and (by construction) the norm on V assigns norm 1 to every state. (We have now fulfilled our earlier promise that V would turn out to be a normed space.)

2.2.1.5. *Faces and Propositions.* To what do these states assign probabilities? While we do not need to answer that question in order to finish the discussion (at this level), it is instructive to do so nonetheless: the 'propositions' to which states assign probabilities are faces of states in the convex set of states (§7.1.10).

In classical mechanics, one can form a face of a convex set by taking the closure, under convex combination, of a set of pure states. In quantum theory, the process of purification will in general add new pure states to the set, and so is essential for the construction of the face. (Recall, as well, that the classical states form a simplex, while the quantum states do not—see §1.2.7.) The physical idea behind the definition of a face in both cases is something like 'the set of all states that one can create (from some initial set) via mixing, plus the set of all states of which the resulting mixed states could, in principle, be a mixture'.

There is a natural connection between faces of states and propositions about, or properties of, a physical system. (Here, as above, we shall use the terms 'property' and 'proposition' interchangeably. Also as before, we begin with the minimal assumption that the set of propositions is a poset, the partial order corresponding to implication.) In particular, let us say that a proposition (about a system, at a time) asserts that a measurement on the system reveals that the state of the system is in some face of states.

Note that an extremal point in the space of states is a (singleton) face, and vice versa. Hence one sort of proposition is of the form 'the system is in the state v', for any pure v. This association makes sense, because pure states are supposed to be, intuitively, states of maximal information, and if propositions correspond to faces, then the most specific propositions are the singleton faces (extremal points).

The idea behind the general association between propositions and faces is the following. Suppose you are handed an ensemble of systems, all in the same state, and

[29] Sketch of a proof: the pure states span V, so write v as a linear combination of the pure states, and separate it into a part with positive coefficients and a part with negative coefficients. The former is clearly in V^+. The latter, multiplied by -1, is also in V^+.

asked to determine in what state the ensemble was prepared. Given the association between maximally specific propositions and pure states, if you can determine that some maximally specific proposition is true of every member of the ensemble, then you are done—the state is the corresponding pure state. But suppose that there is no such proposition, i.e., no maximally specific proposition true of *every* member of the ensemble. Then you have been handed a mixed state, but a mixture of what? In general, you will be able to determine with certainty only that the state was created by mixing states from the smallest face containing the actual mixed state (hence, the smallest proposition to which the actual state assigns probability 1).

In terms of standard quantum theory, the point here is that for any mixed state (density operator) W, the logically strongest proposition that is rendered certain by W is just $\mathrm{ran}\, W$, here generalizing the notion of 'range' to mean 'the image of W under all vectors in the Hilbert space'. (It should come as no surprise that $\mathrm{ran}\, W = \vee_n P_n$, where P_n are the spectral projections of W.) In other words, $\mathrm{ran}\, W$ is in fact the subspace of pure (vector) states that forms the smallest face containing W.

2.2.1.6. *Hilbert Space representations of faces of cones.*

2.2.1.6.a. Homogenous cones. Consider the automorphisms on the set of states (that is, any map from the set of states to itself that preserves the convex structure). Such maps plausibly correspond to the possible state-transitions. Moreover, they are naturally extended to linear maps on V that are automorphisms of V^+ (i.e., such maps take V^+ to itself, and preserve convex structure of V^+; hence they preserve the fact that V^+ is the positive convex cone generated by the pure states). The extension to V^+ is effected simply by the condition that for any such automorphism, f, any real number r, and any state, v, $f(rv) = rf(v)$. Recalling that every element of V can be written as a linear combination of elements of V^+, one can see that f is thus naturally extended to a linear transformation on V.

A cone is said to be 'homogeneous' with respect to this set, T, of transformations if for any two non-extremal points, v, v' inside the cone, there is a transformation in T that takes v to v'. The physical idea here is that there is *some* way for a system to evolve from any non-extremal point to any other non-extremal point.

2.2.1.6.b. The Case of Finitely Many Pure States. While we would not want to assume that there are only finitely many pure states, nonetheless the following theorem is extremely suggestive (Vinberg 1965):

> The faces of a self-dual,[30] homogeneous cone with finitely many extremal points are in one-to-one correspondence with the subspaces of some Hilbert space (over the real numbers, complex numbers, or quaternions).

Hence, if one is convinced that the states of a theory must form a convex set with the properties discussed above, one is, in the sense given by this theorem, committed to the Hilbert space formalism.

2.2.1.6.c. The General Case. Alfsen and Shultz (2003, p. 414) have extended this program to eliminate the unrealistic (and indeed false) asusmption that there are only finitely many pure states. Their main result involves a number of technical assumptions about the structure of the convex set of states that we do not have the space to articulate here. Moreover, in the end, they arrive not quite at a characterization of Hilbert space (at least not directly), but at a characterization of the state spaces of a C^*-algebra, although, via a GNS construction, they can get to Hilbert space from there.[31]

2.2.2. *An Approach in Terms of Pure States and C^*-Algebras.* In this sense (i.e., in dealing ultimately with the state-spaces of C^*-algebras), the theorem of Alfsen and Shultz is similar to a theorem due to Landsman (1998, Theorem 3.9.2 and Corollary 3.9.3), which we will now consider, albeit briefly and (again) in outline.

Landsman argues that the set of pure states should be endowed with two distinct structures, dynamical and probabilistic, and that the two must be connected in the right way. He then adds what is supposed to be the characteristically quantum condition (described below), and arrives at a theorem that characterizes the state spaces of C^*-algebras.

2.2.2.1. *The Poissonian Structure.*

2.2.2.1.a. State Spaces as Poisson Manifolds. The first sort of structure corresponds to the dynamics that we discussed earlier. Recall our discussion of the Heisenberg equation (§1.5.2.3.c), and specifically the fact that (i times the) commutator has the algebraic form of a Poisson bracket. In outline, the general theory of dynamics generated by Poisson brackets goes as follows.

We begin with some space (indeed, manifold—§7.5.2) of states, M. Given M, one defines the evolution of observables in terms of a Poisson bracket defined on the

[30] We pass over the issue of the physical motivation for this condition. See §7.1.10.

[31] For more on C^* algebras, GNS-constructions, and related matters, see Landsman, Ch. 5, Halvorson, Ch. 8, and Emch, Ch. 10, this volume.

infinitely-differentiable real-valued functions on M, $C^\infty(M)$ (the observables), as a bilinear operation, $\{,\}$, on $C^\infty(M)$. (Recall (1.64), and see Butterfield, Ch. 1, §5, this volume, for further discussion.) Together wioth a choice of a scalar function $H : M \to \mathbb{R}$ as the Hamiltonian, one can use this definition of the evolution of observables to define an evolution (of states) on M (analogously to the equivalence between the Heisenberg and Schrödinger pictures). For a given Hamiltonian, the possible dynamical paths from M, thus defined, are the 'Hamiltonian curves' of the system.

Keep in mind that this construction is supposed to be very generic. We are essentially laying down 'what one means' by a certain kind of dynamical system. To this end, Landsman puts additional constraints of 'well-behavedness' on the space of states as a Poisson manifold, which we bypass here.

2.2.2.1.b. Symplectic Leaves. It can happen that some portions of a Poisson manifold are 'inaccessible' from others. In particular, there may be no (piecewise smooth) Hamiltonian curve connecting them, for any Hamiltonian. Let us say that two points, $x, y \in M$ of a Poisson manifold $\langle M, \{,\}\rangle$ are *symplectically equivalent* if for some Hamiltonian they lie on a single (piecewise smooth) Hamiltonian curve. This relation is clearly an equivalence relation, and therefore partitions M into 'symplectic leaves' (see Butterfield, Ch. 1, §5.3.3, this volume).

2.2.2.2. *The Transition Probability Structure.*

2.2.2.2.a. State Spaces as Transition Probability Spaces. The only dynamical structure on a classical Hamiltonian phase space is its Poissonian structure (again see Butterfield, Ch. 1, especially §5.2.4, this volume). However, in standard quantum theory, there are two types of evolution: the continuous, deterministic, evolution described by the Poisson bracket, and the discontinuous, stochastic, 'quantum jumping' from one state to another, often associated with 'measurement'. (See §5.4.3.)

This latter structure is reflected in the fact that, in Landsman's scheme, the quantum state space must be a 'transition probability space', meaning that there must be maps, p, from pairs of elements in the space to $[0, 1]$ that satisfy: $p(v, w) = 1$ if and only if $v = w$; and $p(v, w) = 0$ if and only if $p(w, v) = 0$. (The expression $p(v, w)$ is read as 'the probability of a transition from v to w'.) In addition, we require that these probabilities be symmetric: $p(v, w) = p(w, v)$.

2.2.2.2.b. Sectors. A sector of a transition probability space is a region of the space that is isolated from the rest. That is, for Q a sector of states, $p(v, w) = 0$ for all

$v \in Q$ and all $w \in Q'$ (the complement of Q in the set, S, of all states). Note that the symmetry of p implies that a system cannot make a transition to or from a sector.

2.2.2.2.c. Superpositions. Let $Q \in S$. We define $Q^{\perp}$ by

$$Q^{\perp} := \{v \in S | p(v, w) = 0 \text{ for all } w \in Q\}. \tag{2.1}$$

That is, $Q^{\perp}$ is the set of all states that are inaccessible (via a single probabilistic transition) from every state in Q. We can use this definition to characterize a generic notion of a 'superposition', as follows: the class of all 'superpositions' of the states v and w is $\{v, w\}^{\perp\perp}$.[32]

2.2.2.3. *Landsman's Theorem.* Landsman shows, roughly, that the state space of a C^* algebra is uniquely determined by its Poissonian and transition probability structure, assuming (among other things) that symplectic leaves correspond to sectors. He then characterizes *quantnum* theories as those in which the '2-sphere' property holds, i.e., condition (iv) in the theorem as stated below. *Classical* theories, on the other hand, are characterized by the condition that the transition probabilities are trivial, i.e., $p(w, v) = \delta_{vw}$. Notice that in this case, the sectors are singeltons. We will see a similar characterization of the difference between quantum and classical systems below (§2.3.2).

In effect, then, Landsman (1998, 104–106) proves roughly the following theorem:[33]

> A pure state space, S, is the pure state space of a quantum system if and only if: (i) S is a Poisson manifold; (ii) S is a transition probability space; (iii) the symplectic leaves of S correspond to the sectors of S; (iv) for any $v, w \in S$, $\{v, w\}^{\perp\perp}$ is isomorphic as a transition probability space to the space of statevectors in $\mathbb{C}^2$.

I have already discussed conditions (i) and (ii). Condition (iii) is a requirement that what cannot happen by continuous evolution cannot happen by stochastic evolution either (and vice versa). In other words, if it is 'dynamically impossible' to get from state v to state w, then the probability of a stochastic transition from v to w is 0 (and vice versa).

Condition (iv) is, of course, the one that does a lot of the work getting us to the Hilbert space formalism, for it is essentially the requirement that the set of all 'superpositions' of a pair of states forms a transition probability space that looks like the quantum-mechanical pure states on the space $\mathbb{C}^2$, discussed earlier (§1.3.3.2) in

[32] ADD REF TO BUTTERFIELD 1993 in STUDIES???

[33] The conditions of the theorem are not stated here in full rigor. In addition, the proof of the theorem requires a few other technical assumptions whose immediate physical import is perhaps not clear. I have left them out.

the context of spin. Whether or to what extent this more or less explicit reliance on quantum theory is ultimately satisfactory is a matter of taste, but it is, at any rate, worth noticing that what is 'esssentially quantum' about the Hilbert space formalism can (more or less) be reduced to this assumption.

2.3. **Hardy's Axioms.** Finally, we consider a result from Hardy (2001, 2002), again in the tradition of trying to explain the origins of the quantum formalism. This approach also begins from the notion of a state space, but its framework differs sufficiently from the previous two approaches that we consider it separately.

2.3.1. *The Framework.*

2.3.1.1. *States as Probabilistic Predictors.* Much as has been done already in the previous sections, Hardy takes *states* in a physical theory to be the determiners of probabilities associated with each possible outcome of any measurement that may be performed by a given preparation of the system. (States are thus associated with preparations.) Hence, for a given preparation, knowing the associated state allows one to predict the probabilities for the results of any measurement.

2.3.1.2. *Degrees of Freedom.* One mathematical characterization of a state is as a 'list' of all of these probabilities. (Of course, in general there are at least uncountably many items in the 'list'.) However, in general the state space has some structure that allows states to be somehow characterized by a shorter list of what Hardy calls 'fiduciary' probabilities (which are not the same thing as Fisher's 'fiducial probabilities'). In a given theory, we define the *degrees of freedom*, K, to be the smallest number of fiduciary probabilities that is sufficient to determine the state.

2.3.1.3. *Dimension.* In addition, there may be sets of states that can be distinguished one from another with probability 1 in a single measurement. In other words, for each pair of states, v, w, in the set, if v assigns non-zero probability to some outcome of the measurement, then w assigns probability zero to the same outcome. There will in general be a maximum number, N, of states that are distinguishible in this way. Hardy calls N the *dimension* of the space.

2.3.2. *The Axioms.* Hardy proposes five 'axioms'. The first axiom simply underwrites our earlier assumption that states can be associated with preparations, that is, that there are stable relative frequencies for the outcomes of measurements for a given type of preparation. The remaining axioms are, as adapted from Hardy (2001):

Subspaces: For any integer, N, there exist systems with dimension N. Moreover, all systems of dimension N, and all systems with higher dimension but whose state restricts the system to an 'N-dimensional subspace', have the same properties.

Composite Systems: A composite system consisting of systems A and B, with degrees of freedom and dimensions K_A,K_B and N_A,N_B respectively, have $K = K_A K_B$ degrees of freedom and dimension $N = N_A N_B$.

Continuity: For any dimension, N, any system of dimension N, and any two pure states, v and w, of such a system, there exists a continuous reversible transformation (continuously connected to the identity) from v to w.

Simplicity: For given N, K takes the minimum value consistent with the other axioms.

The motivation for some of these axioms is relatively clear, and for others, not as clear. We will not discuss them all here. 'Simplicity' comes into play because the other axioms imply that $K = N^m$ for some integer m. For $m = 1$, the continuity axiom is violated, and one arrives at classical probability theory. For $m = 2$, one arrives at quantum theory.

The continuity condition is clearly significant, therefore. One understanding of it that Hardy has encouraged is that it expresses the desideratum that 'small changes' in the state should entail 'small changes' in the predictions based on that state. However, it is not clear whether this principle is physically compelling. After all, in Hardy's sense, the state space of classical physics is *not* continuous (cf. the beginning of §2.2.2.3), and yet in general one does not suppose that there is somehow a serious problem with the relationship between changes of state and changes of prediction based on that state.

A different understanding of continuity connects it with superpositions. The basic point is easiest to visualize geometrically, and we will leave the matter at that here. The quantum state space is 'continuous' (in Hardy's sense) because for any two pure states, there is another pure state that is 'between' them, and in fact this 'middle' state is a superposition of the two original states. In other words, continuity holds precisely because the superposition principle holds. Continuity fails in the classical theory because the superposition principle fails there. From this point of view, it is less surprising—though not necessarily less important—that continuity is what makes the difference, in Hardy's framework, between classical and quantum theories.

3. Empirical Content

In this section, we will discuss the issue of how the formalism of quantum theory gets empirical content. That discussion will lead naturally to a discussion (§4) of uncertainty, because the particular way that we will, here, understand how the formalism gets empirical content leads naturally to the uncertainty principle (§4), as it should. Hence this section is not merely expository. Implicitly, we will develop an argument in favor of a particular way of understanding how the formalism gets its empirical content, based on the fact that it leads naturally to uncertainty.

One question about empirical content concerns how measurement (or empirical observation more generally) is modeled in the theory. Without such a model, it is hard to see how the theory can make predictions about empirical observations. Our first task (§3.1) will be to say something about this issue. But the more difficult—and arguably more philosohically interesting—issue concerns how *any* element of the formalism is connected with empirical fact at all. The remainder (and majority) of this section will concern that issue.[34]

There are two elements in the present approach to establishing a connection between the formalism and its empirical content: symmetries, and reference frames. These elements are connected—'legitimate' reference frames are connected to one another via certain symmetry transformations—but I will, for the most part, keep them separated. In this section, I first frame the issue of empirical content in terms of POVMs (§3.2), then discuss the role of symmetries (§3.3) and reference frames §3.4) in the definition of physical quantities, and finally I briefly sketch an account (§3.5) of how the formalism gets its empirical content.

3.1. **Measurement.**

3.1.1. *The Standard Account of Measurement.* One frequently encounters the following account of measurement in quantum theory. Suppose that the state of the system is W (a density operator). Suppose that one measures an observable represented by the POVM $E : \mathcal{B}(S) \rightarrow \mathcal{B}(\mathcal{H})^+$. Then the result of the measurement will be some $\Delta \subset S$, with probability $\mathrm{Tr}[E(\Delta)W]$. Furthermore, the state after measurement is just the 'projection' of the original state onto the result. In the case where the POVM is a PVM, the projection is given by $E(\Delta)WE(\Delta)$ (ignoring normalization). Otherwise, the usual procedure is to define 'measurement' operators $M(\Delta) = \sqrt{E(\Delta)}$,

[34]Much of that material is adapted and revised from Dickson (2004a; 2004b).

in terms of which the 'projection' (a misnomer, in this case) is (again ignoring normalization) $M(\Delta)WM^*(\Delta)$. Notice that the latter prescription is equivalent to the former for PVMs.

While useful for making predictions, this account of measurement is completely unsatisfactory, from a foundational point of view. One problem, as it has been pointed out repeatedly by both physicists and philosophers, is that measurement is itself a physical process, and in particular a physical interaction between two (or more) physical systems, one of which we call an 'apparatus' and the other of which we call the 'measured system'. Hence, if quantum theory is our best theory of interactions amongst physical systems, it should be capable of describing this interaction in such a way that the result—the states of the two systems at the end of the interaction—are as stated above.

3.1.2. *Impulsive Measurement.* There is indeed an account of measurement along those lines.[35] It is the so-called 'impulsive model' of measurement. Suppose we are going to measure an observable represented by the self-adjoint operator, F. For simplicity, suppose that the measured system is initially in the pure state $|\psi\rangle$ and the apparatus is in the pure state $|\chi\rangle$. The compound system is thus in the state $|\Psi\rangle = |\chi\rangle|\psi\rangle$. Suppose that apart from the interaction between them, each of these systems evolves freely, with (free) Hamiltonians H_{S} and H_{M}. The total Hamiltonian for the combined system is thus $H_{\mathrm{total}} = H_{\mathrm{S}} + H_{\mathrm{M}} + H_{\mathrm{I}}$, where H_{I} is the interaction Hamiltonian (i.e., it represents the energy exchanged between the systems). Finally, let Π be the momentum observable for the apparatus.

Now, for our model of measurement, we take $H_I = g(t)\Pi \otimes F$ (henceforth the $\otimes$ is left implicit), where $g(t)$ is an interaction function given by $g(t) = \gamma f(t)$, with γ a constant, the 'interaction strength', and $f(t)$ is a function that is zero except for t between 0 (when the measurement-interaction begins) and τ (when the measurement ends), and $f(t)$ is (purely for convenience) normalized, i.e., $\int_0^\tau f(t)dt = 1$. The Schrödinger equation (1.5.2.3.a) may therefore be written:

$$\frac{d}{dt}|\Psi\rangle = -i\Big(H_{\mathrm{S}} + H_{\mathrm{M}} + \gamma f(t)\Pi F\Big)|\chi(t)\rangle|\psi(t)\rangle, \tag{3.1}$$

where now we have explicitly indicated the time-dependence of the states of the measured system and the measuring device.

For an 'impulsive' measurement, τ is very small and γ is very large. (The interaction is quick and strong.) If we may assume that the apparatus and system have

[35]For an extended classic discussion, which is more or less followed here, see Bohm (1951, ch. 22, especially §5).

low or zero momentum, then during the interval $[0, \tau]$, the influence of the interaction Hamiltonian on the evolution of the compound system completely swamps the influence of the free Hamiltonians, so that, during this period, we have

$$\frac{d}{dt}|\Psi\rangle \simeq -i\gamma f(t)|\chi(t)\rangle|\psi(t)\rangle. \tag{3.2}$$

We can readily solve this equation[36] to get the compound state at the end of the interaction (*immediately* at the end, before the free Hamiltonians take over again):

$$|\Psi(\tau)\rangle \simeq \exp\left[-i\int_0^\tau \gamma f(t)\Pi F\ dt\right]\ |\chi(0)\rangle|\psi(0)\rangle. \tag{3.3}$$

Write $|\psi(0)\rangle$ in terms of the (normalized) eigenstates, $|f_n\rangle$, of F (which, for simplicity, we will assume to be maximal): $|\psi(0)\rangle = \sum_n \langle f_n|\psi(0)\rangle|f_n\rangle$. (See the last paragraph of 7.1.4). Then (3.3) becomes

$$|\Psi(\tau)\rangle \simeq \sum_n \langle f_n|\psi(0)\rangle \exp(-i\gamma f_n\Pi)|\chi(0)\rangle|\psi(0)\rangle, \tag{3.4}$$

where f_n is the eigenvalue of F corresponding to the eigenvector $|f_n\rangle$. Now define

$$|\xi_n\rangle = \exp(-i\gamma f_n\Pi)|\chi(0)\rangle. \tag{3.5}$$

Because γ is large, these states are effectively orthonormal:[37]

$$\langle\xi_n|\xi_m\rangle = \langle\chi(0)|\exp[-i\gamma(f_n - f_m)P|\chi(0)\rangle \simeq \delta_{nm}. \tag{3.6}$$

As we will see below in a more generic context (3.13), because Π is the 'momentum' of a pointer, the states $|\xi_n\rangle$ are 'spatial translations' (i.e., translations in the value of the 'pointer-position observable') by an amount γf_n (and therefore, for large γ, they are macroscopically distinguishable states). The upshot of this discussion is that the final state of the compound system is

$$|\Psi(\tau)\rangle \simeq \sum_n \langle f_n|\psi(0)\rangle|f_n\rangle|\xi_n\rangle. \tag{3.7}$$

Notice that this state is entangled, and indeed that it represents a perfect correlation (§1.2.6.4) between the value of F for the measured system and the position-position.

In order for this model to match the prescription of the previous subsection, we must adopt from that prescription the 'rule' that when the apparatus shows the result $|\xi_n\rangle$, the state of the system 'collapses' to $|f_n\rangle$. (Or, the state of the compound system is projected onto $\mathbb{I} \otimes |\xi_n\rangle\langle\xi_n|$, which, because of the perfect correlation, will have the

[36]The solution in (3.3) follows only because the interaction Hamiltonian is given in terms of a scalar function of time. General time-dependent Hamiltonians cannot be treated in this way. See, e.g., Cohen-Tannoudji (1977, 172–175).

[37]The second equality in (3.6) holds by the Riemann-Lebesque lemma, under the assumption that the apparatus has a continuous spectrum for P.

effect of making the compound state into the product state $|f_n\rangle|\xi_n\rangle$.) I will discuss this 'collapse' rule in §5.4.3.

Does this model apply to all, or indeed to any, actual 'measurements' made by actual physical devices? To a reasonable degree of accuracy, probably. But the point here is not to make such a claim, and certainly not to catalogue the varieties of physical measurement and the ways in which they might be modeled quantum mechanically. Creating and justifying such models is the business of physics. The point, here, is that quantum theory can, to some degree, supply a model of measurement. It need not rely on the non-account of the previous subsection.

One final observation about this model: it models a type of measurement in which the state of the system is in a sense unchanged by the interaction with the measuring device: if the measurement is repeated, the probability of getting the same result the second (and subsequent) times is 1. Notice that, contrary to the occasional declaration by physicists or philosophers, in this sense at least, measurement does *not* necessarily 'disturb' the state of the measured system. Following Pauli (1958), such measurements are often called 'measurements of the first kind'.

Of course, measurement sometimes does disturb the measured system. Indeed, sometimes it destroys the measured system. Measurements in which the state of the system is disturbed by the process of measurement (i.e., measurements that either are not repeatable, or whose results will not necessarily be the same upon repetition) are often called (again following Pauli) 'measurements of the second kind'.

3.1.3. *Weak Measurement.* Once we begin modeling measurement as an actual physical process, it becomes natural to ask what would happen if the physical circumstances were different. One natural case to consider is where the measurement is 'adiabatic'; that is, the interaction is weak, and takes a long time (on some appropriate scale).

One scheme for realizing this idea has been called 'protective measurement' (Aharonov et al., 1993). The idea is nicely illustrated by the case where $|\psi(0)\rangle$ is the ground state of a harmonic oscillator (the crucial feature of which, for us, is that there is a finite energy difference between the possible states). When τ is large and γ is small, of course we cannot ignore the evolution due to the free Hamiltonians. Let the interaction Hamiltonian be $H_{\mathrm{I}} = g(t)QF$. The solution to (3.1) in this case is

$$|\Psi(\tau)\rangle = \exp\left[-i\left(H_{\mathrm{S}}\tau + H_{\mathrm{M}}\tau + \int_0^\tau \gamma f(t')QF\ dt'\right)\right]|\chi(0)\rangle|\psi(0)\rangle \tag{3.8}$$

$$= \textstyle\sum_n \exp[-i(H_{\mathrm{S}}\tau + H_{\mathrm{M}}\tau + \gamma QF)]\langle f_n|\psi(0)\rangle|\chi(0)\rangle|\psi(0)\rangle. \tag{3.9}$$

Now, because there is a finite difference in energy between the ground state and any excited state of the measured system, one must add a finite amount of energy to the

ground state in order to change it. According to the quantnum-mechanical adiabatic theorem (Schiff 1968, 289–291), if the energy added to a system is small enough and spread out over a long enough time, it is not additive (i.e., the total energy added does not get larger and larger), but adiabatically negligible. Indeed, the probability amplitudes for states other than the ground state can be made arbitrarily small, with suitably small γ and large τ. In other words, the term $\exp[-i\gamma QF]$ has no net effect on $|\psi(0)\rangle$, and we need consider its effect only on $|\chi\rangle$. (Note that there *is* a possible effect on $|\chi\rangle$ if we presume that the energy spectrum for $|\chi\rangle$ is continuous, or effectively so.) Hence (3.9) becomes

$$|\Psi(\tau)\rangle \simeq \sum_n |\phi_n(\tau)\rangle|\psi(\tau)\rangle := \ |\chi_n(\tau)\rangle|\psi(\tau)\rangle, \tag{3.10}$$

where we have defined $|\phi_n(\tau)\rangle = \exp[-i(\gamma f_n Q + H_{\mathrm{M}}\tau)]|\chi(0)\rangle$.

Notice that the state (3.10) is a product state—the interaction effects only the apparatus, and does not entangle the measured system with it. To see how this change in the state of the apparatus can be used to gain information about the system, recall (1.59). Taking the expectation value of $\mathbb{I} \otimes \Pi$ (where here Π is the momentum conjugate to Q) in the state $|\Psi(\tau)\rangle$ from (3.10), we find that

$$\frac{d}{dt}\langle\chi(\tau)|\mathbb{I} \otimes \Pi|\chi(\tau)\rangle = -g(t)\langle\psi(\tau)|F \otimes \mathbb{I}|\psi(\tau)\rangle. \tag{3.11}$$

In other words, the expected value of momentum for the apparatus is an 'indicator' of the expected value of F for the measured system. If the system is in a stationary state (i.e., $|\psi(t)\rangle = |\psi(0)\rangle$), then we could, for example, make many 'protective' measurements of F, measure the average value of the momentum of the apparatus after the interaction, and gain information about the expected value of F for the measured system.

There are two important remarks to make about this scenario. First, as many have pointed out, in order for the scheme to work, one must know the state of the measured system in advance. (In particular, in this case one must know that it is the ground state of the harmonic oscillator.) Otherwise, we will not know that it is 'protected' (i.e., will not change state as a result of the interaction). So there is an important sense in which the protective measurements do not yield any new information about the system. Moreover, it has been argued (Uffink 1999a) that only observables, F, that commute with the measured system's Hamiltonian can be measured in this way.

Nonetheless, despite these limitations, there remains the question of what exactly is going on in such an interaction. In the case of impulsive measurements (of the first kind, anyway), if we measure F on a system whose value for F was just measured,

we will gain no new information about the value of F for the measured system. Nonetheless, we might be tempted to explain the result of the second measurement thus: the measuring device interacted with the system in such a way that its indicator-state became correlated with the state of the system, and in particular with its value for F. That is, the second interaction was, again, a physically measurement, albeit one that was bound to give us no new information. One might, then, be tempted to say a similar thing about protective measurements. True, they tell us nothing that we did not already know. But how are we to understand what is going on during such interactions? One plausible understanding is that the apparatus' state is changing because it is somehow sensitive to the expected value of F on the measured system (even though, of course, we already *know* what that expected value is). Indeed, what else *could* explain the change in the apparatus' state?

Second, protective measurements are an explicit model of a more general class of measurements, so-called 'weak measurements'.[38] The general scheme (somewhat in parallel with the general scheme for standard measurements, of which impulsive measurement is one model) is as follows. Consider a quantum system known to be in the state $|\psi_1\rangle$ at time 0, and known to be in the state $|\psi_2\rangle$ at time t. Typically, this knowledge is obtained by what is often called 'pre- and post-selection'. That is, the state $|\psi_1\rangle$ is 'pre-selected' (prior to what will be the 'weak measurement' of F on the system) by performing a standard (e.g., impulsive) measurement on some ensemble of systems and selecting just those for which the result of this first measurement is $|\psi_1\rangle$. (In particular, one might measure the observable corresponding to $|\psi_1\rangle\langle\psi_1|$ then select just those systems for which the result is 1.) Then, *after* the 'weak measurement', one again performs a standard measurement on the ensemble, selecting just those systems for which the result corresponds to $|\psi_2\rangle$. The resulting ensemble of systems is said to be 'pre- and post-selected'. For any given observable represented by the operator F, define the 'weak value of F on the pre- and post-selected ensemble' by

$$\frac{\langle\psi_2|F|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle}. \tag{3.12}$$

Note that in the protective measurement discussed above, we presumed that the state of the measured system was unchanged over time, so that the weak value of F was just its expectation value in the state $|\psi(0)\rangle$.[39]

[38] An early paper on weak measurements is (Aharonov et al., 1987). See also note 39.

[39] The very notion of 'weak values', not to mention the interpretation of these values, is controversial. A recent discussion by the main proponent (and co-author), with plenty of references to prior work, both pro and con, is Aharonov and Botero (2005). Note that Aharonov and other proponents often discuss weak values in the context of a 'two-state-vector' formalism for quantum theory (which concerns pre- and post-selected systems and is supposed to be time-reversal invariant) that

There is, of course, a great deal more to be said about measurement. Later (§5), we will consider perhaps the most important philosophical issue concerning measurement, namely, the measurement problem (which has already made an early appearance in the form of our barely suppressed skepticism about the collapse of the state after an impulsive measurement). For now, however, we will rest content with the observation that, barring problems to arise later, quantum theory provides a rich framework in which to describe measurements.

3.2. **The Issue of Empirical Content in Terms of POVMs.** Describing measurements is part, but only part, of the story about how the formalism is connected with empirical observation. Another part of the story concerns a more general question about the connection between formalism and physical fact. For example, we have been allowing observables such as S_u to 'represent' spin in the u-direction, but what precisely *is* this relationship of 'representation'? How may the connection between formalism and physical fact be made, or understood? (Notice that the accounts of measurement above already presuppose an answer to this, more fundamental, question.)

It is crucial to understand that the issue here is not about how to engineer a spin-measuring device, for example. Rather, it is about what it *means* to 'have' spin-up in the u-direction (for example), and how this meaning is captured in the formalism. Supposing that there are no limitations of engineering, there remains a question about what laboratory procedures correspond to 'measuring S_u', for example. Below, we will offer a partial answer to this question.

Recall that, considered as POVMs, observables are maps from (Borel sets of) 'the possible values' to positive operators. Another way to put the question above, then, is in terms of the empirical meanings of the mathematical elements in the domain of this map. (Once this question is answered, then, for example, the probability calculus associated with the elements in the range of the map becomes a calculus with empirical content.) Indeed, one advantage of conceiving of observables as POVMs (apart from the greater generality of this approach) is that it affords greater precision to a discussion of the issue of which parts of the formalism are, in which contexts, doing the work of 'representation'. For a POVM $E : \mathcal{B}(S) \to \mathcal{B}(\mathcal{H})^+$, the elements of $\mathcal{B}(\mathcal{H})^+$ are doing the representing, and the elements of $\mathcal{B}(S)$ (indeed, ultimately, S) are, in a sense, what is being represented.

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is itself controversial; however, the notion of a weak value is not irrevocably tied to that approach to quantum theory, but only to the idea of a pre- and post-selected ensemble, the operational meaning of which, at any rate, is clear enough.

Why we represent with elements of $\mathcal{B}(\mathcal{H})^+$ is a question that we have addressed already in §2 (though certainly it has not been conclusively answered). But what is the physical interpretation of the domain of the map E, and how are we to understand the statement that some element in its domain is 'represented by' some element in its range? What, in other words, is the relationship between the elements of the mathematical formalism that we have described and physical matters of fact? And finally, why do we pick one map (POVM) rather than another to represent some given physical quantity? The next section (§3.3), on symmetries, and the following one (§3.4), on reference frames, introduce material that will eventually contribute to a sketch of a story (§3.5) about the empirical content of quantum theory, one that addresses, or suggests ways to address, some of the issues raised in this section.

3.3. **Symmetries.**

3.3.1. *Groups as Symmetries.* There is a traditional account of one way that groups have been related to empirical content.[40] Take any group, G, and consider its action on a set, S. If two elements of S are connected by an element of G, then call them 'equivalent'. One can readily verify that G thus paritions S into equivalance classes, and we can say, then, that G is a group of 'symmetries' on S, in the sense that elements of S connected by an element of G are in some important sense 'the same'. (If, according to some theory of the elements of S, distinct elements within the same equivalence class in S can have 'importantly' distinct properties, then in this theory, G would arguably *not* be a symmetry. What one means by 'important' is subtle, but in the context of this discussion it could, for example, mean 'empirical', or 'observable'.) For example, let the elements of S represent the positions of all of the particles in the universe (i.e., S is a configuration space for the universe). A spatial translation of a point in S results in a universe that is, arguably, no different, empirically (because all distances and other spatial relations amongst the particles stay the same).

3.3.2. *Groups and Observables.* The (outline of an) approach to empirical content that we shall propose is given, in part, in terms of groups of transformations on S (considered as the domain of a POVM, E), and the requirement that E in a sense preserve the behavior of S under those transformations.[41] To get this view off of the

[40]Here I am considering groups primarily as transformations on the set of states. One can also think of them in terms of formal transformations of physical laws. For more discussion of this and related points, see Brading and Castellani, Ch. 13, this volume.

[41]The view described here has been strongly influenced by discussions with Scott Tanona, and by Tanona (2002; 2006). (Tanona's view is distinctive, and differs from the one presented here in important ways.) Indeed, it is largely due to those discussions that I began to develop a view about this matter.

ground, we require (at least!) the following two things of a quantum theory: (1) a faithful representation of the relevant groups on a Hilbert space, and (2) a map from $\mathcal{B}(S)$ to $\mathcal{B}(\mathcal{H})^+$ that in some relevant sense 'preserves' the action of these groups.

Let us consider these points in relation an example, the Galilean group, $\mathcal{G}$, and more specifically, the spatial translations, $\mathcal{A}$, and boosts, $\mathcal{V}$ (i.e., ignoring rotations and temporal translations). For any $a \in \mathcal{A}$ and $b \in \mathcal{V}$, let U_a and V_b be the corresponding elements of a faithful representation of $\mathcal{A}$ and $\mathcal{V}$, respectively. It is, at this point, an open question whether there *exist* faithful representations of $\mathcal{A}$ and $\mathcal{V}$ on a given Hilbert space, i.e., whether requirement (1), above, can be met. On an infinite-dimensional space, there are in fact operators P (momentum) and Q (position) that do the job:

$$\begin{aligned} U_a &= e^{-iPa} \\ V_b &= e^{-iQmb}, \end{aligned} \tag{3.13}$$

respectively, where m is the mass of the particle, and appears here because momentum is mass times velocity.[42]

Notice that positions are translated by the action of $\mathcal{A}$ and invariant under the action of $\mathcal{V}$. The converse holds for velocties (and therefore, of course, momenta).

Now, let $E_Q : \mathcal{B}(\mathbb{R}^3) \to \mathcal{B}(\mathcal{H})^+$ be the POVM for position and consider the action (§7.6.7) of $\mathcal{G}$ on $\mathbb{R}$ (or $\mathbb{R}^3$—see note 42), understood as representing *positions* of a particle.[43] In that case, the requirement (2) above amounts to the requirement that the quantum-theoretic representation of position have the same properties:

$$\begin{aligned} U_a E_Q(\Delta) U_a^{-1} &= E_Q(\alpha_a(\Delta)) \quad \text{(covariance)} \\ V_b E_Q(\Delta) V_b^{-1} &= E_Q(\Delta) \qquad\quad \text{(invariance)} \end{aligned} \tag{3.14}$$

for any $a \in \mathcal{A}$ and any $b \in \mathcal{V}$, where α_a is the action of a on $\mathcal{B}(\mathbb{R}^3)$. Corresponding conditions must hold for E_P, the POVM for momentum (i.e., it should be invariant under translations and covariant under boosts). Note that at this point, we do *not* presume that the P and Q in (3.14) are those from (3.13). Indeed, these conditions (3.14) turn out to be powerful enough on their own to determine the maps E_Q and

[42] One normally says that position is the 'generator of translations' and momentum is the 'generator of boosts'. The reason ultimately has to do with the fact that the momentum (position) operator is involved in the expression for an infinitesimal translation (boost). The expressions in the text for finite translations and boosts are essentially integrals of their infinitesimal counterparts. Note, also, that (3.13) is given in one spatial dimension only. Replacing P and Q with the 'vectors' of operators $\vec{P} = (P_x, P_y, P_z)$ and $\vec{P} = (P_x, P_y, P_z)$, and the parameters a and v with vectors (from $\mathbb{R}^3$) $\vec{a}$ and $\vec{b}$, we would have the three-dimensional version. Note, for example, that these changes do not change anything about the definition of a continuously parametrized group in §7.6.4.

[43] $\mathcal{G}$ naturally acts on the phase space in classical physics, but we can of course consider its action on the reduced space of position or velocity.

E_P, as well as the representations (3.13) (up to unitary equivalence). We shall return to this point below. Here, we take it as motivation for *defining* 'position' and 'momentum' in terms of (3.14) (and the corresponding conditions for momentum).

One might have either of two objections at this point. First, where did these requirements come from? Why do we insist that position must have this particular invariance and covariance, even in the classical case, or at all? The answer is that, at least in this discussion, we are taking covariance with respect to $\mathcal{A}$ and invariance with respect to $\mathcal{V}$ to *define what we mean* by position. For example, if the position of a particle is $\vec{x}$ in a reference frame F (below we will consider in more detail the role of reference frames in this discussion), and if frame F is related to frame F' by a spatial translation $\vec{a}$, then the position of the particle in frame F' is $\vec{x} - \vec{a}$. The seeming triviality of this feature of position is, one might suppose, a consequence of the fact that it is part of what 'position' *means*. That is, an observable (POVM) that lacked this feature would, *ipso facto*, not be 'position'. Similar remarks will hold for boosts, and again for the relationship between momentum on the one hand and translations and boosts on the other. Corresponding remarks hold for other observables, such as angular momentum and spin.

Second (objection), by insisting that position 'mean the same thing' in the context of quantum theory (i.e., that the POVM respect, in the relevant sense, the action of the (representation of the) Galilean group), are we not thereby preventing ourselves from learning, perhaps, that position is 'very different' from what we thought it was? (A similar remark holds, of course, for momentum, and indeed for the hosts of other physical quantities that can be defined in this way.) There are two answers to this objection. First, in the context of quantum theory (where the position and momentum POVMs *do* obey the 'correct' invariances and covariances), we *do* in fact learn that position is 'very different' from what we thought it was, and this fact already makes it clear that we have not so narrowly restricted our definition of position that substantial modifications to our existing conception of it become impossible. The second answer is that we should distinguish between, on the one hand, learning something new about an existing physical concept—in which case *something* must make it 'the same' concept both before and after we learned something new about it (and we are proposing that what is the same is its relationship to certain parts of the Galilean group)—and, on the other hand, discovering new physical concepts. We have no proposals to make here about how such discoveries occur, or how to understand them.[44]

[44]Tanona (2006) is particularly helpful on this point.

In the remainder of this subsection, I will discuss in further detail the claims made above, that insisting on the relevant invariances and covariance is sufficient to fix the position and momentum observables. In the next subsection, I will turn to the role of reference frames in the definition of observables.

3.3.2.1. *Systems of Imprimitivity and the 'Uniqueness' of Quantum Observables.* Position and momentum, considered as POVMs, each give rise to a 'system of imprimitivity', a special case of a system of covariance.[45] Generically, a system of covariance is a set $(\mathcal{H}, E, S, \mathcal{G}, \alpha, \{U_g\})$ where $\mathcal{H}$ is a Hilbert space, E is a POVM whose domain is S and whose range is positive operators on $\mathcal{H}$, $\mathcal{G}$ is some group, α is the action of $\mathcal{G}$ on S, and $\{U_g\}$ is a unitary representation of $\mathcal{G}$ on $\mathcal{H}$. If E is a PVM then the set is a *system of imprimitivity*.[46] Systems of imprimitivity have important properties, in part summarized by Mackey's imprimitivity theorem. My discussion here will follow that of Mackey (1996), emphasizing the structure and assumptions of the argument, rather than the mathematical details, which can easily be found in many places.[47]

In a system of imprimitivity, S is often taken to be quite generic — e.g., it might be a generic metric space with $\mathcal{G}$ some locally compact and separable group of isometries (assumed to have a continuous and transitive action on S—see §7.6.7). However, we will immediately specialize, in order to make quicker and more evident contact with real physical concerns. With that goal in mind, it is natural to take S to be $\mathbb{R}^3$ and $\mathcal{G}$ to be, for example, the semi-direct product (§7.6.2) of translations and rotations ($\mathcal{A} \ltimes \mathcal{R}$). However, in preparation for an application of the imprimitivity theorem, it is more useful to take S to be the topological group of translations ($\mathcal{A} = \mathcal{G}/\mathcal{R}$), which is clearly isomorphic to $\mathbb{R}^3$ as a topological space (indeed, as a metric space, given a suitable, and obvious, metric on $\mathcal{A}$). The idea, then, is that elements of S represent 'displacements' from some fixed origin, and thereby represent a position (and so are possible values of a position observable). The subgroup $\mathcal{R}$ describes rotations around this origin.

We now require that a PVM for position, E^Q, be covariant with respect to $\mathcal{G} = \mathcal{A} \ltimes \mathcal{R}$. (The action of any $g \in G$ on $S = \mathcal{A}$ is defined in the obvious way: for $g \in \mathcal{A}$ its action on $a \in \mathcal{A}$ is just ga; $g \in R$ acts as the identity on $\mathcal{A}$. The action thus

[45] See Landsman, Ch. 5, this volume, for further discussion of imprimitivity and its application to issues in quantum theory.

[46] Systems of covariance can be 'dilated' to systems of imprimitivity via the Neumark dilation theorem. See, e.g., Cattaneo (1979). It follows, in essence, that we lose no generality by considering systems of imprimitivity.

[47] In addition to Mackey (1996), see Busch, Grabowski and Lahti (1995) and references therein for additional mathematical details and alternative routes to the same conclusion. Full details are available in Varadarjan (1985).

defined is transitive.) Finally, given a representation $\{U_g\}_{g\in\mathcal{G}}$ of $\mathcal{G}$ on some Hilbert space, $\mathcal{H}$, we have a system of imprimitivity.

In one form, the imprimitivity theorem is the following.

> Theorem (Mackey): Let $\{U_g\}_{g\in\mathcal{G}}$ be a unitary representation of a separable, locally compact, topological group, $\mathcal{G}$, on a separable Hilbert space, $\mathcal{H}$, and let $\mathcal{K}$ be any closed subgroup of $\mathcal{G}$. Let E be a PVM whose domain is $\mathcal{G}/\mathcal{K}$ such that $(\mathcal{H}, \mathcal{G}/\mathcal{K}, \mathcal{G}, \alpha, \{U_g\}, E)$ is a system of imprimitivity (with α a transitive action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{K}$). Then for any representation, $\{V_k\}_{k\in\mathcal{K}}$ of $\mathcal{K}$ on some Hilbert space, $\mathcal{H}'$, the representation of $\mathcal{G}$ induced by $\{V_k\}$ on the Hilbert space $L^2(\mathcal{G}/\mathcal{K}) \otimes \mathcal{H}'$ (and this induced representation exists) is unitarily equivalent to $\{U_g\}$. Moreover, $E(\Delta)$ (or its appropriate unitary transform, if the induced representation is related to $\{U_g\}$ by a non-trivial unitary transformation) must be the tensor product of the multiplication operator χ_Δ on $L^2(\mathcal{G}/\mathcal{K})$ with the identity on $\mathcal{H}'$.

So in the case we are considering, we will let $\mathcal{H}$ be $L^2(\mathbb{R}^3)$.[48] Then let $\{D_r\}_{r\in\mathcal{R}}$ be a representation of $\mathcal{R}$ on some Hilbert space, $\mathcal{H}'$, and consider the associated induced representation of $\mathcal{G}$. This induced representation must be unitarily equivalent to *any* representation of $\mathcal{G}$, and moreover, $E^Q(\Delta)$ is just $\chi_\Delta \otimes \mathbb{I}'$.

Now consider the simplest case, where $\{D_r\}$ is the trivial identity representation (i.e., every element of $\mathcal{R}$ is represented by the identity on a 1-dimensional Hilbert space). Mackey's theorem immediately yields the usual Schrödinger representation of the position operator.[49] More explicitly, for $\Delta \in \mathbb{R}^3$, $E(\Delta) = \chi_\Delta$, where the latter is the operator with action 'multiply an element of $L^2(\mathbb{R}^3)$ by the characteristic function of Δ'. (This operator is a projection.) Integrating over all of S with respect to this PVM yields the usual position operator:

$$Q = \int_{\mathbb{R}^3} \vec{r}\, dE^Q(\vec{r}), \tag{3.15}$$

which is a 'vector' of operators, Q_x, Q_y, Q_z, with the action, for $\vec{r} = (x, y, z)$, $Q_x\psi(\vec{r}) = x\psi(\vec{r})$ for any $\psi(\vec{r}) \in L^2(\mathbb{R}^3, d\vec{x})$, and similarly for Q_y and Q_z.

Notice what has happened, here. We began with the requirement that the PVM representing position have the 'correct' co-variances with respect to translations and rotations—i.e., that it be a part of the relevant system of imprimitivity—and we

[48] More precisely, we should use the topological group of translations in place of $\mathbb{R}^3$, but we already noted that for our purposes these are the same spaces.

[49] The result is a description of a spinless particle—see Mackey (1996). Non-trivial representations of $\mathcal{R}$ result in the description of particles with spin $\frac{\dim(\mathcal{H}')-1}{2}$.)

ended up, via Mackey's theorem, showing that up to unitary transformation, we *must* choose the usual ('Schrödinger') representation of position. Another, somewhat more operational, way of putting the point is this: assuming that observations (or predictions) involving the position observable respect the symmetries (translations and rotations) of the space of possible values of position, all representations of the position operator (PVM) are unitarily equivalent to the Schrödinger representation. In terms of establishing the empirical meaning of the theory, we might say that what it *means* to 'be the position' of a system is to transform in the right way under the action of the relevant pieces of the Galilean group. Mackey's theorem establishes the uniqueness (up to unitary transformation) of position, so defined.

Indeed, Mackey's theorem establishes more. It also establishes the relationship between the representation of (i.e., choice of a POVM for) position (and momentum) and the representation of the Galilean group. To see why (in outline), let $\mathcal{G}$ be any group continuously parametrized by $a \in \mathbb{R}$ (§7.6.4). Now, as we have already discussed, if $\mathcal{G}$ is supposed to be a *symmetry* group, then in general it should be represented, quantum-mechanically, in terms of transformations of the Hilbert space that 'make no difference, physically', and such transformations are often given in terms of unitary operators. (Recall the discussion from §1.5.1.2.) Moreover (recall §1.5.1.4), when the group is continuously parametrized by a (as it will be in the cases of interest for us), these unitary operators are given in terms of a self-adjoint operator, F, on $\mathcal{H}$ such that $U_a = e^{-iFa}$. Hence, for example, the unitary representation of spatial translations and boosts must take this form. Mackey's theorem then implies that, in fact, translations and boosts are given by (3.13) up to unitary equivalence.

It is important to keep in mind that 'up to unitary equivalence' does *not* mean that one can apply *different* unitary transformations to the U_a and V_v (from eq. 3.13) and still satisfy all of the (invariance and covariance) conditions that have been placed on the POVMs for position and momentum. The point here is that the *relationship* between position and momentum is established by those conditions via Mackey's imprimitivity theorem. (On the other hand, one can always apply a *global* unitary transformation, but such a transformation is akin to 'translating the universe five feet to the right' in classical mechanics.)

Indeed, one can establish, from these results, the conclusion that position and momentum must obey the Weyl form of the commutation relations.[50] In particular,

$$U_a V_b = e^{iab} V_b U_a, \quad (a, b \in \mathbb{R}). \tag{3.16}$$

[50]Consult Varadarajan 1985, ch. V or Mackey (1949; 1978) for details.

As we will discuss later (§4.1.2), this expression is a version of the canonical commutation relation (1.24) between P and Q. In other words, the assumption that position and momentum bear the right relation to the Galilean group leads directly to their incompatibility.

Hence the role that the position and momentum *operators* (POVMs) play in the Galilean group, the action of the Galilean group on them, and the (Weyl form of the) commutation relations between them, are fixed as soon as we insist on two things: (1) that position and momentum satisfy the invariance and covariance conditions given above, and (2) that position and momentum, as well as the Galilean group itself, be represented on a Hilbert space. There seems to be little room to deny (1) in a non-relativistic theory, and (2) may be understood as the requirement that our theory be genuinely quantum-mechanical.[51] Finally, one should bear in mind the *generality* of Mackey's theorem. I have discussed it largely in the context of position and momentum, but analogous accounts hold for any observables (POVMs) that form a system of imprimitivity (or covariance) with regard to some group of symmetries. Some examples (among them, angular momentum and spin) can be found throughout Busch et al. (1995). We leave the reader to investigate, and turn now to consider the role of reference frames in the definition of physical quantities in quantum mechanics.

3.4. **Reference Frames.**

3.4.1. *Identification and Role of Reference Frames in Quantum Theory.* It has been claimed (not unreasonably) since the early days of quantum theory that there is no room in the theory for the notion of a reference frame. The apparent difficulty is this: a reference frame, *by definition*, has a well-defined location and state of motion, because locations and states of motion are defined relative to it. But then it is hard to see how there could be any such thing as a quantum reference frame, because, as I have mentioned already (§1.2.5), and shall discuss in some detail below (§4), standard quantum theory cannot describe anything as having a well-defined location

[51] We must be a little careful, here. (Thanks to Jos Uffink for raising this question.) It is well known that classical mechanics can be represented as a theory on a complex Hilbert space. See Bracken (2003) and references therein, the originator of the idea being, apparently, Groenewold (1946). However, the classical observables on Hilbert space form (unsurprisingly) a commutative algebra, in virtue of the definition of a 'non-standard' product of linear operators (the 'odot' product in Bracken (2003)), one that does not make an appearance in quantum theory. (The dynamics, for example, are defined in terms of a Lie bracket defined in terms of this 'odot' product rather than in terms of the usual product of operators given by the composition of their action on the space.) Hence the more careful statement of the claim in the text is that we require position and momentum to be operators in the algebra of operators on a Hilbert space *as standardly understood*, that is, under the usual product of operators (composition).

and state of motion (momentum). Bohr (on one reading[52]) concluded that it is up to *us* to *stipulate* some object (normally, a measuring apparatus) as defining a reference frame, and that this stipulation requires us to treat the object classically, because the stipulation requires the object to be well-defined in position and momentum. (Of course, we can also step back and describe that object quantum-mechanically again, having stipulated some *other* object to serve the role of a reference frame.)

However, there are good and bad stipulations—as Bohr himself emphasized, not every object is reasonably taken to define a frame of reference for a given purpose. One important reason is that there is a prima facie distinction between inertial frames and reference frames. Inertial frames are the frames in which the laws of motion (whether classical or quantum) are valid.[53] It is a matter for empirical investigation to seek out inertial frames. Reference frames, on the other hand, are the frames used to define physical quantities, such as position, momentum, angular momentum, spin, and so on. For the instantaneous definition of these quantities, *any* stipulation of a reference frame is fine.

However, the choice of a reference frame for the description of systems *over time* is far from arbitrary. For example, a rotating coordinate system used as a reference frame will introduce fictitious Coriolis forces. Of course, by 'rotating' one means 'rotating relative to an inertial frame', and herein lies the crucial point: a reference frame that is not inertial will always introduce fictitious forces (i.e., apparent violations of the laws of motion). Hence, in the end, although it is certainly permissible to describe our physical systems with respect to whichever reference frame is most convenient, it is also necessary that we know *how* to describe the system in terms of a reference frame that is inertial, by which we mean, here, one in which the laws of motion—classical or quantum, as the case may be—are true.

Nothing in quantum theory rules out the possibility of an inertial frame in this sense. Indeed, just as classical physics does, quantum physics contains an assumption (usually left implicit) that there is some frame (some system of coordinates) in which the laws are valid. (It does not follow that such a system of coordinates can be

[52]See Bohr (1935). For some (albeit flawed) interpretive remarks on this paper, see Dickson (2002a; 2002b). Recent detailed interpretations of Bohr's insistence on the necessity of classical concepts can be found, for example, in Tanona (2002) and Howard (2003). Of course, many others have also written on this topic. The references in those works will get the interested reader started.

[53] There is a long history behind this understanding of what it means to be inertial. See DiSalle (1990; 2002) and Barbour (1989). It culminates in the idea that Newton's law of inertia should be understood as the claim that there *is* a frame of reference, an 'inertial frame', in which the other two laws are true. This idea can be extended to quantum theory.

used as a reference frame in the *classical* sense, i.e., to define, simultaneously, definite positions and momenta.)

In quantum mechanics, as in classical mechanics, one goes about finding an inertial frame by searching for coordinates in which the dynamical laws are true. In classical mechanics, this search often extends to frames defined in terms of celestial bodies (Ma et al. 1998). In quantum theory, one does not go to such lengths. Typically measuring devices, or the labs that contain them, are sufficient to serve the purpose of defining a reference frame.

3.4.2. *Relational and Absolute Coordinates.* If it is correct that quantum observables are typically defined relative to a frame of reference (whether inertial or otherwise), then typically, in quantum theory, the observationally significant variables will be relational in character.

This point does not apply only to the case of measurements involving position (or $L^2(\mathbb{R}^{3N})$ spaces) explicitly. Consider spin. What, for example, does it mean *observationally* to say that a system is in the state $|\psi\rangle = |z_+\rangle$? If we are not told *which* direction in space counts as z then the claim that a system is in that state is observationally empty.

However, we saw above that when we wish to apply the quantum laws of motion to a system, we must do so in coordinates given by some frame in which the laws are valid. We also mentioned that in quantum theory, this frame is often determined by some macroscopic piece of apparatus. What are the conditions that must be met by this macroscopic piece of apparatus in order for it to serve the purpose of defining coordinates in which the laws are valid? And how are we ever in a position to verify that those conditions are met?

In the end, the answer to the last question is the same as in classical mechanics—we never have access to any 'absolute' frame, one that is known to be inertial, from which we can check the inertiality of other frames (and hence their suitability to define coordinates in which the laws are valid). The best we can do is to determine, empirically, as best we can, that the laws are valid in some particular frame, F, and then justify the use of other frames by reference to F, and in particular by noting that these other legitimate frames are related to F by an appropriate symmetry transformation.

In other words, at least until we have a truly relational theory on hand, the relational coordinates are ultimately defined in terms of the absolute coordinates, and

they are so defined in terms of certain symmetry transformations, as we shall now describe.[54]

Let us begin by recalling that the transformations from one inertial frame to another are (in the present non-relativistic contex) given by the Galilean transformations. (Moreover, we have seen that we have little choice about the mathematical form that these transformations take in quantum theory.) We can use that fact to derive an expression for a transformation from the absolute coordinates of some given inertial frame to coordinates measured relative to some stipulated reference frame.

A helpful way to consider the situation is as follows. Imagine an observer, A, inside a lab and suppose that A measures physical quantities relative to the lab. Now imagine an 'external' observer, B, who has been *given* the information (or has assumed it) that some frame, F, is inertial. B also uses F as a reference frame: as far as B is concerned, the lab and its contents are all described in the coordinates given by F. But suppose B wishes to describe A's measurements *as relational* and *in the frame given by the lab*. How does B transform from the coordinates given by F to the (relational) coordinates given by the lab (which could also be moving relative to F)?

The answer follows more or less immediately from the form of the Galilean transformations. (Of course, we are not simply applying a Galilean transformation in this case. We are also transforming to relational coordinates.) As Aharonov and Kaufherr (1984) point out, the correct transformations are

$$U_{\mathrm{AK}} = e^{-i\sum_{n>0} P_n^B Q_0^B} \tag{3.17}$$

where P_n^B represents the momentum observable used by B to describe the momentum of system n, and Q_n^B similarly represents B's position observable. System 0 is the lab itself. (Note that their result implicitly assumes that the reference body—the 'lab'—moves inertially in F. Notice also that in this situation, the lab drops out of view. A has no coordinates to describe the lab (system 0) because A's coordinates are all defined *relative to* the lab.)

As a quick check, note that

$$U_{\mathrm{AK}} Q_n^A U_{\mathrm{AK}}^{-1} = Q_n^B - Q_0^B \quad (n > 0), \tag{3.18}$$

so that, as expected, what A describes bt Q_n^A, B 'knows to be' $Q_n^B - Q_0^B$. That is, B can describe, in purely quantum-theoretic terms, the fact that A's measurements of position are made relative to the lab.

The main lesson of this discussion, for us, is this: in general, we are in the position of observer A, not that of B. We are not *given* an absolute frame. Instead, our

[54]The discussion here is motivated primarily by the work of Aharonov and Kaufherr (1984).

quantities are measured (and therefore, operationally at least, defined) relationally. Nonetheless, the symmetries that take us from one frame to another are ultimately involved in the definition of those quantities, because in order for our physical theories legitimately to apply to them, we must consider them to be 'really' defined in terms of the coordinates of an 'absolute' inertial frame, and such a definition involves the symmetry transformations that take us from one (inertial) frame to another.

An interesting side-point here is the following. Observer B (whose point of view is, in principle, the truly legitimate one, here), has a particularly interesting way of noticing the incompatibility of position and momentum. Suppose A is going to measure the position of a particle, and writes down an interaction Hamiltonian along the lines of the model of impulsive measurement (§3.1.2). Of course, A just writes down something like $H_{\mathrm{I}} = g(t)\Pi Q$, where Q is A's position operator for the measured system. When B transforms this Hamiltonian to the *correct*, i.e., *relational* coordinates, and solves the equation, B finds that as a result of the measurement, the *lab itself* experiences a shift in momentum, which renders the lab unsuitable (for A) for defining momentum (because A has no way to measure the shift in the momentum of the lab—the lab is the reference frame relative to which any such measurement could be made, for A).[55]

3.5. **A Group-Theoretic Characterization of Empirical Content.**

3.5.1. *Reframing the Issue.* How does the discussion of §3.3 and §3.4 help to establish an account of the empirical content of the quantum formalism? Of all of the questions raised in §3.2, let us focus on the following two: (1) What empirical meaning is attached to the elements in the domain of a POVM? (2) Given an answer to (1), how is the POVM itself chosen appropriately?

The discussion above suggests the following general approach. (a) Observables are frame-dependent quantities that are *defined* in terms of their behavior with regard to some group of symmetries. (b) Having pointed out that legitimate reference frames ought to be related in the right way to inertial frames—more generally, they ought to be suitable for defining the quantities that we indeed wish to define—we ought then seek to attach some empirical, observational, significance to such things. (c) Similarly, having pointed out the role that symmetry transformations play in the definition of physical quantities, we ought then to seek to attach some empirical, observational,

[55] See Dickson (2004b) for details. It is not sufficiently emphasized there that these observations are at best a first step towards understanding the incompatibility of position and momentum. Note, moreover, that nothing in B's calculations implies the uncertainty relations, understood as placing a lower bound on the precision with which position and momentum can be simultaneously measured, or known, or defined.

significance to such things. A more detailed version of (a) would constitute an answer to question (2) above. More detailed versions of (b) and (c) would constitute an answer to question (1) above.

In fact, we have already said something about (a), and considerably more has been said in the literature about the symmetries that are obeyed by (and thus that are definitive of) quantum-mechanical observables. In any case, the basic point is that the (reference-frame-dependent) observables may be uniquely characterized (up to trivial transformations, such as scaling of length, and global unitary transformations) by the invariances and covariances that they obey with respect to some appropriate symmetry groups. In this sense, the very meaning of 'position', 'momentum', and so on, is partially given by these invariances and covariances. I conclude this section with some preliminary thoughts about (b) and (c), followed by the consideration of an objection.

3.5.2. *The Empirical Content of Frames and Transformations.* Concerning (b), the basic proposal here is that reference frames describe the world as witnessed by some observer, and ultimately, some human being. This proposal reflects a point of view according to which theories are, ultimately, human constructions. This point of view does not (necessarily) include the idea that there is any (or much) arbitrariness in physical theory—the world might still dictate how observational creatures such as ourselves are bound to construct theories, if we are to be successful. It does entail that the 'observables' of a theory are intimately connected with the observational capacities of human beings, and with the properties of those capacities.

However, those who would subscribe to such a view must be careful, for at least two related reasons. First, reference frames are typically idealized in various ways that might not apply to actual human observation. They are, for example, typically taken to be entirely rigid (spatially). Second, if a given reference frame is to be used over any stretch of time, then either it must be inertial, or one must know how to relate it to an inertial frame, in order legitimately (and successfully) to apply the law of motion (whether that be Newton's second law, Schrödinger's equation, or something else). But as noted earlier, in actual practice it is very difficult to determine whether a given frame is inertial. Nonetheless, the proposal being floated here is that ultimately reference frames should be understood as 'legitimate (and idealized) human points of view', where the notion of legitimacy is to be spelled out in terms of a known connection with an inertial frame.

Concerning (c), the empirical content of the symmetry transformations themselves, I again offer a kind of anthropocentric view. The suggestion, coming from various

19th century philosophers of geometry (for example Helmholtz, and in a different way, Poincaré), is that such transformations are connected to experience via the physiological-kinesthetic experience of undergoing the change from observing from one frame of reference, to observing from another. (Consider, for example, the physiological-kinesthetic experience associated with a rotation, or spatial translation.) However, the connection of specific groups of transformations with empirical content of various sorts is far beyond the scope of this essay. We merely note that an essential ingredient in the program outlined here is establishing (or understanding) such connections, for the groups that are at the heart of quantum theory.[56]

3.5.3. *'Absolute' Quantities.* What about quantities that are *not* reference-frame dependent? Surely one of the lessons of relativity theory is that while many measured quantities are frame-dependent, some physical quantities—perhaps even the most important ones—are 'absolute', i.e., frame-independent. Think, for example, of the spacetime interval, defined by $\tau^2 := t^2 - (x^2 + x^2 + x^2)$. It is frame-independent, in the sense that *whatever* coordinates one uses to calculate τ, the result is always the same. Indeed, one understanding of *non*-absolute quantities is that they are nothing more than the absolute quantities seen 'from a particular perspective', so that the absolute quantities are somehow fundamental, the frame-dependent ones derived.

But how does one *measure*, or *observe* the value of an absolute quantity? The claim made here is that we cannot help but do so from within some reference frame. While the *result* is not dependent on the reference frame, the measurement still occurs in one. (Consider, for example, how one might measure τ.) If this claim is correct, there there are two responses to the point that some quantities are absolute.

The less radical response is to allow that the absolute quantities might even be more fundamental in some sense. However, we are here concerned with the *observational* content of quantum theory—how does the mathematical formalism connect with experimental observation? If, as claimed above, observation always takes place within a frame, then it is fair enough for an account to consider the empirical content of just the frame-dependent quantities.

The more radical response is to assert the reverse of the point of view mentioned above: the fundamental quantities are the frame-dependent ones, and the 'absolute' quantities are derived (calculated) from them. This view is in fact closely connected with the view, expressed above, that (our) physical theories are fundamentally about

[56]One could adopt an approach completely opposite to the one suggested here, one that takes a theory as somehow 'already' empirically meaningful *prior* to the empirical account of the relevant groups of transformations, and then define the empirical content of those transformations in terms of their consequences within the theory. I shall not follow that idea through here.

the world *as observed by us*. On this view, the role of the absolute quantities is to underwrite the possibility of communication amongst inhabitants of different frames about the values of their frame-dependent quantities; it makes possible some sort of agreement; and that is all.

4. Uncertainty

This section is devoted to an examination of uncertainty in quantum theory. We will begin (§4.1) with the formal origins of the uncertainty relations, namely, the canonical commutation relations. We will then consider the uncertainty relations (§4.2), which, minimally, express the fact that the dispersion of two non-commuting observables (a notion that we will define) cannot be made simultaneously arbitrarily small. We will then (§4.3) consider two fundamentally different ways to understand, or interpret, the uncertainty relations, and finally (§4.4), we will consider in some detail the famous argument by Einstein, Podolsky, and Rosen (1935), intended to cast some form of doubt on the fundamentality of quantum uncertainty, and ultimately on the 'completeness' of quantum theory as a description of physical reality.

4.1. **Canonical Commutation Relations.**

4.1.1. *Representations of the Commutation Relations.* Earlier (§1.2.5) we notes that the quantum-mechanical position (Q) and momentum (P) operators obey the CCRs (1.24). It is perhaps more accurate to say that in quantum theory, Q and P are *chosen* so that (1.24) is obeyed. Choosing operators, Q and P, on some Hilbert space, $\mathcal{H}$, such that (1.24) is satisfied is choosing a *representation* of the CCRs. It is, arguably, the satisfaction of these commutation relations by (the operators that stand for) certain pairs of observables—centrally, position and momentum—that makes a theory truly 'quantum'.

It turns out that for any representation, the operators cannot both be bounded, and therefore $\mathcal{H}$ must be infinite-dimensional (§7.2.2). Heisenberg constructed a representation on the space ℓ^2 (§1.2.1) in terms of infinite-dimensional square matrices:

$$Q=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad P=\frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & -\sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.1}$$

Schrödinger constructed a representation in terms of operators on the space $L^2(\mathbb{R})$ in which Q is the multiplication operator (i.e., $Qf(x) = xf(x)$ for any $f \in L^2(\mathbb{R})$) and $P = -i\frac{d}{dx}$.

These two representations are in fact unitarily equivalent. (I.e., there exists an isomorphism from ℓ^2 to $L^2(\mathbb{R})$, under which the Heisenberg operators go to the Schrödinger operators.) Any representation that is isomorphic to these is called *regular*. Non-regular representations exist.[57]

Note that because (at least one of) the operators in a representation of the CCRs must be unbounded, we must be careful about keeping track of their domains of definition (§7.2.2). The CCRs are thus defined only on some (dense) subset of the space.

4.1.2. *The Weyl Relations.* The fact that P and Q must be unbounded is occasionally bothersome. For example, we just noted that this fact requires one to keep track of their domains of definition. An alternative approach, due to Weyl, avoids the problem. We begin by considering a pair of strongly continuous one-parameter unitary groups of operators U_a and V_b, $a, b \in \mathbb{R}$ (§7.6.4). We will call them a *Weyl pair* if they satisfy the relation (3.16) from §3.3.2.1. By Stone's Theorem (§1.5.1.4), U_a and V_b can be written as

$$U_a = e^{-iaQ}, \qquad V_b = e^{-ibP} \tag{4.2}$$

where Q and P are unbounded selfadjoint operators, defined on a common (dense) domain. (Recall eq. 3.13.) Writing these exponentials formally, in terms of a power series expansion,

$$e^{-iaQ} = \sum_{n=0}^{\infty} \frac{(-iaQ)^n}{n!}, \tag{4.3}$$

(similarly for e^{-ibP}) and substituting into (4.2) we retrieve the CCRs. (If both sides of (4.2) are defined on a common dense subspace, then this procedure is rigorously valid. Otherwise, it is suggestive symbol-manipulation.) Note, finally, that the operators e^{iaQ} and e^{ibP} are bounded, so that, for example, issues about the domain of definition do not arise.

4.1.3. *Von Neumann's Uniqueness Theorem.* There is another nice feature of the Weyl relations: all representations of them are regular. In other words, every Weyl pair is unitarily equivalent to the Weyl pair generated by the Schrödigner position and momentum operators. This result, due to von Neumann (1931, implies that the generators of any Weyl pair must have spectra that are the entire real line.

[57]Here is an easy example. Consider the space of square-integrable functions on the open interval $(0, 1)$. Let Q and P again be the multiplication and differentiation operators (as given above). In this case, Q is in fact bounded (but P is not), and so this representation cannot be unitarily equivalent to the Schrödinger representation, because, in that case, both of the operators are unbounded.

A further question concerns when a representation of the CCRs is regular—i.e., how can one tell, apart from checking for unitary equivalence directly (which can be hard) that a given representation is regular? Answers to this question (beyond the observation just made about spectra) are known (Rellich 1946, Dixmier 1958), but beyond the scope of this essay. See Summers (2001) for discussion and further references.

4.2. **The Uncertainty Relations.** One of the reasons that the CCRs are so important is that they give rise, directly, to the uncertainty relations. Recall that if two bounded operators do not commute, then there are eigenvectors of one that are not eigenvectors of the other. (Similar remarks hold for unbounded operators, but there we must take account of the fact that they might not have any eigenvectors, and speak instead in terms of the non-commutativity of their spectral projections.) It follows that there are states that assign trivial probabilities to the possible values of one observable (i.e., probability 1 for one eigenvalue, and 0 for the others), and non-trivial (not 0 or 1) to at least two possible values of the other. Hence non-commutativity already implies a type of 'uncertainty relation': certainty about the value of one observable can imply uncertainty about the value of another. Below, we shall make this idea more precise, and consider its interpretation.

4.2.1. *The Optical Derivation.* In 1927, and in an improved version in 1930, Heisenberg made the following argument, intended to make some sense of, perhaps even to derive, the uncertainty relations for position and momentum. Suppose we wished to measure the location of a small particle (e.g., an electron), by means of an optical microscope. The resolving power of the microscope with an aperture angle θ is approximately $\lambda/\sin\theta$, where λ is the wavelength of the light. This resolving power determines our uncertainty about the position of the particle after the measurement. On the other hand, in order for us to detect the particle, at least one photon would have to strike it. This photon has momentum[58] h/λ and the angle of impact is uncertain to within the angle θ; hence the amount of momentum transferred to the particle is uncertain to within roughly $(h\sin\theta)/\lambda$, and the *product* of the uncertainty in the position and momentum of the measured particle, after the measurement, is roughly h.

In other words, there is a lower limit on the product of the uncertainty of position and momentum. Notice that this lower bound applies only *after* the measurement. Indeed, we could measure the momentum of the particle with arbitrary precision

[58] The relation $p = h/\lambda$ as applied to photons was introduced as part of Einstein's (1905) explanation of the photoelectric effect, and generalized to material particles by de Broglie (1924).

prior to measuring its position, also with arbitrary precision. We would then have determined its position *and* momentum just *prior* to the moment of impact (of the photon on the particle) with arbitrary precision (though we would still be uncertain about its momentum *after* the measurment).

There are other derivations of the uncertainty relations, derivations that rely more explicitly on the formalism of quantum theory. We now consider two of them.

4.2.2. *The Wavefunctional Derivation.* Position and momentum are related by a Fourier transformation. Indeed, it is often easier, when dealing with momentum in the Schrödinger representation, to apply a Fourier transform (which is a unitary transformation on the Hilbert space), so that the momentum operator becomes a multiplication operator (and the position operator then becomes differentiation). But consider what happens to the wavefunctions. A wavefunction that is very well peaked corresponds to a state in which position is well-defined. That is, most of the probability is concentrated in a relatively small region of space. But the Fourier transform of such a function is very flat, so that the probability is uniformly distributed across all of the real line, which, after the transformation, corresponds to possible momenta of the particle.

This general idea can be made mathematically more precise. Consider a Gaussian 'wavepacket', a wavefunction from $L^2(\mathbb{R})$, which, as a function of x, has appreciable magnitude only in some region of size $2a$:

$$\psi(x) = e^{-x^2/2a^2}. \tag{4.4}$$

The Fourier transform of this wavefunction (i.e., transforming to a 'momentum representation') is:

$$\bar{\psi}(k) = e^{-a^2k^2/2} \tag{4.5}$$

(where k is the 'wave number'; momentum is given by $p = \hbar k$). This Gaussian has width $2/a$. Hence a narrowly peaked wavefunction in position (i.e., a is small) is widely spread in momentum (i.e., $1/a$ is large). Specifically, setting $\Delta x \approx a$ and $\Delta p = \hbar \Delta k \approx 1/a$, we have that $\Delta x \Delta p \approx \hbar$. (This expression is not quite the standard uncertainty relation, but then the derivation here is not meant to be exact.)

4.2.3. *The Algebraic Derivation.* There are in fact many roads from the quantum formalism to the uncertainty relations. Here, we consider just one other commonly found derivation, in part because it will shed some additional light on meaning of 'uncertainty', and in part because unlike the previous two derivations, this one is rigorous, and results in the exact form of the uncertainty relations.

Given an observable, F, define $\Delta F := F - \langle F \rangle$. (The right-hand side is the expectation value of F, where we have left the state unspecified.) The expectation of $(\Delta F)^2$ is the *dispersion* of F. Indeed,

$$\langle (\Delta F)^2 \rangle = \left\langle F^2 - 2F\langle F \rangle + \langle F \rangle^2 \right\rangle = \langle F^2 \rangle - \langle F \rangle^2, \tag{4.6}$$

which is a standard statistical notion of 'dispersion' (often called 'mean-square deviation' by physicists, and 'variance' by statisticians; its square root is the standard deviation). Now, let F and G be observables (self-adjoint operators). Then the Schwarz Inequality (§7.1.3) implies that

$$\langle (\Delta F)^2 \rangle \langle (\Delta G)^2 \rangle \geq |\langle \Delta F \Delta G \rangle|^2. \tag{4.7}$$

Straightforward algebraic manipulations (see, e.g., Sakurai 1985, p. 36) transform (4.7) into the standard *uncertainty relation*:

$$\langle \Delta F \rangle \langle \Delta G \rangle \geq \frac{1}{2} |\langle [F, G] \rangle|. \tag{4.8}$$

for any observables F and G. Notice, for example, that

$$\langle \Delta P \rangle \langle \Delta Q \rangle \geq \frac{1}{2}. \tag{4.9}$$

(or, if we are not setting $\hbar = 1$, then the right-hand side is $\hbar/2$—see note 11).

Above, we said that this derivation would shed some light on the meaning of uncertainty. In particular, we can now see that, rigorously, 'uncertainty' refers to the dispersion (standard deviation) of an observable in a given state, which is normally understood as a measure of the 'spread' in its values over an ensemble of systems all in some given state.

4.2.4. *Limitations and Generalizations.* The derivation of (4.8) makes it clear that *any* two non-commuting operators will give rise to some uncertainty relation. Hence (4.8) is quite general. However, there are also 'uncertainty relations' between quantities that are not represented by operators. The most well-known is the time-energy uncertainty relation, whose interpretation is notoriously problematic precisely because time is not an observable in quantum theory. (There is no self-adjoint operator that represents time.) Another example is phase and photon number. (Again, there is no 'phase' operator.) Various proposals exist for how to understand these other uncertainty relations, but here we merely note the point that they must apparently be understood in some sense other than that given by (4.8).

Moreover, (4.8) faces other problems. First, recall that (4.8) is state-dependent. Indeed, if we choose a state that is an eigenstate of F (or G), then even if F and G do not commute, both sides of (4.8) are zero, which certainly seems to violate the

spirit of 'uncertainty'. (The 'solution', perhaps, is to notice that if F and G do not commute, then in general the dispersion of G for a system in an eigenstate of F will be non-zero.) Second, keep in mind that 'dispersion' itself can be misleading. Even when most of the probability is concentrated on a narrow range of possible values of F, a small amount of probability that is located very far from F's mean can cause its dispersion to become large. There are proposals to handle these shortcomings. (See especially Uffink 1994.)

4.2.5. *'Wave-Particle Duality'.* A quantum-theoretic experiment that is commonly associated with the uncertainty principle is the double-slit experiment (which had been done, in some form, from well before the advent of quantum theory). The set-up is as follows: a source of particles (or monochromatic light, i.e., photons) is placed in front of an opaque barrier with two parallel slits. Behind the barrier is a screen (such as a photographic plate). A particle is fired at the slits in the barrier, and the screen records the location of each particle as it strikes the screen. (See figure 3.)

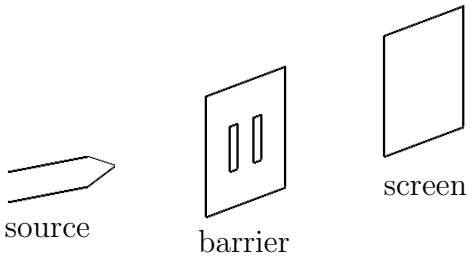
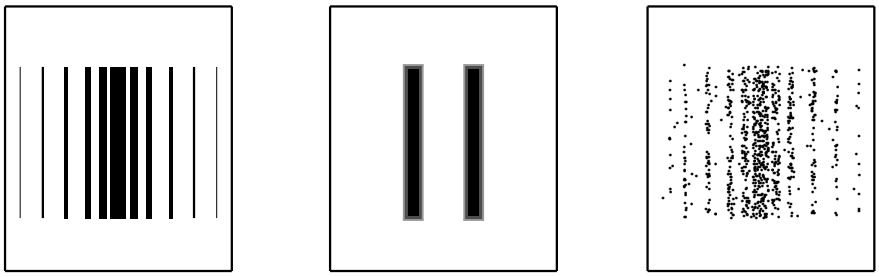


FIGURE 3. Double-Slit Experiment.

The main point is the following. We shine a beam of particles from the source onto the barrier. If both slits are left open and no determination is made about which slit the particle traverses, then an interference pattern develops on the screen, which is just what one would expect if a wave somehow passes through the slits. (See figure 4a.) If, on the other hand, we determine which slit the particles traverse, then no interference pattern shows up on the screen—instead, a pattern that is characteristic of particles (one 'blob' behind each slit) shows up. (See figure 4b.) What is more, one can do the experiment one particle at a time, and in this case, one sees 'dots' on the screen, and yet, if no determination is made about which slit the particles traverse, eventually the dots exhibit an interference pattern. (See figure 4c.)

This experiment illustrates 'wave-particle duality': when we measure a wave-like property of particles (interference), we get wave-like behavior (interference pattern), while when we measure a particle-like property of particles (which slit a particle traverses), we get particle-like behavior (no interference pattern).



(a) (b) (c)

FIGURE 4. Results of the double-Slit Experiment: (a) with the particles passing freely through the slits; (b) while determining which slit each particle traverses; (c) with the particles passing freely through the slits, sent one at a time (simulated, after 800 particles are detected at the screen).

Indirectly, it also illustrates the uncertainty relations. For consider what it takes to determine with reasonable accuracy which slit a particle traverses — in that case, we must measure the particle's position to an accuracy much better than $d/2$, where d is the distance between the slits. If the interference pattern is to be maintained despite this measurement, then the particle's momentum cannot be disturbed so much that, with appreciable probability, it gets deflected from a region of constructive interference (where, from the wave-theoretic point of view, the waves passing through each slit interefere constructively, i.e., a region where many dots show up in figure 4c) to an adjacent (or indeed any) region of destructive inference (i.e., regions where few or no dots show up in figure 4c). A rough trigonometric analysis shows that in fact the product of the uncertainty in our position measurement and the required low uncertainty in momentum must violate the uncertainty relation between position and momentum. In other words, the uncertainty relations appear to require that measuring the position of the particle at the slits well enough to determine (with good accuracy) which slit the particle traverses will tend to wash out the interference pattern (the more so, the more accurate the measurement of position).

4.3. **Interpretation of the Uncertainty Relations.**

4.3.1. *Observational-Epistemological.* Interpretations of the uncertainty relations are helpfully divided into two kinds: those that understand uncertainty purely in terms of observationally obtainable facts about the values of observables, and those that attribute observational uncertainty to a more fundamental, 'ontological', uncertainty, or 'indeterminacy'. I will consider each of these two kinds of interpretation in turn.

4.3.1.1. *Uncertainty as Uncertainty.* The term 'uncertainty', and the understanding of uncertainty as standard deviation (dispersion), strongly suggests an epistemic, even operationalist, understanding of the uncertainty relations. In particular, one is quite tempted to understand the uncertainty relations in terms of post-observational uncertainty about the actual values of observables.

There is a question about whether this uncertainty concerns single systems, or ensembles. In the latter case—i.e., if, as suggested by the usual statistical understanding of standard deviation, we understand uncertainty to reflect dispersion of values (for some observable) in an ensemble—the 'interpretation' of the uncertainty relations does not go beyond a straightforward statistical understanding of standard deviation. Of course, in this case then there is still a sense in which uncertainty applies to single systems, namely, when they drawn at random from such an ensemble. But one might, in addition, suppose that the notion of uncertainty applies to single systems independently of the consideration of any ensemble. Indeed, one might argue that only in this case would we really *, understand* why uncertainty holds at the statistical level. In any case, let us consider that possibility.

The idea, then, is that measurement (more generally, observation) in general reduces our uncertainty about the value of some observable, F, on the (single!) measured system. The uncertainty principle would then be read as asserting that a reduction in uncertainty about the value of F can imply an increase in uncertainty about the values of observables that are incompatible with F.

The meaning of such an assertion is clear enough, but understanding *why* incompatible observables have this feature is another matter. Suppose that F and G are incompatible. Suppose that we measure F. We then measure G. Why should it be the case that this second measurement *ruins* our previous knowledge of the value of F? Indeed, supposing that the second measurement is isolated from the first, and that the value of F was not in any way disturbed in the meantime, *how* could the second measurement ruin our previous knowledge of the value of F?

4.3.1.2. *Einstein's Early Thought Experiment.* Early critics of quantum theory—notably, Einstein—asked something like this question, and indeed proposed (thought) experiments that were apparently intended to show that in fact it *is* possible to 'beat' the uncertainty principle. A famous such experiment, due to Einstein, involves a standard two-slit apparatus (see figure 3) that is mounted on springs. The basic idea is to use the springs to determine which slit the particle traversed, without in fact disturbing the particle itself, by measuring the exchange of momentum between the particle and the barrier. (If the source is located on the plane exactly between the slits, then if

the particle goes through the right-hand slit (and the slits are narrow enough), the barrier will in general experience a kick to the right, and so on.)

4.3.1.3. *Reply to Einstein.* Einstein's challenge to the uncertainty principle, here, is empirical: he is claiming that, contrary to what quantum theory allows, it *is* possible to prepare a system in a state of precise position and momentum (or at any rate, more precise than the uncertainty relations allow). The response must, therefore, be empirical. If indeed it is possible to prepare a system in a precise state of position and momentum, then we ought to be able to use our knowledge of either to make a verifiable prediction about the system. In particular, if the measurement of the position of the particle right after it passes through the slits does not, in fact, disturb the momentum of the particle, then we should still see the same interference pattern on the screen that we see in the standard two-slit experiment. If, on the other hand, the interference pattern 'washes out' (see §4.2.5), and approaches the 'two blobs' (see figure 4b) as the measurement of position becomes more and more precise, then the epistemic version of the uncertainty principle stands.

As far as current experiment can discern, it seems that a measurement of which slit the particle traverses does indeed wash out the interference pattern. Indeed, a remarkable experiment appears to show that not even this much is quite right; rather, what seems to matter is whether a *record* of the result of the measurement is kept, where by a 'record', here, we mean an encoding of the result in a measurable physical state of the universe. The experiment[59] is, in essence, the two slit experiment, with a detector placed behind one of the slits. The detector is, moreover, 'eraseable', in the following sense. Once a particle passes through the detector, the particle leaves a trace in the state of the detector. We may then choose to magnify this 'trace' in order to turn it into a discernible signal indicating the presence of the particle, or we may completely erase it, so that the state of the detector no longer contains any retrievable information about whether the particle was once in the detector.

With this erasable detector in place, now imagine performing the following experiment. Fire the particles at the barrier one at a time. After the particle passes through the barrier, either erase the detector, or not. On the runs where we erase the detector, the particles build up an interference pattern (as in figure 4c). On the runs where we do not erase the detector, the particles do not build up an interference pattern, but instead 'behave as particles' (as in figure 4b). The experiment is relatively recent, and requires further scrutiny, but it strongly suggests, as do other quantum-mechanical experiments, that the epistemic version of the uncertainty principle is a fact of nature.

[59]See, for example, Scully and Walther (1989) and Walborn et al. (2002).

4.3.2. *Ontological.* With the epistemic version of the uncertainty principle apparently intact, we are left wondering *why* it is true. Different interpretations will propose different answers to this question. Some assert a 'disturbance theory', according to which the measurement of one quantity physically disturbs others (corresponding to incompatible observables) in an uncontrollable and unpredictable way. Others assert that reality matches our knowledge, here: we cannot know the value of G (when the value of incompatible F is known) because, in fact, G *has* no value.

There are many versions of this idea. Here we consider two. The first, often claimed to be part of the 'Copenhagen' interpretation of quantum theory, rests on some version of a verificationist, or operationalist, theory of the meanings of physical quantities, according to which a physical quantity has a value if and only if it has been measured (i.e., verified—the appropriate physical operations have been carried out, with the appropriate outcome). One must then argue that it is physically impossible to perform measurements of incompatible observables simultaneously on the same system.

This argument suggests a second ontological version of the uncertainty principle, according to which a physical quantity is well-defined just in case the conditions required for its well-definedness obtain. Of course, stated thus, this view sounds almost tautologous. It's real content comes from arguing that there are indeed non-trivial physical conditions that must be in place in order for certain physical quantities to be well-defined, and that the conditions required for the well-definedness of a given quantity cannot be in place simultaneously with the conditions required for the well-definedness of any incompatible quantity.

This sort of argument has bite, if it does, because of the role of reference frames in the definition of physical quantities (§3.4). Hence, for example, 'momentum' must mean 'momentum relative to X', where 'X' is some physical system that defines a frame of reference. But if 'X' (more precisely, the frame of reference that it defines) is a non-inertial system, then it is not suitable for defining momentum (at least not over any stretch of time), unless we know its relation to some inertial frame F (but then we are *really* defining momentum relative to F)—recall §3.4.1. Similarly, a non-inertial physical system is inappropriate for defining position (with the same caveat as before). These brief points recall the more extensive discussion above, and we shall have to leave the matter at that.

4.4. **The Einstein-Podolsky-Rosen Argument.** Thus far, everything that has been said about the uncertainty relations is consistent with the claim that the loss of certainty about the value of one observable, F, upon measurement of incompatible G,

is a result of an unknown and uncontrollable physical disturbance of the value of F as a consequence of the measurement of G. Indeed, Heisenberg's 'optical' derivation of the uncertainty relations uses precisely this idea. But an argument due to Einstein, Podolsky, and Rosen (EPR)[60] purports to show that in fact the uncertainty relations cannot be understood in this way.

This argument concedes that the epistemic version of the uncertainty principle is true: the values of incompatible observables are not simultaneously verifiable. The point, here, is to question *ontological* versions of the principle.

4.4.1. *Incompleteness.* In particular, the point is to call into question the *completeness* of quantum theory. Recall (§1.2.3.9) the standard interpretation of quantum theory, according to which an observable, F, has a value for a system in a state, W, just in case W assigns probability 1 to some possible value of F (and 0 to the others). This interpretation immediately implies that there are no quantum states that assign simultaneously definite values to all observables, or indeed to any two observables with no common eigenvectors. Hence any argument that successfully shows that two such observables must have definite values implies that quantum theory is incomplete—more precisely, it shows that under the standard interpretation of the theory, quantum states do not describe (and cannot describe) the complete physical state of a system. The EPR argument purports to show exactly this claim.

4.4.2. *The Generic Experiment.* Generically, the experiment considered by EPR involves a pair of particles (call them α and β) prepared in the state

$$|\Psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |a_n\rangle|b_n\rangle \tag{4.10}$$

where the $|a_n\rangle$ and the $|b_n\rangle$ form orthonormal sets.[61] Hence there are observables, A for system α and B for system β (whose eigenvectors are respectively the $|a_n\rangle$ and the $|b_n\rangle$, corresponding to eigenvalues a_n and b_n) that are perfectly correlated (§1.2.6.4) in this state.

So suppose that α and β are in the state $|\Psi_{\mathrm{EPR}}\rangle$ while they are spatially separated. Then the perfect correlation between A and B allows one to discover the value of A without, as EPR say, 'in any way disturbing' α, simply by measuring B on β.

[60]There are very good reasons to believe that the argument as presented by EPR (1935) was not quite what Einstein himself had intended. (The paper was not written by him.) See Fine (1986, esp. chs. 3-5, for example.

[61]In fact, there is a 'continuous' version of (4.10), which is appropriate when we are considering observables, such as position and momentum, with continuous spectra. See §4.4.3.

Now for the crucial point. Consider orthonormal bases $\{|a'_n\rangle\}$ and $\{|b'_n\rangle\}$ obtained from the $|a_n\rangle$ and the $|b_n\rangle$ as follows:

$$|a'_n\rangle = \frac{1}{\sqrt{2}}\Big(|a_n\rangle - i|a_{n+1}\rangle\Big) \qquad |b'_n\rangle = \frac{1}{\sqrt{2}}\Big(|b_n\rangle + i|b_{n+1}\rangle\Big), \tag{4.11}$$

where the sum in subscripted '$n+1$' is modulo-N (i.e., $N+1=1$). In this basis, the state $|\Psi_{\text{EPR}}\rangle$ has exactly the same form:

$$|\Psi_{\text{EPR}}\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}|a'_n\rangle|b'_n\rangle. \tag{4.12}$$

(To verify, plug (4.11) into (4.12) and simplify. The 'cross' terms of the form $-i|a_n\rangle|b_{n+1}\rangle$ and $i|a_n\rangle|b_{n+1}\rangle$ cancel.) Hence there are additional observables, A' and B' (whose eigenvectors are respectively the $|a'_n\rangle$ and the $|b'_n\rangle$) that are *also* perfectly correlated in the state $|\Psi_{\text{EPR}}\rangle$. Moreover, A' does not commute with (is incompatible with) A—indeed, they share no eigenvetors—and similarly for B and B'. Again, we can discover the value of A' on α by measuring B' on β.[62]

Finally, notice that this entire description is *quantum-mechanical*. One sometimes hears the EPR experiment described in more or less classical terms. For example, letting A and B be position and momentum (see §4.4.3), one might be tempted to describe the preparation of the EPR state as follows: fire two particles of equal mass from a common source with equal (in magnitude) but oppositely directed forces. Their positions (distance from the source) and momenta (relative to the source) will then be perfectly correlated. This picture *is* tempting, but it is also completely wrong. Indeed, standard quantum mechanics implies that the state just described cannot be prepared in a way that would allow one to infer the position (or momentum) of α from that of β, because such inferences would require us to know, with precision, the position and momentum of the source, and such knowledge already violates the (epistemic) uncertainty principle.

4.4.3. *Position and Momentum.* Above we assumed that the perfectly correlated observables have a discrete spectrum. In fact, when they consider an explicit example (rather than the abstract case as considered above), EPR refer to position and momentum, and a state that is perfectly correlated in position and momentum (in the sense that it is a simultaneous eigenstate of the sum of the momenta of the two particles, and the difference of their positions). Explicitly,

$$\Psi_{\text{EPR}}(x_1, x_2) = \int_{-\infty}^{+\infty} e^{(2\pi i/h)(x_1 - x_2 + x_0)p}dp, \tag{4.13}$$

[62]The situation as described here is not mathematically exactly the same as that considered by EPR, but the result is the same.

for some fixed x_0. However, it is worth pointing out that in fact this state is not an allowed state of the system at all—it is not a vector in $L^2(\mathbb{R}^2)$. Moreover, this state, even if it could be prepared, necessarily spreads under any time evolution (associated with a finite potential energy), hence immediately becoming a state of *less* than perfect correlation.[63]

To overcome these limitations of EPR's example, one could consider, for example, a narrow Gaussian that is very close to being a state of perfect correlation, but doing so makes the argument to come messy at best. And we should keep in mind that the discussion in terms of position and momentum is in fact only an *example* that EPR give to illustrate the main point.

In the end, in fact, it is easier to consider an example that involves bounded observables. The simplest case is the so-called 'singlet' state of a pair of spin-1/2 particles. Generically, this case corresponds (for an appropriate choice of the $|a_n\rangle$ and the $|b_n\rangle$) to the state $|\Psi_{\mathrm{EPR}}\rangle$ above, for $N = 2$. We will just continue to speak of the observables A, A', B, and B', with the understanding that the perfect correlations discussed above obtain.

4.4.4. *The Argument.* How can this experimental situation be used to generate an argument for the incompleteness of quantum theory? The conclusion that EPR hope to establish is that A and A' both have a definite value simultaneously. Because they share no common eigenvectors, this conclusion is inconsistent with the standard interpretation of quantum states (see §1.2.3.9).

One tempting path to this conclusion involves presuming that once B has been measured on β, thereby establishing the value of A on α, we can measure A' on α and thereby establish its value directly. However, the problem with this suggestion should be clear: the disturbance theory of uncertainty can quickly be applied to conclude that the measurement of A' on α disturbs the previously established value of A. We are thus reminded that the point of this discussion is, in fact, to establish the definiteness of A and A' without 'in any way' disturbing α.

EPR's strategy involves two assumptions. The first, with which at least some versions of the standard interpretation can easily agree, is their 'criterion for physical reality', which asserts that whenever the value of an observable can be predicted with certainty, the observable actually has that value. (Notice that this criterion is inconsistent with the sort of verificationist or operationalist views, mentioned above (§4.3.2), according to which a system has a value only if that value has been obtained

[63] By the state 'spreading', here, we mean, roughly, that it gets closer to a uniform distribution over $\mathbb{R}^2$. See Dickson (2002b) for a discussion of these points, and further references.

as the result of a measurement.) Before we introduce the second assumption, let us see how far we can get with just this one.

Notice that if we measure B on β, we can predict the value of A on α with certainty. Similarly for B' and A'. Of course, we cannot measure *both* B and B'. Hence, instead of considering *actual* measurements of B and B', let us consider non-actual, but possible (i.e., 'counterfactual') measurements of B and B'. We have:

> **Premise 1:** *Possibly*, B is measured on β, and in this case, α has a definite value for A.
>
> **Premise 2:** *Possibly*, B' is measured on β, and in this case, α has a definite value for A'.

From these two premises (which follow from the criterion for physical reality), EPR hope to conclude:

> **Conclusion:** *Possibly*, α has a definite value for both A and A'.

However, the Conclusion does not follow from the Premises 1 and 2. Indeed, the logical problem is, in part, that there is no guarantee that the possible conditions ('B is measured' and 'B' is measured') are co-possible. Indeed, as we know, they are not.[64]

Hence EPR need another premise. They introduce a notion of 'non-disturbance' that is supposed to help patch up the argument: although the conditions (measurement of B) under which we can infer the definiteness of A on α are incompatible with the conditions (measurement of B') under which we can infer the definiteness of A', the difference between them is supposed to make no difference to α, because they only involve a change of circumstances for β, which may be spatially separated from α.

However, not just any such principle will work. Consider, for example, the following:

> **Weak non-disturbance:** if B is measured on β and (therefore, by the criterion for physical reality) A is definite for α, then: had we not measured B on β, α would still have had a definite value for A (and likewise, substituting primed observables for the unprimed ones).

[64] Consider the following analogous argument. (i) It is possible that the paper is burned, and in this case it will be reduced to ashes. (ii) It is possible that the paper is not burned, and in this case it will remain whole. Therefore, (iii) it is possible that the paper is both whole and reduced to ashes. Of course, this argument is invalid.

This principle, which might be taken to deny that measuring B is what brings it about that A has a definite value, is insufficient to get EPR's conclusion. They need, instead:

> **Strong non-disturbance:** if B is measured on β and (therefore, by the criterion for physical reality) A is definite for α, then: had we *instead* measured B' on β, α would still have had a definite value for A (and likewise, substituting primed observables for the unprimed ones).

The weak principle is insufficient to get the conclusion because the introduction of the measurement of B' (as opposed to the mere absence of the measurement of B) could destroy essential features of the situation, and in particular features that permit the inference of properties of α (its value for A) from the results of measurements (of B) on β.[65]

The principle of strong non-disturbance is supposed, of course, to capture some notion of 'locality'. In particular, the idea is supposed to be that *nothing* that is done to β can have any affect on the properties of α, under the assumption that the two particles are space-like separated. Einstein's theory of special relativity is presumably supposed to license this asusmption.[66] In any case, with the principle of strong non-disturbance, EPR have a logically valid argument for the Conclusion, above. Indeed, they could (and seem to claim to) establish a stronger form of the conclusion, replacing 'Possibly' with 'Actually', as follows. Argue, further, that a *measurement* of B (or B') does not bring it about that α has a value for A (A'), so that α must have values for both A and A' even when B (B') is not measured.

4.4.5. *Replies to EPR.* One can, in fact, deny strong non-disturbance by denying locality. We shall consider the status of locality in quantum theory below (§6). Here, we consider two other replies to EPR.

We have already more or less encountered one of these replies: note that the EPR argument has no impact on those verificationist or operationalist views according to which an explicit measurement of a physical quantity is required not merely for us to know its value, but also for it to *have* a value. Of course, such views must deny the criterion for physical reality (which, keep in mind, is only a sufficient, not a necessary condition), and many find this principle quite compelling. (Apart from

[65]The possible-worlds semantics for counterfactuals makes the point easy to see: while the closest 'B is not measured'-worlds to the 'B is measured and A is definite for α'-worlds might all be 'A is definite for α'-worlds, those closest worlds might not contain any 'B' is measured'-worlds, so that the closest 'B is not measured but B' is'-worlds to the 'B is measured and A is definite for α'-worlds need not be 'A is definite for α'-worlds.

[66]See Malament, Ch. 3, this volume.

actually having the value in question, what else could, reasonably, make it be the case that we can predict its value with certainty?)

But Bohr offered a reply that appears to rely neither on this verificationist or operationalist strategy, nor—so he claims—on an explicit endorsement of non-locality. In particular, recall that EPR were aiming to avoid the 'disturbance' account of uncertainty, according to which, for example, a measurement of A' physically disturbs the value of A. Here, because we are only ever performing measurements on β, it is far from clear how a measurement of B', for example, could disturb the value of A on α—after all, such a disturbance would have to be non-local. Bohr's reply denies strong non-disturbance, but without (so the claim goes) endorsing a physically direct (Bohr uses the term 'mechanical') disturbance of (e.g., the value of A for) α as a consequence of any measurement (e.g., of B) on β. Instead, recall the idea from above (§4.3.2), that the very well-definedness of certain physical quantities relies on certain physical conditions being in place.

Indeed, consider the EPR experiment, now, as Bohr (and EPR) did, in terms of position and momentum. We will assume (without loss of generality) that the positions and momenta are defined relative to the source. On Bohr's view, the well-definedness of the *sum* of the momenta for each particle (i.e., the total momentum for the system) is maintained just so long as the system (particles plus source) remains closed, i.e., just as long as total momentum is conserved. But a measurement of the position of β introduces a disturbance of β's momentum. The system is no longer closed: β either loses momentum to or gains momentum from an object (the measuring apparatus) that is external to the system. But then the total momentum (relative to the source, which 'knows nothing' about this external influence on β) is no longer conserved, and thus the conditions for its well-definedness (relative to the source) are no longer in place, and therefore any inferences that we might have made about the well-definedness of α's momentum, based on the well-definedness of the total momentum, are no longer valid. In other words, on Bohr's view, the conditions required (under the circumstances) for the well-definedness of α's momentum are, when we are measuring β's position, no longer in place. Note that the 'old' type of disturbance is still at work here—we are indeed supposing that β's momentum is physically disturbed by a measurement of its position—but in addition there is another type of disturbance at work: the measurement of β's position 'disturbs' the conditions required for the well-definedness of α's momentum (under the circumstances). Thus Bohr believes that he can avoid EPR's conclusion by (in essence) denying strong non-disturbance,

but in a way that does not entail a non-local disturbance, in the sense of an exchange of energy or momentum between β and α.

4.4.6. *Where We Stand.* Regardless, finally, of whether one finds any of these replies to EPR satisfactory (and we should not make any judgment on this point at least until we have considered the issues in §5 and §6, it should be emphasized that none of them shows that the EPR argument is unsound, much less invalid. Indeed, they are, in a sense, defensive maneuvers designed to articulate a view of quantum theory that avoids EPR's conclusion by denying, in a consistent and presumably plausible way, one of the premises of their argument. Their conclusion is thus far from called into question by such replies. Indeed, many philosophers of quantum theory are more or less in agreement with EPR that standard quantum theory is incomplete, if not because of EPR's argument, then because of the so-called 'measurement problem', which we consider next.

5. The 'Measurement Problem'

The problem of measurement is, perhaps, the most discussed issue in the foundations of quantum theory, and has inspired numerous and varied interpretations of the theory, from the brilliant to the bizarre. In this section, I will first review the problem (§5.1), emphasizing its generality. Then I will consider some natural, but in the end unsatisfactory, reactions to the problem (§5.4). In the final section (§5.5), I will consider a few examples of interpretations of the theory, each of them largely directed towards solving the measurement problem.

5.1. **The Basic Problem.**

5.1.1. *'Schrödinger's Cat'.* Recall (1.2.3.2.b) that the superposition principle implies that for any observable, F, if a system can have each (or indeed, just two of) the eigenvalues of F as its values for F, then it is also possible for a system to have (assuming the eigenstate-eigenvalue link, §1.2.3.9) *no* value for F (because it is in a superposition of the eigenstates of F).

If F is some unfamiliar observable of the unfamiliar quantum world, then perhaps one can live with this consequence. But what about observables on middle-sized solid objects that are the bread and butter of our everyday experience? What about 'the (approximate) location of the house', or 'the (approximate) momentum of the horse', and so on? In 1935, Schrödinger illustrated the sort of problem that we face with the following example.

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts. (Schrödinger 1935a)

The point, of course, is that after some stretch of time, the atom is in a superposition of 'decayed' and 'not decayed', and hence the hammer, poison, and ultimately the cat, are in corresponding superpositions—in the case of the cat, a superposition of 'alive' and 'dead'. But of course we never witness cats in such states. So apparently there is a serious problem with quantum theory.

5.2. **Measurement.** Notice that the cat serves, in essence, as a decay-indication device, a kind of crude measuring apparatus. Indeed, it is precisely Schrödinger's point that one can magnify a superposition at the microscopic level (the atom) to a superposition at the macroscopic level (the cat), the level at which one is probably more inclined to deny categorically that superpositions (of at least some sorts, such as of the states 'alive' and 'dead') make any sense, physically, or at any rate that they exist with anything like the frequency that quantum theory apparently predicts. Measurement devices designed to measure quantum-mechanical observables are characteristically of this sort: they 'magnify' the state of some microscopic quantum system into the (indicator, or 'pointer') state of an apparatus that can be directly observed.[67]

Of course, this magnification is a physical process, and as such, it is in principle to be described as a solution to Schrödinger's equation (or some other quantum-theoretic equation of motion). Those equations are *linear*. (In fact, it is precisely this linearity that partially grounds the principle of superposition: any linear combination

[67]Recall my earlier (§1.4.3) skepticism, however, about 'pure' states such as $|\text{cat dead}\rangle$. Does this skepticism suggest a way out of the measurement problem? Alas, it does not. We could just as well speak in terms of mixed states, here, and generate the problem, which relies only on the fact that at the end of a measurement, the state of the apparatus may assign non-trivial probabilities to *all* possible values of the 'pointer-observable'.

of solutions is again a solution.) Indeed, recall that the dynamical evolution of a system can be described in terms of a family of unitary operators, which are, of course, linear.

So let a given measurement-interaction (between some apparatus and some measured system) be described by the operator U. (As an example, recall §3.1.2.) Let the apparatus' 'ready-to-measure' state be $|\Psi_0\rangle$ and let its 'pointer' states be $|\Psi_n\rangle$. For each of the eigenstates, $|f_n\rangle$ of the measured observable, F, we presume that the measurement-interaction results in an accurate indication of the result:

$$U(|f_n\rangle|\Psi_0\rangle) = |f_n\rangle|\Psi_n\rangle \tag{5.1}$$

so that the state $|\Psi_n\rangle$ indicates that the measured system has the value f_n for F. But then, by linearity,

$$U\left(\sum_n k_n|f_n\rangle|\Psi_0\rangle\right) = \sum_n k_n|f_n\rangle|\Psi_n\rangle. \tag{5.2}$$

Now we are in trouble. The eigenstate-eigenvalue link implies that this final state is one in which the pointer-observable for the apparatus in fact *has no value*. The pointer is not indicating *anything*. But at the end of typical successful measurements, the apparatus *does* indicate a result, even when the measured system begins in a superposition of eigenstates of the measured observable. Standard quantum theory seems to contradict this apparent fact of experience. Hence the 'measurement problem'.

5.3. **Generality of the Problem.** There are at least three senses in which the 'measurement problem' as described above does not capture the real problem with sufficient generality. First, it relies on the quite conservative eigenstate-eigenvalue link, and on a very restrictive notion of 'measurement'. Second, it fails to indicate the ubiquity of superpositions. Third, it relies on the questionable assumption that the macroscopic pointer-states of the apparatus are pure states, when in fact they are almost surely mixed. In this subsection I will briefly consider these points. Finally, I will briefly raise a related problem, the problem of the 'classical limit'.

5.3.1. *'No-go' Theorems.* Two important premises in the argument leading up to the measurement problem, above, were the eigenstate-eigenvalue link and the account of what counts as a successful 'measurement'. Both of these assumptions can be weakened considerably.[68] I will consider each in turn.

[68]There is a long history of proofs, increasingly general, that the measurement problem is 'insoluble' in the context of standard quantum theory. A review of the early history, together with arguably the simplest proof of the theorem, is given by Brown (1986). For a more recent survey and extensive discussion of this issue, see Mittestaedt (1998, esp. ch. 4).

5.3.1.1. *Weaker Conditions for Definiteness.* First, we may introduce a weak condition for when an observable has a definite value, roughly as follows: the final state of the apparatus assigns a definite value to the pointer-observable for the apparatus just in case the final state of the compound system is a mixture of states each of which has a definite value for the pointer observable according to the eigenstate-eigenvalue link. In other words, we now allow the adoption of an ignorance interpretation of mixed states of the compound system, accepting that when the compound system has the sort of mixed state just described, it is actually in one of the pure states appearing in the mixture, and therefore, according to the eigenstate-eigenvalue link, the apparatus has a definite value for the pointer-observable.

Note that the condition applies to the compound system, not to the apparatus on its own. The difference is subtle, but crucial. For example, in an ideal measurement, the apparatus by itself will always be in a mixture of eigenstates of the pointer-observable. It does not follow, however, that this mixture can be given an ignorance interpretation. I shall discuss this point in a somewhat different context below (§5.4.5.2.a).

5.3.1.2. *Weaker Accounts of Measurement.* We may also weaken the account of measurement, along the following lines. Let U represent the time-evolution of states of the compound system during the measurement of a POVM, E, by a 'pointer-observable' POVM, Q. We require: for any two states, W and W', of the compound system, if W and W' differ in their probabilities for at least one of the effects in the image of E, then UWU^{-1} and $UW'U^{-1}$ differ in their probabilities for at least one of the effects in the image of Q. Intuitively, U renders the pointer-observable 'somehow sensitive (even if only probabilistically) to the measured observable'.

Note, however, that while this weaker account of measurement, together with the weaker criterion for definiteness (§5.3.1.1), is sufficient to derive the measurement problem (i.e., the non-definiteness of the pointer-observable at the end of a measurement), it is also worth keeping in mind that, by the bi-orthogonal decomposition theorem (§1.2.6.2), there are *some* observables (one for the measured system and one for the apparatus) with respect to which the state of the compound system has the same form that it does in an ideal measurement. (I.e., write the state in its biorthogonal form. The bases for the measured system and apparatus will be bases determined by *some* observable for each.) On the other hand, whether *this* apparatus-observable is one that we antecedently believe to be definite is another question, the answer to which probably depends on the details of the interaction.

5.3.2. *The Ubiquity of Superposition.* The superposition principle asserts that superpositions of possible states are again possible states. But perhaps they are extremely rare. In that case, again perhaps we should not be terribly concerned about the measurement problem.

In fact, however, superpositions at the microscopic level are ubiquitous. To see why, we need only consider the case of spin. Recall that $S_{\vec{u}}$ and $S_{\vec{u}'}$ are incompatible unless $\vec{u} = \vec{u}'$ or $\vec{u} = -\vec{u}'$. Hence *every* spin-1/2 particle is 'in a superposition' with respect to just about every direction of spin. A similar remark holds for the polarization of photons. And there is good experimental and theoretical reason to believe that many fundamental particles may have wavefunctions that are highly de-localized.

Moreover, we need not imagine exotic situations such as that described by Schrödinger in order to believe that these microscopic superpositions may be 'magnified'. While such magnification is an important part of measurement, there is little reason to suppose that it does not occur naturally, and frequently. After all, the human eye, for example, is sensitive to as few as a half-dozen or so photons. It seems plausible to suppose that many other interactions that occur in nature have the effect of correlating the state of some macroscopic object with the state of some microscopic object—and such correlation (even if imperfect) is sufficient for the sort of magnification that will give rise to the measurement problem.

Hence, there is good reason to believe that the measurement problem is highly general, in a few senses: (a) even under apparently weak accounts of when apparatus' have a definite value for some macroscopic observable (§5.3.1.1), quantum theory apparently implies that they do not; (b) the sort of correlation between a macroscopic and microscopic object that is required in order to bring about the situation in (a) is very weak, and moreover perfect correlations are, by the biorthogonal decomposition theorem, ubiquitous (§5.3.1.2); and (c) the sort of superposition, and interaction, that is required give rise to the situation in (b) is plausibly quite common (this section).

5.3.3. *The Classical Limit.* Finally, I will quickly notice a problem that is discussed in detail elsewhere in this volume. In a sense, the measurement problem is the problem that the 'weirdness' of the quantum world is somehow not confined to the microscopic world. In other words, the 'weird' quantum world does not (always) mesh nicely with our more familiar classical world, in the sense that superpositions at the micro-level (where we might be able to live with them) are not confined to that level, but can be made to appear at the macro-level (where, so the argument goes, we cannot live with them). This apparent conflict leads to a more general question: how does quantum theory go over to classical theory (which, after all, works quite well for a large domain

of objects in a wide variety of conditions)? The question is vexed in a number of ways, which, however, we shall not consider here. We refer the reader to Landsman, Ch. 5, this volume (esp. §§5,6).

5.4. **Non-Solutions.** Numerous solutions to the measurement problem have been proposed, and we shall consider some of them in a separate section on interpretations. Here, we consider several proposed solutions that in fact do not work, or at the very least, face extraordinary obstacles to making them work.

5.4.1. *Naïve Realism.*

5.4.1.1. *The Obvious Solution to the Rescue?* The obvious solution to the problem is to give up the eigenstate-eigenvalue link. Indeed, why not interpret the probabilities delivered by quantum theory as entirely epistemic? That is, why not suppose that *every* observable has a definite value *all of the time*, and that incompatibility merely represents the fact that observing the value of one of them disturbs the values of others in an incontrollable way? Something like an epistemic version of Bohr's approach to uncertainty, if it works, could be put to use in the service of this view, both accounting for the uncertainty relations, and avoiding the measurement problem.

5.4.1.2. *The Kochen-Specker Theorem.* Initially attractive, this view founders on a well-known theorem, the Kochen-Specker theorem (Kochen and Specker 1967). Notice that naïve realism seeks to identify a single value for each observable. With one further requirement, one can show that no such assignment is possible.

5.4.1.2.a. Non-Contextuality. This further requirement is specifiable in a variety of ways. Here we put it in terms of 'non-contextuality'. Note that assigning a value to an observable amounts to assigning, to each set of mutually orthogonal subspaces that span the entire space, a '1' to exactly one of the subspaces, and '0' to the others. (These subspaces are the eigenspaces of the observable to which a value is assigned; assigning a 1 to a given eigensubspace is equivalent to assigning the corresponding eigenvalue to the observable.[69]) Proceeding in this way, we will further assume that all observables with exactly the same eigenspaces have 'the same' value, in the sense that they have the eigenvalue associated with a given common eigenspace. This

[69]Here I ignore observables, such as position and momentum, whose spectrum is continuous (so that their possible values do not correspond to subspaces). After all, assigning them values *as well* only makes the naïve realist's life even harder.

assumption amounts to the requirement that for any observable G, if G has the value g, then for any function, $f()$, the observable $f(G)$ has the value $f(g)$.[70]

Non-contextuality includes this assumption, but goes a step further, requiring, in addition, that the assignment of a '0' or '1' to a given subspace P is independent of *which* set of mutually orthogonal and jointly spanning subspaces P is considered to be a member. This requirement amounts to the condition that whenever two observables, F and G, share an eigenspace, P (but do not necessarily share *all* of their eigenspaces), F has the eigenvalue corresponding to P if and only if G does. (The eigenvalues will in general differ, of course.) (Note that in this case F and G in general will not commute.)

5.4.1.2.b. The State-Independent Theorem. Given non-contextuality, naïve realism amounts to the requirement that we be able to find a homomorphism from the lattice, $L(\mathcal{H})$, of subspaces of a Hilbert space, $\mathcal{H}$, to the Boolean lattice $\{0, 1\}$. That is, we require a map, $h : L(\mathcal{H} \rightarrow \{0, 1\}$ such that, for any $P, Q \in L(\mathcal{H}$, $h(P) \leq h(Q)$ if and only if $P \leq Q$. It follows (from the definitions of the meet, joint, and orthocomplement, which are all given in terms of the partial order—§7.4.2), that $h(P) \wedge h(Q) = h(P \wedge Q)$, $h(P) \vee h(Q) = h(P \vee Q)$, and $h(P)^{\perp} = h(P^{\perp})$. But notice that in $\{0, 1\}$, the operators $\wedge$, $\vee$, and $^{\perp}$ behave just as those from classical logic (i.e.: $0 \wedge 0 = 0$, $0 \wedge 1 = 0$, $1 \wedge 1 = 1$, $0 \vee 0 = 0$, $0 \vee 1 = 1$, $1 \vee 1 = 1$, and $0^{\perp} = 1$). In other words, the naïve realist requires that it be possible to consider quantum theory to arise from an underlying (logically) 'classical' theory.

Such a picture *is* available in classical mechanics. Indeed, let Γ be the phase space for a classical system. The physical propositions (cf. §2.1.1) are represented by the lattice of Borel subsets of Γ (where the partial order is given by subset inclusion), and it *is* possible to define a homomorphism from this algebra to $\{0, 1\}$. Indeed, choose a point $x \in \Gamma$, and define the map, δ_x, by, for any Borel subset $S \subseteq \Gamma$, $\delta_x(S) = 1$ if $x \in S$ and 0 otherwise. This δ_x is a homomorphism.

The content of the Kochen-Specker theorem is that for lattices of subspaces of a Hilbert space whose dimension is greater than 2, there are no such homomorphisms. This version of the theorem is sometimes called 'state-independent' because it does not rely on any presumptions about the quantum state of a system, but only on the *structure* of the state space as a whole.

5.4.1.2.c. State-Dependent Theorems. There are others versions of the theorem that begin with the assumption that the system is in some given quantum state. In this

[70]Note that, in general, if $G = \sum_n g_n P_n$ is the spectral decomposition of G, then $f(G) = \sum_n f(g_n) P_n$.

case, we can introduce another condition on the value assignments: they must respect the probabilities generated by the quantum state. For example, if the state assigns probability 0 [1] to a given value, f, for F, then the value-assignment must assign 0 [1] to the corresponding eigenspace. There are some particularly simple state-dependent Kochen-Specker theorems, the most famous being the GHZ (Greenberger-Horne-Zeilinger Greenberger et al. (1989)) theorem, which (in a form nicely described by Mermin (1990)) considers a three-particle system of spin-1/2 particles, and the observables

$$S_x^1,\ S_y^1,\ S_x^2,\ S_y^2,\ S_x^3,\ S_y^3, \tag{5.3}$$

$$S_x^1 \otimes S_y^2 \otimes S_y^3,\quad S_y^1 \otimes S_x^2 \otimes S_y^3,\quad S_y^1 \otimes S_y^2 \otimes S_x^3,\ \text{and}\ S_x^1 \otimes S_x^2 \otimes S_x^3, \tag{5.4}$$

where S_x^1 is shorthand for $S_x^1 \otimes \mathbb{I}^2 \otimes \mathbb{I}^3$ and the superscripts indicate the particles with which each observables is associated. For simplicity, suppose that each of the $S_u^{(n)}$ has been 'normalized' to have eigenvalues ± 1 (rather than the usual $\pm 1/2$). Hence the eigenvalues of all of the observables in (5.3) and (5.4) are ± 1.

Note that the observables in (5.4) commute. Hence we can consider a state, the 'GHZ-state', that is a simultaneous eigenstate of them all. We shall consider an eigenstate with eigenvalues $+1, +1, +1, -1$ respectively. For a system in this quantum state, any ascription of definite values to the observables *must* assign those values to the observables in (5.4). Let $v()$ be a map from the observables in (5.3) and (5.4) to their values, and consider then the following array of values:

$$\begin{array}{lll} v(S_x^1) & v(S_y^2) & v(S_y^3) \\ v(S_y^1) & v(S_x^2) & v(S_y^3) \\ v(S_y^1) & v(S_y^2) & v(S_x^3) \\ v(S_x^1) & v(S_x^2) & v(S_x^3). \end{array} \tag{5.5}$$

The possible values are always ± 1. The product of the values *across* each row must be $+1$ for the first three rows, and -1 for the last, so that the product of all twelve numbers must be -1. But on the other hand, $v(S_u^{(n)})$ for each $n = 1, 2, 3$ and $u = x, y$ appears exactly twice in the array, so that the product of all twelve numbers must be $+1$, contradicting our conclusion above. Hence there is no assignment of values to all ten of the observables in (5.3) and (5.4) that is consistent with the GHZ-state.

Naïve realism is thus in serious trouble. The most straightforward reading of its basic commitments leads, via the Kochen-Specker theorem and its analogues such as the GHZ theorem, to a logical contradiction. While there are proposals to save aspects of naïve realism from this problem, all of them (of course) violate one or more

of the conditions of the Kochen-Specker theorem, and, arguably, are no longer 'naïve' realism.

5.4.2. *Ensemble Interpretations.* Ensemble interpretations (e.g., Ballentine 1970) attempt to avoid the measurement problem altogether by stipulating that quantum states are not *about* individual quantum systems. They are essentially statistical, and thus can be used to describe only ensembles of systems. Hence the state at the end of a measurement ought not be thought of as describing a particular apparatus (and measured system) at the end of a particular measurement, but all such apparatuses (and measured systems) and the end of all such measurements.

There are two strains of this sort of interpretation in the literature, not always distinguished. The first we might call 'minimalist': it is the deflationary view that one can make *sense* of the quantum state (for example, at the end of a measurement) only as the description of an ensemble of similarly prepared systems.

The second strain of the ensemble interpretation makes a stronger claim, namely, something to the following effect: nothing more than the deflationary account of the quantum state is needed; i.e., there is no scientific demand for a theory that describes individual systems. On this view, such a demand is an artifact of a deterministic world view, and the usual 'mysteries' of quantum theory simply disappear once one learns to live with statistical theories. However, most researchers in the philosophy and foundations of quantum theory appear to be unconvinced.

Moreover, even the minimalist claim faces a problem, and the problem also shows why the stronger claims are, at the least, in need of further development. The Kochen-Specker theorem already shows that the probabilities generated by quantum states cannot be understood straightforwardly as classical probabilities. Thus the ensemble interpretations appear to be pushed towards other views, such as the quantum logic view, which will be considered later.

5.4.3. *Collapse Postulate.* The 'standard' solution to the problem of measurement is the so-called 'collapse postulate' (also called the 'projection postulate') of quantum theory, already mentioned in §3.1.2, first discussed in detail by von Neumann (1932, p. 351 and pp. 417-418 of the English translation). The postulate can be found stated in numerous ways, such as:

> **Collapse Postulate:** Upon measurement of the observable F on a system in the state W, the result of the measurement will be an eigenvalue corresponding to some eigenspace, P of F, and the state of the system will then be $PWP/\mathrm{Tr}[PWP]$.

If the state can be written as a vector, $|\psi\rangle$, then the 'collapse' amounts to projecting this vector onto ran P, i.e., onto $P|\psi\rangle$, and renormalizing the result.

There are many reasons to be unhappy with the collapse postulate, but it is worth saying, at the start, that most of the time, for most of the predictions that one wants to make from quantum theory, the postulate *works* very well.

One often-cited problem with the collapse postulate is that it amounts to discontinuous (and irreversible) evolution of a system. The resulting picture of the evolution of quantum systems is thus odd indeed: continuous, deterministic, reversible, unitary evolution (through Hilbert space) according to the Schrödinger equation (or some other quantum equation of motion), punctuated by discontinuous, irreversible, probabilistic, instantaneous, changes of the state.

At the very least, one would like some story about why these punctuations occur. Clearly, in the statement of the postulate, they are connected with measurement. But what is physically special about measurement? Alas, no compelling answer is forthcoming. Indeed, the most obvious problem with the collapse postulate is that it relies on an unanalyzed notion of 'measurement'. Most of the time, reasonable people can agree about when a measurement has occurred, but that fact is entirely unhelpful. For most researchers in the foundations and philosophy of quantum theory seem to agree that if quantum theory is supposed to be a fundamental theory, then it should tell *us* when a measurement occurs (or more generally, what a measurement is), not vice versa. The world should not rely on *us* to 'tell it' when to collapse a state.

5.4.4. *Macro-Micro Distinction.* One answer to the question 'what is a measurement' is suggested by our discussion of Schrödinger's cat (§5.1.1), and is sometimes put forward as a response to the measurement problem. The suggestion is that a measurement occurs when the state of a microscopic system is 'magnified' in such a way that it becomes correlated with the state of some macroscopic system. There is no denying that paradigmatic measurements do have this feature. Alas, this characterization relies on another ill-defined notion, namely, the difference between the microscopic and the macroscopic.

The distinction is also sometimes made between the 'easily reversible' and the 'esssentially irreversible'. Because microscopic systems have few degrees of freedom, their behavior is often easy to reverse, while the behavior of macroscopic systems, with vastly many degrees of freedom, is very difficult if not practically impossible to reverse.

In either case—whether the point is made in terms of size or reversibility—the idea is supposed to be that at an appropriately large (or irreversible) scale, physical

systems behave classically. Indeed, one sometimes hears 'the Copenhagen'[71] interpretation of quantum theory, and especially Bohr's version of it, characterized in this way: one *must* presume that measuring apparatuses are classical objects, and this presumption provides the grounds for determining when a collapse (measurement) occurs.

However, while again providing an often useful practical characterization of measurement and 'the classical', these distinctions are, in most researchers' view, not well enough defined to sit at the foundations of a supposedly fundamental theory.

5.4.5. *Decoherence.* A related idea, but recently more well developed, is that sufficient interaction with 'the environment' serves, in essence, to 'collapse' ('decohere') the state of a system. Here we need to be very careful, because there is a well-studied phenomenon, called 'decoherence', that involves the interaction of a system with its environment. There is no disputing the physical importance of this phenomenon. On the other hand, many have invoked this phenomenon as providing some sort of solution to the measurement problem. This claim requires careful scrutiny.

5.4.5.1. *The Phenomenon of Decoherence.*

5.4.5.1.a. Qualitative Description. It is remarkably difficult to isolate a physical system from the rest of the world (its 'environment'). Particles from almost every corner of the particle zoo are hurling around, and a great many of them can penetrate even very strong barriers (such as lead walls). Even for very small systems (such as a particle of dust), it is well-nigh impossible to prevent significant interaction with the environment.

Physicists have developed both simple and very sophisticated models of this interaction, ranging from assuming that the system interacts with an otherwise unspecified thermal bath to carefully modeling the rate, nature, and strength of interactions that a given system is likely to experience in a given type of environment. With these models, one can estimate (and in very rare, usually highly idealized, cases, explicitly determine) the effect of these interactions on the state of the system.

Notice that these interactions will *entangle* the system with the environment. We are therefore no longer talking, in the first place, about 'the' state of the system, but about the state of the composite 'system-plus-environment'. Of course, typically we will have little or no access to the relevant environmental degrees of freedom.

[71]This 'interpretation'—unlikely, however, to have ever been a single unified view—is so-called because it is typically associated with Niels Bohr and his associates, who worked in Copenhagen. See Cushing (1994, chs. 6,7), Beller (1999), and references therein, as well as Landsman, Ch. 4, §1, this volume.

(Imagine, for example, that a photon bounces off a dust particle, then gets trapped in the atmosphere, or worse, heads out to space. In the former case, recovering the photon will be practically impossible, and in the latter case, it could be literally impossible.) Although these degrees of freedom in a sense 'encode' certain information about the system (such as its location, because typically the interactions are position-dependent), that information is almost always practically, and occasionally in principle, lost to us.

Hence, although the system will become entangled with the environment, we typically have access only to its reduced state, not to the state of the total composite. We obtain this reduced state by tracing out the degrees of freedom of the environment. In many models of the interaction, the result is a reduced state for the system that is approximately diagonal in position; that is, its (mixed) state looks like the state of a system that is well-localized in space (in the sense that each component (spectral projection) of the mixture is well-localized in space). Decoherence may, in this sense, be thought of as 'localizing' the system (but see section 5.4.5.2.a, where we will worry explicitly about the fact that this mixture is improper).

This localization ultimately derives from the fact that in these models, the interaction between the system and the environment is position-dependent. (The environment interacts with the particle only in the vicinity of the particle.) More generally, if the interaction Hamiltonian that describes the exchanges of energy between the system and its environment commutes with some system observable, $Q \otimes \mathbb{I}$, then the reduced state of the system becomes approximately diagonal in the basis picked out by the eigenvectors (or eigenspaces) of Q. The environment is said to 'suppress' the off-diagonal, 'interference' terms. Moreover, because the interaction Hamiltonian commutes with Q, a system that is already in an eigenstate of Q will tend to remain so (assuming, as is often physically reasonable, that the interaction Hamiltonian swamps the effects of the free Hamiltonian for the system).

5.4.5.1.b. Example. A classic and much-studied example of decoherence (e.g., Joos and Zeh 1985), involves a dust particle in the atmosphere. Here is a simplified summary of their argument.

Let $|\psi\rangle$ represent the initial state of the dust particle. Let the states $|\psi_q\rangle$ be a basis of well-localized states for the particle (each centered at a position labeled q). Let $|E_0\rangle$ be the initial state of the environment, and consider the interaction between the particle and a single air molecule in the environment for a particle that is in one of the states $|\psi_q\rangle$: as a result of this single interaction, we will assume that the particle-plus-environment evolves into the state $|\psi_q\rangle \otimes |E_q\rangle$. (Here we assume, in particular, that

the dust particle is much heavier than the air molecule, so that the interaction leaves the state of the dust molecule essentially unchanged). Joos and Zeh show, roughly, that if $|\psi\rangle$ (represented as a wavefunction) is initially in a Gaussian (not necessarily well-localized) state, written in the $|\psi_q\rangle$-basis as $|\psi\rangle = \sum_q c_q|\psi_q\rangle$, then the matrix representation of the reduced density operator for the particle in the $|\psi_q\rangle$-basis (recall §1.2.3.4), after one such interaction, will be:

$$W_{qq'} = c_q c_q^* \langle E_q | E_{q'} \rangle, \tag{5.6}$$

where $|\langle E_q|E_{q'}\rangle| \approx 0$ whenever the distance between q and q' is much larger than the wavelength of the dust particle.[72] In other words, the off-diagonal ($q \neq q'$) terms in $W_{qq'}$ get reduced by a factor of $\langle E_q|E_{q'}\rangle$. The intuition here is that because the states $|E_q\rangle$ and $|E_{q'}\rangle$ correspond to environments where the air molecule has scattered from the dust particle in two different locations (represented by q and q'), if those locations are very different, then the corresponding states of the environment will be 'very different' (i.e., nearly orthogonal). (Of course, if $q = q'$ then this inner product is 1.)

Joos and Zeh then show, for a wide range of different models of the environment, that after *many* such interactions, the off-diagonal terms in the reduced density matrix decay exponentially, at a rate that depends on $q - q'$ (and is zero when $q = q'$). The rate is very fast: according to Joos (1986), the reduced density operator for a dust particle with radius 10^{-5}cm in even a high quality vaccuum will be extremely close to diagonal in position[73] in about one microsecond. One says that the state of the object system (the dust particle) has 'decohered' as a result of its interaction with the environment.

5.4.5.2. *Decoherence and the Measurement Problem.*

5.4.5.2.a. Decoherence Does not Solve the Problem. Decoherence appears to 'localize' systems that interact with their environment, so that perhaps we will not, after all, ever need to worry about encountering cats that are 'smeared' between two macroscopically distinct states (recall §5.1.1). Alas, there is a problem with this conclusion, stemming from the fact that the reduced state of the system is an *improper* mixture (recall §1.2.6.3.b). Let us investigate this point further.

On the one hand, we must acknowledge that it would be practically impossible to perform an *experiment* to determine that the reduced state of a system that has

[72]Here we are relying on de Broglie's relation between momentum and wavelength—recall note 58 of §4.2.1.

[73]In this case, 'extremely close to diagonal in position' means that the reduced density operator can be written as a sum of states that represent the dust particle localized to within about 10^{-13}cm.

decohered is *not* a proper mixture. To see why, consider the very simple case of two perfectly correlated particles (analogous to the object system and its environment, which, as a result of their interaction, have become correlated), as in (4.10). The reduced state for α is $W_\alpha = \sum_n (1/N)|a_n\rangle\langle a_n|$ (analogous to the reduced state of the object system, which, recall, is 'nearly diagonal in position'). This state makes the same predictions for *every* observable on α as the state in (4.10) does. In order to distinguish between proper and improper W_α, we would therefore need to measure some observable on the compound system ($\alpha\&\beta$), and in particular we would need to measure the *correlations* between α and β (for α & β is *not* in an entangled state–and therefore α's state is a *proper* mixture—if and only if there are no non-trivial correlations between them). Now, in the case of (4.10), measuring these correlations is relatively straightforward (assuming that we have many copies of the pair all in the same state), because in fact they are perfectly correlated in many observables. But in the case of interest here—an object system and its environment—things are decidedly more difficult, because the correlations exist only between a very few particles of the environment and the object system. As I noted above, it is in general practically impossible to recover those particles from the environment, and even if we could, in general not just any measurement on them will do the job. Indeed, if we do not have many copies of the total system (and in general we will not!), then the measurement in question becomes even more difficult, because we must measure an observable (analogous to the projection onto the state in eq. 4.10) that is not even a product (i.e., of the form $F \otimes G$).

On the other hand, the practical impossibility of performing such a measurement on the environment does not by itself license the assumption that the mixed state of the decohered system is (or may be treated as if it is) proper. The problem, as ever, is the eigenstate-eigenvalue link. The *true* quantum-mechanical state of the compound system is an entangled state, and in this state, according to the eigenstate-eigenvalue link, the object system does *not* have a definite location (in general). The assumption that the state of the system 'might as well' be a proper mixture is in fact *inconsistent* with this claim. In other words, while the two states are observationally very hard to distinguish, they are (given the eigenstate-eigenvalue link) interpretationally inconsistent.

Of course, one could give up on the eigenstate-eigenvalue link, and many interpretations do. We will discuss some of them shortly. The point here is that, with the eigenstate-eigenvalue link in place, decoherence does not solve the measurement problem.

5.4.5.2.b. Decoherence Does Help with the Problem. Decoherence does, however, help with the most general form of the problem of measurement. Recall (§5.3.2) that the measurement problem is not confined to interactions that would typically be regarded as measurments. *Many* interactions between quantum and macroscopic systems will put the macroscopic system in a state that, given the eigenstate-eigenvalue link, will be contrary to what most presume to be the deliverances of everyday experience. In other words, in its most general form, the 'measurement' problem is just the problem that quantum theory apparently fails to assign definite values to observables that appear, on the basis of everyday observation, to have definite values. In the special case of a typical measurement, we are perhaps already halfway to a solution, for in this case, the apparatus is at least already in a mixture, albeit improper, of 'desired' states.

But what about the more general sorts of interactions, mentioned above, that are not explicitly measurements but still generate 'bad' states for macroscopic objects? Decoherence promises to secure the following: for 'relatively large' systems (for example, at least as big as a dust particle), interaction with the environment will entail that the state of the system will become a mixture, albeit improper, of 'desired' states (or something that is very close to such a mixture—so close that one might be willing to overlook the difference). Even so, however, this strategy has merely reduced the general problem to a problem that continues to elude a satisfactory solution, although it does give one further justification for focusing the discussion specifically on measurements (because decoherence apparently reduces the general situation to one that is at least formally similar to measurement-situations).

5.5. **Interpretations.**

5.5.1. *Ways of Pursuing the Project.* I have covered a number of philosophical and foundational issues arising from quantum theory, and an 'interpretation' of the theory should address all of them in some way or other—from providing an account of the empirical content of the theory, to providing an understanding of incompatibility (and thereby the 'uncertainty relations'), to providing some understanding of the apparent failure of locality in the theory (§6). Nonetheless, most interpretations are aimed primarily at solving the measurement problem. As we saw above, that problem can be characterized (among other ways) as a conflict between (i) a common understanding of the physical properties of familiar physical objects, including the sorts of inferences about them that are valid, and (ii) a minimal account of when observables are empirically well-defined, taken in the context of (iii) quantum theory.

Conceived in this way, there are three ingredients that lead to a contradiction, and therefore three generic strategies for avoiding the contradiction:

(1) Deny that the common understanding of the physical world or inferences made about the physical world are always true or valid. For example, one might deny, contrary to appearance, that 'pointers' at the end of a measurement have a single definite (or nearly definite) position. We will encounter other ways to 'deny common sense' below. Let us call these theories 'uncommon sense interpretations'.

(2) Supplement, or replace, the minimal account of when observables are empirically well-defined. In some cases, extra 'hidden' variables are added to the theory. In all cases, systems are said to have properties that go beyond what standard quantum theory would assign, and hence all such interpretations are often, with admitted stretching of the concept, called 'hidden-variables interpretations'. (Inasmuch as adding new physical variables—and often a dynamics for them—to the theory is tantamount to proposing a new theory, these interpretations are often called 'hidden-variables *theories*'. I shall usually stick with the term 'interpretation', though I mean to be making no particularly substantive claim by doing so, and I shall occasionally use the term 'theory' to conform to custom.)

(3) Supplement (i.e., change) standard quantum theory as thus far described with some additional physics. Such interpretations add a rule (usually conceived as expressing a dynamical law of some sort) that results in systems having the properties, or nearly the properties, that we naïvely take them to have. Because the result of all such rules is a 'collapse' of the state from a superposition of values to a single value, these interpretations are often called 'collapse interpretations'. (Again, they are sometimes instead called 'collapse theories' and I shall ocassionally use that term as well.)

(Of course, the strategies might also be combined, but generally interpretations characterize themselves in terms of one of these strategies, even if one or both of the others fall out as an additional consequence.) I shall accordingly discuss interpretations under these three headings.

There is not space here to consider *all*, or even a *majority*, of the interpretations in each class. I will therefore have to restrict attention to a brief account of just two important representatives from each of the three genres. The reader should keep in mind that there are different versions of even the few interpretations that I shall

discuss here. In most cases, I have chosen to describe a version that strikes me as the least problematic, but this claim is, in most cases, controversial.

My treatment of these interpretations will, necessarily, be brief. I will give a few more references than normal, to get the interested reader started on the literature.

5.5.2. *'Uncommon Sense' Interpretations.*

5.5.2.1. *Quantum Logic Interpretations.* I have already noticed (§2.1.1) that the lattice of subspaces of a Hilbert space, i.e., the lattice of 'propositions' about a physical system, can, *prima facie*, be interpreted logically. Quantum logic interpretations take this idea seriously, and understand quantum theory to necessitate a revolutionary change from classical to quantum logic.[74]

One way to characterize the difference between classical and quantum logic is in terms of the failure of distributivity (§7.4.3)—the classical and quantum lattices of propositions are otherwise structurally (logically) the same. The basic idea of the quantum logic interpretation, then, is that in particular the classical law of distributivity is *invalid*. This invalidity is supposed to allow one to make the sorts of assertion tha, for example, the naïve realist (§5.4.1) had hoped all along to be able to make. For example, let F and G be two incompatible maximal observables sharing no eigenvectors. Denote their eigenspaces by $\{F_n\}$ and $\{G_m\}$. As propositions (elements of the lattice of subspaces of the Hilbert space, interpreted logically), they assert that a system has the corresponding eigenvalue as a value for the observable. Then:

$$\left(\bigvee_n F_n\right) \wedge \left(\bigvee_m G_n\right) = \mathbb{I} \wedge \mathbb{I} = \mathbb{I} \tag{5.7}$$

where $\mathbb{I}$ is the logically true proposition. If we read the first half of this conjunction as asserting that the observable F has *some* value (understanding the existential quantifier in terms of disjunction, as one often does), and similarly for the second hand of the conjunction, then (5.7) asserts that F has a value and G has a value. Indeed, this assertion is *logically* true. (Note that if we could apply distributivity to (5.7), then we could quickly turn it into a logical contradiction, because for any n and m, $F_n \wedge G_m = 0$, the logically false proposition.)

[74] Birkhoff and von Neumann (1936) were among the earliest advocates of something like a quantum logic interpretation. Subsequent work in this area is founded on theirs. Some of the advocates of a quantum logic interpretation have been: Finklestein (1962; 1969), Putnam (1969), Friedman and Putnam (1978), and Bub (1974). See also the collection of papers (Hooker 1975,1979). Some well-known critiques have been made by Dummett (1976) and Gibbons (1987). Many of the standard objections are addressed in Dickson (2001), and occasionally also in the work of the advocates listed above.

Moreover, the corresponding claim is true for a similar conjunction ($\wedge$) of such disjunctions for any number of observables. Hence (understanding the universal quantifier in terms of conjunction) the quantum logic interpretation claims to recover the idea that all observables (for a given system) always have a value (for that system). If so, then the measurement problem is no longer a problem. Of course, this approach also raises some questions. In addition to doubts about whether logic is revisable at all, some have argued that the quantum logic interpretation simply moves the mystery from one place (quantum theory) to another (logic). In any case, it is clear that the quantum logic interpretation needs to make some argument about why classical logic does in fact work within certain domains. (Decoherence could be helpful, here.) It also needs to say something about the success of classical reasoning in mathematics. Both of these projects amount to open questions in the quantum-logic interpretation (Dickson 2001).

5.5.2.2. *Many-Somethings Interpretations.* Our second example of 'uncommon-sense' interpretations is the 'many-somethings' interpretations. These interpretations generally go under the heading 'many-worlds' interpretations, but the notion of a 'world' that is at work here (quite problematic in its own right) is sufficiently far from the usual understanding of that term that a less committal word seems appropriate.[75]

In any case, whereas quantum-logic interpretations deny (or at best re-interpret) the logical *validity* of apparently valid inferences about the properties of physical objects, many-somethings interpretations deny (or at best re-interpret) the *truth* of apparently true claims about the properties of physical objects.

Consider again the right-hand side of (5.2). The problem that this state raised earlier was that, on the standard interpretation (adopting the eigenstate-eigenvalue link), a (compound) system in this state has *no* particular value for the observable $F \otimes Q$ (where Q is the pointer-observable). The many-somethings interpretations go to the opposite extreme, and claim that *every* term on the right-hand side is 'real', corresponds to some reality. The apparatus has neither zero, nor one, of the values for the pointer-observable, but *all* of them.

Immediately, these interpretations face some questions. There is, of course, the obvious question of how systems can manage to 'have' multiple states—it appears

[75]Indeed, we are talking, here, about a rather wide class of interpretations. It is far from clear that the intention of the originator of this class of views, Everett (1957), was to invoke the idea of many worlds, which appears to have been introduced by DeWitt (see his, and Everett's, article in DeWitt and Graham, 1973). A 'many-minds' variant has been proposed by Albert and Loewer (1988) and Donald (1990), among others. See the review by Butterfield (1995) and the book by Barrett (1999) for finer-grained classifications and many references.

to involve a straightforward contradiction. Generically, the reply is to introduce an indexical property that resolves the contradiction—the different values of the pointer-observable, for example, are realized *relative to* different values of the indexical property, which correspond to different *somethings* ('worlds', or in some versions, 'minds').

Everett's original idea, however, seems rather to have been fundamentally relational in character. Indeed, consider a generic two-system state, as in (1.26). If the first system might be said, in some sense, to 'really have' the state $|x_j\rangle$, then *relative to* $|x_j\rangle$, the second system might be said to be in the state $|\psi_{\text{relative to } x_j}\rangle = K_j \sum_m c_{jm}|y_m\rangle$, where K_j is some constant of normalization. The probabilities generated by $|\psi_{\text{relative to } x_j}\rangle$ for the results of measurements on the second system are exactly those generated by the original compound state (for measurements that are restricted to the second system). Everett's original idea seems to have been that systems posses states only 'relative' to the states of other systems. Others seem to adopt something like the view that the first system has various of the $|x_j\rangle$ in different 'worlds', and in those worlds, the second system has the corresponding relative state.

Each of these views raises difficult questions. In the case of Everett's relational view, more needs doing. Consider the analogous case in space-time theories, where (prior to the 20th century), relationalists *asserted* that the only (spatio-temporal) reality is relational, but the only *theory* that was available (Newtonian classical mechanics) was *not* explicitly relational.[76] Everett's relational view seems to be in somewhat of a similar situation. (For example, how and from what 'point of view' does the quantum state for the compound system itself get assigned? And how can *it* be understood relationally?) The many-somethings views face the obvious metaphysical hurdle of making some sense of the plurality of 'somethings' (worlds, minds, whatever).

Here, I will raise just one additional question, commonly raised. The general prescription of these interpretations is to allow that *every* term in a superposition corresponds to a 'reality', in the sense just described. But recall that a given state can be decomposed in many ways, and that in general the terms in the superposition are different, depending on the decomposition. Which decomposition is 'correct'?

One sort of answer to this question will postulate, or argue for, a *preferred basis*, in terms of which the decomposition is to be made. (The argument for a preferred basis is most often made on the grounds that decoherence (§5.4.5) picks it out, but there is a serious question whether the *near*-diagonality in position that decoherence

[76] See, for example, Brown and Pooley (2002) for details of the history, and see Butterfield, Ch. 1, §2.3, this volume, for further discussion of relationalism.

typically effects is sufficient.[77]) This answer is, in any case, highly problematic. For suppose that we had some good reason to believe that there is a preferred basis whose elements represented the true physical properties of all objects. In that case, we would hardly need the extravagance of the many-somethings interpretations to resolve the measurement problem. Relying on the preferred basis, we would simply assert that systems are always to be described in terms of this basis. There would then be no obstacle (for example, no Kochen-Specker contradiction) to defining probabilities over the properties picked out by this basis. (Indeed, below (§5.5.3.1) we will consider an interpretation—the de Broglie-Bohm theory—that can be characterized in precisely these terms.)

Another answer allows *all* decompositions to correspond to realities. Here, decoherence *can* be helpful, because it suggests that *creatures like us*, that is, creatures who interact with their environment in the particular way that we in fact do, will in fact be subject to decoherence, so that our perceptions will be correlated to the 'right' states of the objects of our perception, and we, as perceivers, will, as far as we can tell (with very high probability) be in worlds where the properties are the ones we typically believe objects to have. Of course, there remains to explain the point that in fact there are many 'copies' of each one of us (one for each value of the indexical), but we leave that issue to the advocates of these interpretations, who have their own ways of making sense of this type of plurality.

5.5.3. *'Hidden-variables' Interpretations.* Ths basic idea behind any hidden-variables interpretation is that quantum theory (as EPR argued!—§4.4) is incomplete. These interpretations propose to supplement the state assigned to a system by quantum theory with an additional, 'hidden', state. (The variables need not in any sense be unavailable to observation; they are 'hidden' only from the eyes of quantum theory.) Of course, not just any proposal will do. In particular, such theories must somehow recover the empirical success of quantum theory. Doing so amounts to recovering the quantum probabilities as an average over the hidden states, in much the same way that one hopes to recover classical thermodynamics from classical statistical mechanics by averaging over the micro-states of the system. (See Uffink, Ch. 9, this volume.)

Hence, for example, label the hidden states by λ (where for simplicity we will assume that $\lambda \in \mathbb{R}$). Let $\rho_W()$ be the distribution of the hidden states in the (quantum) state W. Finally, let $\Pr_{\lambda,W}(F = f_k)$ be the probability assigned by the hidden state λ to

[77]It is worth noticing as well that a state's being *nearly* diagonal in position does not entail that the basis in which is *is* diagonal is anything close to position. See Bacciagaluppi (2000). In these cases, which basis are we to say is 'picked out' by decoherence?

the result (eigenvalue) f_k of a measurement of F on a system in the state W. (In the deterministic case, all such probabilities are of course 0 or 1.) Empirical adequacy then requires that

$$\int_{\mathbb{R}} \rho_W(\lambda) \mathrm{Pr}_{\lambda,W}(F = f_k) d\lambda = \mathrm{Tr}[W P_{f_k}], \tag{5.8}$$

where P_{f_k} is the eigenspace of F corresponding to the eigenvalue f_k.

In this section, I will consider the de Broglie-Bohm theory (§5.5.3.1), a deterministic hidden-variables theory, and modal interpretations (§5.5.3.2), which are in general indeterministic.

5.5.3.1. *The Theory of de Broglie and Bohm.* The de Broglie-Bohm theory is a deterministic theory of the trajectories of particles, somewhat in the mold of classical Newtonian (better, Hamiltonian) dynamics.[78] In its original form, the hidden-variables theory was formulated explicitly by dividing the Schrödinger equation (including a potential, V, in the Hamiltonian) into a real and complex part, so that the solution, $\psi(x,t)$, is written as $\psi(x,t) = R(x,t)e^{iS(x,t)}$. The complex part has the form of Hamilton's equations of motion, with an extra term in the expression for the potential energy, the 'quantum potential':

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{1}{2m}\frac{\nabla^2 R}{R} = 0 \tag{5.9}$$

This equation has the form of Hamilton's equation, with the potential given by $V + U$ where U is the 'quantum' potential $-\frac{1}{2m}\frac{\nabla^2 R}{R}$. The real part has the form of a continuity equation, and implies the conservation of probability:

$$\frac{\partial p}{\partial t} + \nabla \cdot p\dot{q} = 0, \tag{5.10}$$

where $p(x,t) = R^2(x,t) = |\psi(x,t)|^2$ is understood (stipulated to be) the probability of finding a particle at the point x (at time t) and

$$\dot{q}(x,t) = \nabla S(x,t)/m \tag{5.11}$$

is interpreted as the velocity of the particle (when it is at the point x and time t).

Hence, one can understand the one-particle Schrödinger equation as describing an ensemble of particles (each of whose 'quantum state' is the same), distributed (in space) according to $p(x,t)$. By (5.10), this distribution is conserved over time, and thus may be understood properly as a probability distribution. The individual

[78]The basic idea was first suggested by de Broglie, most famously in his (1927). Later (after de Broglie had been convinced that the theory does not work) it was developed by Bohm (1952), and has seen much further development in the past few decades. For a variety of approaches to and viewpoints on the theory, see Cushing, Fine, and Goldstein (1996).

particles move as if they were classical particles governed essentially by Newton's laws with the usual classical potential plus the additional 'quantum potential', U.

This theory is important for several reasons. Here I will mention three.

First, it shows that the eigenstate-eigenvalue link is far from a required principle of interpretation. Indeed, the eigenstate-eigenvalue link is quite strongly denied in this theory. The position observable does not even *have* eigenstates, and yet, on this theory, *every* particle *always* has a definite position.

Second, it provides a clear counter-example to oft-cited lessons of quantum theory, to the effect that quantum mechanics implies a lack of determinacy at the fundamental level, the impossibility of fundamental determinism, the impossibility of definite trajectories for particles, the impossibility of picture quantum phenomena, and so on. The theory clearly describes particles with definite trajectories, moving deterministically under the potential $V + U$.

Third, this theory is probably the best-developed interpretation, with applications. There are compelling accounts of how particles behave during measurements of spin, for example (Dewdney, Holland, and Kyprianidis, 1986). Moreover, the theory has occasionally been used to make headway on applications and problems that seemed much more difficult from the standard point of view. A nice example involves the prediction of how long a particle that will tunnel through a potential barrier will spend in the barrier. This quantity is conceptually tricky (at best) in standard quantum theory, because the 'particles' have no definite trajectories, and there is no time observable in quantum theory. But the problem is conceptually straightforward within the de Broglie-Bohm theory (Leavens 1990), precisely because one can essentially display the possible trajectories of a particle under the influence of the potential barrier (*and* the quantum potential), and then simply take the average of the times spent in the barrrier.

However, we should not ignore some difficult issues faced by the theory. Setting aside (as outside the purview of this article) the (very significant) difficulty of extending the basic idea to relativistic quantum field theory, there are other questions that it faces. Here I will mention one.

Initial appearances notwithstanding, the ontology of the theory can stretch the imagination a bit. Indeed, one faces the following dilemma. One the one hand, if we take the 'guiding field' (quantum potential) to be a *real* potential field of some sort, it is highly non-local. (In the literature, this point is sometimes put in terms of the fact that it 'lives' not in $\mathbb{R}^3$, but in the configuration space for the multi-particle system; of course, for a one-particle system they are the same.) This point gets reflected in

the form of the 'guidance condition' (5.11) for many particles: $\dot{q}_i(x_1, x_2, \ldots x_n, t) = \nabla_i S(x_1, x_2, \ldots x_n, t)/m_i$. That is, the *velocity* of particle i depends not only on *its* location, but also on the location of the *other* particles. Finally, notice that the particle does not respond to the *intensity* of the quantum potential in its region, but to the *form*. That is, increasing the intensity of the quantum potential (multiplying R in (5.9) by some constant > 1) does not change its effect on the particles at all. All of these (and more) observations will, for some, add up to the conclusion that the quantum potential is, at best, in need of some serious metaphysical clarification, and at worst, too bizarre to countenance.

On the other hand, one might then deny its reality. A particle moving according to the guidance condition, with $V = 0$ (zero 'classical' potential) will in general still 'deviate' from the *classically* expected trajectory, but are we thereby *required* to suppose that this 'deviation' is due to the presence of some additional, non-classical, potential? Perhaps, instead, we can take the guidance condition as a fundamental rewriting of Newton's laws.[79] In this case, we may avoid the oddness of the quantum potential as a physical entity, but the theory becomes extremely reductionistic—the *only* real property of particles is their spatio-temporal trajectory. Such a theory is forced to eschew providing explanations (for phenomena such as bonding between particles) that one might wish to have, and in general we *do* have in standard quantum theory.[80]

5.5.3.2. *'Modal' Interpretations.* Model interpretations[81] are in many ways similar to the de Broglie-Bohm theory: they also postulate hidden-variables (denying the eigenstate-eigenvalue link) and they also (at least, they *can* formulate a dynamics for these hidden-variables. There are two major differences: (1) in general model interpretations will allow that the physical quantities that are 'definite' for a system are state-dependent, and therefore they can change in time; (2) in general (and typically as a consequence of the choice they make for which physical quantities are definite) model interpretations are stochastic. Many such interpretations have been

[79]Indeed, the basic idea here is to reform the very notion of an 'inertial' trajectory. See Pitowsky (1991.

[80]See Bedard (1999 and Dickson (2000.

[81]There are many variants on the general theme of modal interpretations. The term itself was coined by van Fraasen (1972). A renaissance of sorts occurred in the 1980s, with, for example, key works by Kochen (1985), Dieks (1988), and Healey (1989), and a later related but somewhat different view put forward by Bub (1997). A helpful monograph is Vermaas (2000), and a helpful collection is Dieks and Vermaas (1998).

proposed. Here we concentrate one class of such interpretations ('spectral modal interpretations') that has seen much development, and briefly refer to some more recent ideas.

Recall that every density operator can be uniquely decomposed as a weighted sum of its spectral projections. The central claim of spectral modal interpretations is that the set of observables with a definite value is the *largest* set that can consistently be assigned values, with the restriction that the spectral projections of the density operator must have values (0 or 1, in this case). In other words, treat the density operator as an observable, and assign it a value; then assign values to as much else as one can without running into a Kochen-Specker-type contribution. It turns out that, under some reasonable assumptions, this idea leads, for any given state (density operator), to a unique set of observables that will get assigned a definite value.[82] Because the state (of course) changes in time—and keep in mind that we are talking about the *reduced* density operator, so that it need not change unitarily—the definite-valued observables generally change in time.

Many (though not all) modal theorists aim to define some sort of dynamics for the definite properties of a system. The problem is complicated because there are two sorts of dynamics occurring 'together': (deterministic) changes in the set of definite-valued properties (observables) for a system, and (stochastic) transitions from one definitely possessed property to another. It was shown by Bacciagaluppi and Dickson (1999) that the problem has a solution, and they exhibit some examples. However, just as in Bohm, there arises a serious difficulty about whether such a dynamics can be made Lorentz-invariant. Straightforward generalizations from the non-relativistic to the relativistic case are known to be unavailable, but it remains an open question whether a 'natively' relativistic version of the approach can work. The resolution of this issue is connected with the more general issue of whether, and if so how, the basic idea of the modal interpretation ('make as many observables definite as possible, given certain constraints') can be extended to quantum field theory. If it can, it seems that the most likely venue will be that of algebraic quantnum field theory, for even the non-relativistic quantum-mechanical versions of the modal interpretation lend themselves nicely to an algebraic formulation.[83]

[82]As an example of this genre of theorem, see Clifton (1995). Bub and Clifton (1996) prove a similar theorem, but geared towards Bub's (1997) interpretation.

[83]For example, see Halvorson (1999). For an initial foray into modal interpretations of quantum field theory, see Clifton (2000). For a discussion of this foray especially as it relates to the issue of Lorentz-invariance, see Earman and Ruetsche (2006), which also includes references to the earlier work on Lorentz-invariance in modal interpretations.

Do modal theories solve the measurement problem? If so, they do because their prescription for choosing the definite-valued observables manages to pick out the observables—such as pointer observables—whose definiteness is threatened by the measurement problem. The de Broglie-Bohm theory (as well as the modal interpretation of Bub 1997) accomplishes this feat more or less by fiat. So long as the properties we believe pointers (and cats, and so on) to have can ultimately be understood to supervene on the trajectories of the constituent particles, the de Broglie-Bohm theory has a convincing solution to the measurement problem. Modal interpretations must in general argue their way to this point. For example, is definiteness of the spectral projections of the reduced density operator enough to secure definiteness of the properties that we believe macroscopic objects have? This question, as it turns out, is quite difficult to answer. In the case of an idealized impulsive measurement, the reduced density operator for the apparatus *does* have the definite pointer-states as its spectral projections. But in realistic (non-ideal) measurements, there is potential trouble. At one time, it was thought that decoherence saves the day, but it is no longer clear whether decoherence does the work needed here.[84]

5.5.4. *Collapse Interpretations.*

5.5.4.1. *Non-Dynamical Collapse Theories.* The idea that the quantum state must, in some sense or other, 'collapse', is quite natural as soon as one interprets it in terms of probability. And various suggestions have been made, from time to time, about when this collapse occurs. See, for example, Dirac (1930)

> Consider an observation, consisting of the measurement of an observable α, to be made on a system in the state ψ. The state of the system after the observation must be an eigenstate of α, since the result of a measurement of α for this state must be a certainty.

That is, collapse occurs upon observation (i.e., measurement). The argument that Dirac is making here is the following. If we repeat the measurement of α on the system, we will get the same result that we got the first time (regardless of ψ, and assuming, of course, that the measurement was of the first kind—§) with probability 1. Suppose that result was a. There is only one state that assigns probability 1 to a, namely, the eigenstate corresponding to a.

[84] See Bacciagaluppi and Hemmo (1996) for the former attitude, and Bacciagaluppi (2000) for the latter.

I have already suggested (§5.4.3) that an account along these lines is, at best, in need of help.[85] Otherwise, it is little more than a statement of the collapse postulate, with no clear account of what makes the difference between 'measurements' and other interactions.

One obstacle to drawing this distinction in a principled way is that if too *few* interactions count as measurements, then we might, after all, still be stuch with the measurement problem (because collapse might not occur in situations where we need it to get the definiteness of properties that we experience as definite). However, if too *many* interactions count as measurements, we might end up with a theory that is empirically false. (For example, if the state of a photon passing through the double-slit apparatus (figure 3) *always* collapses after it passes through the barrier, then we will never see an interference pattern, contrary to experimental results.)

One approach to drawing the distinction (between cases where collapse occurs and cases where no collapse occurs) that arguably avoids both of these pitfalls, and arguably does begin to smell like interpretation of the theory, is due to Wigner (1961). Wigner is motivated by the following argument, often referred to by the name 'Wigner's friend'.

Wigner asks us to imagine the following scenario. A measuring apparatus, μ, measures some observable, F (by means of a pointer-observable, M), on an object-system, σ. Meanwhile, both Wigner (α) and his friend (β) are in a position to observe μ (by means of 'observation-observables' A and B). Let us use $|m_n\rangle$ to refer to the eigenstates of M, and similarly for the rest.[86] Now, suppose that the measurement of σ by μ has occurred, while neither α nor β has observed μ. If σ was originally in the state $\sum_n c_n|f_n\rangle$ then we may write the state of the total system as

$$\sum_n c_n|f_n\rangle|m_n\rangle|a_0\rangle|b_0\rangle \tag{5.12}$$

According to the eigenstate-eigenvalue link, in this state, μ and σ are *not* in a definite states of $M \otimes F$. At this stage, standard quantum theory will say to invoke the collapse postulate, so that the state becomes $|f_k\rangle|m_k\rangle|a_0\rangle|b_0\rangle$ for some k. In other words, in absence of specific knowledge about the outcome, the state becomes the (ignorance-interpretable) mixture

$$\sum_n |c_n|^2 P_{f_n} \otimes P_{m_n} \otimes P_{a_0} \otimes P_{b_0}. \tag{5.13}$$

[85]And, alas, such help has only rarely been forthcoming. Hence I have no list of references to offer the reader, here. Perhaps the most famous attempt to make sense of non-dynamical collapse is precisely the one discussed here, due to Wigner.

[86]There is a serious question whether a physical state such as 'observes the pointer to be in the state $|M_n\rangle$ is pure, and therefore whether there *are* observables such as A and B. See §1.4.3.

However, as noticed above (§5.4.5.2.a), there is a real empirical difference between (5.12) and (5.13), even if in general it is practically impossible to detect. Wigner finds nothing in the nature of σ, μ, or the interaction between them, to warrant the assumption that this physical change of state occurs.

Notice, moreover, that the observers α and β have not yet entered the picture in a substantial way. But it is *their* definite experiences that we are (in this argument) obligated to secure. Wigner considers himself (α), and notices that at *this* stage of the process (i.e., once the state is given by eq. 5.12), nothing threatens the definiteness of his experiences—in fact, he remains in the 'has not yet observed' state ($|a_0\rangle$) with probability 1. So consider the state after α observes the apparatus:

$$\sum_n c_n |f_n\rangle|m_n\rangle|a_n\rangle|b_0\rangle. \tag{5.14}$$

Now Wigner feels compelled to assent to collapse, because otherwise (by the eigenstate-eigenvalue link) his own state would be indefinite.

But what about Wigner's friend (β)? Suppose that β observes the apparatus before α. Wigner argues that, given a principle of charity about β's reports of prior mental states, he is obligated to collapse the state upon observation by β, even if he (α) has not yet observed μ. For suppose that α then (after β) observes μ, and then asks β: After you observed μ, did you feel that you experienced a definite outcome? Did you observe it to be in a definite state? Surely β will answer 'yes', and assuming (as Wigner does—this assumption is the principle of charity) that we are to believe β's report, we must assume that the state collapse upon observation by β.

Hence any observer to whom the principle of charity applies will effect a collapse of the state, upon observation. Wigner believe that the principle extends to anything that is, or has, a 'mind' (a term that is largely uncircumscribed, here). Moreover, given a fairly strong distinction between physical bodies and minds (i.e., given some form of dualism), one can point to the distinctness of minds as somehow the *reason* that observation by a mind collapses the state, while observation by a non-mind (e.g., a typical measuring apparatus) does not collapse the state.

Setting aside the question of the plausibility of the premises of Wigner's argument, notice that we have arrived at this conclusion through the backdoor. In other words, Wigner does not give us an argument that proceed from the nature of minds, and appeals to that nature to argue that observation by a mind will collapse the physical state. Indeed, Wigner's view faces the same difficulty that all dualisms face: what is the connection between mind and matter? *How* does mind collapse the *physical*

state of a system? Various extraordinarily speculative ideas have been floated, none particularly convincing. (See Atmanspacher 2004 for a review.)

5.5.4.2. *Dynamical Reduction Theories.* Another strategy for making collapse well-defined is to describe it as a physical process. Of course, in one sense it already *has* been characterized as a physical process, for it is a change in the physical state of a system. Here, however, I mean something more, namely, formulating a single equation of motion that somehow incorporates *both* the continuous 'Schrödinger' evolution *and* collapse, in a unified way. There have been (and continue to be) many proposed schemes along these lines.[87] Here, again, I consider just one example.

5.5.4.2.a. Intuitive Account of Continuous Spontaneous Localization. The example that I will consider is the continuous version of a theory originally proposed by Ghirardi, Rimini, and Weber (1986). The continuous version, called 'continuous spontaneous localization' (CSL), is due to Pearle (1989). It is easiest to state in terms of wavefunctions. The basic idea is that each physical system in the universe experiences, simultaneously, two types of evolution: the 'normal', deterministic, 'Schrödinger evolution', and a stochastic tendency towards localization. By the latter, we mean that the system's wavefunction (in configuration space) experiences random (but infinitesimally small) fluctuations, the net result of which is, on average and with overwhelming probability, to tend to make the wavefunction more localized—more of the probability gets concentrated in some localized region.

The strength of this stochastic evolution, which determines how quickly the localization occurs, is determined by the number of particles in the system.[88] For systems with microscopic numbers of particles, the stochastic part of the evolution is swamped by the Schrödinger evolution, so that the evolution is, for such systems, nearly identical to that given by standard quantum theory. But for systems with macroscopic (e.g., around 10^{23}) numbers of particles, if the system is initially in a superposition of two (or more) localized states whose centers of localization are far apart ($> 10^{-5}$cm or so), then the stochastic element of the evolution acts very quickly to suppress (continuously) all but one of the terms in the superposition. The parameters of this stochastic process are arranged so that the probability of reduction to one state or the other is equal to the quantum probability assigned to that state (by the initial superposition).

[87]See, for just a few examples, Diosi (1992), Ghirardi, Rimini, and Weber (1986), Gisin (1984), and Primas (1990).

[88]There are schemes in which the strength of the stochastic evolution depends on other things, for example, mass.

5.5.4.2.b. Mathematical Account of Continuous Spontaneous Localization. Here are a few mathematical details. In CSL, the evolution of the wavefunction (here written in one spatial dimension) is given in terms of the (non-unitary[89] operator:

$$U(t) = \exp[-iHt] \exp\left[-\frac{\gamma}{2} t \int N^2(x) dx\right] \exp\left[\int N(x) B_x(t) dx\right]. \tag{5.15}$$

The first term is just the usual quantum-mechanical evolution operator (for a time-independent Hamiltonian, H). The $N(x)$ are a family of 'number density' operators that, intuitively, indicate roughly the number of particles inside some region centered on x (or, in three dimensions, a ball of some fixed radius around x—this fixed radius is part of the definition of the $N(x)$, which we set aside here). The $B_x(t)$ is a field of stochastic processes (i.e., a continuous family of stochastic processes, each evolving at a point x). Hence the evolution operator in (5.15) is 'stochastic'—it depends on the values that the $B_x(t)$ take. Notice, though, that for a *given* realization of the $B_x(t)$, one can plug the realization into (5.15) and it would then be deterministic.

The second term in (5.15) is a 'decay' term, where the rate of decay depends on γ (and the particle-density). The third term in (5.15) is a 'growth' term, where the growth depends on the evolution of the stochastic process $B_x(t)$ (and the particle-density). These two terms 'fight' against one another, with the ultimate result tending (with extremely high probability) to be an overall decay of the wavefunction, except in some localized region, where it grows.

$B_x(t)$ is defined so that the likelihood of a growth in any given region is directly proportional to the amplitude of the wavefunction in that area. Hence the probability that the growth term will eventually cause the wavefunction to grow in a given area is indeed the same as the quantum-mechanical probability that a collapse would occur in that region. Notice that as the wavefunction begins to collapse onto some region, the probability that it will continue to collapse onto that region grows, because the amplitude of the wavefunction will be higher there, and therefore the stochastic processes in that region will be more likely to grow.

The final 'trick' is to choose γ so that the claim of the previous section is true, namely: for systems with a small number of particles, the first term in (5.15) dominates the other two (because $N(x)$ will be small everywhere), while for systems with a very large number of particles, the second and third terms will come into play. There are empirical constraints on the choice of γ, but it is in fact possible to find values for γ that are consistent with known experimental fact.

[89] In general, the norm of the wavefunction is not preserved by its evolution in CSL. However, it is a straightforward matter to renormalize the the wavefunction at any time.

5.5.4.2.c. Evaluation. CSL has the obvious virtue of being a well-defined, mathematically precise, theory that manages to describe 'collapse' as a physical process in an unambiguous way. Nonetheless, several questions have been raised about it. Here I mention two.

The first concerns perhaps the biggest theoretical obstacle faced by the theory, namely, the formulation of a relativistic version. While progress has been made in various directions (e.g., Ghirardi, Grassi and Pearle 1990), serious problems remain. Moreover, these problems are directly related to the characteristic feature of CSL, the introduction of the family stochastic processes $B_x(t)$, which, in the relativistic context, gives rise to infinities that do not appear in the standard theory (and cannot be 'renormalized away' in the same way).

The second issue concerns the fact that the collapse in CSL is never complete. That is, the state never becomes one whose support is *entirely* contained within a localized region. Instead, it always has 'tails'—the wavefunction always has non-zero amplitude at points outside the region in which the system has supposedly been localized.

The question here is whether one is still licensed to call such a system 'localized'. If we maintain the eigenstate-eigenvalue link then the answer might well be 'no'. After all, a wavefunction with tails is *not* an eigenstate of the position observable, nor any coarse-graining of it. The response on the part of at least some advocates of CSL has indeed been a new understanding of the wavefunction, as some sort of direct representation of, as Bell (1990) put it, 'density of stuff'. This view, while avoiding the immediate problem, does have some metaphysically disturbing consequences, such as the fact that apparently there will, on this view, be very 'faint' (low-density) 'copies' of all objects located in many places.

6. Non-locality

Recall (§1.2.6.4) that compound systems in a non-factorizable state may exhibit correlations in the propreties that they exhibit (upon measurement at least). These correlations may be 'non-local'. Non-locality involves, first, the spatial separation of the particles (and the measurement-events on them that exhibit the correlated properties). There is the question, then, whether correlations exhibited by spatially separated systems can be explained in terms of a 'common cause', a single event in the history of the two systems that could explain the correlation between them.

In §6.1 I will clarify this question (§6.1.1), and then consider some theorems that show, under reasonable assumptions, that in fact no such common cause explanation is available (§6.1.2–6.1.3). Hence quantum theory–and indeed the physical world

itself!—appears to be 'non-local', in some sense or other. In §6.2 I will consider several reactions to these theorems, and the implications for the prospects of finding a relativistically invariant and also conceptually satisfactory interpretation of quantum theory.

6.1. **No-Go Theorems.**

6.1.1. *Non-local Correlations.*

6.1.1.1. *Statistical Correlation.* The term 'correlation' comes from statistics, and it is being used here in that sense. Given two random variables, A and B, their correlation is defined to be:

$$r_{AB} = \frac{\mathrm{E}[(A-\bar{A})(B-\bar{B})]}{\sigma(A)\sigma(B)}, \tag{6.1}$$

where $\mathrm{E}[\cdot]$ is the expected value and $\sigma(\cdot)$ is the standard deviation. The correlation is 1 if $A = kB + m$ for positive k (and any fixed m, and -1 for negative k. The numerator of (6.1) is called the 'covariance'. The denominator is present essentially for normalization.

It should be clear that correlation is a measure of the 'dependence' of the value of one variable on the other. Non-zero correlations will generally be a consequence of the fact that, for at least some possible values a and b of the random variables, $\Pr(A = a|B = b) \neq \Pr(A = a)$, i.e., A and B are not statistically independent.

6.1.1.2. *Correlations in the Singlet State.* For a pair of spin-1/2 particles, the state

$$|\psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}\Big(|z_+\rangle|z_-\rangle - |z_-\rangle|z_+\rangle\Big). \tag{6.2}$$

is called the 'singlet' state. (Notice that it is a two-particle analogue of the state in eq. 4.10.) Consider the observables S_z on each of the two particles. (One can think of these observables as random variables, because the state in (6.2) generates probabilities for the possible values of these observables, or more precisely, for the observables $S_z^{(1)} \otimes \mathbb{I}^{(2)}$ and $\mathbb{I}^{(1)} \otimes S_z^{(2)}$, where the superscripts number the particles.) The covariance of these observables (normalized to have eigenvalues ± 1 in order to bypass calculating standard deviations) in the state $|\psi\rangle$ above is:

$$\begin{aligned} r_{S_z^{(1)}S_z^{(2)}} &= \langle\psi|\Big(S_z^{(1)} \otimes S_z^{(2)}\Big)|\psi\rangle \\ &= \tfrac{1}{2}\Big(\langle z_+|\langle z_-| - \langle z_-|\langle z_+|\Big)\Big((-1)|z_+\rangle|z_-\rangle - (-1)|z_-\rangle|z_+\rangle\Big) \\ &= -1. \end{aligned} \tag{6.3}$$

The values are perfectly anti-correlated. Indeed, a similar calculation for $S_u^{(1)} \otimes S_u^{(2)}$ (for an arbitrary direction u) shows that $r_{S_u^{(1)}S_u^{(2)}} = -1$; the perfect anti-correlation

holds in every direction. For distinct directions, u and u' (but, without loss of generality given the spherical symmetry of the state, setting $\phi = \phi' = 0$ in (1.36) and again 'normalizing' the observables), we calculate:

$$r_{S_u^{(1)} S_{u'}^{(2)}} = -\cos\theta\cos\theta' - \sin\theta\sin\theta' = -\cos(\theta - \theta'). \tag{6.4}$$

This correlation is predicted by quantum theory, and well confirmed by a variety of experimental tests.

6.1.1.3. *Common-Cause Accounts of Correlations.* Of course, in one sense, non-local correlations are ubiquitous. Consider: there is a correlation in the tides at different locations on the planet; there is a (negative) correlation between the average daily temperature in Sydney, Australia and London, England; there is a (postive, near-perfect) correlation in the number of words appearing in the newspaper at my front door each morning, and the one appearing at my neighbor's front door each morning. None of these correlations are surprising, even though they are correlations between the properties of spatially separated objects. The reason is clear: these 'non-local' correlations have a *local* explanation, in terms of a common cause.

Following Reichenbach (1956, 158-159), one can formalize the notion of a common cause in terms of conditional probabilities. Suppose that A and B are correlated. In this case, $\Pr(A \wedge B) \neq \Pr(A)\Pr(B)$. A common cause for the (let us say, simultaneously occurring) events A and B is an event, C, such that: (i) $\Pr(A|C) > \Pr(A|\neg C)$; (ii) $\Pr(B|C) > \Pr(B|\neg C)$; (iii) $\Pr(A \wedge B|C) = \Pr(A|C)\Pr(B|C)$; and (iv) $\Pr(A \wedge B|\neg C) = \Pr(A|\neg C)\Pr(B|\neg C)$. Conditions (i) and (ii) say that C is probabilistically relevant to the occurrence A and B respectively, while conditions (iii) and (iv) say that C 'screens A off from B' (and vice versa)—that is, C completely accounts for the correlation between A and B.[90]

Hence, for example, we may observe that the newspaper at my front door (A) was printed from the same master (C) as the one at my neighbor's front door (B). The same earth whose inclination towards the sun (C) partially determines temperature in Sydney (A) also partially determines temperature in London (B). And so on.

And yes, thanks to Einstein's theory of general relativity, even the moon's gravitational pull on the oceans is a *local* common cause: the gravitational 'force' propagates locally, as a wave, through space. Indeed, physics has, for at least the past century, been pushing towards *local* theories, and indeed theories that are, in some appropriate sense, Lorentz-invariant. (The precise meaning of 'Lorentz-invariant' varies, but the

[90]This formulation is not entirely satisfactory, but illustrates the notion of a common cause well enough for present purposes. See Uffink (1999b) and references therein.

rough idea is that the theory does not permit the transmission of matter and energy faster than light, or does not permit a signal to be sent faster than light, or at the very least, does not permit matter to be accelerated from subluminal to superluminal speeds. See Malament, Ch. 3, §2, this volume.)

Quantum theory permits non-local correlations. The question we face is whether those correlations can also, like the others mentioned here, be given a (local) common cause explanation. In particular, can the correlations implied by (6.4) be explained by a common cause?

6.1.2. *Bell's Theorem.* Bell's theorem essentially answers 'no' to this question. Moreover, Bell was able to derive some constraints on the predictions made by any local theory. These constraints are quite convincingly violated by experiment.

6.1.2.1. *The Experiment.* The experiment considered by Bell is essentially the one considered by EPR (§4.4), in a version described by Bohm (1951, ch. 22). There is a source of pairs of spin-1/2 particles, each pair in the state $|\psi_{\text{singlet}}\rangle$ (6.2); the particles are directed towards Stern-Gerlach devices, each oriented in some direction (labeled u and u'). The particles arrive at these devices in such a way that the measurements are made at space-like separation. Indeed, even the choices of the directions u and u' are made at space-like separation. That is, the choice of measurement, and the measurements themselves, are made sufficiently far apart in space, and sufficiently close together in time, that a signal of any sort from one measurement-event to the other would have to travel faster than light. After many runs of the experiment, the results are collected, and correlations determined.

In fact, in order to get Bell's Theorem underway, we need consider only three possible directions (u_a, u_b, and u_c) on each side. The measurement-devices thus choose, at the 'last second', which of these three directions to measure, and then they record the result.

6.1.2.2. *The Issue in Terms of 'Hidden Variables'.* One way to think about the question about common causes that we raised above is in terms of so-called 'hidden-variables theories' (§5.5.3) and indeed Bell conceived of the issue in these terms). Recall that hidden variables describe complete states of the particles, states that include information not provided by the quantum state. In the deterministic case, these complete states fix the values of S_u for each value of u, for each particle. The picture, then, is that, back at the source, the pair of particles about to be released towards the two measurement-devices has some 'hidden' state. This state already determines,

for each particle, and each possible direction u_a, u_b, and u_c, a value ($+1$ or -1) for spin.

Indeed, recall the EPR argument (§4.4), and their conclusion that quantum mechanics is 'incomplete'. The point there was that locality (plus the criterion for physical reality) forces one to conclude, in the context of this experiment, that each system must in some way *already have* values for all S_u— they must have 'established' the correct correlations back at the source (for they cannot do so later, at spacelike separation, by locality).

Note that for a given direction, the hidden state must determine the particles to have *opposite* spin, because of the perfect anti-correlations. (These hidden states can thus be partitioned into eight classes, two possible assignments of spin for each of the three directions.) Apart from that, the only requirement is that it be possible to recover the quantum-theoretic correlations by averaging over some distribution of the hidden states, as in (5.8).

6.1.2.3. *Probabilistic Hidden States.* A slightly more general approach is to allow the hidden states to determine spins only probabilistically. The theory will remain 'local' as long as the hidden state of the particles back at the source screens off the measurement-event at one side from the results at the other side (§6.1.1.3). Letting $\Pr_\lambda$ be the probabilities for measurement-results prescribed by the hidden state λ (and we are assuming, now, that the quantum state is fixed as $|\psi_{\text{singlet}}\rangle$), the screening-off condition is that, for any u, u', $k = \pm 1$, and $k' = \pm 1$:

$$\Pr_\lambda(x = k | i = u, j = u', y = k') = \Pr_\lambda(x = k | i = u), \tag{6.5}$$

where x is the result for particle 1, y is the result for particle 2, i is the direction of spin measured on particle 1, and j is the direction of spin measured on particle 2. The intuitive idea, here, is that the result for particle 1 depends only on the direction of spin measured (and λ), and is thus independent of the result for particle 2, and the direction in which spin was measured on particle 2. This condition, or something similar, is often called 'Bell locality'.

So, finally, the question we face is whether there can be hidden states, λ, and a distribution over them, $\rho()$, such that the probabilties, $\Pr_\lambda$ both obey Bell locality (6.5) and reproduce the quantum correlations (6.4) via averaging as in (5.8). (Note that a similar condition will hold for the single-wing probabilities, i.e., those obtained for just *one* of the two particles, which will also, of course, be obtained by averaging over hidden states.)

In general, as I mentioned earlier (§5.5.3), the distribution $\rho()$ would depend on the quantum state (which, recall, we are presuming here to be $|\psi_{\text{singlet}}\rangle$). But it must not depend on anything that would require a non-local influence on the source, because $\rho()$ is supposed to represent the distribution of hidden states amongst pairs produced at the source. For example, we will presume that the choice of measurement-direction and the results of the measurements d not influence $\rho()$.

6.1.2.4. *Bell's Theorem.* Bell's theorem shows that in fact no common-cause explanation of the sort outlined above exists.

> **Theorem** (Bell): For any choice of directions u_a, u_b, and u_c, any hidden-variables theory that satisfies (i) Bell locality (6.5) and (ii) independence of $\rho()$ from the measurement-events will obey the inequality
>
> $$|r_{ac} - r_{bc}| - r_{ab} \leq 1. \tag{6.6}$$
>
> (where $r_{ab} := r_{S^{(1)}_{u_a} S^{(2)}_{u_b}}$ and so on).[91]

Given the quantum-mechanical prediction (6.4), this inequality is violated by many choices for u_a, u_b, and u_c. For example, defining each u_k in terms of Euler angles ϕ_k and θ_k, choose $\phi_k = 0$ for $k = a, b, c$ (so that these directions are co-planar), $\theta_a = 0$, $\theta_b = \pi/4$, and $\theta_c = \pi/2$. In this case, the quantity on the left-hand side of (6.6) is $\sqrt{2}$.

6.1.2.5. *Experimental Violation of Bell's Inequality.* Bell's Inequality is violated by experiment. Of course, as is always the case with experimental realizations of theory, there are complications. For example, actual sources often do not produce particles that move away from the source in precisely opposite directions. One must either filter out those that do not, or take account of the fact that they do not move in exactly opposite directions. Neither solution is completely straightforward. And there are other issues with existing experimental tests of Bell's inequality, although experimentalists are increasingly able to settle these issues in favor of the violation of the inequality in the world, i.e., the confirmation of the quantum correlations.

There is one issue that will not be settled by experiment, at least not entirely, and it involves the fact that for any given pair of particles, only *one* pair of measurements (one on each particle) can be made. In other words, for each given pair, we are not *really* probing the entire hidden state, but only its implications for one pair of measurements. We must therefore *presume* that the sample we are getting of ensemble of hidden states is representative. To put the point another way, consider all of the

[91] Numerous other inequalities have since been derived, with various virtues, especially regarding experimental testability. See Clauser and Shimony (1978).

pairs where we measure, for example, u_a, u_b, and then all the pairs where we measure u_b, u_c. In order to test Bell's inequality, we will calculate the correlations amongst these two groups of pairs, then plug the results into (6.6). In doing so, we are assuming that *had* we measured u_b, u_c on the u_a, u_b-pairs (i.e., *instead* of measuring u_a, u_b on them), we *would* have gotten the same correlation that we in fact got for our actual u_b, u_c-measurements. Most consider this counterfactual innocuous—in large part because violating it seems to involve one in some rather odd conspiracy theories, many of which would in any case be non-local.[92] It is worth pointing out, however, that the counterfactual nature of the assumption is not due to contingent facts about what we happened to measure (as the corresponding counterfactuals often would be in classical physics), but rather it is due to the in principle incompatibility of the measurements involved—S_{u_a} and S_{u_b} *cannot* be measured at the same time on the same particle, and, moreover, in accordance with the uncertainty principle (§4.2), measuring one destroys whatever previous knowledge we have of the other.

Finally, note that although Bell's Theorem is framed in terms of so-called 'hidden' states, it is ultimately *not* a theorem about hidden-variables theories, but about *locality*. After all, there is nothing stopping us from taking the λ to be the quantum state itself and following through Bell's derivation. Of course, in doing so, we will make an assumption (Bell locality) that is violated by quantum theory, and we could have checked this violation directly if we had wanted to. Either way the conclusion is the same: in the sense discussed here (Bell locality), standard quantum theory is itself a non-local theory.

6.1.3. *Other Bell-Like Theorems.* Plenty of other theorems are floating around that also bear on the issue of locality. A couple of them we have already seen (§5.4.1.2 and §5.4.1.2.c), and so will be mentioned only briefly here.

6.1.3.1. *The Kochen-Specker Theorem as a Theorem about Locality.* The non-contextuality condition in the Kochen-Specker theorem can, in the right circumstances, be seen as a locality condition, and the sought-for homomorphisms (recall §5.4.1.2.b) from the lattice of subspaces to the Boolean lattice $\{0, 1\}$ are the 'hidden states'. However, rather than develop this point in the context of the general Kochen-Specker theorem,

[92]For example, it will not do to suppose that only a 'certain kind' of hidden state is ever involved in a u_b, u_c-measurement, because this theory would violate the condition that the distribution at the source be independent of the choice of measurement. However, models that rely on detector inefficiencies can assert—however oddly—that detectors have a preference for revealing only 'certain kinds' of hidden states, based on which measurement was performed. Fine's 'prism models' are of this sort—see Fine (1991 and Szabó and Fine (2002). Note that detector efficiencies continue to improve, and could eventually rule out such theories on purely empirical grounds.

let us just consider the point in the context of the GHZ theorem, which in fact was originally (and correctly) characterized by its authors as a Bell-like theorem without inequalities—one need only suppose that the three particles are space-like separated.

In this context, the condition of non-contextuality becomes this: the value assigned to each of the observables in (5.3) must be assigned independently of which observables (5.4) one considers it to be a part, that is, independently of the 'context of measurement', which is given by the set of observables measured on each of the three particles. In other words, the value that particle 1, for example, has for $S_x^{(1)}$ cannot depend on whether we are measuring $S_x^{(1)} \otimes S_y^{(2)} \otimes S_y^{(3)}$ or $S_x^{(1)} \otimes S_x^{(2)} \otimes S_x^{(3)}$. This condition is a consequence of locality, because such a dependence would imply that the result of the measurement on particle 1 would depend on which observable we decided to measure on particles 2 and 3, even if those decisions were made at space-like separation from the measurement-event on particle 1.

6.1.3.2. *Hardy's Argument.* GHZ simplified the conclusion of Bell's argument, getting rid of the inequality, but at the expense of introducing another particle. Hardy (1992) managed to make an argument not based on inequalities, but considering only two particles.[93] He considers the experimental arrangement depicted in figure 5, which shows a 'double-interferometer' (called 'double' because there are two interferometers, 'overlapping' at the point A). The electron (e^-) and positron (e^+) enter the

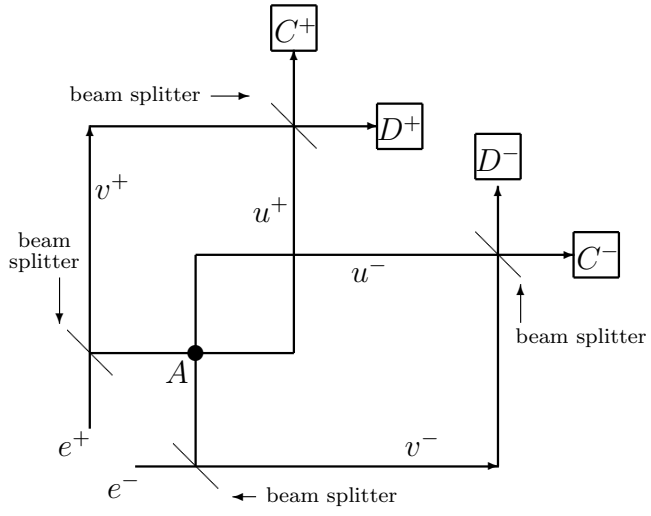


FIGURE 5. Hardy's Experiment.

interferometers on the paths s^- and s^+ repectively. They encounter a beam-splitter, and as a result, follow either the 'u' path or the 'v' path, with equal probability.

[93]The exposition here follows that in Dickson (1998, pp. 209–211.

If they both follow their u path then they meet at A and annihilate one another. Otherwise, they both reach a second beam splitter and the positron then goes on to either the detector C^+ or the detector D^+, with equal probability. The electron is in the corresponding situation.

Hardy assumes that a hidden-variable theory will assign a definite path (u or v) to each particle. He claims further that the question whether a particle is on a given path ought to have a Lorentz-invariant answer—its path ought not depend on an observer's frame of reference. Otherwise the hidden-variable theory is not Lorentz-invariant, and is, in that sense at least, non-local. Here, we will consider three frames of reference. In the first, F^+, the detection of the positron (at C^+ or D^+) occurs before the electron has passed through its beam splitter. In the second, F^-, the order of these events is reversed. And in the third, the detections are simultaneous.

The initial state of the electron and positron is $|s^-\rangle|s^+\rangle$ respectively, indicating that (before they reach the initial beam-splitter) they are on the paths s^- and s^+ respectively. The initial beam splitters effect the evolution

$$|s^\pm\rangle \rightarrow (1/\sqrt{2})(i|u^\pm\rangle + |v^\pm\rangle). \tag{6.7}$$

Similarly, the second set of beam splitters effect the evolution

$$|u^\pm\rangle \rightarrow (1/\sqrt{2})(|c^\pm\rangle + i|d^\pm\rangle) \tag{6.8}$$

$$|v^\pm\rangle \rightarrow (1/\sqrt{2})(i|c^\pm\rangle + |d^\pm\rangle). \tag{6.9}$$

Using (6.7), the state after the system passes the point A is (in frame F)

$$\tfrac{1}{2}\Big(- |\gamma\rangle + i|u^+\rangle|v^-\rangle + i|v^+\rangle|u^-\rangle + |v^+\rangle|v^-\rangle\Big), \tag{6.10}$$

where $|\gamma\rangle$ is the state after an annihilation. One can calculate the state at various times in the frames F, F^+, and F^- by applying one or more of (6.7–6.9). I leave these calculations to the reader.

Now, in F^+, if the positron is detected at D^+, then the electron must be on the path u^- (with probability 1). Similarly, in F^-, if the electron is detected at D^-, then the positron is on the path u^+. However, in F, the state of the pair before either particle goes through its second beam splitter is orthogonal to a state where the particles take the paths u^+ and u^- (because of the annihilation in that case).

Consider, then, a run of the experiment in which detectors D^+ and $D-$ both register a hit. (This outcome does indeed have non-zero probability.) In that case, an observer in F^+ will conclude that the electron took the path u_-. An observer in F^- will conclude that the positron took the path u^+. *Both* of these statements must be true for an observer in F, but as we saw above, they cannot both be true.

Hence, Hardy concludes, a theory that assigns definite paths to the particles cannot be Lorentz-invariant.

Note, finally, that Hardy has implicitly assumed non-contextuality as well. A hidden-variables theory is obliged only to reproduce the *experimental* predictions of quantum theory. If the theory is contextual, then in particular the path of a particle may depend on the presence (or not) of a device on the path that will detect the particle (or not) on the path. Hence, for example, such a theory is *not* obliged to assign the path u^+ to the positron if the electron is detected at D^- (in F^-) *unless* there is a detector in place to determine whether the positron is on u^+. But in this case, the detector will interact with the positron, and everything changes, including (most importantly) the quantum-theoretic calculations that are behind Hardy's argument. In the absence of such a detector, and such an interaction, a hidden-variables theory need not respect the quantum probabilities. But agreed, a non-contextual theory *does* always have to respect those probabilities, because such a theory cannot alter the (hidden) state (i.e., the path) that it assigns to a particle depending on whether a detector is in place along the path u^+.

6.2. **Reactions to the Theorems.** What should we make of the failure of locality? In this section, I shall consider four more precise versions of that question.[94] Can the failure of locality be used to send signals from one measurement-station to the other (§6.2.2)? Does the failure of locality imply the existence of some causal connection between the measurement-stations (§6.2.3)? Does the failure of locality imply (or can it be understood in terms of the claim that) the two systems are somehow not really distinct (§6.2.4)? Finally, and perhaps most important from the point of view of theoretical physics, what does the failure of locality imply about the possibilities for a fully Lorentz-invariant (relativistic) quantum theory (§6.2.5)?

Prior to asking these questions, I shall review (§6.2.1) an important analysis of locality, one that some have claimed helps resolve some of these questions.

6.2.1. *Parameter Independence and Outcome Independence.* Once we have accepted the non-locality of quantum theory, and of empirically adequate hidden-variables theories—and it is surprising how far some will go to resist this conclusion—the question becomes how to understand Bell locality. The most famous analysis of this condition is in terms of two other conditions, often called 'parameter independence' and 'outcome independence'. The former expresses the idea that the parameters—the

[94]Extensive discussion of these issues and more is available in many places. Cushing and McMullin (1989), Butterfield (1992), Maudlin (1994), and Dickson (1998, chs. 6-9) will get the interested reader started.

settings on (direction of spin to be measured by) the measurement-apparatus—at one measurement-station do not affect the outcomes at the other measurement-station. The latter expresses the idea that the outcomes at one measurement-station do not affect the outcomes at the other.

These conditions are, like Bell locality, statements of probabilistic independence:[95]

Parameter Independence: for all i, j, k, k', λ,

$$\Pr_\lambda(x = k | y = k', i = u, j = u') = \Pr_\lambda(x = k | y = k', i = u) \tag{6.11}$$

and similarly, reversing the roles of particles 1 and 2.

Outcome Independence: for all i, j, k, k', λ,

$$\begin{aligned} &\Pr_\lambda(x = k, y = k' | i = u, j = u') \\ &\quad = \Pr_\lambda(x = k | i = u, j = u') \times \Pr_\lambda(y = k' | i = u, j = u'). \end{aligned} \tag{6.12}$$

(See 6.1.2.3 for the notation.) The conjunction of Parameter Independence and Outcome Independence yields Bell locality with just some trivial applications of probability theory.

One of the values of this analysis is that it reveals, more precisely, how quantum theory violates Bell locality. In particular, quantum theory violates Outcome Independence, but satisfies Parameter Independence. Indeed, the derivation of the correlation (6.1.1.2) more or less proves this point.[96] Moreover, standard quantum theory satisfies Parameter Independence. We will see one way of proving this point in the next subsection.

6.2.2. *Signaling.*

6.2.2.1. *Locality and Signaling.* Part of the original intention behind the analysis of Bell locality into Parameter Independence and Outcome Independence was to isolate a part of Bell Locality that does *not* involve a violation of relativity theory, in some sense or other. In particular, the claim is often made that a failure of Outcome

[95]There are subtly different versions of these conditions, taking into account, for example, hidden variables in the apparatus, and so on. We are skating over such distinctions here. In addition, there are hosts of other locality conditions one might introduce. See Dickson (1998, chs. 6-9). Jarrett (1984) is responsible for the first clear statement of this distinction. The conditions as stated here, and the terminology used here, is due to Shimony (1986).

[96]Strictly speaking, in order to implement the probabilities in (6.12), one should introduce the apparatuses as two additional systems, and define the apparatus-settings (choice of direction of spin) as states of these additional systems. Hence one would have a state in a four-fold tensor product Hilbert space, and would consider the probabilities generated by that state. Following through the analysis in this way, however, is not particularly enlightening, and yields the same result, namely, the failure of Outcome Independence.

Independence is somehow consistent with relativity, while a failure of Parameter Independence is not.

In particular, one might understand relativity theory as prohibiting superluminal signaling. A violation of Outcome independence, so the argument goes, does not involve the possibility signaling, because even though it implies that the outcome at one measurement-station depends probabilistically on the outcome at the other (and of course the measurement-events are space-like separated), the outcomes themselves are probabilistic. That is, an experimenter cannot control the outcomes. But controlling the outcomes would be necessary in order to use the dependence between outcomes to send a signal.

On the other hand, experimenters *are* in control of parameters—they are in fact normally assumed to be the result of a free choice of the experimenter. Hence, so the argument goes, a violation of Parameter Independence implies that an experimenter can (probabilistically) influence the outcomes at the other station by manipulating the parameter (direction in which spin is measured) at the local station.

However, keep in mind that the probabilities in Parameter Independence and Outcome Independence are those generated by the hidden state, λ. If the experimenter is not in control of these hidden states, then a failure of Parameter Independence will *also* not imply the possibility of signaling. Moreover, control of the hidden states would mean that in fact a violation of Outcome Independence *also* implies the possibility of signaling, so long as the probabilities for the outcomes generated by different hidden states are different. In this case, an experimenter *can* in fact locally manipulate the outcomes in the probabilistic sense of controlling the hidden state in order to make a given outcome more or less likely, and thus, via the probabilistic dependence between outcomes, influence (probabilistically) the outcomes at the other station.

An illuminating example of the significance of being able to control the hidden state is provided by the de Broglie-Bohm theory. It is clear that the theory satisfies Outcome Independence, because it is a fully deterministic theory. That is, given the parameters i and j, and the initial state (which, in this theory, is the initial positions of the particles, plus the quantum wavefunction), the results of the measurements are fixed. Hence, in particluar, the result of one measurement, being fixed by λ, i, and j, does not depend on the result of the other measurement. On the other hand, the de Broglie-Bohm theory *does* violate Parameter Independence, for the result of a measurement on one particle *does* in general depend on the direction in which spin is measured on the other. (Changing the setting on the apparatus changes the wavefunction for the compound system in a way that ultimately changes the

quantum potential, and hence the trajectories for *both* particles.) However, recall that the precise positions of the particles are *not* controllable in this theory. Hence the violation of Parameter Independence cannot be used for signaling.

Indeed, as Cushing (1994) points out in this context, Valentini (1991a; 1991b has shown that in the de Broglie-Bohm theory, signaling is possible if and only if the distribution of the particles is *different* from that given by the usual quantum-mechanical probability. Hence, given that Bohm's theory *does* respect this distribution, we have a case where, phenomenologically, the theory violates Outcome Independence and respects Parameter Independence, while at the level of the (uncontrollable!) hidden-variables (trajectories of particles), the reverse is true.[97] Hence one should be very cautious about identifying a failure of Parameter Independence with the possibility of signaling.

6.2.2.2. *The No-Signaling Theorem.* While the distinction between Outcome Independence and Parameter Independence is, as explained above, not the same as the distinction between the impossibility and possibility of signaling, it is true that quantum theory (which, recall, obeys Outcome Independence but not Parameter Independence) does not permit signaling from one measurement-station to the other (or, indeed, non-local signaling in general). This claim is the conclusion of the quantum no-signaling theorem.

There are (at least) two routes to this conclusion, one focused on the effect that local generic interactions can have on the states of spatially separated systems, and on focused on the effect that measurements can have on the results of measurements on spatially separated systems. I shall outline each in turn. In both cases, we will consider a two-particle system in the (generally entangled) state W, and suppose that the two particles are spatially separated.

6.2.2.2.a. Local Interactions and Reduced States. Now consider the evolution of the compound system by two different unitary evolutions (generated by two different Hamiltonians), $U^{(1)} \otimes U^{(2)}$ and $U^{(1)} \otimes \tilde{U}^{(2)}$. The no-signaling theorem says that the reduced state for particle 1 is the same at the end of the evolution, whether the system evolves according to $U^{(1)} \otimes U^{(2)}$ or $U^{(1)} \otimes \tilde{U}^{(2)}$. In fact, under the first evolution, the

[97] In fact, Valentini's (1991a; 1991b view, supported by a kind of quantum analogue of the classical H-Theorem (see Uffink, ch. 9, §I.4.2, this volume), is that the standard quantum distribution is a kind of 'equilibrium' distribution, to which the universe naturally relaxes. It is possible (indeed quite likely), on this view, that the universe is not *quite* in the equilibrium distribution, which means that in principle one could signal, if one could find an ensemble of particles not in the equilibrium distribution.

reduced state for particle 1 is (recall §1.2.6.3.a)

$$W^{(1)} = \mathrm{tr}^{(2)}[U^{(1)} \otimes U^{(2)} W (U^{(1)} \otimes U^{(2)})^{-1}] = \sum_n \langle e_n | U^{(1)} \otimes U^{(2)} W (U^{(1)} \otimes U^{(2)})^{-1} | e_n \rangle. \tag{6.13}$$

where $\{|e_n\rangle\}$ is some orthonormal basis in the Hilbert space for particle 2. But recall that the (partial) trace functional does not depend on the choice of this basis. Hence, when figuring the reduced state under the alternative evolution, $\mathrm{tr}^{(2)}[U^{(1)} \otimes \tilde{U}^{(2)} W (U^{(1)} \otimes \tilde{U}^{(2)})^{-1}]$, we need only choose a basis, $\{|\tilde{e}_n\rangle\}$, such that $\tilde{U}^{(2)}|\tilde{e}_n\rangle = U^{(2)}|e_n\rangle$. Because $U^{(2)}$ and $\tilde{U}^{(2)}$ are unitary, such a basis always exists.

In other words, no interaction with just one of the systems can effect the reduced state of the other, and because the marginal probabilities for particle 1 (the probabilities for outcomes of measurements of observables on particle 1 only) depend only on the reduced state $W^{(1)}$, no interaction that involves just particle 2 can change the statistics of measurement-results for particle 1 (and, of course, vice versa).

6.2.2.2.b. Single-System Measurements. But what about measurements? Consider a measurement of an observable $\mathbb{I} \otimes G$, i.e., a measurement of G on system 2, leaving system 1 untouched. Let $G = \sum_n g_n P_n$. Of course, we have already seen that if we project the state of the compound system onto the result of this measurement, then in general (for example, if W is the singlet state), the probabilities for results of measurements on system 1 will change. That fact is just a rehearsal of the existence of correlations between the two systems. But keep in mind that we are talking about the possibility of *signaling*, here. An observer who is restricted to the vicinity of system 1 will not know the results of measurements on system 2, but will know only that a measurement of $\mathbb{I} \otimes G$ might or might not occur. To such an observer, the measurement of $\mathbb{I} \otimes G$ on particle 2 is a 'non-selective' measurement, meaning that the most that this observer can say about the state after the measurement is that the compound system is now in the state

$$\sum_n (\mathbb{I} \otimes P_n) W (\mathbb{I} \otimes P_n). \tag{6.14}$$

It is instructive to see why (6.14) is the state after a non-selective measurement. Suppose that the result were in fact g_k. In that case, applying the collapse postulate, the state would be $P_k W P_k / \mathrm{Tr}[W P_k]$. (Compare this expression with Lüder's Rule, eq. 1.19.) Now consider that the probability of the result P_k is $\mathrm{Tr}[W P_k]$, so that if we do not know the result of the measurement (i.e., the measurement is non-selective) then the state is a weighted sum of all possibly results, the weights given by the probabilities of the various results. I.e., the state is (6.14).

The idea here is that an experimenter ('B') in the vicinity of particle 2 will attempt to send a signal to an experimenter ('A') in the vicinity of particle 1 by choosing to measure G (on particle 2) or not. The question, then, is whether A can detect any change in the statistics of results for measurements on particle 1 as a result of B's measurement on particle 2.

The answer is 'no'. To see why, consider an observable, $F\otimes\mathbb{I}$, and let $F = \sum_n f_n Q_n$. In the state W, the probability for the result f_m (of a measurement of $F \otimes \mathbb{I}$) is

$$\mathrm{Tr}[(Q_m \otimes \mathbb{I})W]. \tag{6.15}$$

Suppose, instead, that the state is (6.14), i.e., a non-selective measurement of $\mathbb{I} \otimes G$ has been made. In that case, the probability for the result f_m (of a measurement of $F \otimes \mathbb{I}$) is

$$\mathrm{Tr}\left[(Q_m \otimes \mathbb{I}) \sum_n (\mathbb{I} \otimes P_n) W (\mathbb{I} \otimes P_n)\right]. \tag{6.16}$$

By the linearity of $Q_m \otimes \mathbb{I}$, it can be taken inside the sum, and by the linearity of the trace functional, the trace of a sum then becomes a sum of traces. Moreover, because $Q_m \otimes \mathbb{I}$ commutes with $\mathbb{I} \otimes P_n$ (for any n, m), this sum becomes:

$$\sum_n \mathrm{Tr}\left[(\mathbb{I} \otimes P_n)(Q_m \otimes \mathbb{I}) W (\mathbb{I} \otimes P_n)\right]. \tag{6.17}$$

Using (1.12) together with the fact that $PP = P$ for any projection P, (6.17) is

$$\sum_n \mathrm{Tr}\left[(\mathbb{I} \otimes P_n)(Q_m \otimes \mathbb{I}) W\right]. \tag{6.18}$$

Taking the sum back inside the trace functional and observing that the $\mathbb{I} \otimes P_n$ constitute a resolution of the identity (i.e., $\sum_n \mathbb{I} \otimes P_n = \mathbb{I} \otimes \mathbb{I}$), we find that (6.18) is exactly (6.15). In other words, no measurement that A can make on particle 1 can determine whether the compound state is the original W, or the state (6.14)—B's non-selective measurement of $\mathbb{I} \otimes G$ has no effect on the probabilities for outcomes of measurements performed by A on particle 1. Hence B cannot send a signal to A by means of such a measurement.

6.2.3. *Causation.* Does a failure of Bell locality (or one of its constituent conditions) imply a causal connection between the two measurement-stations? This question is much-discussed and disputed, and I certainly will not settle it here. But let us consider a few approaches.

Consider a counterfactual approach to the analysis of causation, one that, for example, takes as a sufficient condition for causal connection the truth of counterfactuals such as 'if it had been that C then it would have been that E' and 'if it had not

been that C, then it would not have been that E'.[98] It would appear, adopting some such understanding of causation, that causal connections *do* exist between outcomes, or between parameters and outcomes, in a deterministic hidden-variables theory that violates Outcome Independence, or Parameter Independence, respectively. If the hidden-variables theory is probabilistic, then perhaps one would conclude instead that there is a relation of probabilistic causality, though of course one would have to formulate appropriate probabilistic counterfactual conditions, for example, replacing 'it would [not] have been the case that E' with 'the probability of E would have been higher [lower]'.

On the other hand, others prefer an account of causation that requires that causal connections are underwritten by 'causal processes' that can transmit a 'mark'.[99] That is, impressing some sort of 'mark' on the cause should leave a trace 'from the cause to the effect', and ultimately in the effect itself. There are different understandings of what this account really means, but on at least some understandings—most clearly, those that require the mark to travel continuously in space—the violation of Parameter Independence and Outcome Independence does *not* imply the existence of a causal connection between the measurement-stations.

Finally, some, e.g., Collier (1999), have argued that causation amounts to the transfer of information, in the information-theoretic sense. Maudlin (1994, ch. 6) has argued that there *is* a transfer of information in the Bell-type experiments. Hence, if one buys these arguments, there *is* causation between the measurement-stations, on this account of causation.

6.2.4. *Holism.* The issues surrounding Bell's Theorem have led some to a radical view of entangled states as representing 'holistic' properties of entangled pairs (or n-tuples) of particles. On the one hand, we can agree that the properties represented by, for example, the (projection onto the) singlet state are not reducible to, and do not supervene on, properties of the individual particles. (Otherwise, a local hidden variables theory *would*, after all, be possible.) On the other hand, some claim that these holistic properties *explain* our inability to provide a common cause explanation for the quantum correlations in a way that does not violate any version of locality that we should care about, and does not, in particular, imply any violation of relativity theory. The basic idea is clear enough: the 'two' (or more) particles involved in a

[98]Causation has frequently been associated with counterfactuals such as these. Lewis (1973) is famous for such an analysis, and as his work clearly shows, one must add several additional conditions to this basic idea—for example, conditions about the similarity of possible worlds, where those conditions will effect the semantics, hence truth-values, of the counterfactuals involved.

[99]See, for example, Salmon (1984).

Bell-type experiment are not *really* 'two' particles; they are really one object, and therefore, no matter how far apart they may be, there is no question of an 'influence from one to the other', for there is no 'one and other' but only the whole thing.

There are, however, some prima facie problems with this proposal. First, it is not clear that it really explains anything; perhaps it only restates the problem in the form of an equally mysterious doctrine. Moreover, it is unclear what the real difference is, as regards the apparent conflict with relativity theory, between a space-like influence between two distinct objects and a space-like 'connection' between different 'parts' of the same object. (Of course, we must be very cautious about the meaning of 'part' here.) Indeed, the events that are involved in Bell-type experiments are well-defined, localized, events in space-time, viz., macroscopic pointer-readings. There are correlations between these events that cannot be explained in terms of a local common cause. That fact is enough to raise the question about nonlocality and the compatibility with relativity, regardless of whether one wishes to invoke some 'holistic' property of the pair that somehow gives rise to these correlations. In other words, we might simply have to conclude that holism itself is incompatible with relativity. (See Butterfield 1992.)

Second, as discussed earlier in the context of the measurement problem (§5.3.2), entanglement is ubiquitous—even the objects of our everyday experience are likely in entangled states. Can we make sense of the idea that these apparently distinct objects are in fact not independent objects at all, but somehow 'parts' of some holistic object? Perhaps, but at the very least, some serious metaphysical work is going to have to be done here, if we are to recover the obvious facts of ordinary experience.

6.2.5. *Relativity Theory.* While some might find non-locality to be contrary to intuition, the real problem that it poses, arguably, is an apparent incompatibility with the theory of relativity. Indeed, supposing that the non-locality of quantum theory were completely and unambiguously compatible with the theory of relativity, it is hard to see how one could have any serious objections to it.

And indeed, on the one hand, one might note that the quantum no-signaling theorem (§6.2.2.2) suggests some kind of compatibility of quantum theory with relativity theory. Hence, there may be, as some have called it, a 'peaceful co-existence' between the two theories.[100] While the quantum correlations seem to imply a failure of locality, and while collapse of the quantum state occurs instantaneously, neither of these facts entails the capacity to send a signal faster than light. More generally, neither of these features of quantum theory entails an *experimental*, or *observational* conflict

[100]The term was introduced by Shimony (1978).

with relativity. It is sometimes said that quantum theory respects the letter of the law of relativity (but violates the spirit of the law).

Moreover, even the claim that superluminal signaling is incompatible with relativity is controversial. The argument for the incompatibility relies on the idea that such signaling generates inconsistent causal loops because it allows for causal propagation into the past. And some will point out that even superluminal causal processes (for example, the superluminal transmission of matter or energy) is not inconsistent with relativity.

So what *does* relativity demand? One thing seems clear, and is, at least, agreed by the vast majority of physicists: a theory must be Lorentz-invariant to be relativistic. Here, we encounter the real problem. The collapse postulate of standard quantum theory is not Lorentz-invariant, nor is it clear how to make it so, while maintaining empirical adequacy. (See Aharonov and Albert 1981.)

There are, of course, relativistic quantum theories. (See 't Hooft, Ch. 7 and Halvorson, Ch. 8, this volume.) Their equations of motion are Lorentz-invariant in the requisite way. However, the collapse postulate, the need for which does not disappear in relativistic quantum theories, is not Lorentz-invariant—it specifies an *instantaneous* collapse.

Moreover, it is far from clear whether the interpretations of quantum theory that most straightforwardly and obviously solve the measurement problem (recall §5.5) can be made Lorentz-invariant. We do not *know* that all hidden-variables theories *must* violate Lorentz-invariance, but the evidence for this claim is very good.

On the other hand, the no-signaling theorem seems to imply that quantum theory itself is observationally consistent with relativity. Hence *any* theory, and in particular any hidden-variables theory, that is observationally indistinguishable from quantum theory, will be observationally consistent with relativity. Hence, while many hidden-variables theories explicitly violate Lorentz-invariance at the level of the evolution of the values of the hidden variables, this violation is not empirically accessible. Another way to put the point is this: these theories require a preferred frame of reference (as quantum theory with the collapse postulate itself does), and yet there is no way to determine, experimentally, which is the preferred frame. The question of whether they are thus satisfactory therefore has a lot do with the status one attaches to Lorentz-invariance.

7. Mathematical Appendix

These notes are intended as quick reminders of several definitions and standard mathematical facts that are assumed in the text. All proofs are omitted.

7.1. **Hilbert Spaces.**

7.1.1. *Vector Spaces.* A *vector space*, V, is a set that is closed under addition and 'multiplication by a scalar' from a field, K. These operations must satisfy, for any $u, v, w \in V$ and $k, k' \in K$: (commutativity) $v + w = w + v$; (vector associativity) $u + (v + w) = (u + v) + w$; (additive identity) $\exists \vec{0} \in V \ \forall v \in V, v + \vec{0} = v$; (additive inverses) $\forall v \in V, \exists - v \in V, v + (-v) = \vec{0}$, and one generally writes $v - w$ for $v + (-w)$; (scalar associativity) $k(k'v) = (kk')v$; (scalar identity) for $1 \in K$ (the identity in K) $1v = v$; (vector distributivity) $k(v + w) = kv + kw$; (scalar distributivity) $(k + k')v = kv + k'v$. (Notes: commutativity actually follows from the others; additive inverses are (provably) unique; $\forall k \in K, v \in V, k\vec{0} = 0v = \vec{0}$.) In all of the cases that we consider, $K = \mathbb{R}$ or $K = \mathbb{C}$. The set V is used interchangeably to refer to the entire vector space, and to the underlying set of vectors. Sometimes for clarity one speaks of 'a vector space over K'.

7.1.2. *Bases and Dimension.* Given a set of vectors $\{v_n\} \subseteq V$, any vector of the form $v = \sum_{i=1}^{N} k_n v_n$ (with $k_n \in K$) is called a *linear combination* of the v_n. The set $\{v_n\}$ is called *linearly independent* if none of the v_n can be written as a linear combination of the others. (In this case, the set may have infinite cardinality, but notice that linear combinations are always finite sums.) Any maximal linearly independent set in V is called a *basis* for V. All such sets provably have the same cardinality, which is called the *dimension* of the space, denoted $\dim V$. All vector spaces, V, V', with the same dimension are isomorphic. That is, there exists a 1-1 map $m : V \to V'$ such that, for any $k \in K$ and any $v, w \in V$, $m[k(v + v')] = km(v) + km(v')$. (This last condition makes m a *linear map*, and being 1-1 in addition makes it an isomorphism.)

7.1.3. *Inner Product Spaces.* An *inner product* on a vector space, V over K, is a map from $V \times V$ to K, denoted $\langle \cdot, \cdot \rangle$, and satisfying, for all $u, v, w \in V$ and all $k \in K$: (non-negativity) $\langle v, w \rangle \geq 0$; (non-degeneracy) $\langle v, v \rangle = 0$ iff $v = \vec{0}$; (sesquilinearity) $\langle u, k(v + w) \rangle = k\langle u, v \rangle + k\langle u, w \rangle$ and $\langle v, w \rangle = \langle w, v \rangle^*$, where * indicates conjugation in K (so, complex conjugation if $K = \mathbb{C}$ and the identity map if $K = \mathbb{R}$). (Note: it follows from the last two properties that $\langle k(v + w), u \rangle = k^* \langle v, u \rangle + k^* \langle w, u \rangle$; hence the name '*sesqui*linearity'.) A vector space with an inner product is called an 'inner product space.' In an inner product space, w is orthogonal to w', written $w \perp w'$, if

and only if $\langle w, w' \rangle = 0$. In an inner product space, a basis is called 'orthogonal' if its elements are mutually orthogonal.

An important fact about inner products is the *Schwarz Inequality*: for any $v, w \in V$, $\langle v, v \rangle \langle w, w \rangle \geq |\langle v, w \rangle|^2$.

7.1.4. *Norms and Orthonormal Bases.* A *norm* on a vector space, V, over K is a function from V to $\mathbb{R}$, denoted $|| \cdot ||$, satisfying, for all $v, w \in V$ and $k \in K$: $||v|| = 0$ iff $v = 0$; $||kv|| = |k| \, ||v||$; and $||v + w|| \leq ||v|| + ||w||$. (Note: it follows that $||v|| \geq 0$ for all $v \in V$.) The norm defines a topology (see §7.5) on V in the obvious way: the open balls are sets of the form $\{x| \; ||x - v|| < r\}$ for some $v \in V$ and $r \in \mathbb{R}$. Another way to put the point is this: a sequence $\{v_n\} \subseteq V$ converges to a vector $v \in V$ 'in the norm topology' just in case $||v_n - v||$ converges to 0 (in $\mathbb{R}$). Inner products define a norm by: $||v|| = \sqrt{\langle v, v \rangle}$. An important fact about normed complex vector spaces in which the norm is derived from an inner product is the 'polarization identity':

$$\langle w, v \rangle = \tfrac{1}{4}\big(||w + v||^2 - ||w - v||^2 + i||w + iv||^2 - i||w - iv||^2\big). \tag{7.1}$$

In other words, in such spaces, the inner product is also derived from the norm.

In an inner product space, an orthogonal basis (§7.1.3) is called *orthonormal* if its elements all have norm-1. Given any basis for an inner product space, it is possible to construct an orthonormal basis by 'Gram-Schmidt orthogonalization', the details of which we will skip. Note that in an inner product space, there is a convenient expression for a given vector, v, in terms of some orthonormal basis, $\{e_n\}$, as: $v = \sum_n \langle e_n, v \rangle e_n$.

7.1.5. *Subspaces.* A subset, W, of a vector space, V, is a *subspace* if it is a vector space in its own right, under the operations inherited from V. Similar definitions hold for inner product spaces and Hilbert spaces (defined below, §7.1.7). In an inner product space, one subspace, W, is *orthogonal* to another, W', when, for any $w \in W$ and any $w' \in W'$, $w \perp w'$.

7.1.6. *Direct Sums.* A *direct sum* of vector spaces V and V' (over the same field, e.g., $\mathbb{R}$ or $\mathbb{C}$) is the vector space whose elements are taken from the Cartesian product $V \times V'$, with the vector space operations defined component-wise. If $V = V' \oplus V''$, for example, then V' and V'' are disjoint (or, in an inner-product space, orthogonal) subspaces of V, and V is their span.

7.1.7. *Banach Space and Hilbert Space.* A *Banach space* is a normed vector space that is complete with respect to the norm topology (meaning that the limit of any sequence of vectors is itself contained in the space). A *Hilbert space* is an inner product space

that is complete with respect to the norm topology (the norm here being the one given by the inner product). The three 'classical' types of Hilbert space are those over the real numbers, complex numbers, and quaternions.

7.1.8. *Dual Space.* Given a vector space, V, the dual space (sometimes denoted V^*, but the * here is *not* complex conjugation) is the space of linear functionals on V, that is, the space of linear maps from V to $\mathbb{R}$. When V has topological structure (such as when V is a Hilbert space), we restrict to the continuous linear functionals. The norm of a (continuous) linear functional, ψ, on V is

$$||\phi|| = \sup_{v \in V} \{|\phi(v)| \mid ||v|| \leq 1\}. \tag{7.2}$$

Every finite-dimensional vector space has the same dimension as its dual. The Riesz Representation Theorem states that for any Hilbert space, $\mathcal{H}$ (finite-dimensional or infinite-dimensional), and any continuous linear functional, ϕ, in its dual $\mathcal{H}^*$, there is a unique $v \in \mathcal{H}$ such that, for all $w \in \mathcal{H}$, $\phi(w) = \langle v, w \rangle$. Conversely, each $v \in \mathcal{H}$ obviously generates a continuous linear functional, $\langle v, \cdot \rangle$, on $\mathcal{H}$. In other words, there is a 1-1 map, $\Phi : \mathcal{H} \to \mathcal{H}^*$, from a Hilbert space to its dual. Moreover, Φ is an isometry ($||v|| = ||\Phi(v)||$) and an 'anti-isomorphism' (in particular, $\Phi(kv) = k^*\Phi(v)$ for any $v \in V$ and $k \in \mathbb{C}$). This last property follows from the fact that the inner product is sesquilinear.

7.1.9. *Tensor Products.* The tensor-product of two Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, both over K, is a third Hilbert space, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ over K, constructed from $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows. First choose bases $\{e_n\}$ for $\mathcal{H}_1$ and $\{f_n\}$ for $\mathcal{H}_2$. Then form the Cartesian product of $\{e_n\}$ and $\{f_n\}$. This set contains all pairs of the form (e_n, f_m), and it is stipulated to be a basis for the tensor product space $\mathcal{H}$. Hence, at this stage, $\mathcal{H}$ consists of all formal linear combinations (over K) of the (e_n, f_m). Now, let the inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ be denoted $\langle , \rangle_1$ and $\langle , \rangle_2$. Define the inner product on $\mathcal{H}$ by $\langle v \otimes w, x \otimes y \rangle = \langle v, x \rangle_1 \langle w, y \rangle_2$ for all $v, w \in \mathcal{H}_1$ and $x, y \in \mathcal{H}_2$, and extend to all of $\mathcal{H}_1 \otimes \mathcal{H}_2$ by linearity. Finally, complete $\mathcal{H}_1 \otimes \mathcal{H}_2$ in the norm topology induced by this inner product.

Note that $\dim \mathcal{H} = \dim \mathcal{H}_1 \times \dim \mathcal{H}_2$. One can (if $\dim \mathcal{H}$ is not prime) 'factorize' a given Hilbert space into a tensor product, typically in many ways.

7.1.10. *Convex Sets and Cones.* A subset, X, of a real vector space, V, is *convex* if for any $x, y \in X$, $rx + (1-r)y \in X$ for all r in the real interval $[0, 1]$. In other words, the 'line segment' connecting x and y is also in X. A point, x, in a convex set X is an *extreme point* if it is not a convex combination of other points from X. A convex set,

X, is a *simplex* if every non-extreme point has a *unique* decomposition as a convex combination of points from X.

A *positive cone* in V is a set $C \subseteq V$ such that $rx \in C$ for all real $r \geq 0$ and all $x \in C$. (A negative cone requires instead $r \leq 0$.) A *convex cone* is a cone that is also convex. Given a set, $S \subseteq V$, one can form the *convex set generated by* S, in the obvious way, by closing under the required condition. Equivalently, one can take the intersection of all convex sets containing S. A similar point holds for cones, and of course for convex cones.

Given a positive cone, C, in V, and an inner product, $\langle \cdot, \cdot \rangle$ on V, the *dual* of C with respect to this inner product is: $C^* = \{y | \langle x, y \rangle \geq 0, x \in C\}$. If $C = C^*$ we say that C is *self-dual*.

A *face*, $F \subseteq X$, of a convex set, X, is a convex subset that is closed under 'purification', meaning that for any $v \in F$, if $v = pv_1 + (1-p)v_2$ then v_1 and v_2 are in F as well.

7.2. **Operators.**

7.2.1. *Basic Definitions Regarding Operators.* A *linear operator*, F, on a vector space, V over K, is a map from V to itself that preserves the linear structure of V, i.e., for any $v, w \in V$ and any $k \in K$: $F(v+w) = F(v) + F(w)$; and $F(kv) = kF(v)$. One normally just writes Fv, and here we refer to F as an 'operator' (as we will not be discussing non-linear operators). Two operators, F and G, are said to *commute* if their *commutator*, $[F, G] =_{\text{df}} FG - GF$ is 0 (i.e., the 'zero operator'). The *identity operator*, denoted $\mathbb{I}$, is the operator on V such that $\mathbb{I}v = v$, $\forall v \in V$.

Given an operator, F, any vector, v, such that $Fv = kv$ for some $k \in K$ is called an *eigenvector* of F, and k is its associated *eigenvalue*. Operators need not have any eigenvectors. The zero vector is not normally counted amongst an operator's eigenvectors.

An operator, F, on the vector space V is *invertible* just in case there is an operator, G, such that $FG = \mathbb{I}$, where $\mathbb{I}$ is the identity on V. The operator G is denoted F^{-1}. It is unique if it exists.

On a tensor-product space $V = V_1 \otimes V_2$, consider two operators, F on V_1 and G on V_2. The tensor-product operator $F \otimes G$ can be defined as follows. Choose any bases, $\{e_n\}$ and $\{f_m\}$ for V_1 and V_2. Define $(F \otimes G)(e_n \otimes f_m) = (Fe_n) \otimes (Gf_m)$, and extend to all of V by linearity.

7.2.2. *Boundedness and Continuity.* If V has a norm, $|| \cdot ||$, then F is *bounded* if and only if there is some $r \in \mathbb{R}$ such that $||Fv|| \leq r||v||$ for all $v \in V$. When V has a

topology (as when it is a Hilbert space), one says that an operator F is *continuous* if and only if it is continuous as a function on v as a topological space. An operator is bounded if and only if it is continuous. Moreover, if the dimension of V is finite, then all operators on V are bounded, i.e., continuous. Note that unbounded operators do not have the entire space as their domain, so that one must keep track of their domains of definition.

7.2.3. *Adjoints.* When $\mathcal{H}$ is finite-dimensional, we define adjoints and related notions as follows. The *adjoint*, F^*, of an operator, F, on $\mathcal{H}$ satisfies $\langle Fw, v\rangle = \langle w, F^*v\rangle$. An operator, F, is self-adjoint if $F = F^*$. The proof that adjoints exist is non-trivial.

When $\mathcal{H}$ is infinite-dimensional, we must be slightly more careful, because v might not be in the domain of F, if F is unbounded (see §7.2.2). In the infinite-dimensional case, then, we define the adjoint, F^*, of F as follows. The domain of F^* is all $v \in \mathcal{H}$ such that there is a $v' \in \mathcal{H}$ satisfying $\langle Fw, v\rangle = \langle w, v'\rangle$ for all w in the domain of F. For each such v, define $F^*v = v'$. (One must show that F^* is thus an operator.) Finally, in order to make relevant distinctions in the infinite-dimensional case, we say that an operator, F, is *symmetric* if $\langle Fw, v\rangle = \langle w, Fv\rangle$ for all v, w in the domain of F. We say that F is *self-adjoint* if $F = F^*$. The difference is, provably, that a symmetric but non-self-adjoint operator will have a domain that is a proper subset of the domain of its adjoint.

7.2.4. *Normal Operators.* A *normal* operator, F, is one that commutes with its adjoint: $FF^* = F^*F$. Given the comments above, it is clear that one must be aware of issues relating to the domains of operators if $\mathcal{H}$ is infinite-dimensional. Clearly all self-adjoint operators are normal, but the converse is not true. Consider $F = 2i\mathbb{I}$.

7.2.5. *Projection Operators.* An operator, P, is *idempotent* just in case $PP = P$, i.e., $P(Pv) = Pv$ for any $v \in V$. An operator, P, on V is a *projection operator* just in case it is self-adjoint and idempotent. Each projection operator, P, corresponds to a closed subspace, namely, the subspace of vectors, v, for which $Pv = v$. Note that $PQ = 0$ if and only if the corresponding subspaces are orthogonal.

7.2.6. *Unitary Operators.* An *automorphism* of a vector space, V, is a map from V to itself that 'preserves the structure of V', and in particular the linear, inner-product, and topological structures (the latter two if they exist in V). Let U be a (linear) operator on the Hilbert space $\mathcal{H}$ such that: U is invertible (hence U is 1-1); and U preserves inner products (i.e., for any $v, w \in \mathcal{H}$, $\langle Uw, Uv\rangle = \langle w, v\rangle$). Such an operator is called 'unitary', and clearly implements an automorphism of $\mathcal{H}$. In that case, of course U also preserves norms, i.e., $||Uv|| = ||v||$ for all v.

It is readily shown that for any unitary operator, U, $U^* = U^{-1}$. Conversely, any invertible linear operator with that property is unitary. (If, instead, $\langle Uw, Uv\rangle = \langle w, v\rangle^*$ and $U(kv) = k^*Uv$, then U is *anti-unitary.*)

7.3. **The Hilbert Space $\mathbb{C}^2$.** The space of complex column-vectors with 2 components is denoted $\mathbb{C}^2$. The elements of $\mathbb{C}^2$ are added component-wise:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}. \tag{7.3}$$

The (linear) operators on this space can be represented by 2×2 complex matrices. A matrix operates on a vector according to the rule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}. \tag{7.4}$$

The product of two matrices (which can in fact be derived from the rule above) is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}. \tag{7.5}$$

The inner product on the space is given by

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = a^*c + b^*d. \tag{7.6}$$

The norm, or 'length', of the vector with components a, b is then just $\sqrt{a^*a + b^*b}$. (Compare this expression with the Euclidean length of a vector in the real vector space $\mathbb{R}^2$.)

7.3.1. *The Pauli Matrices.* The Pauli Matrices (operators on $\mathbb{C}^2$) are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.7}$$

These matrices have many nice properties, which the reader might wish to verify. For example:

$$\begin{gathered} \sigma_x^2 = \sigma_x^2 = \sigma_x^2 = \mathbb{I} \\ \mathrm{Tr}[\sigma_x] = \mathrm{Tr}[\sigma_y] = \mathrm{Tr}[\sigma_z] = 0 \\ \sigma_x\sigma_y = i\sigma_z \\ [\sigma_x, \sigma_y] = 2i\sigma_z. \end{gathered} \tag{7.8}$$

The last two properties generalize: they remain true under cyclic permutations of the indices.

One often sees the expression $\vec{\sigma}$, which is to be understood as a 'vector' whose components are the three Pauli matrices, so that, for example, $\vec{r}\cdot\vec{\sigma}$ is a shorthand for a linear combination of the three Pauli matrices, with the coefficients given by the components of (the real vector) $\vec{r}$.

7.4. **Posets and Lattices.**

7.4.1. *Posets.* A *partially ordered set* (normally, *poset*) is a set, L, together with a relation, $\leq$, that obeys, for all $a, b, c \in L$: (reflexivity) $a \leq a$; (anti-symmetry) if $a \leq b$ and $b \leq a$ then $a = b$; and (transitivity) if $a \leq b$ and $b \leq c$ then $a \leq c$. The relation $\leq$ is called a *partial order* on L. Note, in particular, that in general there will be $a, b \in L$ such that neither $a \leq b$ nor $b \leq a$. (If there are no such a and b in L, then $\leq$ is a total order on L.) As elsewhere, I let L denote both the poset itself as well as its underlying set.

7.4.2. *Lattices.* Let L be a poset. Define the *join* of two elements, $a, b \in L$ as the least upper bound of a and b, i.e., the smallest (under $\leq$) c such that $a \leq c$ and $b \leq c$. The join of a and b (which need not exist in general) is denoted $a \vee b$. Define the *meet* of two elements, $a, b \in L$ as the greatest lower bound of a and b, i.e., the largest (under $\leq$) c such that $c \leq a$ and $c \leq b$. The meet of a and b (which need not exist in general) is denoted $a \wedge b$. A poset in which every pair (hence every finite set) of elements has a meet and a join is called a *lattice*.

A lattice, L, is *complemented* if every $a \in L$ has a *complement*, $a^\perp \in L$, satisfying: $a \wedge a^\perp = 0$; and $a \vee a^\perp = 1$. The operation $^\perp$ is in this case a *complement*. A complemented lattice L is called *orthocomplemented*, or an *ortholattice*, if for all $a, b \in L$: $a \leq b$ implies $b^\perp \leq a^\perp$; and $a^{\perp\perp} = a$. The operation $^\perp$ is in this case an *orthocomplement*.

7.4.3. *Distributivity.* A lattice, L, is *distributive* if, for all $a, b, c \in L$: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, and similarly, swapping $\wedge$ with $\vee$. In general, lattices are not distributive. The *center*, $Z(L)$, of a lattice L is the set of all $z \in L$ such that, for any $p \in L$, $p = (p \wedge z) \vee (p \wedge z^\perp)$.

7.4.4. *Direct Products and Reducibility.* Let L_1 and L_2 be ortholattices. Then we can form a third ortholattice, L, which is the 'direct product' of L_1 and L_2. As a set, L is the direct (Cartesian) product of L_1 and L_2 as sets. Then define, for $a, b \in L$ where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, $a \leq b$ just in case $a_1 \leq b_1$ and $a_2 \leq b_2$. Meets, joins, and orthocomplements are similarly (therefore) defined componentwise. An ortholattice, L is 'irreducible' if it is not isomorphic to the direct product of non-trivial ortholattices.

7.4.5. *Atomicity and the Covering Property.* The 'top' (or 'identity') of a lattice (if it exists) is the element $\mathbb{I} \in L$ such that $a \leq \mathbb{I}$ for all $a \in L$. The 'bottom' (or 'zero') of a lattice (if it exists) is the element $0 \in L$ such that $0 \leq a$ for all $a \in L$. An *atom*

in a poset, L, is a non-zero element $a \in L$ such that, for any $b \in L$, if $b \leq a$ then either $b = 0$ or $b = a$. A poset is *atomic* if every non-zero element contains (under $\leq$) an atom. Finally, a lattice is *complete* if *every* collection of elements from L has a meet and join. In a complete atomic lattice, every element (apart from 0) is either an atom or the join of atoms.

Finally, we say that a lattice L has the *covering property* if, for every atom $a \in L$ and any $b \in L$ where $a \wedge b = 0$, $a \vee b$ 'covers' b; that is, there is no element strictly between b and $a \vee b$.

7.5. **Topology and Measure.**

7.5.1. *Topological Spaces.* A *topological space* is a set, S, together with a collection, T, of subsets of S satisfying: $\emptyset, S \in T$ (with $\emptyset$ the null set); the union of any collection of sets in T is in T; the intersection of any pair of sets in T is in T. T is the *topology* of S. The elements of T are the *open sets*, and their complements (in S) are the *closed sets*. A function, f, from one topological space to another is *continuous* if the inverse image under f of every open set is again an open set.

A *base*, B, for a topology, T, is a collection of open sets in T such that every open set in T can be written as a union of elements of B. A common example involves spaces (such as vector spaces) with a norm, where one can define a base as the collection of 'open balls', that is, the collection of sets of the form $\{x | ||x - y|| < \epsilon\}$ where x and y are points (e.g., vectors) in the space, and ϵ is a real number.

A topological space is *compact* if every sequence of points has a subsequence that converges to some point in the space. The space is *locally compact* if, roughly, each point in the space has a neighborhood that is compact—roughly, each small part of the space 'looks like' a small part of a compact space.

Given two topological spaces, S_1 and S_2, we can form their Cartesian product, $S_1 \times S_2$ as sets. We then define the *product topology* on this Cartesian product as follows. Let B_1 and B_2 be bases for the topologies on S_1 and S_2 respectively. The product topology on $S_1 \times S_2$ is the topology whose base is the Cartesian product $B_1 \times B_2$. (The definition of the product topology on a Cartesian product of more than two spaces is more convoluted. We will not need it here.)

7.5.2. *Manifolds.* A *manifold* is a topological space that is 'locally Euclidean', meaning that there is a neighborhood around each point that is topologically the same as the open unit ball in $\mathbb{R}^n$ (for some n—and n is then the *dimension* of the manifold). An open set, S, of the manifold M, together with a homeomorphism between S and an open set of $\mathbb{R}^n$ is called a *coordinate chart*. A collection of charts that covers M is

an *atlas*. Now consider the region, S, in which two charts overlap. We thus have two distinct maps from S to $\mathbb{R}^n$, which defines a map, μ, from a subset of $\mathbb{R}^n$ (the range of the first chart, applied to S) to some other subset of $\mathbb{R}^n$ (the range of the second chart, applied to S). If all of the μ generated by all overlapping charts in the atlas are infinitely differentiable, then the manifold is a *smooth manifold*.

7.5.3. *Weak Operator Topology.* The weak operator topology over the operators on a Hilbert space, $\mathcal{H}$ is the weakest topology on the set, $\mathcal{B}(\mathcal{H})$, of bounded operators on $\mathcal{H}$, such that the map $F \mapsto \langle w, Fv\rangle$ is continuous for any vectors $v, w \in \mathcal{H}$ and any $F \in \mathcal{B}(\mathcal{H})$. In the weak operator topology, a sequence, $\{F_n\}$, of operators converges to the operator F just in case $|\langle w, F_n v\rangle - \langle w, Fv\rangle|$ converges to 0 for every $v, w \in \mathcal{H}$.

7.5.4. *Lebesuqe Measure.* On the real line, $\mathbb{R}$, the *Lebesque measure* is the natural extension of the usual measure of distance (size of intervals) to more complicated sets of points. For example, given any open set, S, that is the union of disjoint intervals, the Lebesque measure of S is the sum of the size of the intervals. Any countable union of individual points in $\mathbb{R}$ has Lebesque-measure zero. The measure is extended to volumes in $\mathbb{R}^3$ in the obvious way.

7.5.5. *Borel Sets.* Borel sets of real numbers are definable as follows. Given some set, S, a *σ-algebra over S* is a family of subsets of S closed under complement, countable union and countable intersection. The *Borel algebra* over $\mathbb{R}$ is the smallest σ-algebra containing the open sets of $\mathbb{R}$. (One must show that there is indeed a smallest.) A *Borel set* of real numbers is an element of the Borel algebra over $\mathbb{R}$. Note that not every subset of real numbers is a Borel set, though the ones that are not are somewhat exotic. All open and closed sets are Borel. The importance of Borel algebras (hence Borel sets) lies in the fact that certain measure-theoretic results apply only to them. On the other hand, in many cases one can extend the important results and definitions to a wider class of sets, for example, all sets that are the image of a Borel set under a continuous function. However, we shall not continue to make note of such points.

7.5.6. *Probability Measures.* Let X be a set (a 'sample space') of basic events and $\mathcal{A}$ a σ-algebra over X. The (Kolmogorovian) axioms of probability theory may then be stated as follows. Let $p : \mathcal{A} \to [0, 1]$ be a map from $\mathcal{A}$ to the interval $[0, 1]$. This p is a *probability measure* on $\mathcal{A}$ just in case: (normalization) $p(X) = 1$; (negation) $p(E') = 1 - p(E)$ for any $E \in \mathcal{A}$; and (additivity) for any countable disjoint sequence $\{E_k\}$ of elements of $\mathcal{A}$, $p(\cup_k E_k) = \sum_k p(E_k)$.

7.6. **Groups.**

7.6.1. *Groups and Homomorphisms.* A *group* is a non-empty set, $\mathcal{G}$, with a binary operation, $*$ (called the 'product') on that set satisfying: (associativity) for all $a, b, c \in \mathcal{G}$, $(a*b)*c = a*(b*c)$; (identity) there is an element, $e \in \mathcal{G}$, such that for any $a \in \mathcal{G}$, $e*a = a*e = a$; (inverse) for all $a \in \mathcal{G}$, there exists $b \in \mathcal{G}$ such that $a*b = b*a = e$ (this b is usually denoted a^{-1}); (closure) for all $a, b \in \mathcal{G}$, $a*b \in \mathcal{G}$. A structure satisfying all of these properties except for the existence of inverses is called a 'semi-group'. A map, $m : \mathcal{G} \to \mathcal{G}'$, from one group, $\mathcal{G}$, to another, $\mathcal{G}'$, is a *homomorphism* if, for all $a, b \in \mathcal{G}$, $m(a*b) = m(a)*m(b)$. (Note that the product on the left-hand side is taken in $\mathcal{G}$, while the product on the right-hand side is taken in $\mathcal{G}'$.) It follows that m preserves inverses and maps the identity in $\mathcal{G}$ to the identity in $\mathcal{G}'$. The map m is an isomorphism if it is a homomorphism and it is 1-1. An isomorphism from $\mathcal{G}$ to itself is an *automorphism* of $\mathcal{G}$. Normally the group product operator $*$ is left implicit; hence we will from now on write $a*b$ as ab and so on.

7.6.2. *Subgroups and Products.* Given a group, $\mathcal{G}$, the subgroup $\mathcal{H}$ is *normal* if and only if $g\mathcal{H}g^{-1} \subset \mathcal{H}$ for any $g \in \mathcal{G}$ (where $g\mathcal{H}g^{-1}$ is the set $\{ghg^{-1} | h \in \mathcal{H}\}$). Given two groups, $\mathcal{H}$ and $\mathcal{K}$, the group $\mathcal{G}$ is their *direct product* if and only if: (i) $\mathcal{H}$ and $\mathcal{K}$ are (isomorphic to) normal subgroups of $\mathcal{G}$; (ii) $\mathcal{H} \cap \mathcal{K} = e$, the identity in $\mathcal{G}$; and (iii) as a set, $\mathcal{G}$ is (isomorphic to) $\{hk : h \in \mathcal{H}, k \in \mathcal{K}\}$. We write $\mathcal{G} = \mathcal{H} \times \mathcal{K}$. The usual way to construct a direct product of groups is to let $\mathcal{G}$ as a set be the Cartesian product of $\mathcal{H}$ and $\mathcal{K}$ as sets, and define $(h, k)(h', k') = (hh', kk')$ for all $h, h' \in \mathcal{H}$ and $k, k' \in \mathcal{K}$. (Notice in this case that $\mathcal{H}$ is isomorphic to the subgroup of elements of the form (h, e) for any $h \in \mathcal{H}$ and e the identity in $\mathcal{G}$, and similarly for $\mathcal{K}$. Both $\mathcal{H}$ and $\mathcal{K}$ in this case are also normal subgroups.) If only $\mathcal{K}$ is a normal subsgroup of $\mathcal{G}$, while $\mathcal{H}$ is a non-normal subgroup, then $\mathcal{G}$ is the *semi-direct product* of $\mathcal{H}$ by $\mathcal{K}$, and we write $\mathcal{G} = \mathcal{H} \ltimes \mathcal{K}$.

7.6.3. *Cosets and Quotients.* Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$. We define the *(left) cosets of $\mathcal{H}$ in $\mathcal{G}$* as the set $\{g\mathcal{H} | g \in \mathcal{G}\}$, where $g\mathcal{H} = \{gh | h \in \mathcal{H}\}$. (Note that for some of the $g, g' \in \mathcal{G}$ we will have $g\mathcal{H} = g'\mathcal{H}$.) The left cosets of $\mathcal{H}$ in $\mathcal{G}$ partition $\mathcal{G}$. They also themselves form a group, with the multiplication rule $(g\mathcal{H})(g'\mathcal{H}) = (gg')\mathcal{H}$. This group is called the *quotient of $\mathcal{G}$ by $\mathcal{H}$*, typically written $\mathcal{G}/\mathcal{H}$.

7.6.4. *Continuously Parametrized Groups.* To say that the group, $\mathcal{G}$, is *continuously parametrized* by a means, in particular, that the group as a set is indexed by a, and: $g_0 = I$ (the identity on $\mathcal{G}$); $g_{a+b} = g_a g_b$; $\lim_{n\to\infty} a_n = b$ implies $\lim_{n\to\infty} g_{a_n} = g_b$. In the third condition, the limit on the right requires that the group be a *topological group*, i.e., the group is also a topological space.

7.6.5. *Lie Groups.* A *Lie group* is a smooth manifold (§7.5.2) that is also a group, where the group operations of multiplication and inversion are continuous maps on the manifold. (In fact, Lie groups are often defined as analytic manifolds, but we will not bother with that point here.)

7.6.6. *Vector Space Representations.* It is a fundamental theorem of group theory that every group, $\mathcal{G}$, is (isomorphic to) a subgroup of the group of permutations on some set. Another common type of representation of $\mathcal{G}$ is a *vector space representation*, a group-homomorphism from $\mathcal{G}$ to $\mathrm{GL}(V)$, the 'general linear group' of transformations of the vector space V, that is, the group of invertible linear operators on V.

A representation $m : \mathcal{G} \to \mathrm{GL}(V)$ of the group $\mathcal{G}$ is *faithful* just in case m is 1-1. Non-faithful representations ignore structure in the represented group. Representations can also *introduce* structure, in the following sense. A proper subspace W of V that is invariant under the group (that is, the group is an automorphism of W) carries a 'subrepresentation' of $\mathcal{G}$ in the sense that the restriction of the representation of $\mathcal{G}$ on V to W is itself a representation of $\mathcal{G}$. When a (proper) subrepresentation of a representation, m, of $\mathcal{G}$ exists, m is called 'reducible'. Otherwise, it is 'irreducible'. If a representation is reducible, then the vector space on which is it represented is in a sense 'bigger than it needs to be' to represent the group.

7.6.7. *Group Action.* For any set, S, a *group action* of $\mathcal{G}$ on S is a map $\mu : \mathcal{G} \times S \to S$ satisfying: (a) $\mu(g, \mu(h, s)) = \mu(gh, s)$ for all $g, h \in \mathcal{G}$ and $s \in S$; and (b) $\mu(e, s) = s$ for all $s \in S$ (where e is the identity in $\mathcal{G}$). $\mu(g, s)$ is often written $\mu_g s$. Each μ_g is in fact a bijection on S, so that we may also define a group action as a group homomorphism from $\mathcal{G}$ to the group of bijections on S. Sometimes μ_g is called the 'action of g on S'. A group action, μ, of $\mathcal{G}$ on S is *transitive* if and only if for any $s, t \in S$, there exists $g \in \mathcal{G}$ such that $\mu_g x = y$. If both $\mathcal{G}$ and S have a topological structure, then the action of $\mathcal{G}$ on S is *continuous* if the map μ is continuous with respect to the product topology (§7.5.1) of $\mathcal{G} \times S$.

7.6.8. *Unitary Representations.* Given a Hilbert space, $\mathcal{H}$, any unitary operator, U, on $\mathcal{H}$ implements an automorphism of $\mathrm{GL}(\mathcal{H})$. In particular, map $\mathrm{GL}(\mathcal{H})$ to itself via $F \mapsto U^{-1}FU$ for every $F \in \mathrm{GL}(\mathcal{H})$. Note that, in particular, $U^{-1}FGU = (U^{-1}FU)(U^{-1}GU)$ and $U^{-1}F^{-1}U = (U^{-1}FU)^{-1}$ (assuming F is invertible), for any $F, G \in \mathrm{GL}(\mathcal{H})$ (using the fact that $(AB)^{-1} = B^{-1}A^{-1}$ for any invertible operators A and B). Moreover, note that this map is 1-1: for any operator F, there is a unique operator G such that $U^{-1}GU = F$, namely, $G = UFU^{-1}$.

Note also that every unitary operator is an element of $\mathrm{GL}(\mathcal{H})$. Hence we have here an example of a general construction, the *group inner isomorphism*, by which one maps a group, $\mathcal{G}$, to itself via $h \mapsto ghg^{-1}$ for some $g \in \mathcal{G}$ and all $h \in \mathcal{G}$.

7.6.9. *Induced Representations.* Given a group, $\mathcal{G}$, and a subgroup, $\mathcal{H}$, a representation of $\mathcal{H}$ on a vector space, W, 'induces' a representation of $\mathcal{G}$, in the following sense. One can construct, from the representation of $\mathcal{H}$ on W, a vector space, V, which is in fact the direct sum of copies of W, and a representation of $\mathcal{G}$ on V. Each of the copies of W inside V carries a representation of $\mathcal{H}$.

Here is a rough description of how the construction goes. Let σ be a representation of $\mathcal{H}$ on W. The general idea behind constructing the representation of $\mathcal{G}$ induced by σ is to construct a vector space V that is the direct sum of copies of W, i.e., $V = \bigoplus_n W_n$, where each W_n is a copy of W, and each copy of W corresponds to an element of $\mathcal{G}/\mathcal{H}$. The representation, ρ, of $\mathcal{G}$ induced by σ is defined as follows. Fix a representative, g_n, for each coset, n, of $\mathcal{G}/\mathcal{H}$. Note that for any $g \in \mathcal{G}$, $gg_n = g_m h$ for some $h \in \mathcal{H}$ and some $m \in \mathcal{G}/\mathcal{H}$. Let w_n be an arbitrary vector from W_n, corresponding (under some isomorphism) to $w \in W$. (Note that if we define an operator on a basis for each of the W_n then we have defined it for all of V, by linearity.) Define the $\rho(g)$ (for any $g \in \mathcal{G}$) by $\rho(g)w_n = (\sigma(h)w)_m$, where h and m are given as above. The expression on the right should be read as 'let $\sigma(h)$ act on $w_n \in W_n$ in the same way that it acts on $w \in W$, then map the result to the corresponding vector in W_m.' Note that this entire prescription presupposes a set of isomorphisms between W and the W_n. Finally, it can be shown that none of the above depends on the choice of representatives g_n, in the sense that a different choice produces an isomorphic representation.

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