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Nonrenormalization Theorem for Gauge Coupling in $2 + 1D$

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Abstract

We prove that β-function of the gauge coupling in $2 + 1D$ gauge theory coupled to any renormalizable system of spinor and scalar fields is zero. This result holds both when the gauge field action is the Chern-Simons action and when it is the topologically massive action.

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1 Introduction.

It is characteristic of $2+1D$ that one can construct a local first-order action which is invariant with respect to infinitesimal gauge transformations. This is the Chern-Simons action

$$S = \frac{k}{4\pi}\int Tr(AdA + \frac{2}{3}A^3).$$

One can easily see that the theory of scalar or spinor fields interacting with the Chern-Simons field is renormalizable. On spacetime with topology $\mathcal{R}\times\mathcal{M}$ where $\mathcal{M}$ is simply connected, the Chern-Simons field has no physical degrees of freedom and can be expressed through other fields. If we add the Maxwell term to the action then the gauge field will have one physical degree of freedom describing a massive vector particle. Such gauge field is called topologically massive. The theory of the topologically massive gauge field interacting with spinors, or scalars, is also renormalizable. (The theory with Maxwell term only is not good since it has terrible infrared divergencies [1] .)

On general grounds one could expect that $g^2 = 4\pi/k$ requires infinite renormalization, and therefore g experiences a nontrivial renormalization group (RG) flow driven by the corresponding β-function

$$\frac{dg(t)}{dt} = \beta(g).$$

However, there are strong reasons to suspect that in fact $\beta(g)$ is zero in any order of perturbation theory. The first (trivial) observation is that in the nonabelian case the Chern-Simons action is not invariant with respect to large gauge transformations. Rather it is shifted by a number proportional to the winding number of the gauge transformation [2]. To ensure that e^{iS} be invariant, k has to be quantized ($k \in \mathcal{Z}$ for $G = SU(N)$), otherwise the theory will have a global anomaly. Nontrivial renormalization of k would spoil the quantization of k imposed on the classical level and introduce the global anomaly. However, this argument does not prove that $\beta = 0$, rather it shows that in case $\beta \neq 0$, the theory probably will not be nonperturbatively consistent. Another argument comes from direct perturbative calculations which were performed by several authors. Chen et al., [3] showed that g is not renormalized up to two-loop order in the theory of the nonabelian Chern-Simons field interacting with scalars or spinors. Avdeev et al., [4] calculated two-loop RG-functions in the most general abelian theory including scalars and spinors interacting with each other and with the Chern-Simons field. They also found that $\beta(g) = 0$.

As for exact results, Delduc et al., [5] proved that g is not renormalized in any order of loop expansion in the pure Chern-Simons theory (without matter fields). Moreover, they showed that the anomalous dimension of the gauge field also vanishes, so the theory is finite; the latter fact is not true if matter fields are present [3]. One may say that the pure Chern-Simons theory is not a very good example of the generic behaviour since it has no physical degrees of freedom and in the axial gauge is reduced to a free field theory [6, 7]. Later it was shown [8] that in the abelian Chern-Simons theory with scalar fields β-function also vanishes.

In our paper [9] we proposed a proof of the nonrenormalization theorem for the gauge coupling in the theory of spinors interacting with the Chern-Simons field. The proof was based on the study of the conformal anomaly and resembled that in [8]. Here we want to extend the arguments of [9] to the general renormalizable $2+1D$ theory of scalars and spinors interacting with each other and with the gauge field. The gauge field action is assumed to be the Chern-Simons action or the topologically massive action. The proof is valid in both abelian and nonabelian cases. The paper is organised as follows. In Section 2 we derive certain useful expressions for the trace of the renormalized energy-momentum tensor and use them to calculate the conformal anomaly in our theory. In Section 3 vanishing of β-function is proved for the case when the gauge field action is the Chern-Simons action. In Section 4 we prove that in the topologically massive gauge theory with matter fields both β-function of the gauge coupling and the anomalous dimension of the gauge field vanish.

2 Conformal anomaly in Chern-Simons theory with matter fields.

There is an intimate connection between the conformal anomaly and RG-functions. Indeed, it is well known that the trace of the energy-momentum tensor can be expressed through the scale variation of the action:

$$\int T^{\mu}_{\mu} dx = \delta S - \sum_i d_i \int \frac{\delta S}{\delta \phi_i} dx, \tag{1}$$

where $\delta S = \frac{d}{dt}|_{t=1} S[\phi^t]$, $\phi_i^t(x) = t^{d_i}\phi_i(tx)$, d_i is the canonical dimension of ϕ_i. The quantum analogue of this formula is

$$\left\langle \int T^{\mu}_{ren\mu} dx \right\rangle_j = \delta\Gamma_{ren} + \sum_i d_i \int j_i \phi_i dx, \tag{2}$$

where $j_i = -\delta\Gamma_{ren}/\delta\phi_i$. Renormalization alters the scaling properties of Γ which results in the appearance of the anomalous part in the renormalized trace of the energy-momentum tensor. So one can hope to express the conformal anomaly through RG-functions. A well known example is a formula for conformal anomaly in $4D$ QCD derived in [10, 11] :

$$T^{\mu}_{ren\mu} = \frac{2\beta(g)}{g}\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)_{ren} + (1+\delta(g))m\Big(\bar{\psi}\psi\Big)_{ren}. \tag{3}$$

In general the behaviour of Γ_{ren} under scale transformation is determined by RG equations. As is known their explicit form depends on the subtraction scheme. Suppose we consider a theory parametrized with N dimensionless couplings $g_1, \dots, g_N$ and a mass m. We will use a mass independent subtraction scheme, i.e., we will make all subtractions at $p^2 = \mu^2$. Then standard RG arguments [12] lead to the following homogeneous RG equation (first derived by Weinberg [13]):

$$\delta\Gamma_{ren} = \sum_{a=1}^{N} \beta_a \frac{\partial \Gamma_{ren}}{\partial g_a} - (\delta + 1) m \frac{\partial \Gamma_{ren}}{\partial m} + \sum_i \gamma_i \int \phi_i j_i \, dx \tag{4}$$

Here $\beta_a, \gamma_i, \delta$ depend not only on $g_1, \dots, g_N$, but also on m/μ [13] and are defined as

$$\beta_a = g_a \mu \frac{\partial}{\partial \mu} \ln Z_a, \quad \gamma_i = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z_i, \quad \delta = \mu \frac{\partial}{\partial \mu} \ln Z_m \tag{5}$$

Substituting (4) into (2) we get

$$\left\langle \int T^{\mu}_{ren\mu} dx \right\rangle_j = \sum_{a=1}^{N} \beta_a \frac{\partial \Gamma_{ren}}{\partial g_a} - (\delta + 1) m \frac{\partial \Gamma_{ren}}{\partial m} + \sum_i (\gamma_i + d_i) \int \phi_i j_i \, dx \tag{6}$$

Let us use (6) to compute conformal anomaly in the most general renormalizable theory of scalars and spinors interacting with the Chern-Simons field:

$$\begin{aligned} \mathcal{L} = \; & \tfrac{1}{2} \epsilon^{\mu\nu\rho} \left(A^a_\mu \partial_\nu A^a_\rho + \tfrac{1}{3} g f^{abc} A^a_\mu A^b_\nu A^c_\rho \right) + b \partial_\mu D^\mu c + B \partial^\mu A_\mu \\ & + (D_\mu \varphi)^\dagger D^\mu \varphi - m_\varphi^2 \varphi^\dagger \varphi + \bar{\psi} \left(i \not{D} - m \right) \psi + \mathcal{V}(\psi, \varphi). \end{aligned} \tag{7}$$

We chose Landau gauge for definiteness. The most general potential $\mathcal{V}$ allowed by renormalizability is

$$\mathcal{V} = a_1 \bar{\psi} \psi \varphi + a_2 \bar{\psi} \psi \varphi \varphi + a_3 \varphi^3 + a_4 \varphi^4 + a_5 \varphi^5 + a_6 \varphi^6. \tag{8}$$

Here $\bar{\psi}\psi\varphi$ is a trilinear invariant linear in $\bar{\psi}, \psi$ andφ, φ^3 is a trilinear invariant built of φ and $\varphi^\dagger$ and so on. The existence of such invariants depends on the gauge group and the matter field representations.

The theory described by the Lagrangian (7) is renormalizable by power counting provided gauge anomaly is absent. To show the absence of the anomaly it suffices to exhibit a BRST-invariant regularization scheme. Evidently higher covariant derivatives supplemented by the Pauli-Villars regularization of the one-loop diagrams serve our aim. To apply formula (6) to our theory let us rescale the parameters of the Lagrangian (7) in the following way:

$$m_\varphi^2 = \alpha_0 m^2, \qquad a_b = \alpha_b m^{dim\, a_b} \quad \text{for} \quad b = 1, \dots, 6. \tag{9}$$

Now the Lagrangian is parametrized with eight dimensionless couplings g, $\alpha_0, \ldots \alpha_6$ and a mass m, just as required for formula (6) to be applicable. The renormalized Lagrangian is

$$\mathcal{L}_{ren} = \frac{Z_3}{2}\epsilon^{\mu\nu\rho}\left(A^a_\mu\partial_\nu A^a_\rho + \frac{1}{3}g\frac{\tilde{Z}_1}{\tilde{Z}_3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right) + \tilde{Z}_3 b\,\partial_\mu D^\mu_{ren}c + B\partial^\mu\ A_\mu$$
$$+ Z_{2\varphi}\left((D_{ren\,\mu}\varphi)^\dagger D^\mu_{ren}\varphi - \bar{Z}_0 Z_m^2\alpha_0 m^2\varphi^\dagger\varphi\right)$$
$$+ Z_2\bar{\psi}\,(i\not{D} - Z_m m)\,\psi + \mathcal{V}_{ren}(\psi,\varphi). \qquad (10)$$

Here D^μ_{ren} is obtained from D^μ by a change $g \to \frac{\tilde{Z}_1}{\tilde{Z}_3}g$ and $\mathcal{V}_{ren}$ is

$$\mathcal{V}_{ren} = \bar{Z}_1 Z_m Z_2 Z_{2\varphi}^{1/2}\alpha_1 m^{1/2}\bar{\psi}\psi\varphi + \bar{Z}_2 Z_{2\varphi}\alpha_2\bar{\psi}\psi\varphi\varphi + \bar{Z}_3 Z_m^{3/2}Z_{2\varphi}^{3/2}\alpha_3 m^{3/2}\varphi^3$$
$$+ \bar{Z}_4 Z_m Z_{2\varphi}^2\alpha_4 m\varphi^4 + \bar{Z}_5 Z_m^{1/2}Z_{2\varphi}^{5/2}\alpha_5 m^{1/2}\varphi^5 + \bar{Z}_6 Z_{2\varphi}^3\alpha_6\varphi^6. \qquad (11)$$

RG functions are defined according to

$$\beta = g\mu\frac{\partial}{\partial\mu}\ln\frac{Z_3^{1/2}\tilde{Z}_3}{\tilde{Z}_1}, \qquad \beta_a = \alpha_a\mu\frac{\partial}{\partial\mu}\ln\bar{Z}_a \quad \text{for} \quad a = 0,\ldots,6,$$
$$\delta = \mu\frac{\partial}{\partial\mu}\ln Z_m, \qquad \gamma_A = \mu\frac{\partial}{\partial\mu}\ln Z_3^{1/2},$$
$$\gamma_c = \gamma_b = \mu\frac{\partial}{\partial\mu}\ln\tilde{Z}_3^{1/2}, \quad \gamma_\psi = \mu\frac{\partial}{\partial\mu}\ln Z_2^{1/2}, \quad \gamma_\varphi = \mu\frac{\partial}{\partial\mu}\ln Z_{2\varphi}^{1/2}. \qquad (12)$$

The expectation value of the zero momentum-transfer part of $T^\mu_{ren\,\mu}$ is given by the general formula (6) where the first sum extends over all eight couplings $g, \alpha_0, \ldots, \alpha_6$. We can represent the derivatives of Γ_{ren} entering (6) as expectation values of renormalized operator insertions using rules of [14]. After this we may remove the angular brackets since the sources j are arbitrary and get the following identity for renormalized operator insertions:

$$\int T^\mu_{ren\,\mu}\,dx = -\int\frac{\beta}{g}\epsilon^{\mu\nu\rho}\left(A^a_\mu\partial_\nu A^a_\rho + \frac{1}{3}gf^{abc}A^a_\mu A^b_\nu A^c_\rho\right)_{ren}dx$$
$$+ \int\left[\sum_{a=1}^{6}\beta_a\left(\frac{\partial\mathcal{V}}{\partial\alpha_a}\right)_{ren} - \beta_0 m^2\left(\varphi^\dagger\varphi\right)_{ren}\right]dx$$
$$+ (1+\delta)\int\left(m\bar{\psi}\psi + 2m^2\alpha_0\varphi^\dagger\varphi - m\frac{\partial\mathcal{V}}{\partial m}\right)_{ren}dx$$
$$+ \sum_i A_i\int\left(\phi_i\frac{\delta S}{\delta\phi_i}\right)_{ren}dx, \qquad (13)$$

where A_i are finite coefficients which can be easily computed (for our aims their concrete values are not important). This formula is a $2+1D$ analogue of (3).

3 Nonrenormalization theorem for the Chern-Simons coupling.

In this Section we are going to show that one of the consequences of (13) is $\beta = 0$. First let us find the most general form of $T^{\mu}_{ren\mu}$ following the approach of [11]. To this end we consider the regularized trace $T^{\mu}_{reg\mu}$. Since the regularized action is a sum of the classical action and the regulator terms, the regularized trace contains a classical piece and a piece coming from regulator terms. The classical part of $T^{\mu}_{reg\mu}$ is obviously an operator of dimension 3 and ghost number 0. Moreover, it is a gauge invariant operator up to terms of the form $\phi_i \frac{\delta S}{\delta \phi_i}$. A well-known result of the theory of operator mixing is that after regularization such an operator can mix only with local operators of the same ghost number and the same or lower dimension. It is shown in [15] that BRST-invariance restricts further the set of operators which can mix, namely a gauge invariant operator of ghost number 0 can mix only with gauge invariant operators and operators of the form $\phi_i \frac{\delta S}{\delta \phi_i}$. (The proof uses only BRST-invariance and antighost equation of motion, therefore it is valid in any gauge field theory provided the gauge anomaly is absent. It was shown above that this is the case for $2+1D$ gauge theories.) One can easily see that the same is true for the "regulator" part of T^{μ}_{μ}. Indeed the regulator term consists of the Pauli-Villars-ghost term and the higher-covariant-derivative term. They are both gauge invariant by definition and have ghost number 0. The Pauli-Villars term has dimension 3. The higher-covariant-derivative term has dimension d greater than three. However, it is multiplied by Λ^{3-d} (Λ is the cutoff), hence it is an irrelevant operator in RG-terminology [12], that is in the limit $\Lambda \to \infty$ it is reduced to a linear combination of operators of dimension 3 and lower. Hence the "regulator" term in $T^{\mu}_{reg\mu}$ also mixes only with gauge invariant operators of dimension 3 and lower and with operators of the form $\phi_i \frac{\delta S}{\delta \phi_i}$.

One can see that there are only eight linearly independent gauge invariant operators with ghost number 0 and dimension 3 or lower which are not reduced to $\phi_i \frac{\delta S}{\delta \phi_i}$ with some i. These are

$$O_0 = \varphi^{\dagger}\varphi, \quad O_1 = \bar{\psi}\psi\varphi, \quad O_2 = \bar{\psi}\psi\varphi\varphi, \quad O_3 = \varphi^3,$$
$$O_4 = \varphi^4, \quad O_5 = \varphi^5, \quad O_6 = \varphi^6, \quad O_7 = \bar{\psi}\psi. \tag{14}$$

These are just the operators out of which the Lagrangian (7) is built. However, the Chern-Simons density does not appear here (though it appears in (7)) because it is not gauge invariant. So $T^{\mu}_{ren\mu}$ is a linear combination of $(O_i)_{ren}\,, i = 0, \ldots, 7$. Using this information we can reconstruct $T^{\mu}_{ren\mu}$ from its zero momentum-transfer part found in Section 2 just as it was done in [11]. However here an important difference arises as compared to the $3+1D$ case. Namely (13) contains a term (the Chern-Simons term) which cannot be present in $T^{\mu}_{ren\mu}$ because it is not one of O_i. Hence its coefficient β must be zero. This is the desired result. From the definition (12) it is clear that this means that Z_g is finite. Hence the corresponding Callan-Simanzik function also vanishes. (Our β is *not* the Callan-Simanzik function because we chose a mass-independent subtraction scheme. Neither is it the Gell-Mann–Low function because our m does not determine the physical pole of the propagator.)

4 Topologically massive gauge theory.

The case of the topologically massive gauge theory is much simpler. We add to the Lagrangian (7) the Maxwell term $1/MTrF_{\mu\nu}^2$ where M is a new mass parameter. Power counting shows that the gauge field self-energy diverges at most linearly and the 3-gluon vertex diverges at most logarithmically. Hence no counterterm of the form $TrF_{\mu\nu}^2$ is needed. Then Z_1/Z_3 must be finite. Indeed, let Z_{M3} and Z_{M1} be the renormalization constants for the 2- and 3-gluon Maxwell vertices correspondingly. Ward identities require that $Z_1/Z_3 = Z_{M1}/Z_{M3}$. But we have just seen that Z_{M1} and Z_{M3} are finite, therefore Z_1/Z_3 is finite and $\beta = 2\gamma_A$.

Now let us show that Z_3 is finite. The Schwinger-Dyson equation for the gluon self-energy is represented graphically on Fig. 1. We are interested only in the part of the gluon self-energy which has superficial degree of divergency $\omega = 1$. Gauge invariance tells us that the divergent piece is proportional to $\epsilon^{\mu\nu\rho}p_\rho$. But in the ultraviolet the irreducible vertices in these diagrams have the same form as the tree vertices apart from logarithmic factors (Weinberg's theorem and Lorenz-invariance), therefore they are P-even (proportional to γ_μ in the case of the $\bar{\psi}\psi A$ vertex). The same is true about exact propagators; in particular the gluon propagator has Maxwell asymptotics. Hence the parts of the diagrams which have $\omega = 1$ are P-even (contain even number of γ-matrices in the case of diagrams (b) and (d)) and cannot give the divergency proportional to $\epsilon^{\mu\nu\rho}p_\rho$. We conclude that all these diagrams are in fact finite and hence Z_3 is finite and $\beta = \gamma_A = 0$ (essentially the same argument was used in [1] to prove the ultraviolet fineteness of the topologically massive gauge theory without matter fields).

We showed that the gauge coupling g is not renormalized both in the Chern-Simons gauge theory and in the topologically massive gauge theory. In the topologically massive case the reason for this is just the presence of the Maxwell term which serves as an ultraviolet regulator, so the gauge field part of the action is not renormalized. In the Chern-Simons theory with matter fields β-function vanishes because it turns out to be a coefficient which determines the mixing between the trace of the energy-momentum tensor and the Chern-Simons density. This mixing is in fact forbidden because the Chern-Simons density is not a gauge invariant operator, therefore β must be zero. In particular vanishing of β means that the global anomaly is absent.

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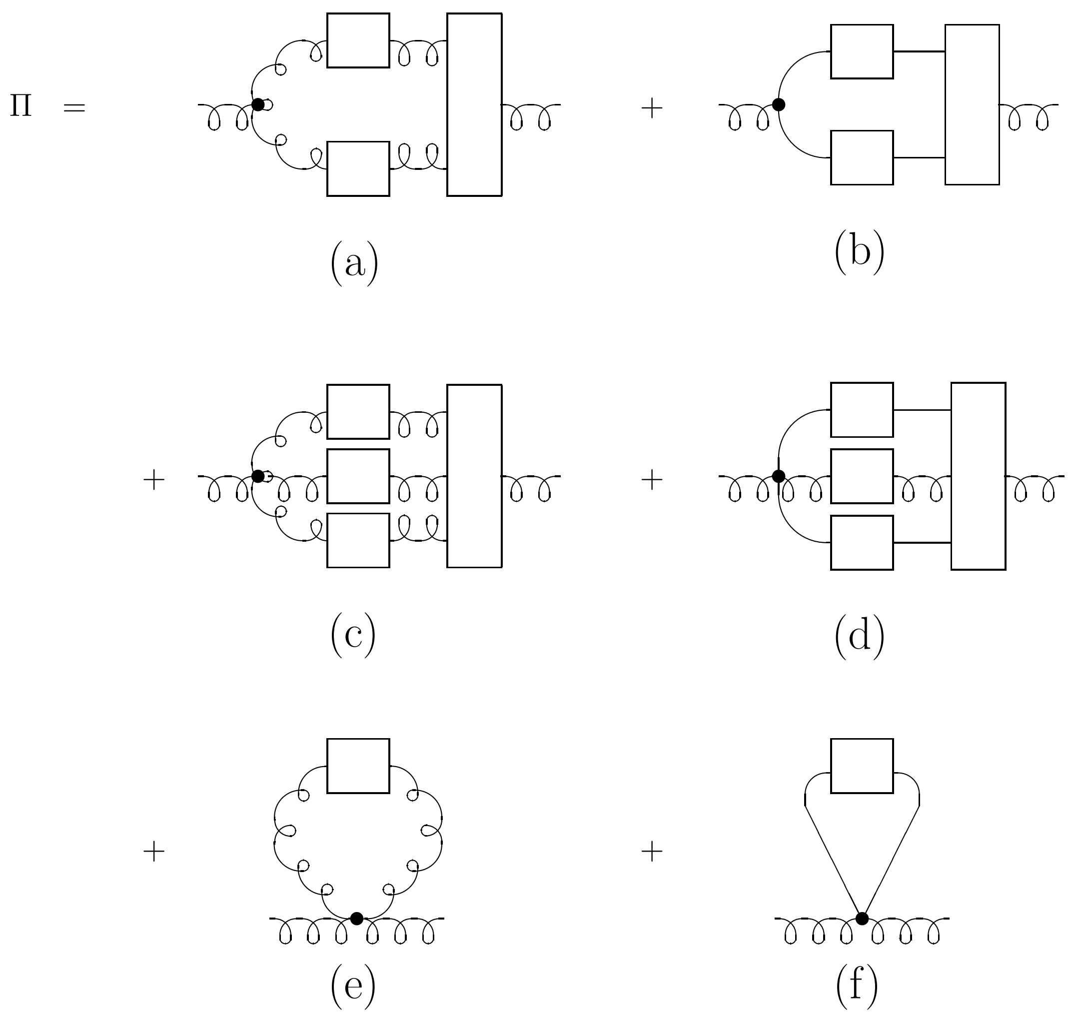

Figure 1: Schwinger-Dyson equation for gluon self-energy. Boxes denote exact propagators and irreducible vertices. Curly lines correspond to gluons and solid lines correspond to ghosts and matter fields.