Ann. Henri Poincaré 5 (2004) 1 – 73

1424-0637/04/010001-73
DOI 10.1007/s00023-004-0160-1

Annales Henri Poincaré

Non-Selfadjoint Perturbations of Selfadjoint Operators in 2 Dimensions I

Michael Hitrik and Johannes Sjöstrand

Abstract. This is the first in a series of works devoted to small non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2. In the present work we treat the case when the classical flow of the unperturbed part is periodic and the strength ϵ of the perturbation is $\gg h$ (or sometimes only $\gg h^2$) and bounded from above by h^δ for some $\delta > 0$. We get a complete asymptotic description of all eigenvalues in certain rectangles $[-1/C, 1/C] + i\epsilon[F_0 - 1/C, F_0 + 1/C]$.

1 Introduction

In [20], A. Melin and the second author observed that for a wide and stable class of non-selfadjoint operators in dimension 2 and in the semi-classical limit ($h \to 0$), it is possible to describe all eigenvalues individually in an h-independent domain in $\mathbf{C}$, by means of a Bohr-Sommerfeld quantization condition. This result is quite remarkable since the corresponding conclusion in the selfadjoint case seems to be possible only in dimension 1 or under strong (and unstable) assumptions of complete integrability. The underlying reason for this result is the absence of small denominators which allows us to avoid the usual trouble with exceptional sets in the KAM theorem.

As a next step, the second author noticed ([22]) that for non-selfadjoint operators of the form $P(x, hD_x) + i\epsilon Q(x, hD_x)$ it is possible to find a similar result, when P is selfadjoint, $\epsilon > 0$ small and fixed and the classical bicharacteristic flow is periodic on each real energy surface. (Again, it is important that we are in dimension 2.) The method is similar to the one in [20] and uses non-linear Cauchy-Riemann equations, now in an "ϵ-degenerate" form. (See also [24] for a different extension.)

It soon became quite clear that we run into a fairly vast program, and that logically one should start with even smaller perturbations, say $\epsilon = \mathcal{O}(h^\delta)$, for some $\delta > 0$. The present work is planned to be the first in a series, devoted to small perturbations of selfadjoint operators in dimension 2. In addition to the challenge of doing plenty of things in dimension 2, that can usually only be done in dimension 1, we have been motivated by recent progress around the damped wave equation ([19], [2], [25], [14]), as well as the problem of barrier top resonances for the semi-classical Schrödinger operator ([17]) where more complete results than the corresponding ones for eigenvalues of potential wells ([26], [3], [21]) seem possible. One long term goal of this series is to get improved results on the distribution of resonances for strictly convex obstacles in $\mathbf{R}^3$. See [30] (and references given there) for a first result on Weyl asymptotics for the real parts inside certain bands. In the case of analytic obstacles, much more can probably be said, especially in dimension 3 (and 2).

Let M denote $\mathbf{R}^2$ or a compact real-analytic manifold of dimension 2.

When $M = \mathbf{R}^2$, let

$$P_\epsilon = P(x, hD_x, \epsilon; h) \tag{1.1}$$

be the Weyl quantization on $\mathbf{R}^2$ of a symbol $P(x, \xi, \epsilon; h)$ depending smoothly on $\epsilon \in \mathrm{neigh}\,(0, \mathbf{R})$ with values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4$, with

$$|P(x, \xi, \epsilon; h)| \leq Cm(\mathrm{Re}\,(x, \xi)) \tag{1.2}$$

there. Here m is assumed to be an order function on $\mathbf{R}^4$, in the sense that $m > 0$ and

$$m(X) \leq C_0 \langle X - Y\rangle^{N_0} m(Y),\ X, Y \in \mathbf{R}^4. \tag{1.3}$$

We also assume that

$$m \geq 1. \tag{1.4}$$

We further assume that

$$P(x, \xi, \epsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\epsilon}(x, \xi) h^j,\ h \to 0, \tag{1.5}$$

in the space of such functions. We make the ellipticity assumption

$$|p_{0,\epsilon}(x, \xi)| \geq \frac{1}{C} m(\mathrm{Re}\,(x, \xi)),\ |(x, \xi)| \geq C, \tag{1.6}$$

for some $C > 0$.

When M is a compact manifold, we let

$$P_\epsilon = \sum_{|\alpha| \leq m} a_{\alpha,\epsilon}(x; h)(hD_x)^\alpha, \tag{1.7}$$

be a differential operator on M, such that for every choice of local coordinates, centered at some point of M, $a_{\alpha,\epsilon}(x; h)$ is a smooth function of ϵ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$. We further assume that

$$a_{\alpha,\epsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\epsilon,j}(x) h^j,\ h \to 0, \tag{1.8}$$

in the space of such functions. The semi-classical principal symbol in this case is given by

$$p_{0,\epsilon}(x, \xi) = \sum a_{\alpha,\epsilon,0}(x)\xi^\alpha, \tag{1.9}$$

and we make the ellipticity assumption

$$|p_{0,\epsilon}(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^m,\ (x,\xi) \in T^*M,\ |\xi| \geq C, \tag{1.10}$$

for some large $C > 0$. (Here we assume that M has been equipped with some Riemannian metric, so that $|\xi|$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ are well defined.)

Sometimes, we write p_ϵ for $p_{0,\epsilon}$ and simply p for $p_{0,0}$. Assume

$$P_{\epsilon=0} \text{ is formally selfadjoint.} \tag{1.11}$$

In the case when M is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$ for some positive real-analytic density $\mu(dx)$ on M.

Under these assumptions, P_ϵ will have discrete spectrum in some fixed neighborhood of $0 \in \mathbf{C}$, when $h > 0, \epsilon \geq 0$ are sufficiently small, and the spectrum in this region will be contained in a band $|\mathrm{Im}\, z| \leq \mathcal{O}(\epsilon)$. The purpose of this work and later ones in this series, is to give detailed asymptotic results about the distribution of individual eigenvalues inside such a band.

Assume for simplicity that (with $p = p_{\epsilon=0}$)

$$p^{-1}(0) \cap T^*M \text{ is connected.} \tag{1.12}$$

Let $H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$ be the Hamilton field of p. In this work, we will always assume that for $E \in \mathrm{neigh}\,(0, \mathbf{R})$:

$$\begin{array}{l}\text{The } H_p\text{-flow is periodic on } p^{-1}(E) \cap T^*M \text{ with} \\ \text{period } T(E) > 0 \text{ depending analytically on } E.\end{array} \tag{1.13}$$

Let $q = \frac{1}{i}(\frac{\partial}{\partial \epsilon})_{\epsilon=0} p_\epsilon$, so that

$$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2 m), \tag{1.14}$$

in the case $M = \mathbf{R}^2$ and $p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2\langle\xi\rangle^m)$ in the manifold case. Let

$$\langle q\rangle = \frac{1}{T(E)}\int_{-T(E)/2}^{T(E)/2} q \circ \exp tH_p dt \text{ on } p^{-1}(E) \cap T^*M. \tag{1.15}$$

Notice that $p, \langle q\rangle$ are in involution; $0 = H_p\langle q\rangle =: \{p, \langle q\rangle\}$. In Section 3, we shall see how to reduce ourselves to the case when

$$p_\epsilon = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2), \tag{1.16}$$

near $p^{-1}(0) \cap T^*M$. An easy consequence of this is that the spectrum of P_ϵ in $\{z \in \mathbf{C}; |\mathrm{Re}\, z| < \delta\}$ is confined to $]-\delta, \delta[+i\epsilon]\langle \mathrm{Re}\, q\rangle_{\mathrm{min},0} - o(1), \langle \mathrm{Re}\, q\rangle_{\mathrm{max},0} + o(1)[$, when $\delta, \epsilon, h \to 0$, where $\langle \mathrm{Re}\, q\rangle_{\mathrm{min},0} = \min_{p^{-1}(0)\cap T^*M}\langle \mathrm{Re}\, q\rangle$ and similarly for $\langle q\rangle_{\mathrm{max},0}$. We will mainly think about the case when $\langle q\rangle$ is real-valued but we will work under the more general assumption that

$$\mathrm{Im}\,\langle q\rangle \text{ is an analytic function of } p \text{ and } \mathrm{Re}\,\langle q\rangle, \tag{1.17}$$

in the region of T^*M, where $|p| \leq 1/|\mathcal{O}(1)|$.

Let $F_0 \in [\langle \mathrm{Re}\, q\rangle_{\min,0}, \langle \mathrm{Re}\, q\rangle_{\max,0}]$. The purpose of the present work is to determine all eigenvalues in a rectangle

$$\left]-\frac{1}{|\mathcal{O}(1)|}, \frac{1}{|\mathcal{O}(1)|}\right[+ i\epsilon \left]F_0 - \frac{1}{|\mathcal{O}(1)|}, F_0 + \frac{1}{|\mathcal{O}(1)|}\right[\,, \tag{1.18}$$

for

$$h \ll \epsilon \le \mathcal{O}(h^\delta), \tag{1.19}$$

where $\delta > 0$ is any fixed number. (When the subprincipal symbol of P is zero, we can treat even smaller values of ϵ: $h^2 \ll \epsilon \le \mathcal{O}(h^\delta)$.) We will achieve this under the general assumption that

$$T(0) \text{ is the minimal period of every } H_p\text{-trajectory in } \Lambda_{0,F_0}, \tag{1.20}$$

where

$$\Lambda_{0,F_0} := \{\rho \in T^*M;\, p(\rho) = 0,\, \mathrm{Re}\,\langle q\rangle(\rho) = F_0\}, \tag{1.21}$$

in the following three cases:

I) The first case is when

$$dp,\ d\mathrm{Re}\,\langle q\rangle \text{ are linearly independent at every point of } \Lambda_{0,F_0}. \tag{1.22}$$

This implies that every connected component of Λ_{0,F_0} is a two-dimensional Lagrangian torus. For simplicity, we shall assume that there is only one such component. Notice that in view of (1.20), the space of closed orbits in $p^{-1}(0) \cap T^*M$;

$$\Sigma := (p^{-1}(0) \cap T^*M)/\sim,$$

where $\rho \sim \mu$ if $\rho = \exp tH_p\mu$ for some $t \in \mathbf{R}$, becomes a 2-dimensional symplectic manifold near the image of Λ_{0,F_0}, and (1.22) simply means that $\mathrm{Re}\,\langle q\rangle$, viewed as a function on Σ, has non-vanishing differential along the image of Λ_{0,F_0}. The image of Λ_{0,F_0} is just a closed curve. The main results in this case are Theorems 6.2, 6.4 and they show that the eigenvalues form a distorted lattice.

II) The second case is when $F_0 \in \{\langle \mathrm{Re}\, q\rangle_{\min,0}, \langle \mathrm{Re}\, q\rangle_{\max,0}\}$. In this case, we again view $\langle \mathrm{Re}\, q\rangle$ as a smooth function on Σ near the image of Λ_{0,F_0} and assume that

$$\begin{aligned}&\text{The Hessian of } \langle \mathrm{Re}\, q\rangle \text{ is non-degenerate (positive}\\ &\text{or negative) at every point } \rho \in \Sigma, \text{ with } \langle \mathrm{Re}\, q\rangle(\rho) = F_0.\end{aligned} \tag{1.23}$$

The main results in this case are given by Theorems 6.6, 6.7 which tell us that the eigenvalues form a distorted half-lattice.

III) The third natural case would be when F_0 is a critical value of $\mathrm{Re}\,\langle q\rangle$ corresponding to a saddle point. We hope to study this case in the near future.

The analyticity assumptions are introduced, because the optimal spaces are deformations of the usual L^2-space obtained by adding exponential weights with exponents that are $\mathcal{O}(\epsilon)$, and there are closely related Fourier integral operators with complex phase some of which have associated complex canonical transformations that are ϵ-perturbations of the identity. When $\epsilon \sim h^{\delta}$, $0 < \delta < 1$, appropriate Gevrey type assumptions would probably suffice, but in the case $\epsilon \sim h$ we seem to need analyticity assumptions at one point, even though standard C^{∞}-microlocal analysis would suffice for most of the steps. At the opposite extreme, ϵ small but independent of h, the analyticity assumptions seem necessary, and in order to avoid technicalities, we have chosen to assume analyticity independently of the size of ϵ.

In the selfadjoint case there have been many works about operators whose associated classical flow is periodic ([31], [8], [5], [11], [9], [16]), and we follow one of the main ideas in those works, namely to use some sort of averaging procedure in order to reduce the dimension by one unit, so that in our case, we come down to a one-dimensional problem. The implementation of this is more complicated in our case because of the need to work in modified exponentially weighted spaces (after suitable FBI-transforms). It should also be pointed out that in the case when ϵ is small but independent of h ([22]), this does not seem to work and the problem remains two-dimensional. The same seems to be the case (for the whole scale of ϵ) in other situations, when the H_p-flow is completely integrable without being periodic, or more generally when the energy surface $p^{-1}(0) \cap T^*M$ contains certain invariant Lagrangian tori. We intend to treat such situations later in this series.

The plan of the paper is the following:

In Section 2, we reexamine the Egorov theorem in a form suitable for us, and complete some observations of [13] about the two term version of this result.

In Section 3 we perform dimension reduction by averaging.

In Section 4 we make a complete reduction in the torus case (I) and determine the corresponding quasi-eigenvalues.

In Section 5 we do the analogous work in the extreme case (II).

In Section 6 we justify the earlier computations by treating an auxiliary global (Grushin) problem, and we obtain the two main results.

In Section 7, we give a first application to barrier top resonances.

In the appendix, we review some standard facts about FBI-transforms on manifolds.

The next work(s) in this series (in addition to [22]) will remain in the case when the classical flow of the unperturbed part is periodic. We intend to study the saddle point case (III), and the case when $\langle q \rangle$ vanishes.

2 Quantization of canonical transformations between non-simply connected domains in phase space

We first give an affirmative answer to a question asked in Appendix A of [13]. Let $\kappa : \mathrm{neigh}\,((y_0,\eta_0),T^*\mathbf{R}^n) \to \mathrm{neigh}\,((x_0,\xi_0),T^*\mathbf{R}^n)$ be an analytic canonical transformation and consider a corresponding Fourier integral operator

$$Uu(x) = h^{-\frac{n+N}{2}} \iint e^{i\phi(x,y,\theta)/h} a(x,y,\theta;h)u(y)dyd\theta, \tag{2.1}$$

with $a = a_0 + \mathcal{O}(h)$, a classical symbol in $S^{0,0}$ (see the appendix), and ϕ non-degenerate phase function in the sense of Hörmander [15] (without the homogeneity requirement in θ) which generates the graph of κ. (Since we work microlocally, ϕ, a are assumed to be defined near a fixed point (x_0,y_0,θ_0) with $\phi'_\theta(x_0,y_0,\theta_0) = 0$, $(x_0,\xi_0) = (x_0,\phi'_x(x_0,y_0,\theta_0))$, $(y_0,\eta_0) = (y_0,-\phi'_y(x_0,y_0,\theta_0))$.) We require U to be unitary:

$$U^*U = 1, \text{ microlocally near } (y_0,\eta_0), \tag{2.2}$$

and we are interested in the improved Egorov property:

$$\begin{gathered}\text{If } PU = UQ, \text{ where } P = P^w,\ Q = Q^w \text{ are } h\text{-pseudodifferential} \\ \text{operators of order 0, then } P\circ\kappa = Q + \mathcal{O}(h^2).\end{gathered} \tag{2.3}$$

Here and in what follows we use the same letter to denote an operator and a corresponding Weyl symbol. In Appendix A of [13], it was shown that such U's exist and we shall answer the question raised there, by establishing the following proposition. (We learned from C. Fefferman that Jorge Silva has obtained essentially the same result in the framework of classical Fourier integral operators.)

Proposition 2.1 *Within the class of operators satisfying* (2.1) *and* (2.2), *the property* (2.3) *is equivalent to:*

$$a_0|_{C_\phi} \text{ has constant argument.} \tag{2.4}$$

Here ϕ is defined in some open set $\mathcal{D}(\phi) \subset \mathbf{R}^{2n+N}$ and

$$C_\phi = \{(x,y,\theta) \in \mathcal{D}(\phi);\ \phi'_\theta(x,y,\theta) = 0\}.$$

Proof. We first consider the special case of pseudodifferential operators, i.e., the case when κ is the identity. Then a_0 is the principal symbol and (2.2) implies that $|a_0| = 1$ (after inserting an additional factor $(2\pi)^{-n}$ in front of the integral and taking the standard phase $\phi = (x-y)\cdot\theta$). Write

$$U^{-1}PU = P + U^{-1}[P,U].$$

We see that (2.3) holds iff $\{p,a_0\} = 0$ for all p, i.e., iff $a_0 = \mathrm{Const}$. The proposition follows in the case of pseudodifferential operators since we also know in general that the property (2.4) is invariant under changes of (ϕ, a) in the representation of the given operator.

When ϕ is quadratic and a is constant, we have a metaplectic operator and κ is linear. In that case, we know that (2.3) holds, and using the special case of h-pseudodifferential operators, we see that we have equivalence between (2.3) and (2.4) in the case when κ is linear.

Consider a smooth deformation of canonical transformations $[0,1] \ni t \mapsto \kappa_t$, with a deformation field $H_{a(t)}$, so that $\partial_t \kappa_t(\rho) = H_{a(t)}(\kappa_t(\rho))$ where $a(t) = a(t, x, \xi)$ is smooth and independent of h. Let $A(t) = a^w(x, hD_x)$ and consider a corresponding family of Fourier integral operators $U(t)$ associated to κ_t:

$$hD_t U(t) + A(t) \circ U(t) = 0. \tag{2.5}$$

Since $A(t)$ are selfadjoint, unitarity of $U(t)$ is conserved under the flow of (2.5). Let $U(t)$ be such a unitary family.

Proposition 2.2 *We have* (2.3) *for one value of t iff we have it for all values of t.*

Proof. Suppose we have (2.3) for $U(0)$. From (2.5) we get

$$hD_t(U(t)^{-1}) = U(t)^{-1} A(t).$$

Consider a family $P(t) = U(t) P U(t)^{-1}$. Then

$$hD_t P(t) + [A(t), P(t)] = 0,$$

and on the level of Weyl symbols, we get

$$\partial_t P(t) + \{a(t), P(t)\} = \mathcal{O}(h^2),$$

or in other words,

$$(\partial_t + H_{a(t)}) P(t) = \mathcal{O}(h^2).$$

This means that

$$P(t) \circ (\kappa_t(\rho)) = P(0) \circ \kappa_0 + \mathcal{O}(h^2) = P(\rho) + \mathcal{O}(h^2),$$

where we used (2.3) for $U(0)$ in the last step. Then $P(t)$ fulfills (2.3) for all t. $\square$

On the other hand, if $U(t)$ fulfills (2.5), we know, using that the subprincipal symbol of $A(t)$ is 0, that if we represent

$$U(t) = h^{-\frac{n+N}{2}} \iint e^{\frac{i}{h}\phi_t(x,y,\theta)} a_t(x, y, \theta; h) u(y) dy d\theta,$$

with ϕ_t, a_t depending smoothly on t, then the argument of $a_{t,0}|_{C_{\phi_t}}$ is constant along every curve in $\{(t, x, \theta);\, (x, \theta) \in C_{\phi_t}\}$ corresponding to a $H_{a(t)}$-trajectory:

$t \mapsto (\kappa_t(\rho), \kappa_0(\rho))$. This can be seen either by a direct computation leading to a real transport equation for the leading symbol, (using that

$$e^{-i\phi(x)/h} \circ a^w(x, hD_x) \circ e^{i\phi(x)/h} = (a(x, \phi'(x) + hD_x))^w + \mathcal{O}(h^2),$$

see Appendix A in [13]), or by using Hörmander's definition ([15]) of the principal symbol of a Fourier integral operator, as well as a result of Duistermaat-Hörmander giving a real transport equation for the principal symbol for the evolution problem (2.5).

In particular, if $a_{t|_{C_{\phi_t}}}$ has constant argument for one value of t, the same holds for all other values.

For a given U associated to κ, choose κ_t and $U(t)$ as in (2.5), so that κ_0 is linear and $U(1) = U$. (We may assume for simplicity that $(y_0, \eta_0) = (x_0, \xi_0) = (0, 0)$ and take $\kappa_t(y, \eta) = \frac{1}{t}\kappa(t(y, \eta))$.) Then using Proposition 2.2 and the above remark, we get the equivalences: [U satisfies (2.3).] $\Leftrightarrow$ [$U(0)$ satisfies (2.3).] $\Leftrightarrow$ [The principal symbol of $U(0)$ has constant argument.] $\Leftrightarrow$ [The principal symbol of U has constant argument.] This gives Proposition 2.1. $\square$

Let X, Y be analytic manifolds of dimension n equipped with analytic integration densities $L(dx) = L_X(dx)$, $L(dy) = L_Y(dy)$. Let

$$\kappa : \Omega_Y \to \Omega_X$$

be a canonical transformation (and diffeomorphism), analytic for simplicity, where

$$\Omega_Y \subset\subset T^*Y,\ \Omega_X \subset\subset T^*X,$$

are connected, open with smooth boundary. We do not assume Ω_X, Ω_Y to be simply connected, so we may have finitely many closed cycles $\gamma_1, \ldots, \gamma_N \subset \Omega_Y$ which generate the homotopy group of Ω_Y.

Let $S : L^2(X) \to H_\Phi(\widetilde{X})$, $T : L^2(Y) \to H_\Psi(\widetilde{Y})$ be corresponding FBI-transforms as in the appendix, where $\widetilde{X}, \widetilde{Y}$ denote tubular complex neighborhoods of X, Y and with associated canonical transformations:

$$\kappa_S : T^*X \cap \{|\xi| < C\} \to \Lambda_\Phi,\ \kappa_T : T^*Y \cap \{|\eta| < C\} \to \Lambda_\Psi,$$

where we equip H_Φ, H_Ψ with the scalar products that make S, T unitary, and we can have $C > 0$ as large as we like. Choose C large enough, so that κ_S, κ_T are well defined on Ω_X, Ω_Y respectively, and let

$$\widetilde{\Omega}_X = \pi_x \kappa_S \Omega_X \subset \widetilde{X},\ \widetilde{\Omega}_Y = \pi_y \kappa_T \Omega_Y \subset \widetilde{Y}.$$

Let $\widetilde{\kappa} : \Lambda_\Psi \to \Lambda_\Phi$ be the lift of κ, so that $\widetilde{\kappa} = \kappa_S \circ \kappa \circ \kappa_T^{-1}$. Here $\Lambda_{\Phi,\Psi}$ are restricted to $\widetilde{\Omega}_{X,Y}$: $\Lambda_\Psi = \{(y, \frac{2}{i}\partial_y \Psi);\ y \in \widetilde{\Omega}_Y\}$, $\Lambda_\Phi = \{(x, \frac{2}{i}\partial_x \Phi);\ x \in \widetilde{\Omega}_X\}$.

We shall define a multi-valued "Floquet periodic" Fourier integral operator $U : L^2(Y) \to L^2(X)$ which is only microlocally defined from Ω_Y to Ω_X and associated to κ. Requiring that U be microlocally unitary with the improved Egorov property, we will see that we can have the Floquet periodicity:

$$\gamma_* U = e^{i\theta(\gamma)} U, \tag{2.6}$$

where γ is a closed loop in Ω_Y joining some point ρ to itself, U denotes the operator U as it is defined near ρ and the left-hand side of (2.6) denotes the operator obtained from U by following the loop γ. We will then achieve (2.6) with $\theta(\gamma) = h^{-1}S(\gamma) + k(\gamma)\pi/2$, where $S(\gamma) = \int_{\kappa\circ\gamma} \xi dx - \int_\gamma \eta dy$ is the difference of the actions of $\kappa \circ \gamma$ and γ, and $k(\gamma) \in \mathbf{Z}$ is a "Maslov index", both quantities depending only on the homotopy class of γ. (Requiring only the unitarity of U, we could take $\theta(\gamma) = S(\gamma)/h$.)

When discussing the improved property (2.3), recall from [13] and [29], that on a manifold with a preferred positive density, we can define the Weyl symbol of a 0-th order h-pseudodifferential operator modulo $\mathcal{O}(h^2)$ by taking the ordinary Weyl symbol for some system of local coordinates $x_1, \dots, x_n$ for which the preferred density reduces to the Lebesgue measure. Clearly Proposition 2.1 extends to this situation.

We first notice that if

$$Vu(x) = h^{-\frac{n+N}{2}} \iint e^{i\phi(x,y,\theta)/h} a(x,y,\theta;h) u(y) dy d\theta$$

is an elliptic Fourier integral operator with leading symbol $a_0(x,y,\theta) \neq 0$ on C_ϕ, then we can obtain $V^*V = 1 + \mathcal{O}(h)$ by multiplying a_0 by a positive real-analytic function.

The same remark applies to

$$\widetilde{V} : H^{\mathrm{loc}}_{\Psi}(\widetilde{\Omega}_Y) \to H^{\mathrm{loc}}_{\Phi}(\widetilde{\Omega}_X),$$

where we put $\widetilde{V} = S \circ V \circ T^{-1}$ and represent it as in [20] by

$$\widetilde{V}u(x) = h^{-n} \int e^{i\psi(x,y)/h} b(x,y;h) u(y) e^{-2\Psi(y)/h} L(dy). \tag{2.7}$$

Here $\psi(x,y)$ is the multi-valued grad-periodic function near $\pi_{x,y}\Gamma$, with

$$\partial_{\overline{x},y}\psi = 0, \ \partial_{\overline{x},y} b = 0 \text{ near } \pi_{x,y}(\Gamma),$$

$$\partial_x \psi(x,y) = \frac{2}{i}\partial_x \Phi(x), \ \partial_{\overline{y}}\psi(x,y) = \frac{2}{i}\partial_{\overline{y}}\Psi(y) \text{ on } \pi_{x,y}(\Gamma),$$

$$\Phi(x) + \Psi(y) + \operatorname{Im}\psi(x,y) \sim \operatorname{dist}\left((x,y), \pi_{x,y}(\Gamma)\right)^2,$$

where Γ denotes the graph of $\widetilde{\kappa}$. (In [20] the first equation holds only to infinite order on $\pi_{x,y}(\Gamma)$ and the present improvement follows from the analyticity of $\widetilde{\kappa}$.)

Recall that $\operatorname{Im}\psi$ is single-valued, and that

$$\operatorname{var}_{(\widetilde{\kappa}\circ\gamma,\gamma)}\psi = \int_{\widetilde{\kappa}\circ\gamma} \xi dx - \int_{\gamma} \eta dy, \tag{2.8}$$

is the action difference, when γ is a closed curve in Λ_Ψ and $(\widetilde{\kappa}\circ\gamma,\gamma)$ denotes the curve $t\mapsto(\widetilde{\kappa}(\gamma(t)),\gamma(t))$. Here we also identify $\Lambda_\Psi, \Lambda_\Phi$ with $\widetilde{\Omega}_Y, \widetilde{\Omega}_X$ whenever so is convenient.

Thus after multiplying $b|_{\pi_{x,y}(\Gamma)}$ by a positive real-analytic function, we may assume that

$$V^*V = 1 + \mathcal{O}(h). \tag{2.9}$$

In order to have the improved Egorov property, we further need that locally on $\pi_{x,y}(\Gamma)$:

$$\arg b_0(x,y) = K(y) + \text{Const.},\ \ (\text{notice that } x = x(y) \text{ on } \Gamma), \tag{2.10}$$

where $K(y)$ is a grad-periodic function on $\pi_{x,y}(\Gamma)$, that we do not try to compute here, but whose existence we infer from Proposition 2.1 and the computation of $\widetilde{V}$ as $S\circ V\circ T^{-1}$, with V written microlocally with a real phase as in (2.1).

We can find b_0 satisfying (2.10) everywhere if we accept that $b_0|_{\pi_{x,y}(\Gamma)}$ is multi-valued. More precisely, K is not globally well defined on $\pi_{x,y}(\Gamma)\simeq\Omega_Y$, but $\omega = dK$ is a well defined closed real 1-form on Ω_Y and we can find $b_0|_{\pi_{x,y}(\Gamma)}$, unique up to a constant factor of modulus 1, such that (2.9), (2.10) hold, though b_0 will be multi-valued:

$$\gamma_* b_0 = \exp\left(i\int_\gamma \omega\right) b_0, \tag{2.11}$$

where $\gamma_* b_0$ denotes the new locally defined symbol obtained by following b_0 around the closed loop γ in $\pi_{x,y}(\Gamma)\simeq\Omega_Y$.

Proposition 2.3 *We have $\int_\gamma \omega = k(\gamma)\frac{\pi}{2}$ for some integer $k(\gamma)\in\mathbf{Z}$, for every closed loop $\gamma\subset\pi_{x,y}(\Gamma)$.*

Proof. Let γ be a closed loop and cover γ by small open topologically trivial sets $\widetilde{\Omega}_0, \widetilde{\Omega}_1, \dots, \widetilde{\Omega}_{N-1}$ with increasing index corresponding to the orientation of γ in the natural way. Let $\widetilde{\Omega}_N = \widetilde{\Omega}_0$. Let Ω_j be the corresponding regions in Ω_Y. In Ω_j, we represent V by

$$V_j u(x) = h^{-\frac{n+N_j}{2}} \iint_{\theta\in\mathbf{R}^{N_j}} e^{i\phi_j(x,y,\theta)/h} a_j(x,y,\theta;h) u(y)dyd\theta. \tag{2.12}$$

For a given point in $\Omega_j\cap\Omega_{j+1}$, we have

$$\phi_j = \phi_{j+1}, \quad a_{j+1} = r_{j+1,j} e^{i\alpha_{j+1,j}\pi/2} a_j + \mathcal{O}(h), \quad r_{j+1,j} > 0,\ \alpha_{j+1,j}\in\mathbf{Z},$$

at the corresponding points in C_{ϕ_j}, $C_{\phi_{j+1}}$, provided that we require all the fiber-variable dimensions N_j to have the same parity. (Cf. [15].) This last property is easy to achieve since we can always add one fiber-variable. We conclude that

$$\int_\gamma \omega = \frac{\pi}{2}(\alpha_{1,0} + \alpha_{2,1} + \cdots + \alpha_{N,N-1}),$$

and the proposition follows. □

Take V as above with $b = b_0$ in (2.7), so that (2.9), (2.10) hold. Put

$$U = V(V^*V)^{-\frac{1}{2}}. \tag{2.13}$$

Then $\widetilde{U} = SUT^{-1}$ is of the form (2.7) with $b = b_0 + \mathcal{O}(h)$. We have $U^*U = 1$ and U satisfies (2.3). Since the unitarization is a local operation which commutes with multiplication by a constant factor of modulus 1, (2.11) becomes valid also for b:

$$\gamma_* b = e^{ik(\gamma)\frac{\pi}{2}} b. \tag{2.14}$$

Here we also used Proposition 2.3.

Summing up, we get

Theorem 2.4 *Under the assumptions above on κ, we can find a microlocally defined multi-valued Fourier integral operator U associated to κ, and a corresponding lift $\widetilde{U} = SUT^{-1}$ of the form* (2.7)*, such that U is unitary: $U^*U = 1 + \mathcal{O}(e^{-1/(Ch)})$, satisfies the improved Egorov property* (2.3)*, and*

$$\gamma_* U = e^{i(S(\gamma)/h + k(\gamma)\pi/2)} U,$$

for every closed loop in Ω_Y, where $k(\gamma) \in \mathbf{Z}$ and

$$S(\gamma) = \int_{\kappa\circ\gamma} \xi dx - \int_\gamma \eta dy.$$

3 Reduction by averaging along trajectories

Let P, M be as in the introduction. We work in a neighborhood of $p^{-1}(0) \cap T^*M$, and recall that $P = P_\epsilon$ has the semi-classical principal symbol

$$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2), \tag{3.1}$$

in a complex neighborhood of $p^{-1}(0) \cap T^*M$. Let G_0 be an analytic function defined near $p^{-1}(0) \cap T^*M$ such that

$$H_p G_0 = q - \langle q \rangle, \tag{3.2}$$

where $\langle q\rangle$ is the trajectory average, defined in (1.15). We may take

$$G_0 = \frac{1}{T(E)}\int_{-T(E)/2}^{T(E)/2} (1_{\mathbf{R}_-}(t)(t+\frac{T(E)}{2}) + 1_{\mathbf{R}_+}(t)(t-\frac{T(E)}{2}))q\circ \exp tH_p dt, \quad (3.3)$$

on $p^{-1}(E)$.

We replace $\mathbf{R}^4$ by the new IR-manifold

$$\Lambda_{\epsilon G_0} = \exp{(i\epsilon H_{G_0})}(\mathbf{R}^4), \quad (3.4)$$

which is defined in a complex neighborhood of $p^{-1}(0)\cap T^*M$. Writing $(x,\xi) = \exp{(i\epsilon H_{G_0})}(y,\eta)$, and using $\rho = (y,\eta)$ as real symplectic coordinates on $\Lambda_{\epsilon G_0}$, we get

$$\begin{aligned} p_{\epsilon}|_{\Lambda_{\epsilon G_0}} &= p_\epsilon(\exp{(i\epsilon H_{G_0})}(\rho)) \quad (3.5)\\ &= \sum_{k=0}^{\infty}\frac{(i\epsilon H_{G_0})^k}{k!}(p_\epsilon) = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2). \end{aligned}$$

Iterating this procedure, or looking more directly for $G(x,\xi,\epsilon)$ as an asymptotic sum

$$G\sim \sum_0^\infty \epsilon^k G_k(x,\xi) \quad (3.6)$$

in some complex neighborhood of $p^{-1}(0)\cap T^*M$, we see that we can find G_1, $G_2\ldots$ such that if

$$\Lambda_{\epsilon G} = \exp{(i\epsilon H_G)}(\mathbf{R}^4), \quad (3.7)$$

and we again write $\Lambda_{\epsilon G}\ni (x,\xi) = \exp{(i\epsilon H_G)}(y,\eta)$ and parametrize by the real variables (y,η), then

$$p_{\epsilon}|_{\Lambda_{\epsilon G}} = p + i\epsilon\langle q\rangle + \epsilon^2 q_2 + \epsilon^3 q_3 + \cdots, \quad (3.8)$$

where $q_j = \langle q_j\rangle$, $j\geq 2$. This means that we can transform p_ϵ to $p_\epsilon\circ\exp{(i\epsilon H_G)}$ in such a way that we get a new leading symbol which Poisson commutes with the unperturbed leading symbol.

As is well known in the selfadjoint case, this construction can be extended to the level of operators, and we may develop this globally in another paper. In the present work we will do it only after a reduction to a torus-like situation.

After replacing p_ϵ by $p_\epsilon\circ\exp{(i\epsilon H_{G_0})}$ and correspondingly P_ϵ, by $U_\epsilon^{-1}\circ P_\epsilon\circ U_\epsilon$, where U_ϵ is the Fourier integral operator $U_\epsilon = e^{-\frac{i}{h}i\epsilon G_0(x,hD_x)} = e^{\frac{\epsilon}{h}G_0(x,hD_x)}$ (defined microlocally near $p^{-1}(0)\cap T^*M$), we may assume that our operator P_ϵ is microlocally defined near $p^{-1}(0)\cap T^*M$ and has the h-principal symbol

$$p_\epsilon = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2). \quad (3.9)$$

This can be done in such a way that $P_{\epsilon=0}$ remains the original unperturbed operator. We refer to the beginning of Section 6 for the construction of U_ϵ by means of an FBI-transform.

Let $\gamma_0 \subset p^{-1}(0) \cap T^*M$ be a closed H_p-trajectory and assume that $T(0)$ is the minimal period of γ_0. Let $g : \mathrm{neigh}\,(0, \mathbf{R}) \to \mathbf{R}$ be the analytic function defined by

$$g'(E) = \frac{T(E)}{2\pi},\ g(0) = 0. \tag{3.10}$$

Then $H_{g\circ p} = g'(p)H_p$ has a 2π-periodic flow and the same closed trajectories as H_p. Clearly 2π is the minimal period of γ_0 when viewed as a $H_{g\circ p}$-trajectory.

Proposition 3.1 *There exists an analytic canonical transformation* $\kappa : \mathrm{neigh}\,(\{\tau = x = \xi = 0\}, T^*(S^1_t \times \mathbf{R}_x)) \to \mathrm{neigh}\,(\gamma_0, T^*M)$*, mapping* $\{\tau = x = \xi = 0\}$ *onto* γ_0*, such that* $g \circ p \circ \kappa = \tau$*.*

Proof. Fix a point $\rho_0 \in \gamma_0$ and choose local symplectic coordinates $(t, \tau; x, \xi)$ centered at ρ_0, with $g \circ p = \tau$. This means that

$$\{\xi, x\} = 1,\ \{t, x\} = \{t, \xi\} = 0 \tag{3.11}$$

$$H_\tau t = 1,\ H_\tau x = H_\tau \xi = 0. \tag{3.12}$$

Now extend the definition of t, τ, x, ξ to a full neighborhood of γ_0, by putting $\tau = g \circ p$ and requiring t, x, ξ to solve (3.12). Since the H_τ-flow is 2π-periodic (with 2π as the minimal period) near γ_0, we see that x, ξ are well defined single-valued functions, while t becomes multi-valued in such a way that it increases by 2π each time we make a loop in the increasing time direction. (3.11) extends to a full neighborhood of γ_0. This is equivalent to the proposition. □

Notice that

$$p \circ \kappa = f(\tau), \tag{3.13}$$

where $f := g^{-1}$. From (3.9) we infer that

$$p_\epsilon \circ \kappa = f(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2), \tag{3.14}$$

for a new function $\langle q\rangle$ which is independent of t (and obtained from the earlier one by composition with κ).

If we let the Fourier integral operator U quantize κ as in Section 2, we get a new operator $U^{-1}P_\epsilon U$ with leading semi-classical symbol $p_\epsilon \circ \kappa$ as in (3.14). (Here P_ϵ is the new version of P_ϵ; $P_{\epsilon,\mathrm{new}} = U^{-1}P_{\epsilon,\mathrm{old}}U$.)

Now write simply p, p_ϵ, P_ϵ for the transformed objects. Then

$$P_\epsilon = P(t, x, hD_{t,x}, \epsilon; h)$$

is the formal Weyl quantization of a symbol $P(t, x, \tau, \xi, \epsilon; h)$ which has an asymptotic expansion (1.5) in the space of holomorphic functions in a fixed complex neighborhood of $\{\mathrm{Im}\ t = \tau = x = \xi = 0\}$ in $T^*(\widetilde{S}^1 \times \mathbf{C})$, with $\widetilde{S}^1 = S^1 + i\mathbf{R}$, and we will use the same notation as in Section 1. (An exact value of the new symbol $P(t, x, \tau, \xi, \epsilon; h)$ cannot be easily defined, but we know how to define it mod $\mathcal{O}(e^{-1/(Ch)})$. We shall however avoid using the full power of analytic pseudodifferential operators, and content ourselves with the knowledge of P mod $\mathcal{O}(h^\infty)$.)

Now look for $G^{(1)} = \epsilon G_1(t, \tau, x, \xi) + \epsilon^2 G_2(t, \tau, x, \xi) + \cdots$ such that

$$p_\epsilon \circ \exp i\epsilon H_{G^{(1)}} = f(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2)$$

is independent of t. Here the left-hand side can be written

$$\sum_{k=0}^{\infty} \frac{1}{k!}(i\epsilon H_{G^{(1)}})^k p_\epsilon,$$

and we get

$$\begin{aligned} &p_\epsilon + i\epsilon^2 H_{G_1}(f(\tau)) + \mathcal{O}(\epsilon^3) = \\ &f(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) - i\epsilon^2 f'(\tau)\frac{\partial}{\partial t}G_1 + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^3), \end{aligned}$$

where the $\mathcal{O}(\epsilon^2)$ term is the same as in (3.14). It is clear that we can find G_1 so that the ϵ^2-term in this expression is independent of t. Looking at the $\mathcal{O}(\epsilon^3)$-term we then determine G_2 and so on. (In this construction, we could have applied κ at the very beginning before replacing q by $\langle q\rangle$ by averaging, and then incorporated G_0 into the expression $G = G_0 + \epsilon G_1 + \cdots$, and as already indicated, this could also have been done entirely (and in a full neighborhood of $p^{-1}(0) \cap T^*M$), before applying κ.)

After replacing p_ϵ by $p_\epsilon \circ \exp i\epsilon H_{G^{(1)}}$, we are now reduced to the case when

$$p_\epsilon = f(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2) \tag{3.15}$$

is independent of t, up to $\mathcal{O}(\epsilon^\infty)$.

Finally we remove the t-dependence from the lower order terms. After conjugating P_ϵ by a Fourier integral operator V_ϵ, which quantizes $\exp i\epsilon H_{G^{(1)}}$, we may assume that p_ϵ in (3.15) is the principal symbol of P_ϵ (and that it is independent of t). Look for an h-pseudodifferential operator $A(t, x, hD_{t,x}, \epsilon; h)$ with symbol

$$A(t, x, \tau, \xi, \epsilon; h) \sim \sum_{k=1}^{\infty} a_k(t, x, \tau, \xi, \epsilon)h^k, \tag{3.16}$$

such that the full (Weyl) symbol of

$$e^{\frac{i}{h}A}P_\epsilon e^{-\frac{i}{h}A} = e^{\frac{i}{h}\mathrm{ad}_A}P_\epsilon = \sum_{k=0}^{\infty}\frac{1}{k!}(\frac{i}{h}\mathrm{ad}_A)^k P_\epsilon \tag{3.17}$$

is independent of t. Since $A = \mathcal{O}(h)$ we know that $\frac{i}{h}\mathrm{ad}_A$ lowers the order in h by one (with the convention that a symbol $= \mathcal{O}(h^{-j})$ is of order j), so (3.17) makes sense asymptotically. The subprincipal symbol of (3.17) is

$$h(p_{1,\epsilon}(x,\xi) + \{p_\epsilon, a_1\}) = h(p_{1,\epsilon}(x,\xi) + f'(\tau)\frac{\partial}{\partial t}a_1(t,\tau,x,\xi,\epsilon) + \mathcal{O}(\epsilon)),$$

and we make this independent of t by successively determining the coefficients in the asymptotic series

$$a_1(t,\tau,x,\xi,\epsilon) = \sum_{j=0}^{\infty} a_{1,j}(t,\tau,x,\xi)\epsilon^j.$$

After that we return to (3.17) and see that the construction of a_2, a_3, $\dots$ is essentially the same.

Actually, we do not have to do this construction in 2 steps, and we can view $\epsilon G^{(1)}$ above as (a constant factor times) the leading symbol $a_0 = \mathcal{O}(\epsilon^2)$ in

$$A \sim \sum_{k=0}^{\infty} a_k(t,x,\tau,\xi,\epsilon)h^k, \tag{3.18}$$

such that if P_ϵ denotes the very first operator we get on $S^1 \times \mathbf{R}$, then the left-hand side of (3.17) has a symbol which is well defined as an asymptotic series in (ϵ, h) and is independent of t, up to $\mathcal{O}(h^\infty)$. This can be seen by first determining a_0 from (3.17) (leading to a repetition of what we already did) and then the other terms. (When ϵ is small but fixed, the problem becomes more subtle and the break-up into two steps is more natural, with the first step being the one containing the new difficulties.)

Summing up the discussion of this section, we have

Proposition 3.2 *Let P, M be as in Section* 1. *Let $\gamma_0 \subset p^{-1}(0) \cap T^*M$ be a closed H_p-trajectory where $T(0)$ is the minimal period and let κ be the canonical transformation of Proposition* 3.1. *Let U be a corresponding elliptic Fourier integral operator as in Section* 2. *Then there exist $G(x,\xi,\epsilon)$ (independent of γ_0, κ, U) with the asymptotic expansion* (3.6) *in the space of holomorphic functions in some fixed complex neighborhood of $p^{-1}(0) \cap T^*M$ and a symbol $A(t,x,\tau,\xi,\epsilon;h)$ as in* (3.16), *where*

$$a_k \sim \sum_{j=0}^{\infty} a_{k,j}(t,x,\tau,\xi)\epsilon^j \tag{3.19}$$

in the space of holomorphic functions in a fixed complex neighborhood of $\mathrm{Im}\ t = \tau = x = \xi = 0$ *in $T^*(\widetilde{S}^1 \times \mathbf{C})$), such that if G, A also denote the corresponding Weyl quantizations, the operator*

$$\widetilde{P}_\epsilon = e^{\frac{i}{h}A}U^{-1}e^{-\frac{\epsilon}{h}G}P_\epsilon e^{\frac{\epsilon}{h}G}Ue^{-\frac{i}{h}A} = \mathrm{Ad}_{e^{\frac{i}{h}A}U^{-1}e^{-\frac{\epsilon}{h}G}}P_\epsilon \tag{3.20}$$

has a symbol

$$\widetilde{P}_\epsilon(x,\tau,\xi,\epsilon;h) \sim \sum_0^\infty \widetilde{p}_k(x,\tau,\xi,\epsilon)h^k \tag{3.21}$$

independent of t *(up to* $\mathcal{O}(h^\infty)$*). Here each* $\widetilde{p}_k = \widetilde{p}_k(x,\tau,\xi,\epsilon) \sim \sum_{j=0}^\infty \widetilde{p}_{k,j}(x,\tau,\xi)\epsilon^j$ *in the space of holomorphic functions in a fixed complex neighborhood of* $\tau, x, \xi = 0$. *Moreover*

$$\widetilde{p}_{0,\epsilon} = f(\tau) + i\epsilon\langle q\rangle(\tau,x,\xi) + \mathcal{O}(\epsilon^2). \tag{3.22}$$

If $\langle q\rangle$ has a non-degenerate extreme value along γ_0, then the proposition is directly applicable (see Section 5), while in other situations (such as in Section 4), it is not global enough.

4 Normal forms and quasi-eigenvalues in the torus case

Let $P, M, p, q, \langle q\rangle, \Lambda_{0,F_0}$ be as in Section 1. After replacing q by $q - F_0$, we may assume that $F_0 = 0$, so we consider

$$\Lambda_{0,0} : \; p = 0, \, \mathrm{Re}\,\langle q\rangle = 0. \tag{4.1}$$

Notice that $\Lambda_{0,0}$ is invariant under the H_p-flow. We assume that $T(0)$ is the minimal period for all the closed trajectories in $\Lambda_{0,0}$ and that

$$dp, d\langle \mathrm{Re}\, q\rangle \text{ are independent at the points of } \Lambda_{0,0}, \tag{4.2}$$

so that $\Lambda_{0,0}$ is a Lagrangian manifold and also a union of tori. Assume for simplicity that $\Lambda_{0,0}$ is connected, so that it is equal to one single Lagrangian torus. In this section we work microlocally near $\Lambda_{0,0}$ and proceed somewhat formally. In Section 6 we follow up with suitable function spaces and see how to justify the computation of the spectrum via a global Grushin problem. We have seen that we can reduce ourselves to the case when

$$p_\epsilon = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2). \tag{4.3}$$

Assume from now on that $\langle q\rangle$ is real-valued or more generally that $\langle q\rangle$ is a function of p and $\mathrm{Re}\,\langle q\rangle$. We can make a real canonical transformation

$$\kappa : \mathrm{neigh}\,(\xi = 0, T^*\mathbf{T}^2) \to \mathrm{neigh}\,(\Lambda_{0,0}, T^*M), \;\; \mathbf{T}^2 = (\mathbf{R}/2\pi\mathbf{Z})^2, \tag{4.4}$$

such that $p \circ \kappa = p(\xi_1)$, $\langle q\rangle \circ \kappa = \langle q\rangle(\xi)$ (with a slight abuse of notation).

Recall that this can be done in the following way: Let $\Lambda_{E,F}$ be the Lagrangian torus given by $p = E, \mathrm{Re}\,\langle q\rangle = F$, for $(E,F) \in \mathrm{neigh}\,(0,\mathbf{R}^2)$. Let $\gamma_1(E,F)$ be the cycle in $\Lambda_{E,F}$ corresponding to a closed H_p-trajectory with minimal period, and let $\gamma_2(E,F)$ be a second cycle so that γ_1, γ_2 form a fundamental system of cycles on the torus $\Lambda_{E,F}$. Necessarily γ_2 maps to the simple loop given by $\mathrm{Re}\,\langle q\rangle = F$ in the abstract quotient manifold $p^{-1}(E)/\mathbf{R}H_p$. Now it is classical (see [1]) that we can find a real analytic canonical transformation $\kappa : \mathrm{neigh}\,(\eta = 0, T^*\mathbf{T}^2) \ni (y,\eta) \mapsto (x,\xi) \in \mathrm{neigh}\,(\Lambda_{0,0}, T^*M)$, such that

$$\eta_j = \frac{1}{2\pi}(\int_{\gamma_j(E,F)} \xi dx - \int_{\gamma_j(0,0)} \xi dx),$$

where E, F depend on (x,ξ) and are determined by $(x,\xi) \in \Lambda_{E,F}$, i.e., by $E = p(x,\xi), F = \mathrm{Re}\,\langle q\rangle(x,\xi)$. We also know that here $\eta_1 = \eta_1(E)$ is a function of E only.

Let us also recall that κ can be constructed as follows: We start by taking a first canonical transformation $\kappa_0 : \mathrm{neigh}\,(\xi = 0, T^*\mathbf{T}^2) \to \mathrm{neigh}\,(\Lambda_{0,0}, T^*M)$ such that the zero section is mapped to $\Lambda_{0,0}$ and the lines $\{x_2 = \mathrm{Const},\ \xi = 0\}$ are mapped onto the closed H_p-trajectories in $\Lambda_{0,0}$. Then using κ_0, we can consider $p, \langle q\rangle$ as living on $T^*\mathbf{T}^2$. $\Lambda_{E,F}$ is then given by

$$\xi = \phi'_x,\ \phi = \phi_{\mathrm{per}}(x,E,F) + \eta_1 x_1 + \eta_2 x_2,\ \text{with}\ \det \phi''_{x,(E,F)} \neq 0,$$

with $\eta_j = \eta_j(E,F)$ as above (now being the actions/2π with respect to ξdx), and ϕ_{per} being $(2\pi\mathbf{Z})^2$-periodic. Moreover, $\phi'_x(x,\eta) = 0$, $\eta = 0$ for $E = F = 0$. It is easy to check, using that our functions are real-valued, that $(E,F) \mapsto (\eta_1(E,F), \eta_2(E,F))$ is a local diffeomorphism, so we can use η_1, η_2 as new parameters replacing E, F, and write $\phi = \phi(x,\eta)$. Consider

$$\kappa_1 : (\frac{\partial\phi}{\partial\eta}, \eta) \mapsto (x, \frac{\partial\phi}{\partial x})$$

which maps the zero section to itself. Then $\kappa := \kappa_0 \circ \kappa_1$ has the required properties.

Let U be a corresponding Fourier integral operator, implementing κ, so that if we denote by P_ϵ also the conjugated operator $U^{-1}P_\epsilon U$, we have a new operator with leading symbol

$$p_\epsilon = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2). \tag{4.5}$$

For the conjugated operator, we still have the property that $P_{\epsilon=0}$ is selfadjoint. From the assumption (4.2) about linear independence, we get

$$\partial_{\xi_1} p(0) \neq 0,\ \partial_{\xi_2} \mathrm{Re}\,\langle q\rangle(0) \neq 0. \tag{4.6}$$

As in the preceding section, we can find an h-pseudodifferential operator A with symbol $\sum_{\nu=0}^{\infty} h^\nu a_\nu(x,\xi,\epsilon)$, $a_0 = \mathcal{O}(\epsilon^2)$, such that formally

$$e^{\frac{i}{h}A} P_\epsilon e^{-\frac{i}{h}A} = e^{\frac{i}{h}\mathrm{ad}_A}(P_\epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!}(\frac{i}{h}\mathrm{ad}_A)^k(P_\epsilon) =: \widetilde{P}_\epsilon, \tag{4.7}$$

with $\widetilde{P}_\epsilon(x,\xi,\epsilon;h)$ independent of x_1, and leading symbol

$$\widetilde{p}_\epsilon = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2)$$

also independent of x_1. We recall that the symbol $A(x,\xi,\epsilon;h)$ is a formal power series both in ϵ and h with coefficients all holomorphic in the same complex neighborhood of $\xi = 0$. This construction can be done in such a way that $\widetilde{P}_{\epsilon=0}$ is selfadjoint.

We next look for a further conjugation that eliminates the x_2-dependence in the symbol.

a) We start by considering the general case, when the subprincipal symbol of $P_{\epsilon=0}$ is not necessarily 0, so that the complete symbol of $\widetilde{P}_\epsilon$ takes the form

$$\widetilde{P}_\epsilon(x_2,\xi;h) = \sum_{\nu=0}^{\infty} h^\nu \widetilde{p}_\nu(x_2,\xi,\epsilon), \tag{4.8}$$

with

$$\widetilde{p}_0(x_2,\xi,\epsilon) = \widetilde{p}_\epsilon = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2), \tag{4.9}$$

and $\widetilde{p}_1(x_2,\xi,0)$ not necessarily identically equal to 0.

The easiest case is when $h/\epsilon \le \mathcal{O}(h^{\delta_1})$ for some $\delta_1 > 0$, so that we can consider h/ϵ as an asymptotically small parameter. Look for

$$B(x_2,\xi,\epsilon,\frac{h}{\epsilon},h) = \sum_{\nu=0}^{\infty} h^\nu b_\nu(x_2,\xi,\epsilon,\frac{h}{\epsilon}), \tag{4.10}$$

with $b_\nu = \mathcal{O}(\epsilon + h/\epsilon)$, such that on the operator level (with hD_x instead of ξ),

$$e^{\frac{i}{h}B}\widetilde{P}_\epsilon e^{-\frac{i}{h}B} =: \widehat{P}_\epsilon(hD_x,\epsilon,\frac{h}{\epsilon},h) \tag{4.11}$$

has a symbol independent of x. Notice that $B(x_2,hD_x,\epsilon;h)$ and $p(hD_{x_1})$ commute. On the symbol level we write

$$\begin{aligned}\widetilde{P}_\epsilon &= p(\xi_1) + \epsilon(i\langle q\rangle(\xi) + \mathcal{O}(\epsilon) + \frac{h}{\epsilon}\widetilde{p}_1(x_2,\xi,\epsilon) + h\frac{h}{\epsilon}\widetilde{p}_2(x_2,\xi,\epsilon) + \cdots) \qquad (4.12)\\ &= p(\xi_1) + \epsilon(r_0(x_2,\xi,\epsilon,\frac{h}{\epsilon}) + hr_1(x_2,\xi,\epsilon,\frac{h}{\epsilon}) + \cdots),\end{aligned}$$

with

$$r_0(x_2,\xi,\epsilon,\frac{h}{\epsilon}) = i\langle q\rangle(\xi) + \mathcal{O}(\epsilon) + \frac{h}{\epsilon}\widetilde{p}_1 = i\langle q\rangle(\xi) + \mathcal{O}(\epsilon) + \mathcal{O}(\frac{h}{\epsilon}),$$

$$r_1 = \frac{h}{\epsilon}\widetilde{p}_2(x_2,\xi,\epsilon),\ \ldots$$

Notice that $r_j = \mathcal{O}(h/\epsilon)$ for $j \ge 1$. We shall treat h/ϵ as an independent parameter.

We use this and develop (4.11) to get, with $\mathrm{ad}_b c$ denoting the symbol of

$$\mathrm{ad}_{b(x,hD_x)}c(x,hD_x) = [b(x,hD),c(x,hD)],$$

$$p(\xi_1)+\epsilon\sum_{k=0}^{\infty}\sum_{j_1=0}^{\infty}\cdots\sum_{j_k=0}^{\infty}\sum_{\ell=0}^{\infty}h^{\ell+j_1+\cdots+j_k}\frac{1}{k!}(\frac{i}{h}\mathrm{ad}_{b_{j_1}})\ldots(\frac{i}{h}\mathrm{ad}_{b_{j_k}})r_\ell=p(\xi_1)+\epsilon\sum_{n=0}^{\infty}h^n\widehat{r}_n,$$

with $\widehat{r}_n$ being equal to the sum of all coefficients for h^n resulting from all the expressions

$$h^{\ell+j_1+\cdots+j_k}\frac{1}{k!}(\frac{i}{h}\mathrm{ad}_{b_{j_1}})\cdots(\frac{i}{h}\mathrm{ad}_{b_{j_k}})r_\ell, \tag{4.13}$$

with $\ell+j_1+\cdots+j_k\le n$.

The first term is

$$\widehat{r}_0=\sum\frac{1}{k!}H_{b_0}^k r_0=r_0\circ\exp(H_{b_0}),$$

where we want $\widehat{r}_0$ to be independent of x_2 (in addition to x_1). We get with $b_0=\mathcal{O}(\epsilon+h/\epsilon)$:

$$\widehat{r}_0=i\langle q\rangle(\xi)+\mathcal{O}(\epsilon+h/\epsilon)-i\partial_{\xi_2}\langle q\rangle\partial_{x_2}b_0+\mathcal{O}((\epsilon,\frac{h}{\epsilon})^2), \tag{4.14}$$

and using that $\partial_{\xi_2}\langle q\rangle\neq 0$, it is clear how to construct $b_0=\mathcal{O}(\epsilon+h/\epsilon)$ as a formal Taylor series in $\epsilon, h/\epsilon$, so that $\widehat{r}_0=i\langle q\rangle(\xi)+\mathcal{O}(\epsilon+h/\epsilon)$ is independent of x (modulo a term $\mathcal{O}(h^\infty)$).

Assume for simplicity that the conjugation by $e^{\frac{i}{h}b_0(x_2,hD_x,\epsilon,h/\epsilon)}$ has already been carried out, so that we are reduced to the case when $r_0=i\langle q\rangle(\xi)+\mathcal{O}(\epsilon+h/\epsilon)$ is independent of x_2, and $r_j=\mathcal{O}(\epsilon+h/\epsilon)$ for $j\ge 1$. Then look for a new conjugation $\exp\frac{i}{h}\mathrm{ad}_B$, with $B(x_2,\xi,\epsilon,h/\epsilon;h)=\sum_{\nu=1}^{\infty}h^\nu b_\nu(x_2,\xi,\epsilon,\frac{h}{\epsilon})$. The new expression for the left-hand side of (4.11) becomes

$$p(\xi_1)+\epsilon\sum_{k=0}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{j_k=1}^{\infty}\sum_{\ell=0}^{\infty}h^{\ell+j_1+\cdots+j_k}\frac{1}{k!}(\frac{i}{h}\mathrm{ad}_{b_{j_1}})\ldots(\frac{i}{h}\mathrm{ad}_{b_{j_k}})r_\ell=p(\xi_1)+\epsilon\sum_{n=0}^{\infty}h^n\widehat{r}_n, \tag{4.15}$$

with $\widehat{r}_n$ equal to the sum of all coefficients for h^n resulting from the expressions (4.13) with $\ell+j_1+\cdots+j_k\le n$ *and* $j_\nu\ge 1$. Then $\widehat{r}_0=r_0$, $\widehat{r}_1=r_1+H_{b_1}r_0=r_1-H_{r_0}b_1,\ldots,\widehat{r}_n=r_n-H_{r_0}b_n+s_n$, where s_n only depends on $b_1,\ldots,b_{n-1}$ and is the sum of all coefficients of h^n arising in the expressions (4.13) with $\ell+j_1+\cdots+j_k\le n$, $j_1,\ldots,j_k,\ell<n$, $j_\nu\ge 1$.

It is therefore clear how to find $b_1,b_2,\ldots$ successively with $b_j=\mathcal{O}(\epsilon+h/\epsilon)$, such that all the $\widehat{r}_j$ are independent of x and $=\mathcal{O}(\epsilon+h/\epsilon)$. This completes the proof of (4.11).

Summing up the discussion so far, if we do not make any assumption on the subprincipal symbol of $P_{\epsilon=0}$ and restrict the attention to $h/\epsilon\le\mathcal{O}(h^{\delta_1})$ for some $\delta_1>0$, then we can find

$$B_0=b_0(x_2,hD_x,\epsilon,h/\epsilon),\ b_0=\mathcal{O}(\epsilon+h/\epsilon),$$

and

$$B_1=\sum_{\nu=1}^{\infty} b_\nu(x_2,hD_x,\epsilon,h/\epsilon)h^\nu,\ \ b_\nu=\mathcal{O}(\epsilon+h/\epsilon),$$

such that

$$\widehat{P}_\epsilon := e^{\frac{i}{h}\mathrm{ad}_{B_1}}e^{\frac{i}{h}\mathrm{ad}_{B_0}}\widetilde{P}_\epsilon \tag{4.16}$$

has a symbol independent of x:

$$\widehat{P}_\epsilon = p(\xi_1)+\epsilon(r_0(\xi,\epsilon,\frac{h}{\epsilon})+hr_1(\xi,\epsilon,\frac{h}{\epsilon})+\cdots), \tag{4.17}$$

with $r_0=i\langle q\rangle(\xi)+\mathcal{O}(\epsilon+h/\epsilon)$, and $r_\nu=\mathcal{O}(\epsilon+h/\epsilon)$ for $\nu\geq 1$.

Remaining in the general case, without any assumption on the lower order terms, we now assume merely that $h/\epsilon\leq\delta_0$ for some sufficiently small $\delta_0>0$. This means that we can no longer construct b_0 by a formal Taylor series in h/ϵ, and we shall replace $e^{\frac{i}{h}b_0(x_2,hD_x,\epsilon,h/\epsilon)}$ by a Fourier integral operator, constructed directly.

Look for $\phi=\phi(x_2,\xi,\epsilon,h/\epsilon)$ solving

$$r_0(x_2,\xi_1,\xi_2+\partial_{x_2}\phi,\epsilon,\frac{h}{\epsilon})=\langle r_0(\cdot,\xi,\epsilon,\frac{h}{\epsilon})\rangle, \tag{4.18}$$

where $\langle\cdot\rangle$ denotes the average with respect to x_2. By the implicit function theorem, (4.18) has a solution with $\partial_{x_2}\phi$ single-valued and $\mathcal{O}(\epsilon+h/\epsilon)$. If we Taylor expand (4.18), we get

$$(\partial_{\xi_2}r_0)(x_2,\xi,\epsilon,\frac{h}{\epsilon})\partial_{x_2}\phi+(r_0(x_2,\xi,\epsilon,\frac{h}{\epsilon})-\langle r_0(\cdot,\xi,\epsilon,\frac{h}{\epsilon})\rangle)=\mathcal{O}((\frac{h}{\epsilon},\epsilon)^2),$$

and using also that

$$\partial_{\xi_2}r_0(x_2,\xi,\epsilon,\frac{h}{\epsilon})=i\partial_{\xi_2}\langle q\rangle(\xi)+\mathcal{O}(\epsilon+\frac{h}{\epsilon}),$$

we get,

$$\phi=\phi_{\mathrm{per}}+x_2\zeta_2,$$

with $\zeta_2=\zeta_2(\xi,\epsilon,\frac{h}{\epsilon})=\mathcal{O}((\epsilon,h/\epsilon)^2)$, and $\phi_{\mathrm{per}}=\mathcal{O}((\epsilon,h/\epsilon))$ periodic in x_2. Put $\eta=\eta(\xi,\epsilon,h/\epsilon)=(\xi_1,\xi_2+\zeta_2)$, and

$$\psi(x,\eta,\epsilon,\frac{h}{\epsilon})=x\cdot\eta+\phi_{\mathrm{per}},$$

where ϕ_{per} is viewed as a function of η rather than ξ.

Consider the canonical transformation

$$\kappa:(\psi'_\eta,\eta)\mapsto(x,\psi'_x),$$

which is $(\epsilon + h/\epsilon)$-close to the identity and can be viewed as a family of transforms depending analytically on the parameter ξ_1. With $\xi = \xi(\eta, \epsilon, \frac{h}{\epsilon})$, we have by construction:

$$(r_0 \circ \kappa)(y, \eta, \epsilon, \frac{h}{\epsilon}) = \langle r_0(\cdot, \xi, \epsilon, \frac{h}{\epsilon})\rangle = \langle r_0(\cdot, \eta, \epsilon, \frac{h}{\epsilon})\rangle + \mathcal{O}(\epsilon^2 + (\frac{h}{\epsilon})^2), \tag{4.19}$$

and this is a function of (y, η) which is independent of y. Notice that $p(\xi_1)$ is unchanged under composition with κ.

We can quantize κ as a Fourier integral operator U and after conjugation by this operator, we may assume that we have a new operator $\widetilde{P}_\epsilon$ as in (4.12) with $r_0 = i\langle q\rangle(\xi) + \mathcal{O}(\epsilon + h/\epsilon)$ independent of x and with $r_j = \mathcal{O}(\epsilon + h/\epsilon)$.

As before, we can then make a further conjugation $e^{\frac{i}{h}\mathrm{ad}_{B_1}}$ in order to remove the x-dependence completely and the conclusion is that if we make no assumption on the subprincipal symbol and restrict the attention to $h/\epsilon \le \delta_0$, for $\delta_0 > 0$ small enough, then we can find a Fourier integral operator,

$$U^{-1}u(x; h) = \frac{1}{(2\pi h)^2} \iint e^{\frac{i}{h}(\psi(x,\eta) - y\cdot\eta)} a(x, \eta; h)u(y)dyd\eta, \tag{4.20}$$

with $\psi(x, \eta) = x \cdot \eta + \phi_{\mathrm{per}}(x_2, \eta, \epsilon, h/\epsilon)$, $\phi_{\mathrm{per}} = \mathcal{O}(\epsilon + h/\epsilon)$, and

$$B_1 = \sum_{\nu=1}^{\infty} b_\nu(x_2, hD_x, \epsilon, \frac{h}{\epsilon})h^\nu,\ b_\nu = \mathcal{O}(\epsilon + \frac{h}{\epsilon}),$$

such that

$$\widehat{P}_\epsilon := e^{\frac{i}{h}\mathrm{ad}_{B_1}} \mathrm{Ad}_U \widetilde{P}_\epsilon$$

has a symbol independent of x as in (4.17), with the same estimates as there.

b) We now assume that in the original problem, $P_{\epsilon=0}$ has subprincipal symbol 0. Then after a first time averaging, transportation to the torus, and the elimination of the x_1-dependence, we may assume that

$$\widetilde{P}(x_2, \xi, \epsilon; h) = \sum_{\nu=0}^{\infty} h^\nu \widetilde{p}_\nu(x_2, \xi, \epsilon), \tag{4.21}$$

with $\widetilde{p}_0$ independent of x mod $\mathcal{O}(\epsilon^2)$:

$$\widetilde{p}_0(x_2, \xi, \epsilon) = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2), \tag{4.22}$$

$$\widetilde{p}_1(x_2, \xi, 0) = 0. \tag{4.23}$$

(Recall from Section 2 and the references given there, that the canonical transformations can be quantized in such a way that Egorov's theorem holds modulo $\mathcal{O}(h^2)$.) In analogy with (4.12), we have with $\widetilde{p}_1(x_2, \xi, \epsilon) = \epsilon q_1(x_2, \xi, \epsilon)$,

$$\begin{aligned}\widetilde{P}_\epsilon &= p(\xi_1) + \epsilon(i\langle q\rangle(\xi) + \mathcal{O}(\epsilon) + hq_1(x_2, \xi, \epsilon) + \frac{h^2}{\epsilon}\widetilde{p}_2 + h\frac{h^2}{\epsilon}\widetilde{p}_3 + \cdots)\\ &= p(\xi_1) + \epsilon(r_0(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}) + hr_1(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}) + h^2 r_2 + \cdots),\end{aligned} \tag{4.24}$$

with

$$\begin{aligned} r_0(x_2,\xi,\epsilon,\frac{h^2}{\epsilon}) &= i\langle q\rangle(\xi)+\mathcal{O}(\epsilon)+\frac{h^2}{\epsilon}\widetilde{p}_2,\\ r_1(x_2,\xi,\epsilon,\frac{h^2}{\epsilon}) &= q_1(x_2,\xi,\epsilon)+\frac{h^2}{\epsilon}\widetilde{p}_3,\\ r_2(x_2,\xi,\epsilon,\frac{h^2}{\epsilon}) &= \frac{h^2}{\epsilon}\widetilde{p}_4,\dots \end{aligned}$$

We first consider the case when

$$\frac{h^2}{\epsilon}\le h^{\delta_1}, \tag{4.25}$$

for some fixed $\delta_1>0$. A first conjugation by $e^{\frac{i}{h}b_0(x_2,hD_x,\epsilon,\frac{h^2}{\epsilon})}$, with $b_0=\mathcal{O}(\epsilon+h^2/\epsilon)$, allows us to make r_0 independent of x_2, and we still have (4.24) with $r_j=\mathcal{O}(1)$ for $j\ge 1$.

Then we look for a new conjugation $\exp\frac{i}{h}\mathrm{ad}_{B_1}$ with

$$B_1(x_2,\xi,\epsilon,\frac{h^2}{\epsilon};h)=\sum_{\nu=1}^{\infty}h^\nu b_\nu(x_2,\xi,\epsilon,\frac{h^2}{\epsilon}). \tag{4.26}$$

The conjugated operator (4.11) can be expanded as in (4.15) and as after that equation it is clear how to get $b_\nu=\mathcal{O}(1)$ for $\nu\ge 1$, such that the resulting $\widehat{r}_n$ are independent of x_2, with $\widehat{r}_0(\xi,\epsilon,h^2/\epsilon)=i\langle q\rangle(\xi)+\mathcal{O}(\epsilon+h^2/\epsilon)$.

Summing up the discussion so far, if we assume that the subprincipal symbol of $P_{\epsilon=0}$ vanishes, and restrict the attention to the range (4.25) for some fixed $\delta_1>0$, then we can find $B_0=b_0(x_2,hD_x,\epsilon,\frac{h^2}{\epsilon})$ with $b_0=\mathcal{O}(\epsilon+\frac{h^2}{\epsilon})$ and $B_1(x_2,hD_x,\epsilon,\frac{h^2}{\epsilon};h)$ with symbol (4.26), and $b_\nu=\mathcal{O}(1)$, such that

$$e^{\frac{i}{h}\mathrm{ad}_{B_1}}e^{\frac{i}{h}\mathrm{ad}_{B_0}}\widetilde{P}_\epsilon=\widehat{P}_\epsilon$$

has the symbol

$$p(\xi_1)+\epsilon(r_0(\xi,\epsilon,\frac{h^2}{\epsilon})+hr_1(\xi,\epsilon,\frac{h^2}{\epsilon})+\cdots) \tag{4.27}$$

independent of x and with

$$r_0=i\langle q\rangle(\xi)+\mathcal{O}(\epsilon+\frac{h^2}{\epsilon}),\ r_\nu=\mathcal{O}(1),\ \nu\ge 1. \tag{4.28}$$

If we replace (4.25) by the weaker assumption,

$$\frac{h^2}{\epsilon}\le\delta_0,\ \delta_0\ll 1, \tag{4.29}$$

then again we have to replace the conjugation by $e^{\frac{i}{h}B_0}$ by that by a Fourier integral operator constructed as earlier: We solve (4.18) (with h/ϵ replaced by h^2/ϵ) and get $\partial_{x_2}\phi$ single-valued and $\mathcal{O}(\epsilon + h^2/\epsilon)$.

Taylor expanding (4.18) and using that

$$\partial_{\xi_2} r_0(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}) = i\partial_{\xi_2}\langle q\rangle(\xi) + \mathcal{O}(\epsilon + \frac{h^2}{\epsilon}),$$

we get

$$\phi = \phi_{\rm per} + x_2\zeta_2,$$

with $\zeta_2 = \zeta_2(\xi, \epsilon, \frac{h^2}{\epsilon}) = \mathcal{O}((\epsilon, \frac{h^2}{\epsilon})^2)$ and $\phi_{\rm per} = \mathcal{O}(\epsilon + h^2/\epsilon)$ periodic in x_2. Again we put $\eta = \eta(\xi, \epsilon, h^2/\epsilon) = (\xi_1, \xi_2 + \zeta_2)$ and

$$\psi(x, \eta, \epsilon, \frac{h^2}{\epsilon}) = x\cdot\eta + \phi_{\rm per}.$$

The canonical transformation $\kappa : (\psi'_\eta, \eta) \mapsto (x, \psi'_x)$ is $(\epsilon + h^2/\epsilon)$-close to the identity and with $\xi = \xi(\eta, \epsilon, h^2/\epsilon)$, we have by construction

$$(r_0 \circ \kappa)(y, \eta, \epsilon, \frac{h^2}{\epsilon}) = \langle r_0(\cdot, \xi, \epsilon, \frac{h^2}{\epsilon})\rangle = \langle r_0(\cdot, \eta, \epsilon, \frac{h^2}{\epsilon}\rangle + \mathcal{O}((\epsilon, \frac{h^2}{\epsilon})^2), \tag{4.30}$$

which is a function independent of y. Let U^{-1} be the corresponding Fourier integral operator as before. Then after replacing $\widetilde{P}_\epsilon$ by $\mathrm{Ad}_U\widetilde{P}_\epsilon$, we still have (4.24), where now $r_0 = i\langle q\rangle(\xi) + \mathcal{O}(\epsilon + h^2/\epsilon)$ is independent of x and $r_j = \mathcal{O}(1)$ for $j \geq 1$.

We can then make a further conjugation by $e^{\frac{i}{h}B_1}$ as before, and we get the following conclusion: Assume that the subprincipal symbol of $P_{\epsilon=0}$ vanishes and restrict the attention to the range (4.29). Then we can find an elliptic Fourier integral operator U^{-1} of the form (4.20) with ψ as above and $B_1(x_2, hD_x, \epsilon, h^2/\epsilon; h)$ with symbol (4.26), and $b_\nu = \mathcal{O}(1)$, such that

$$e^{\frac{i}{h}\mathrm{ad}_{B_1}} Ad_U\widetilde{P}_\epsilon = \widehat{P}_\epsilon(hD_x, \epsilon, h^2/\epsilon; h) \tag{4.31}$$

has a symbol $\widehat{P}_\epsilon(\xi, \epsilon, h^2/\epsilon; h)$ of the form (4.27), such that (4.28) holds.

We finish this section by discussing what spectral results can be expected from the reductions above. The first reduction (as in Section 3) was to conjugate the original operator P by a Fourier integral operator $e^{iG(x,hD,\epsilon)/h}$, with $G(x, \xi, \epsilon) \sim \epsilon(G_0(x, \xi) + \epsilon G_1(x, \xi) + \cdots)$, defined in some complex neighborhood of $p^{-1}(0) \cap T^*M$, to achieve that the leading symbol of the conjugated operator is of the form $p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2)$ and Poisson commutes with p. At least formally, the new operator also acts on $L^2(M)$ and we have no Floquet type conditions to worry about. Geometrically, this corresponds to the fact that a canonical transformation $\kappa = \exp H_G$ with a single-valued generator $G = \mathcal{O}(\epsilon)$ preserves actions along closed loops: $\int_{\kappa\circ\gamma} \xi dx = \int_\gamma \eta dy$, for every closed loop γ.

The second reduction was to take κ in (4.4) and to conjugate by the inverse of the corresponding Fourier integral operator U. Let $\alpha_1(=\gamma_0)$ and α_2 be the fundamental cycles in $\Lambda_{0,0}$ given by $\alpha_j = \kappa \circ \beta_j$, where β_1, β_2 are the fundamental cycles in $\mathbf{T}^2 \simeq \{(x,0) \in T^*\mathbf{T}^2\}$, given by $x_2 = 0$ and $x_1 = 0$ respectively. Put

$$S_j = \int_{\alpha_j} \xi dx, \tag{4.32}$$

so that S_j is the difference of actions, $\int_{\kappa\circ\beta_j} \xi dx - \int_{\beta_j} \eta dy$, $j = 1, 2$. Since κ is a canonical transformation we know that if β is a closed loop homotopic to β_j, then $\int_{\kappa\circ\beta} \xi dx - \int_\beta \eta dy = S_j$.

As in [20] or as in Theorem 2.4, we see (at least formally) that if we want Uu to be single-valued on M (possibly defined only microlocally near $\Lambda_{0,0}$), then u should not necessarily be periodic on $\mathbf{R}^2$ (i.e., a function on $\mathbf{T}^2$) but a Floquet periodic function with

$$u(x - \nu) = e^{\frac{i\nu\cdot S}{2\pi h} + \frac{i\nu\cdot k_0}{4}} u(x),\ \nu \in (2\pi\mathbf{Z})^2,\ S = (S_1, S_2),\ k_0 \in \mathbf{Z}^2. \tag{4.33}$$

The conjugated operator $\mathrm{Ad}_{U^{-1}e^{\frac{i}{h}G}} P_\epsilon$ should therefore act on Floquet periodic functions as in (4.33).

The further conjugations are by operators on the torus that conserve the property (4.33). This is clear from the definitions, and corresponds to the fact that a canonical transformation: $(y,\eta) \mapsto (x,\xi)$, generated by $\psi(x,\eta) = x\cdot\eta + \phi_{\mathrm{per}}(x,\eta)$ and close to the identity, conserves actions. Indeed, on the graph of the transform, we have $\xi dx + y d\eta = d\psi$, so

$$\xi dx - \eta dy = d(\psi - y\cdot\eta) = d((x-y)\cdot\eta + \phi_{\mathrm{per}}(x,\eta)),$$

and $(x-y)\cdot\eta + \phi_{\mathrm{per}}(x,\eta)$ is single-valued on the graph. On the other hand the space of Floquet periodic functions as in (4.33), equipped with the L^2-norm over a fundamental domain of $\mathbf{T}^2$, has the ON basis:

$$e_k(x) = e^{\frac{i}{h}x\cdot(h(k-\frac{k_0}{4}) - \frac{S}{2\pi})},\ k \in \mathbf{Z}^2, \tag{4.34}$$

and applying our reductions down to the operator $\widehat{P}_\epsilon$ in the cases (a) and (b) above, we get formally (in the sense that we do not define the notion of quasi-eigenvalue):

Proposition 4.1 *Recall that we took $F_0 = 0$ and that S, k_0 are the actions and the Maslov indices in* (4.32), (4.33).
a) In the general case, P_ϵ has the quasi-eigenvalues in $]-\frac{1}{|\mathcal{O}(1)|}, \frac{1}{|\mathcal{O}(1)|}[+i\epsilon]-\frac{1}{|\mathcal{O}(1)|}, \frac{1}{|\mathcal{O}(1)|}[$ for $\epsilon = \mathcal{O}(h^\delta)$, $h/\epsilon \ll 1$:

$$\widehat{P}\left(h(k - \frac{k_0}{4}) - \frac{S}{2\pi}, \epsilon, \frac{h}{\epsilon}; h\right),\ k \in \mathbf{Z}^2, \tag{4.35}$$

where $\widehat{P}(\xi, \epsilon, \frac{h}{\epsilon}; h)$ is holomorphic in $\xi \in \mathrm{neigh}\,(0, \mathbf{C}^2)$, smooth in $\frac{h}{\epsilon}, \epsilon \in \mathrm{neigh}$ $(0, \mathbf{R})$ and has the asymptotic expansion (4.17)*, when $h \to 0$.*

b) If we assume that $P_{\epsilon=0}$ has subprincipal symbol 0*, then P_ϵ has the quasi-eigenvalues in* $]-\frac{1}{|\mathcal{O}(1)|},\frac{1}{|\mathcal{O}(1)|}[+i\epsilon]-\frac{1}{|\mathcal{O}(1)|},\frac{1}{|\mathcal{O}(1)|}[$ *for* $\epsilon=\mathcal{O}(h^\delta)$*,* $h^2/\epsilon\ll 1$*:*

$$\widehat{P}\left(h(k-\frac{k_0}{4})-\frac{S}{2\pi},\epsilon,\frac{h^2}{\epsilon};h\right),\ k\in\mathbf{Z}^2, \tag{4.36}$$

where $\widehat{P}(\xi,\epsilon,h^2/\epsilon;h)$ *is holomorphic in* $\xi\in\mathrm{neigh}\,(0,\mathbf{C}^2)$*, smooth in* ϵ *and* $h^2/\epsilon\in\mathrm{neigh}\,(0,\mathbf{R})$ *and has the asymptotic expansion* (4.27)*,* (4.28)*, when* $h\to 0$*.*

5 Quasi-eigenvalues in the extreme cases

We make the assumptions of the case II in the introduction and assume, in order to fix the ideas, that

$$0=F_0=\langle\mathrm{Re}\,q\rangle_{\min,0}. \tag{5.1}$$

Apply Proposition 3.2 and reduce P_ϵ near γ_0 to $\widetilde{P}_\epsilon=\widetilde{P}(x,hD_{t,x},\epsilon;h)$ with symbol described in that proposition. Recall that $\widetilde{P}_\epsilon$ has the leading symbol

$$\widetilde{p}_\epsilon=f(\tau)+i\epsilon\langle q\rangle(\tau,x,\xi)+\mathcal{O}(\epsilon^2), \tag{5.2}$$

where $\langle q\rangle(\tau,x,\xi)$ is equal to the original averaged function $\langle q\rangle$, composed with the canonical transformation κ of Proposition 3.1. The assumptions (1.23) and (5.1) imply that

$$\mathrm{Re}\,\langle q\rangle(0,x,\xi)\sim|(x,\xi)|^2 \tag{5.3}$$

on the real domain. Also recall that we have the assumption (1.17) which with (5.3) implies that

$$\langle q\rangle(\tau,x,\xi)=g(\tau,\mathrm{Re}\,\langle q\rangle(\tau,x,\xi)) \tag{5.4}$$

on the real domain, for some analytic function $g(\tau,q)$ with $g(0,0)=0$, $\mathrm{Re}\,g(\tau,q)=q$.

We conclude that $(x,\xi)\mapsto i\langle q\rangle(\tau,x,\xi)+\mathcal{O}(\epsilon)$, appearing in (5.2), has a non-degenerate critical point $(x(\tau,\epsilon),\xi(\tau,\epsilon))=\mathcal{O}(|\tau|+\epsilon)$ depending analytically on τ,ϵ and real when $\tau\in\mathbf{R}$, $\epsilon=0$. After composition with the (τ,ϵ)-dependent (symplectic) translation $(x,\xi)\mapsto(x-x(\tau,\epsilon),\xi-\xi(\tau,\epsilon))$ and subtracting the corresponding critical value, we may assume that the critical point is $(0,0)$ and hence that

$$\widetilde{p}_\epsilon(\tau,x,\xi)=f(\tau)+i\epsilon q(\tau,x,\xi,\epsilon), \tag{5.5}$$

with

$$\mathrm{Re}\,q(\tau,x,\xi,\epsilon)\sim|(x,\xi)|^2 \tag{5.6}$$

on the real domain, and

$$q(\tau,x,\xi,0)=g(\tau,\mathrm{Re}\,q(\tau,x,\xi,0)), \tag{5.7}$$

on the real domain, where $g(\tau,0)=0$, $\mathrm{Re}\,g(\tau,q)=q$.

We shall next construct a (τ, ϵ)-dependent canonical transformation in the x, ξ-variables, which reduces $\widetilde{p}_\epsilon(\tau, x, \xi)$ to a function of $\tau, \epsilon, \frac{1}{2}(x^2 + \xi^2)$. In doing so, we essentially follow Appendix B of [13], where the model was $x\xi$ rather than $p_0 := \frac{1}{2}(x^2 + \xi^2)$. These two quadratic forms are equivalent up to a constant factor and composition by a linear complex canonical transformation, so the only difference is that the real domains are not the same.

Let $p(x, \xi) \sim (x, \xi)^2$ be real and analytic in a neighborhood of (0,0).

Lemma 5.1 *There exists a real and analytic function $f(E)$ defined near $E = 0$, with $f(0) = 0$, $f'(0) > 0$, such that the Hamilton flow of $f \circ p$ is 2π-periodic, with 2π as its minimal period except at $(0, 0)$.*

Proof. Consider, first for $0 < E \ll 1$, the action

$$I(E) = \int_{p^{-1}(E)} \xi dx = E \int_{q_E^{-1}(1)} \eta dy,$$

where $q_E(y, \eta) = \frac{1}{E} p(\sqrt{E}(y, \eta))$, so that q_0 is a positive quadratic form (in the limit $E \to 0$). Then q_E is an analytic function of $\sqrt{E}$ in a neighborhood of 0 and consequently we have the same fact for $I(E)$. If we let E describe a simple closed loop around 0 in $\mathrm{neigh}\,(0, \mathbf{C}) \setminus \{0\}$, then $q_E(y, \eta)$ transforms into $\widetilde{q}_E(y, \eta) = q_E(-y, -\eta)$ and it follows that $I(E)$ transforms into itself. It follows that $I(E)$ is analytic as a function of E. The period $T(E)$ of the H_p-flow is given by $T(E) = I'(E)$ and the period of the $H_{f \circ p}$-flow is $T(E)/f'(E)$. It suffices to choose f with $f'(E) = T(E)/2\pi$ and $f(0) = 0$. □

In the following discussion, we replace p by $f \circ p$, so that we get a reduction to the case when the H_p-flow is 2π-periodic. After composition with a real linear canonical transformation, we may assume that $p(x, \xi) = p_0(x, \xi) + \mathcal{O}((x, \xi)^3)$, even though that is not really needed for the argument to follow. Consider the involution $\iota = \exp(\pi H_p)$ with $\iota^2 = \mathrm{id}$. Correspondingly, we have $\iota_0 = \exp(\pi H_{p_0})$, so that $\iota_0(\rho) = -\rho$. Let $h(x, \xi)$ be a real-valued analytic function defined near $(0, 0)$ with $dh(0, 0) \neq 0$, and put $g = \frac{1}{2}(h - h \circ \iota)$. Then $dg(0) = dh(0, 0) \neq 0$, and

$$g \circ \iota = -g. \tag{5.8}$$

$\Gamma := g^{-1}(0)$ is a real curve passing through the origin, invariant under the action of ι. Let Γ also denote a corresponding complexification. If g_0, Γ_0 are the corresponding objects for p_0, we may assume (though this is not essential), that $dg(0, 0) = dg_0(0, 0)$ so that Γ, Γ_0 are tangent at $(0, 0)$.

Since Γ is a curve, we have $p_{|_\Gamma} = q^2$ for some analytic function q, and similarly $p_0{}_{|_{\Gamma_0}} = q_0^2$. (We may assume that $dq_0 = dq \neq 0$ at 0.) Let $\alpha : \Gamma_0 \to \Gamma$ be the analytic diffeomorphism given by $q \circ \alpha = q_0$, so that $p \circ \alpha = p_0$ on Γ_0. For

$$\mathrm{neigh}\,((0, 0), \mathbf{C}^2) \ni \rho = \exp t H_{p_0}(\nu), \ \nu \in \Gamma_0, \, t \in \mathbf{C}, \tag{5.9}$$

we put

$$\kappa(\rho) = \exp tH_p(\alpha(\nu)). \tag{5.10}$$

With the precautions taken above, it is easy to see that the definition of $\kappa(\rho)$ does not depend on how we choose $\nu \in \Gamma_0$ (unique up to the action of ι_0) and t (unique mod (2π), once ν has been chosen.) As in [13], we see that some exceptional points $\rho \in \mathrm{neigh}\,((0,0), \mathbf{C}^2)$ cannot be represented as in (5.9), namely the ones $\neq (0,0)$ in the stable outgoing and incoming complex (Lagrangian) curves for the iH_{p_0}-flow, and if ρ converges to one of these lines, then in general $|t| \to \infty$ for the t in (5.9), so a priori it is not clear then that the right-hand side of (5.10) is defined. These difficulties were analyzed and settled in [13], and at this point there is no difference with our situation, so we conclude that κ is a well-defined analytic map in a neighborhood of $(0,0)$:

Lemma 5.2 *With f, p as in Lemma 5.1, there exists an analytic canonical transformation $\kappa :$ $\mathrm{neigh}\,((0,0), \mathbf{R}^2) \to \mathrm{neigh}\,((0,0), \mathbf{R}^2)$, with $f \circ p \circ \kappa = p_0$.*

If p depends smoothly (analytically) on some real parameters, and fulfills the assumptions above, then f, κ can be chosen to depend smoothly (analytically) on the same parameters. If $p = p_\epsilon = \mathcal{O}((x,\xi)^2)$ is analytic in (x,ξ), depends smoothly on $\epsilon \in \mathrm{neigh}\,(0, \mathbf{R})$ and satisfies the assumptions above for $\epsilon = 0$, then we get $f_\epsilon(E), \kappa_\epsilon(x,\xi)$, holomorphic in E and x, ξ, depending smoothly on ϵ with $f_\epsilon \circ p_\epsilon \circ \kappa_\epsilon = p_0$, but $f_\epsilon, \kappa_\epsilon$ are no more necessarily real when $\epsilon \neq 0$. Clearly $\mathrm{Im}\, f_\epsilon(E) = \mathcal{O}(\epsilon)$, $\mathrm{Im}\, \kappa_\epsilon(x,\xi) = \mathcal{O}(\epsilon)$ when E, x, ξ are real. In our case the parameters are τ, ϵ and the above discussion gives:

Proposition 5.3 *For $\widetilde{p}_\epsilon(\tau, x, \xi)$ in (5.5), we can find a canonical transformation $(x,\xi) \mapsto \kappa_{\tau,\epsilon}(x,\xi)$ depending analytically on τ and smoothly on ϵ with values in the holomorphic canonical transformations:* $\mathrm{neigh}\,((0,0), \mathbf{C}^2) \to \mathrm{neigh}\,((0,0), \mathbf{C}^2)$, *and an analytic function $g_\epsilon(\tau, q)$ depending smoothly on ϵ such that*

$$\kappa_{\tau,\epsilon}(0,0) = (x(\tau,\epsilon), \xi(\tau,\epsilon)), \tag{5.11}$$

$$\widetilde{p}_\epsilon(\tau, \kappa_{\tau,\epsilon}(x,\xi)) = f(\tau) + i\epsilon g_\epsilon(\tau, \frac{1}{2}(x^2 + \xi^2)). \tag{5.12}$$

Moreover, $\kappa_{\tau,0}$ is real when τ is real and

$$\frac{\partial}{\partial q} \mathrm{Re}\, g_\epsilon(0,0) > 0. \tag{5.13}$$

As a matter of fact, as in Section 4, we will apply this result to a modification of $\widetilde{p}_\epsilon$, containing also the leading lower order symbol. Before doing so, we recall how to treat lower order symbols in general for operators with leading symbol modelled on the 1-dimensional harmonic oscillator (similarly to what we did in Section 3 and as in [26]).

Consider a formal h-pseudodifferential operator $Q(x, hD_x; h)$ with symbol

$$Q(x, \xi; h) \sim q_0(x, \xi) + hq_1(x, \xi) + \cdots , \tag{5.14}$$

defined in a neighborhood of $(0, 0) \in \mathbf{R}^2$. As usual, $q_0, q_1, \dots$ are supposed to be smooth and we assume

$$q_0(x, \xi) = g_0(p_0(x, \xi)), \tag{5.15}$$

where $g_0 \in C^\infty(\mathrm{neigh}\,(0, \mathbf{R}))$ satisfies $g_0(0) = 0$, $g_0'(0) \neq 0$. (We do not assume g_0 to be real-valued.) As in Section 3 we find a smooth function $a_0(x, \xi)$, defined in a neighborhood of $(0, 0)$, such that

$$H_{q_0} a_0 = q_1 - \langle q_1 \rangle, \tag{5.16}$$

where $\langle q_1 \rangle$ is the trajectory average $\frac{1}{2\pi} \int_0^{2\pi} q_1 \circ \exp(tH_{p_0}) dt$. Adding lower order corrections, we see that there exists

$$A(x, \xi; h) \sim a_0(x, \xi) + ha_1(x, \xi) + \cdots \tag{5.17}$$

with all a_j smooth in some common neighborhood of $(0, 0)$, such that

$$e^{iA(x,hD_x;h)} Q(x, hD_x; h) e^{-iA(x,hD_x;h)} =: \widehat{Q}(x, hD_x; h) \tag{5.18}$$

has a symbol $\widehat{Q} \sim \widehat{q}_0 + h\widehat{q}_1 + \cdots$, with $\widehat{q}_0 = q_0$ and

$$H_{q_0} \widehat{q}_j = 0, \ \forall j. \tag{5.19}$$

This means that $\widehat{q}_j$ is a smooth function of $p_0(x, \xi)$ and as is well known (and exploited for instance in [26]), the facts (5.18), (5.19) can be reformulated by saying that we have found A as in (5.17) such that

$$e^{iA(x,hD;h)} Q(x, hD; h) e^{-iA(x,hD;h)} = g(p_0(x, hD); h),$$

where $g(E; h) \sim \sum_0^\infty g_j(E) h^j$ in $C^\infty(\mathrm{neigh}\,(0, \mathbf{R}))$, with g_0 as before. When g_0, q_j are holomorphic in fixed neighborhoods of $E = 0$ and $(x, \xi) = (0, 0)$, we get the corresponding holomorphy for $g_k, \widehat{q}_\ell$.

Now return to the operator $\widetilde{P}_\epsilon$ of the beginning of this section. Write the full symbol as

$$\widetilde{P}_\epsilon(\tau, x, \xi, \epsilon; h) \sim \widetilde{p}_\epsilon(\tau, x, \xi) + h\widetilde{p}_1(\tau, x, \xi, \epsilon) + h^2 \widetilde{p}_2(\tau, x, \xi, \epsilon) + \cdots \tag{5.20}$$

a) Consider first the general case without any assumptions on the subprincipal symbol, and assume that

$$h \ll \epsilon < h^\delta, \tag{5.21}$$

for some fixed $\delta > 0$. Following the strategy of Section 4, we rewrite (5.20) as

$$\widetilde{P}_\epsilon(\tau, x, \xi; h) = f(\tau) + \epsilon[(i\langle q \rangle(\tau, x, \xi) + \mathcal{O}(\epsilon) + \frac{h}{\epsilon} \widetilde{p}_1(\tau, x, \xi)) + h\frac{h}{\epsilon} \widetilde{p}_2 + h^2 \frac{h}{\epsilon} \widetilde{p}_3 + \cdots]. \tag{5.22}$$

As before, we now treat h/ϵ as an additional small parameter. Proposition 5.3 extends to the case when $\widetilde{p}_\epsilon$ is replaced by $\widetilde{p}_\epsilon + \epsilon\frac{h}{\epsilon}\widetilde{p}_1$, so we have a canonical transformation $(x,\xi) \mapsto \kappa_{\tau,\epsilon,h/\epsilon}(x,\xi)$ depending analytically on τ and smoothly on ϵ, $\frac{h}{\epsilon}$, equal to $\kappa_{\tau,\epsilon}$ when $\frac{h}{\epsilon} = 0$, such that

$$(\widetilde{p}_\epsilon + \epsilon\frac{h}{\epsilon}\widetilde{p}_1)(\tau, \kappa_{\tau,\epsilon,\frac{h}{\epsilon}}(x,\xi)) = f(\tau) + i\epsilon g_{\epsilon,\frac{h}{\epsilon}}(\tau, \frac{1}{2}(x^2+\xi^2)),$$

with $g_{\epsilon,0} = g_\epsilon$ appearing in Proposition 5.3.

As in Section 4, we therefore obtain an elliptic Fourier integral operator $U_{\epsilon,h/\epsilon}$, which is a convolution in t, and such that the Fourier transform with respect to t, $\widehat{U}_{\epsilon,h/\epsilon}(\tau)$, is a 1-dimensional Fourier integral operator in x quantizing $\kappa_{\tau,\epsilon,h/\epsilon}$. After conjugation of $\widetilde{P}_\epsilon$ by $U_{\epsilon,h/\epsilon}$, we get a new operator $\widetilde{P}_\epsilon$ of the same type, with symbol

$$\widetilde{P}(\tau,x,\xi,\epsilon,h/\epsilon;h) = f(\tau) + \epsilon[ig_{\epsilon,\frac{h}{\epsilon}}(\tau,\frac{1}{2}(x^2+\xi^2)) + h\widetilde{p}_2 + h^2\widetilde{p}_3 + \cdots], \quad (5.23)$$

where $\widetilde{p}_2, \widetilde{p}_3, \ldots$ also depend on h/ϵ.

After a further conjugation by $e^{iA(hD_t,x,hD_x,\epsilon,\frac{h}{\epsilon};h)}$, where each term A_j in the h-asymptotic expansion:

$$A(\tau,x,\xi,\epsilon,\frac{h}{\epsilon};h) \sim A_0(\tau,x,\xi,\epsilon,\frac{h}{\epsilon}) + hA_1(\tau,x,\xi,\epsilon,\frac{h}{\epsilon}) + \cdots$$

is holomorphic in τ, x, ξ in a fixed neighborhood of $(0,0,0) \in \mathbf{C}^3$ and smooth in $\epsilon, h/\epsilon$, we get a new operator of the form

$$\widetilde{P}_\epsilon = f(hD_t) + i\epsilon G(hD_t, \frac{1}{2}(x^2 + (hD_x)^2), \epsilon, \frac{h}{\epsilon}; h), \quad (5.24)$$

where

$$G(\tau,q,\epsilon,\frac{h}{\epsilon};h) \sim \sum_0^\infty G_j(\tau,q,\epsilon,\frac{h}{\epsilon})h^j, \quad (5.25)$$

with G_j holomorphic in τ, q in a j-independent neighborhood of $(0,0)$ and smooth in $\epsilon, h/\epsilon$. Moreover G_0 is equal to the term $g_{\epsilon,h/\epsilon}(\tau,q)$ in (5.23). Recalling that $\frac{1}{2}(x^2+(hD_x)^2)$ has the eigenvalues $h(\frac{1}{2}+k_2)$, $k_2 \in \mathbf{N}$, we get the conclusion:

Proposition 5.4 *Make the assumptions of case* II *in the introduction, and assume that* $F_0 = \langle \operatorname{Re} q \rangle_{\min,0}$ *(the case when* F_0 *is a maximum being analogous). Then in a rectangle* $]-\frac{1}{|\mathcal{O}(1)|}, \frac{1}{|\mathcal{O}(1)|}[+i\epsilon]F_0 - \frac{1}{|\mathcal{O}(1)|}, F_0 + \frac{1}{|\mathcal{O}(1)|}[$, P_ϵ *has the quasi-eigenvalues:*

$$f\left(h(k_1 - \frac{k_0}{4}) - \frac{S_1}{2\pi}\right) + i\epsilon G\left(h(k_1 - \frac{k_0}{4}) - \frac{S_1}{2\pi}, h(\frac{1}{2}+k_2), \epsilon, \frac{h}{\epsilon}; h\right),$$
$$(k_1,k_2) \in \mathbf{Z}\times\mathbf{N}. \quad (5.26)$$

Here $f(\tau)$ is real-valued with $f(0) = 0$, $f'(0) > 0$. The function G has the properties described in and after (5.25) *and* $\mathrm{Re}\, G_0(0,0,0,0) = F_0$, $\frac{\partial}{\partial q}\mathrm{Re}\, G_0(0,0,0,0) > 0$. *Finally, k_0 is a fixed integer.*

b) We next consider the case when the subprincipal symbol of $P_{\epsilon=0}$ vanishes, and assume that

$$h^2 \ll \epsilon < h^\delta, \tag{5.27}$$

for some fixed $\delta > 0$. According to the improved Egorov theorem of Section 2, we know that $\widetilde{p}_1$ in (5.20) vanishes for $\epsilon = 0$, so we can write $\frac{h}{\epsilon}\widetilde{p}_1(\tau, x, \xi, \epsilon) = h\widehat{p}_1(\tau, x, \xi, \epsilon)$ in (5.22) and treat this term as a lower order term, while we now allow $\frac{h^2}{\epsilon}\widetilde{p}_2$ to be a correction to the leading terms. As in the corresponding case in Section 4, we get h^2/ϵ as an additional small parameter instead of h/ϵ, and the same procedure as in case a) now leads to (5.24), (5.25) with h/ϵ replaced by h^2/ϵ.

Proposition 5.5 *Make the assumptions of Proposition* 5.4 *and assume in addition that the subprincipal symbol of $P_{\epsilon=0}$ vanishes. Then for ϵ in the range* (5.27), *P_ϵ has the quasi-eigenvalues as described in the preceding proposition, with the only difference that "h/ϵ" in* (5.26) *should be replaced by "h^2/ϵ".*

6 Global Grushin problem

Let P_ϵ be as in Section 1. In Sections 4 and 5 we have constructed microlocal normal forms for P_ϵ near a Lagrangian torus and near a closed H_p-trajectory, respectively. The purpose of this section is to justify the preceding microlocal constructions and computations, and to show that the quasi-eigenvalues of Proposition 4.1 and Propositions 5.4 and 5.5 give, modulo $\mathcal{O}(h^\infty)$, all of the true eigenvalues of P_ϵ, in suitable regions of the complex plane. This will be achieved by studying an auxiliary global Grushin problem, well posed in a certain h-dependent Hilbert space, and the first and the main step for us will be to define this space globally. The actual setup of the Grushin problem and some of the details of the computations will be closely related to the corresponding analysis in [20].

When constructing the Hilbert space, we shall inspect all the steps of the microlocal reductions of Sections 3–5, and implement each step of the construction. In doing so, for simplicity, we shall concentrate on the case when $M = \mathbf{R}^2$. In view of the results of the appendix, it will be clear how to extend the following discussion to the case of compact real-analytic manifolds. Also, in order to simplify the presentation, we shall assume throughout the section that the order function m, introduced in (1.2), is equal to 1. Again, it will be clear that the discussion below will extend to the case of a general order function. Throughout this section we shall assume that $\epsilon = \mathcal{O}(h^\delta)$, for some fixed $\delta > 0$.

Let $G = G(x, \xi, \epsilon)$ be as in (3.6). We shall introduce an IR-manifold $\Lambda_{\epsilon G} \subset \mathbf{C}^4$, which in a complex neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$ is equal to $\exp{(i\epsilon H_G)}(\mathbf{R}^4)$, and further away from $p^{-1}(0) \cap \mathbf{R}^4$ agrees with the real phase space $\mathbf{R}^4$. The manifold $\Lambda_{\epsilon G}$ will be ϵ-close to $\mathbf{R}^4$, and when defining it, it will be convenient to work on the FBI transform side. We shall use the FBI-Bargmann transform

$$Tu(x) = Ch^{-3/2} \int e^{i\varphi(x,y)/h} u(y)\,dy, \quad x \in \mathbf{C}^2, \quad C > 0, \tag{6.1}$$

where $\varphi(x, y) = i/2(x - y)^2$. Associated to T there is a complex linear canonical transformation κ_T, given by

$$\mathbf{C}^4 \ni (y, -\varphi'_y(x, y)) \longmapsto (x, \varphi'_x(x, y)) \in \mathbf{C}^4.$$

It is well known, see [28], that κ_T maps $\mathbf{R}^4$ onto

$$\Lambda_{\Phi_0} := \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \right), x \in \mathbf{C}^2 \right\}, \quad \Phi_0(x) = \frac{(\operatorname{Im} x)^2}{2}.$$

The IR-manifold $\Lambda_{\epsilon G}$ has already been defined near $p^{-1}(0) \cap \mathbf{R}^4$, and when constructing it globally, we require that the IR-manifold $\kappa_T(\Lambda_{\epsilon G})$ should agree with Λ_{Φ_0} outside a bounded set and that it is ϵ-close to that manifold everywhere. We define therefore $\Lambda_{\epsilon G}$ so that the representation

$$\kappa_T(\Lambda_{\epsilon G}) = \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi}{\partial x} \right), x \in \mathbf{C}^2 \right\} =: \Lambda_\Phi \tag{6.2}$$

holds true. Here the function $\Phi \in C^\infty(\mathbf{C}^2; \mathbf{R})$ is uniformly strictly plurisubharmonic, and is such that

$$\Phi(x) = \Phi_0(x) + \epsilon g(x, \epsilon),$$

with $g(x, \epsilon) \in C^\infty$ in both arguments and with a uniformly compact support with respect to x.

Associated to $\Lambda_{\epsilon G}$ we then introduce the corresponding Hilbert space $H(\Lambda_{\epsilon G})$ which agrees with $L^2(\mathbf{R}^2)$ as a space, and which we equip with the norm $||\, u \,|| := ||\, Tu \,||_{L^2_\Phi}$. Here $L^2_\Phi = L^2(\mathbf{C}^2; e^{-2\Phi/h} L(dx))$, with $L(dx)$ being the Lebesgue measure on $\mathbf{C}^2$.

Performing a contour deformation in the integral representation of P_ϵ on the FBI-Bargmann transform side, as in [20], [28], we see that

$$P_\epsilon = \mathcal{O}(1) : H(\Lambda_{\epsilon G}) \to H(\Lambda_{\epsilon G}), \tag{6.3}$$

and the leading symbol on the FBI transform side is then $p_\epsilon \circ \kappa_T^{-1}\Big|_{\Lambda_\Phi}$. Continuing to work on the FBI-Bargmann transform side, as in Section 2 of [20], we introduce a microlocally unitary semiclassical Fourier integral operator

$$e^{\epsilon G(x, hD_x, \epsilon)/h} : L^2(\mathbf{R}^2) \to H(\Lambda_{\epsilon G}), \tag{6.4}$$

microlocally defined near $p^{-1}(0)\cap\mathbf{R}^4$, and associated to the complex canonical transformation $\exp(i\epsilon H_G):\mathbf{R}^4\to\Lambda_{\epsilon G}$. The operator in (6.3) is then microlocally near $p^{-1}(0)$, unitarily equivalent to the operator

$$e^{-\epsilon G(x,hD_x,\epsilon)/h}P_\epsilon e^{\epsilon G(x,hD_x,\epsilon)/h}:L^2\to L^2,$$

with the principal symbol

$$p\circ\exp(i\epsilon H_G)=p+i\epsilon\langle q\rangle+\mathcal{O}(\epsilon^2). \tag{6.5}$$

This averaging procedure allows us therefore to reduce the further analysis to an operator P_ϵ, microlocally defined near $p^{-1}(0)\cap\mathbf{R}^4$, which has the principal symbol (6.5), where $\langle q\rangle$, as well as the $\mathcal{O}(\epsilon^2)$-term, are in involution with p. As explained in Section 4, at this stage the operator P_ϵ acts on single-valued functions in $L^2(\mathbf{R}^2)$.

In the first part of this section we shall concentrate on the torus case of Section 4. We assume therefore that dp and $d\mathrm{Re}\,\langle q\rangle$ are linearly independent on the set

$$\Lambda_{0,0}:\ p=0,\ \mathrm{Re}\,\langle q\rangle=0. \tag{6.6}$$

We recall also the assumption that $T(0)$ is the minimal period of every closed H_p-trajectory in the Lagrangian torus $\Lambda_{0,0}$, and notice that in a neighborhood of $\Lambda_{0,0}$, p and $\mathrm{Re}\,\langle q\rangle$ form a completely integrable system. Introduce a new Lagrangian torus $\widetilde{\Lambda}_{0,0}\subset\Lambda_{\epsilon G}$ defined by

$$\widetilde{\Lambda}_{0,0}:\ p\circ\exp(-i\epsilon H_G)=0,\ \mathrm{Re}\,\langle q\rangle\circ\exp(-i\epsilon H_G)=0. \tag{6.7}$$

In what follows we shall often identify the tori $\Lambda_{0,0}$ and $\widetilde{\Lambda}_{0,0}$ by means of $\exp(i\epsilon H_G)$, and we shall continue to write $\Lambda_{0,0}$ for $\widetilde{\Lambda}_{0,0}$ when there is no risk of confusion. Combining $\exp(i\epsilon H_G)$ with the canonical transformation κ, introduced in (4.4), and given by the action-angle coordinates associated with p, $\mathrm{Re}\,\langle q\rangle$, we get a smooth canonical diffeomorphism

$$\kappa_\epsilon:\mathrm{neigh}\left(\xi=0,T^*\mathbf{T}^2\right)\to\mathrm{neigh}\left(\Lambda_{0,0},\Lambda_{\epsilon G}\right), \tag{6.8}$$

so that $\kappa_\epsilon=\exp(i\epsilon H_G)\circ\kappa$. As in (4.32), we set

$$S_j=\int_{\alpha_j}\xi\,dx,\quad j=1,2,$$

where α_1 and α_2 are the fundamental cycles in $\Lambda_{0,0}$, with α_1 corresponding to a closed H_p-trajectory of the minimal period $T(0)$. Introduce also the "Maslov indices" $k_0(\alpha_j)\in\mathbf{Z}$, $j=1,2$, of the cycles α_j, defined as in Proposition 2.3. Let $L^2_\theta(\mathbf{T}^2)$ be the subspace of $L^2_{\mathrm{loc}}(\mathbf{R}^2)$ consisting of Floquet periodic functions $u(x)$, satisfying

$$u(x-\nu)=e^{i\theta\cdot\nu}u(x),\quad \nu\in(2\pi\mathbf{Z})^2,\quad\text{where }\theta=\frac{S}{2\pi h}+\frac{k_0}{4}. \tag{6.9}$$

Here $S = (S_1, S_2)$ and $k_0 = (k_0(\alpha_1), k_0(\alpha_2)) \in \mathbf{Z}^2$. An application of Theorem 2.4 allows us to conclude that there exists a microlocally unitary multi-valued Fourier integral operator

$$U : L^2_\theta(\mathbf{T}^2) \to L^2(\mathbf{R}^2), \tag{6.10}$$

microlocally defined from a neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$ to a neighborhood of $\Lambda_{0,0}$ in $\mathbf{R}^4$, and associated to κ in (4.4). Moreover, U satisfies the improved Egorov property (2.3). The composition $e^{\epsilon G(x,hD_x,\epsilon)/h} \circ U$ is then associated with κ_ϵ in (6.8), and we have a Egorov's theorem, still with the improved property (2.3). The operator P_ϵ, acting in $H(\Lambda_{\epsilon G})$ is therefore unitarily equivalent to an h-pseudodifferential operator microlocally defined near $\xi = 0$, acting in $L^2_\theta(\mathbf{T}^2)$, and which has the leading symbol

$$p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2),$$

independent of x_1. We shall continue to write P_ϵ for the conjugated operator on $\mathbf{T}^2$. From Section 4 we next recall that there exists an elliptic pseudodifferential operator of the form $e^{iA/h}$, acting on $L^2_\theta(\mathbf{T}^2)$, such that after a conjugation by it, the full symbol of P_ϵ becomes independent of x_1. Recall also that A is constructed as a formal power series in ϵ and h, with coefficients holomorphic in a fixed complex neighborhood of the zero section of $T^*\mathbf{T}^2$. These formal power series are then realized as C^∞-symbols, in view of our basic assumption $\epsilon = \mathcal{O}(h^\delta)$, $\delta > 0$.

Summing up the discussion so far, we have now achieved that, microlocally near $\Lambda_{0,0}$, the operator

$$P_\epsilon : H(\Lambda_{\epsilon G}) \to H(\Lambda_{\epsilon G})$$

is equivalent to an operator of the form

$$\widetilde{P}_\epsilon(x_2, \xi, \epsilon; h) \sim \sum_{\nu=0}^{\infty} h^\nu \widetilde{p}_\nu(x_2, \xi, \epsilon) \tag{6.11}$$

acting on $L^2_\theta(\mathbf{T}^2)$. Here $\widetilde{p}_\nu(x_2, \xi, \epsilon)$ are holomorphic in a ν-independent complex neighborhood of $\xi = 0$, and

$$\widetilde{p}_0 = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2).$$

Furthermore, $\widetilde{P}_{\epsilon=0}$ is selfadjoint.

Remark. It follows from the construction together with Theorem 2.4 that if the subprincipal symbol of $P_{\epsilon=0}$ vanishes, then $\widetilde{p}_1(x_2, \xi, 0) = 0$.

We must now implement the final conjugation of $\widetilde{P}_\epsilon$, which removes the x_2-dependence in the full symbol. In doing so, we shall first assume that we are in the general case, so that the subprincipal symbol of $P_{\epsilon=0}$ does not necessarily vanish. We shall work under the assumption

$$\frac{h}{\epsilon} \le \delta_0 \ll 1. \tag{6.12}$$

As in Section 4, we write

$$\widetilde{P}_\epsilon = p(\xi_1) + \epsilon\left(r_0\left(x_2,\xi,\epsilon,\frac{h}{\epsilon}\right) + hr_1\left(x_2,\xi,\epsilon,\frac{h}{\epsilon}\right) + \cdots\right),$$

where

$$r_0\left(x_2,\xi,\epsilon,\frac{h}{\epsilon}\right) = i\langle q\rangle(\xi) + \mathcal{O}(\epsilon) + \frac{h}{\epsilon}\widetilde{p}_1(x_2,\xi,\epsilon),$$

and $r_j = \mathcal{O}_j(h/\epsilon)$, $j \geq 1$. Let us introduce a complexification of the standard 2-torus, $\widetilde{\mathbf{T}^2} = \mathbf{T}^2 + i\mathbf{R}^2$. From the constructions of Section 4 we know that there exists a holomorphic canonical transformation

$$\begin{aligned}\widetilde{\kappa} &: \mathrm{neigh}\left(\mathrm{Im}\, y = \eta = 0, \widetilde{\mathbf{T}^2}\times\mathbf{C}^2\right) \\ &\ni (y,\eta) \mapsto (x,\xi) \in \mathrm{neigh}\left(\mathrm{Im}\, x = \xi = 0, \widetilde{\mathbf{T}^2}\times\mathbf{C}^2\right)\end{aligned} \tag{6.13}$$

with the generating function of the form

$$\psi\left(x,\eta,\epsilon,\frac{h}{\epsilon}\right) = x\cdot\eta + \phi_{\mathrm{per}}\left(x_2,\eta,\epsilon,\frac{h}{\epsilon}\right), \quad \phi_{\mathrm{per}} = \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right), \tag{6.14}$$

and such that

$$(r_0\circ\widetilde{\kappa})\,(y,\eta,\epsilon,h/\epsilon) = \langle r_0(\cdot,\eta,\epsilon,h/\epsilon)\rangle + \mathcal{O}\left(\left(\epsilon,\frac{h}{\epsilon}\right)^2\right) \tag{6.15}$$

is independent of y – see (4.19). It follows from (6.14) that $\widetilde{\kappa}$ is $(\epsilon + h/\epsilon)$-close to the identity, and has the expression

$$(y_1,\eta_1;y_2,\eta_2) \longmapsto (x_1(y_2,\eta),\eta_1;x_2(y_2,\eta),\xi_2(y_2,\eta)).$$

In particular it is true that

$$\mathrm{Im}\, x = \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right), \quad \mathrm{Im}\,\xi_2 = \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right), \quad \mathrm{Im}\,\xi_1 = 0,$$

on the image of $T^*\mathbf{T}^2$. We introduce now an IR-manifold $\widetilde{\Lambda} \subset \widetilde{\mathbf{T}^2}\times\mathbf{C}^2$, which is equal to $\widetilde{\kappa}\left(T^*\mathbf{T}^2\right)$ in a complex neighborhood of the zero section of $T^*\mathbf{T}^2$, and outside another complex fixed neighborhood of $\xi = 0$, coincides with $T^*\mathbf{T}^2$. In the intermediate region, we shall construct $\widetilde{\Lambda}$ in such a way that it remains an $(\epsilon + h/\epsilon)$-perturbation of $T^*\mathbf{T}^2$, and such that everywhere on $\widetilde{\Lambda}$ we have the property

$$(x_1,\xi_1;x_2,\xi_2) \in \widetilde{\Lambda} \Longrightarrow \mathrm{Im}\,\xi_1 = 0. \tag{6.16}$$

When constructing $\widetilde{\Lambda}$ and describing the conjugation of $\widetilde{P}_\epsilon$ by a Fourier integral operator associated to $\widetilde{\kappa}$, it is convenient to work on the FBI transform side. As in Section 3 of [20], we notice that the FBI-Bargmann transformation introduced in (6.1) generates an operator from $L^2_\theta(\mathbf{T}^2)$ to the space of Floquet periodic holomorphic functions on $\mathbf{C}^2$. We continue to denote this operator by T. Then after the application of the canonical transformation κ_T, associated to T, the cotangent space $T^*\mathbf{T}^2$ becomes an IR-manifold $\Lambda_{\Phi_1} \subset \widetilde{\mathbf{T}^2} \times \mathbf{C}^2$ given by

$$\Lambda_{\Phi_1} : \xi = \frac{2}{i}\frac{\partial \Phi_1}{\partial x} = -\mathrm{Im}\, x, \quad \Phi_1(x) = \frac{(\mathrm{Im}\, x)^2}{2}.$$

Since T is a convolution operator acting separately in y_1 and y_2, we see that

$$\kappa_T(\widetilde{\Lambda}) = \Lambda_\Phi, \quad \Lambda_\Phi : \xi = \frac{2}{i}\frac{\partial \Phi}{\partial x},$$

where Φ is an $(\epsilon + h/\epsilon)$-perturbation of Φ_1 with the property that $\xi_1 = (2/i)\frac{\partial \Phi}{\partial x_1}$ is real. It follows that $\Phi = \Phi(\mathrm{Im}\, x_1, x_2)$ is independent of $\mathrm{Re}\, x_1$. Using a standard cutoff function around $\mathrm{Im}\, x = 0$, we modify Φ away from $\mathrm{Im}\, x = 0$ to obtain a strictly plurisubharmonic function Φ which coincides with Φ_1 further away from $\mathrm{Im}\, x = 0$, in such a way that Φ remains an $(\epsilon + h/\epsilon)$-perturbation of Φ_1 and is still a function independent of $\mathrm{Re}\, x_1$. We then define the global IR-manifold $\widetilde{\Lambda} = \kappa_T^{-1}(\Lambda_\Phi)$.

Associated to $\widetilde{\kappa}$, there is a Fourier integral operator U^{-1} introduced in (4.20),

$$U^{-1} = \mathcal{O}(1) : L^2(\mathbf{T}^2) \to H(\widetilde{\Lambda}),$$

such that the action of $\widetilde{P}_\epsilon$ on $H(\widetilde{\Lambda})$ is microlocally near $\xi = 0$ unitarily equivalent to the operator

$$U\widetilde{P}_\epsilon U^{-1} : L^2(\mathbf{T}^2) \to L^2(\mathbf{T}^2),$$

whose Weyl symbol has the form

$$p(\xi_1) + \epsilon\left(r_0\left(\xi, \epsilon, \frac{h}{\epsilon}\right) + hr_1\left(x_2, \xi, \epsilon, \frac{h}{\epsilon}\right) + \cdots\right). \tag{6.17}$$

Here

$$r_0 = i\langle q\rangle(\xi) + \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right)$$

is independent of x, and

$$r_j = \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right), \quad j \geq 1.$$

The corresponding statement is also true when considering the action on $L^2_\theta(\mathbf{T}^2)$, since U^{-1} preserves the Floquet property (6.9).

Associated to the IR-deformation $\widetilde{\Lambda}$ on the torus side, there is an IR-manifold $\widehat{\Lambda}_\epsilon \subset \mathbf{C}^4$ which is an $(\epsilon + h/\epsilon)$-perturbation of $\Lambda_{\epsilon G}$ near $\Lambda_{0,0}$, obtained by replacing $\exp(i\epsilon H_G) \circ \kappa(T^*\mathbf{T}^2)$ there by

$$\exp(i\epsilon H_G) \circ \kappa \circ \widetilde{\kappa}(T^*\mathbf{T}^2) = \exp(i\epsilon H_G) \circ \kappa(\widetilde{\Lambda}).$$

In such a way we get a globally defined IR-manifold $\widehat{\Lambda}_\epsilon$, which is $(\epsilon + h/\epsilon)$-close to $\Lambda_{\epsilon G}$ and agrees with $\mathbf{R}^4$ outside a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$. Associated with $\widehat{\Lambda}_\epsilon$ we then have a Hilbert space $H(\widehat{\Lambda}_\epsilon)$, defined similarly to $H(\Lambda_{\epsilon G})$, and obtained by modifying the standard weight $\Phi_0(x)$ on the FBI-Bargmann transform side. We also get a corresponding new Lagrangian torus $\widehat{\Lambda}_{0,0} \subset \widehat{\Lambda}_\epsilon$, with the property that microlocally near $\widehat{\Lambda}_{0,0}$, the original operator

$$P_\epsilon : H(\widehat{\Lambda}_\epsilon) \to H(\widehat{\Lambda}_\epsilon)$$

is equivalent to an operator on $L^2_\theta(\mathbf{T}^2)$, whose complete symbol has the form (6.17). Taking into account the conjugation by an elliptic operator $e^{iB_1/h}$ on the torus side, which was constructed in Section 4 and which eliminates the x_2-dependence also in the terms r_j with $j \geq 1$, we get the following result.

Proposition 6.1 *We make all the assumptions of case* I *in the introduction, and recall that we also take $F_0 = 0$. Assume that $\epsilon = \mathcal{O}(h^\delta)$, $\delta > 0$ is such that $h/\epsilon \leq \delta_0$, $0 < \delta_0 \ll 1$. There exists an IR-manifold $\widehat{\Lambda}_\epsilon \subset \mathbf{C}^4$, and a smooth Lagrangian torus $\widehat{\Lambda}_{0,0} \subset \widehat{\Lambda}_\epsilon$, such that when $\rho \in \widehat{\Lambda}_\epsilon$ is away from a small neighborhood of $\widehat{\Lambda}_{0,0}$ in $\widehat{\Lambda}_\epsilon$, we have*

$$|\mathrm{Re}\, P_\epsilon(\rho, h)| \geq \frac{1}{|\mathcal{O}(1)|} \quad \textit{or} \quad |\mathrm{Im}\, P_\epsilon(\rho, h)| \geq \frac{\epsilon}{|\mathcal{O}(1)|}. \tag{6.18}$$

The manifold $\widehat{\Lambda}_\epsilon$ is an $\left(\epsilon + \frac{h}{\epsilon}\right)$-perturbation of $\mathbf{R}^4$ in the natural sense, and it is equal to $\mathbf{R}^4$ outside a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$. We have

$$P_\epsilon = \mathcal{O}(1) : H(\widehat{\Lambda}_\epsilon) \to H(\widehat{\Lambda}_\epsilon).$$

There exists a smooth canonical transformation

$$\kappa_\epsilon : \mathrm{neigh}\,(\widehat{\Lambda}_{0,0}, \widehat{\Lambda}_\epsilon) \to \mathrm{neigh}\,(\xi = 0, T^*\mathbf{T}^2),$$

such that $\kappa_\epsilon(\widehat{\Lambda}_{0,0}) = \mathbf{T}^2 \times \{0\}$. Associated to κ_ϵ, there is a Fourier integral operator

$$U = \mathcal{O}(1) : H(\widehat{\Lambda}_\epsilon) \to L^2_\theta(\mathbf{T}^2),$$

which has the following properties:

1) *U is concentrated to the graph of κ_ϵ in the sense that if $\chi_1 \in C_0^\infty(\widehat{\Lambda}_\epsilon)$, $\chi_2 \in C_0^\infty(T^*\mathbf{T}^2)$, are such that*

$$(\mathrm{supp}\chi_2 \times \mathrm{supp}\chi_1) \cap \overline{\{(\kappa_\epsilon(y,\eta), y, \eta); (y,\eta) \in \mathrm{neigh}(\widehat{\Lambda}_{0,0}, \widehat{\Lambda}_\epsilon)\}} = \varnothing,$$

then

$$\chi_2(x, hD_x) \circ U \circ \chi_1(x, hD_x) = \mathcal{O}(h^\infty) : H(\widehat{\Lambda}_\epsilon) \to L^2_\theta(\mathbf{T}^2).$$

2) *The operator* U *is microlocally invertible: there exists an operator* $V = \mathcal{O}(1) : L^2_\theta(\mathbf{T}^2) \to H(\widehat{\Lambda}_\epsilon)$ *such that for every* $\chi_1 \in C_0^\infty(\mathrm{neigh}(\widehat{\Lambda}_{0,0}, \widehat{\Lambda}_\epsilon))$, *we have*

$$(VU - 1)\,\chi_1(x, hD_x) = \mathcal{O}(h^\infty) : H(\widehat{\Lambda}_\epsilon) \to H(\widehat{\Lambda}_\epsilon).$$

For every $\chi_2 \in C_0^\infty(\mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2))$, *we have*

$$(UV - 1)\,\chi_2(x, hD_x) = \mathcal{O}(h^\infty) : L^2_\theta(\mathbf{T}^2) \to L^2_\theta(\mathbf{T}^2).$$

3) *We have Egorov's theorem: Acting on* $L^2_\theta(\mathbf{T}^2)$, *there exists* $\widehat{P}\left(hD_x, \epsilon, \frac{h}{\epsilon}; h\right)$ *with the symbol*

$$\widehat{P}\left(\xi, \epsilon, \frac{h}{\epsilon}; h\right) \sim p(\xi_1) + \epsilon \sum_{j=0}^{\infty} h^j r_j\left(\xi, \epsilon, \frac{h}{\epsilon}\right), \quad |\xi| \leq \frac{1}{|\mathcal{O}(1)|},$$

with

$$r_0 = i\langle q\rangle(\xi) + \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right),$$

and

$$r_j = \mathcal{O}_j\left(\epsilon + \frac{h}{\epsilon}\right), \quad j \geq 1,$$

such that $\widehat{P}U = UP_\epsilon$ *microlocally, i.e.,*

$$\left(\widehat{P}U - UP_\epsilon\right)\chi_1(x, hD_x) = \mathcal{O}(h^\infty), \quad \chi_2(x, hD_x)\left(\widehat{P}U - UP_\epsilon\right) = \mathcal{O}(h^\infty),$$

for every χ_1, χ_2 *as in* 2).

Remark. The estimate (6.18) holds true thanks to the property (6.16) of the final deformation, since then the term $p(\xi_1)$ does not contribute to the imaginary part of the symbol on the torus side. The bound (6.18) will allow us to reduce the spectral analysis of P_ϵ to a small neighborhood of the Lagrangian torus $\widehat{\Lambda}_{0,0}$.

Using Proposition 6.1, we shall now proceed to describe the spectrum of P_ϵ in a rectangle of the form

$$R_{C,\epsilon} = \left\{z \in \mathbf{C};\ |\mathrm{Re}\, z| < \frac{1}{C}, \quad |\mathrm{Im}\, z| < \frac{\epsilon}{C}\right\}, \tag{6.19}$$

for a sufficiently large constant $C > 0$. We shall show that the eigenvalues in (6.19) are given by the quasi-eigenvalues of Proposition 4.1, modulo $\mathcal{O}(h^\infty)$. In doing so, let us consider the set of the quasi-eigenvalues, introduced in (4.35),

$$\Sigma(\epsilon, h) = \left\{\widehat{P}\left(h(k - \theta), \epsilon, \frac{h}{\epsilon}; h\right); k \in \mathbf{Z}^2\right\} \bigcap R_{C,\epsilon}, \quad \theta = \frac{S}{2\pi h} + \frac{k_0}{4}.$$

Then the distance between 2 elements of $\Sigma(\epsilon,h)$ corresponding to $k,l\in\mathbf{Z}^2$, $k\neq l$, is $\geq \epsilon h\,|k-l|\,/|\mathcal{O}(1)|$. Introduce

$$\delta := 1/4\inf_{k\neq l}\mathrm{dist}\,(\widehat{P}(h(k-\theta),\epsilon,\frac{h}{\epsilon};h),\widehat{P}(h(l-\theta),\epsilon,\frac{h}{\epsilon};h))>0,$$

and consider the family of open discs

$$\Omega_k(h):=\left\{z\in R_{C,\epsilon};\left|z-\widehat{P}(h(k-\theta),\epsilon,\frac{h}{\epsilon};h)\right|<\delta\right\},\quad k\in\mathbf{Z}^2.$$

The sets $\Omega_k(h)$ are then disjoint, and dist $(\Omega_k(h),\Omega_l(h))\geq \epsilon h\,|k-l|\,/|\mathcal{O}(1)|$. As a warm-up exercise, we shall first show that Spec (P_ϵ) in the set (6.19) is contained in the union of the $\Omega_k(h)$.

When $z\in\mathbf{C}$ is in the rectangle (6.19), let us consider the equation

$$(P_\epsilon - z)\,u=v,\quad u\in H(\widehat{\Lambda}_\epsilon). \tag{6.20}$$

We notice here that the symbol of

$$\mathrm{Im}\,P_\epsilon=\frac{P_\epsilon-P_\epsilon^*}{2i},$$

taken in the operator sense in $H(\widehat{\Lambda}_\epsilon)$, is $\mathcal{O}(\epsilon)$, and from Proposition 6.1 we know that away from any fixed neighborhood of $\widehat{\Lambda}_{0,0}$ in $\widehat{\Lambda}_\epsilon$ it is true that $|\mathrm{Im}\,P_\epsilon(\rho,h)|>\epsilon/C$, provided that $|\mathrm{Re}\,P_\epsilon(\rho,h)|\leq 1/C$, where $C>0$ is sufficiently large. Here we are using the same letters for the operators and the corresponding (Weyl) symbols, and

$$\mathrm{Re}\,P_\epsilon=\frac{P_\epsilon+P_\epsilon^*}{2}:H(\widehat{\Lambda}_\epsilon)\to H(\widehat{\Lambda}_\epsilon).$$

We shall also write p to denote the leading symbol of $P_{\epsilon=0}$, acting on $H(\widehat{\Lambda}_\epsilon)$.

Let us introduce a smooth partition of unity on the manifold $\widehat{\Lambda}_\epsilon$,

$$1=\chi+\psi_{1,+}+\psi_{1,-}+\psi_{2,+}+\psi_{2,-}.$$

Here $\chi\in C_0^\infty(\widehat{\Lambda}_\epsilon)$ is such that $\chi=1$ near $\widehat{\Lambda}_{0,0}$, and supp χ is contained in a small neighborhood of $\widehat{\Lambda}_{0,0}$ where $UP_\epsilon=\widehat{P}U$. The functions $\psi_{1,\pm}\in C_0^\infty(\widehat{\Lambda}_\epsilon)$ are supported in regions, invariant under the H_p-flow, where $\pm\mathrm{Im}\,P_\epsilon>\epsilon/C$, respectively. Finally $\psi_{2,\pm}\in C_{\mathrm{b}}^\infty(\widehat{\Lambda}_\epsilon)$ are such that

$$\mathrm{supp}\psi_{2,\pm}\subset\left\{\rho;\pm\mathrm{Re}\,P_\epsilon(\rho,h)>1/C\right\}.$$

Moreover, we arrange so that the functions $\psi_{1,\pm}$ Poisson commute with p on $\widehat{\Lambda}_\epsilon$. We shall prove that

$$||\,(1-\chi)u\,||\leq\mathcal{O}\left(\frac{1}{\epsilon}\right)||\,v\,||+\mathcal{O}(h^\infty)||\,u\,||, \tag{6.21}$$

where we let $||\cdot||$ stand for the norm in $H(\widehat{\Lambda}_\epsilon)$. In doing so, we shall first derive a priori estimates for $\psi_{1,+}u$.

When $N \in \mathbf{N}$, let

$$\psi_0 \prec \psi_1 \prec \cdots \prec \psi_N, \quad \psi_0 := \psi_{1,+},$$

be cutoff functions in $C_0^\infty(\widehat{\Lambda}_\epsilon; [0,1])$, supported in an H_p-flow invariant region where $\operatorname{Im} P_\epsilon \sim \epsilon$, and which are in involution with p. Here standard notation $f \prec g$ means that $\operatorname{supp} f$ is contained in the interior of the set where $g = 1$. It is then true that in the operator norm,

$$[P_\epsilon, \psi_j] = [P_{\epsilon=0}, \psi_j] + \mathcal{O}(\epsilon h) = \mathcal{O}(h^2) + \mathcal{O}(\epsilon h) = \mathcal{O}(\epsilon h), \quad 0 \leq j \leq N, \tag{6.22}$$

since $\epsilon \geq h$. For future reference we notice that in the case when the subprincipal symbol of $P_{\epsilon=0}$ vanishes, the Weyl calculus shows that $[P_{\epsilon=0}, \psi_j] = \mathcal{O}(h^3)$, and since $\epsilon \geq h^2$, we still get (6.22). Here we have also used that the subprincipal symbol of ψ_j is 0, $0 \leq j \leq N$.

Near the support of ψ_j it is true that $\operatorname{Im} P_\epsilon \sim \epsilon$, and an application of the semiclassical Gårding inequality allows us therefore to conclude that

$$(\operatorname{Im}(P_\epsilon - z)\psi_j u|\psi_j u) \geq \frac{\epsilon}{\mathcal{O}(1)} || \psi_j u ||^2 - \mathcal{O}(h^\infty) || u ||^2.$$

Here the inner product is taken in $H(\widehat{\Lambda}_\epsilon)$. On the other hand, we have

$$(\operatorname{Im}(P_\epsilon - z)\psi_j u|\psi_j u) = \operatorname{Im}\Big((\psi_j (P_\epsilon - z)u|\psi_j u) + ([P_\epsilon, \psi_j]u|\psi_j u) \Big),$$

and since in the operator sense $\psi_j(1 - \psi_{j+1}) = \mathcal{O}(h^\infty)$, we see that the absolute value of this expression does not exceed

$$\mathcal{O}(1) || (P_\epsilon - z)u || \, || \psi_j u || + \mathcal{O}(\epsilon h) || \psi_{j+1} u ||^2 + \mathcal{O}(h^\infty) || u ||^2.$$

We get

$$\begin{aligned} &\frac{\epsilon}{C} || \psi_j u ||^2 \leq \mathcal{O}(1) || (P_\epsilon - z)u || \, || \psi_j u || + \mathcal{O}(\epsilon h) || \psi_{j+1} u ||^2 + \mathcal{O}(h^\infty) || u ||^2 \\ \leq \quad &\frac{\epsilon}{2C} || \psi_j u ||^2 + \frac{\mathcal{O}(1)}{\epsilon} || (P_\epsilon - z)u ||^2 + \mathcal{O}(\epsilon h) || \psi_{j+1} u ||^2 + \mathcal{O}(h^\infty) || u ||^2, \end{aligned}$$

and hence,

$$|| \psi_j u ||^2 \leq \frac{\mathcal{O}(1)}{\epsilon^2} || (P_\epsilon - z)u ||^2 + \mathcal{O}(h) || \psi_{j+1} u ||^2 + \mathcal{O}(h^\infty) || u ||^2.$$

Combining these estimates for $j = 0, 1, \ldots, N$, we get

$$|| \psi_0 u ||^2 \leq \frac{\mathcal{O}(1)}{\epsilon^2} || (P_\epsilon - z)u ||^2 + \mathcal{O}_N(1) h^N || \psi_N u ||^2 + \mathcal{O}(h^\infty) || u ||^2,$$

and therefore

$$\| \psi_{1,+}u \| \leq \frac{\mathcal{O}(1)}{\epsilon} \| v \| + \mathcal{O}(h^\infty) \| u \|.$$

The same estimate can be obtained for $\psi_{1,-}u$, microlocally concentrated in a flow invariant region where $\operatorname{Im} P_\epsilon \sim -\epsilon$, and a fortiori such estimates also hold in regions where $\operatorname{Re} P_\epsilon \sim 1$ and $\operatorname{Re} P_\epsilon \sim -1$. The bound (6.21) follows.

Write next

$$(P_\epsilon - z)\, \chi u = \chi v + w, \ \ w = [P_\epsilon, \chi]u, \tag{6.23}$$

where w satisfies

$$\| w \| \leq \mathcal{O}\left(\frac{h}{\epsilon}\right) \| v \| + \mathcal{O}(h^\infty) \| u \|.$$

Here we have used (6.21) with a cutoff closer to $\widehat{\Lambda}_{0,0}$. Applying the operator U of Proposition 6.1 to (6.23), we get

$$\left(\widehat{P} - z\right) U\chi u = U\chi v + Uw + T_\infty u,$$

where

$$T_\infty = \mathcal{O}(h^\infty) : H(\widehat{\Lambda}_\epsilon) \to L^2_\theta(\mathbf{T}^2).$$

Using an expansion in Fourier series (6.25) below, we see that the operator $\widehat{P} - z : L^2_\theta(\mathbf{T}^2) \to L^2_\theta(\mathbf{T}^2)$ is invertible, microlocally in $|\xi| \leq 1/|\mathcal{O}(1)|$, with a microlocal inverse of the norm $\mathcal{O}(1/\epsilon h)$, provided that $z \in R_{C,\epsilon}$ avoids the discs $\Omega_k(h)$. Using also the uniform boundedness of the microlocal inverse V of U, we get

$$\| \chi u \| \leq \frac{\mathcal{O}(1)}{\epsilon h} \| v \| + \mathcal{O}(h^\infty) \| u \|. \tag{6.24}$$

Combining (6.21) and (6.24), we see that when $z \in R_{C,\epsilon}$ is in the complement of the union of the $\Omega_k(h)$, the operator

$$P_\epsilon - z : H(\widehat{\Lambda}_\epsilon) \to H(\widehat{\Lambda}_\epsilon)$$

is injective. Since the ellipticity assumption (1.6) implies that it is a Fredholm operator of index zero, we know that $P_\epsilon - z : H(\widehat{\Lambda}_\epsilon) \to H(\widehat{\Lambda}_\epsilon)$ is bijective.

We shall now let z vary in the disc $\Omega_k(h) \subset R_{C,\epsilon}$, for some $k \in \mathbf{Z}^2$. We shall show that $z \in \Omega_k(h)$ is an eigenvalue of P_ϵ if and only if $z = \widehat{P}(h(k-\theta), \epsilon, \frac{h}{\epsilon}; h) + r$, where $r = \mathcal{O}(h^\infty)$. In doing so, we shall study a globally well-posed Grushin problem for the operator $P_\epsilon - z$ in the space $H(\widehat{\Lambda}_\epsilon)$.

As a preparation for that, we shall introduce an auxiliary Grushin problem for the operator $\widehat{P} - z$, defined microlocally near $\xi = 0$ in $T^*\mathbf{T}^2$. From (4.34), let us recall the functions

$$e_l(x) = \frac{1}{2\pi} e^{i(l-\theta)x} = \frac{1}{2\pi} e^{\frac{i}{h}\left(h\left(l - \frac{k_0}{4}\right) - \frac{S}{2\pi}\right)x},$$

which form an ON basis for the space $L^2_\theta(\mathbf{T}^2)$, so that when $u \in L^2_\theta(\mathbf{T}^2)$, we have a Fourier series expansion,

$$u(x) = \sum_{l\in\mathbf{Z}^2} \widehat{u}(l-\theta) e_l(x). \tag{6.25}$$

We also remark that $e_l(x)$ are microlocally concentrated to the region of the phase space where $\xi \sim h\left(l - \frac{k_0}{4}\right) - S/2\pi$.

Introduce rank one operators $\widehat{R}_+ : L^2_\theta(\mathbf{T}^2) \to \mathbf{C}$ and $\widehat{R}_- : \mathbf{C} \to L^2_\theta(\mathbf{T}^2)$, given by $\widehat{R}_+ u = (u|e_k)$ and $\widehat{R}_- u_- = u_- e_k$. Here the inner product in the definition of $\widehat{R}_+$ is taken in the space $L^2_\theta(\mathbf{T}^2)$. Using (6.25), it is then easy to see that the operator

$$\widehat{\mathcal{P}} := \begin{pmatrix} \widehat{P} - z & \widehat{R}_- \\ \widehat{R}_+ & 0 \end{pmatrix} : L^2_\theta(\mathbf{T}^2) \times \mathbf{C} \to L^2_\theta(\mathbf{T}^2) \times \mathbf{C}, \tag{6.26}$$

defined microlocally near $\xi = 0$, has a microlocal inverse there, which has the form

$$\widehat{\mathcal{E}} = \begin{pmatrix} \widehat{E}(z) & \widehat{E}_+ \\ \widehat{E}_- & \widehat{E}_{-+}(z) \end{pmatrix}. \tag{6.27}$$

The following localization properties can be inferred from the construction of $\widehat{\mathcal{E}}$: if $\psi \in C_b^\infty(T^*\mathbf{T}^2)$ has its support disjoint from $\xi = 0$, then it is true that $\psi \widehat{E}_+ = \mathcal{O}(h^\infty) : \mathbf{C} \to L^2_\theta$, and $\widehat{E}_- \psi = \mathcal{O}(h^\infty) : L^2_\theta \to \mathbf{C}$. We also find that

$$\widehat{E}_{-+}(z) = z - \widehat{P}\left(h(k-\theta), \epsilon, \frac{h}{\epsilon}; h\right). \tag{6.28}$$

Using (6.25), we furthermore see that the following estimates hold true,

$$\widehat{E} = \frac{\mathcal{O}(1)}{\epsilon h} : L^2_\theta(\mathbf{T}^2) \to L^2_\theta(\mathbf{T}^2),$$

$$\widehat{E}_+ = \mathcal{O}(1) : \mathbf{C} \to L^2_\theta(\mathbf{T}^2), \quad \widehat{E}_- = \mathcal{O}(1) : L^2_\theta(\mathbf{T}^2) \to \mathbf{C},$$

$$\widehat{E}_{-+} = \mathcal{O}(\epsilon h) : \mathbf{C} \to \mathbf{C},$$

so that

$$\epsilon h \| u \| + \| u_- \| \le \mathcal{O}(1) \left(\| v \| + \epsilon h \| v_+ \| \right), \tag{6.29}$$

when

$$\widehat{\mathcal{P}} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}.$$

In (6.29), the norms of u and v are taken in $L^2_\theta(\mathbf{T}^2)$ and those of u_- and v_+ in $\mathbf{C}$.

Passing to the case of P_ϵ, we define $R_+ : H(\widehat{\Lambda}_\epsilon) \to \mathbf{C}$ and $R_- : \mathbf{C} \to H(\widehat{\Lambda}_\epsilon)$ by

$$R_+ u = \widehat{R}_+ U \chi u = (U\chi u|e_k), \quad R_- u_- = V \widehat{R}_- u_- = u_- V e_k. \tag{6.30}$$

It is then true that

$$\chi R_- = R_- + \mathcal{O}(h^\infty) : \mathbf{C} \to H(\widehat{\Lambda_\epsilon}), \tag{6.31}$$

decreasing the support of χ if necessary. We now claim that for $z \in \Omega_k(h)$, the Grushin problem

$$\begin{cases} (P_\epsilon - z)\, u + R_- u_- = v, \\ R_+ u = v_+ \end{cases} \tag{6.32}$$

has a unique solution $(u, u_-) \in H(\widehat{\Lambda}_\epsilon) \times \mathbf{C}$ for every $(v, v_+) \in H(\widehat{\Lambda}_\epsilon) \times \mathbf{C}$, with an a priori estimate,

$$\epsilon h \|\, u \,\| + \|\, u_- \,\| \leq \mathcal{O}(1) \left(\|\, v \,\| + \epsilon h \|\, v_+ \,\| \right). \tag{6.33}$$

Here the norms of u and v are taken in $H(\widehat{\Lambda}_\epsilon)$, and those of u_- and v_+ in $\mathbf{C}$. To verify the claim, we first see that as in (6.21), we have

$$\|\, (1-\chi) u \,\| \leq \mathcal{O}\left(\frac{1}{\epsilon}\right) \|\, v \,\| + \mathcal{O}(h^\infty) \left(\|\, u \,\| + \|\, u_- \,\| \right). \tag{6.34}$$

Here we have also used (6.31).

Applying χ to the first equation in (6.32) we get

$$\begin{cases} (P_\epsilon - z)\, \chi u + R_- u_- = \chi v + w + R_{-\infty} u_-, \\ R_+ u = v_+, \end{cases} \tag{6.35}$$

where $w = [P_\epsilon, \chi] u$ satisfies

$$\|\, w \,\| \leq \mathcal{O}\left(\frac{h}{\epsilon}\right) \|\, v \,\| + \mathcal{O}(h^\infty) \left(\|\, u \,\| + \|\, u_- \,\| \right),$$

and $R_{-\infty} = \mathcal{O}(h^\infty)$ in the operator norm. Applying U to the first equation in (6.35) and using (6.30), we get

$$\begin{cases} (\widehat{P} - z) U \chi u + \widehat{R}_- u_- = U \chi v + U w + w_- \\ \widehat{R}_+ U \chi u = v_+. \end{cases} \tag{6.36}$$

where the $L^2_\theta(\mathbf{T}^2)$-norm of w_- is $\mathcal{O}(h^\infty) \left(\|\, u \,\| + \|\, u_- \,\| \right)$. We therefore get a microlocally well-posed Grushin problem for $\widehat{\mathcal{P}}$ in (6.26), and in view of (6.29) we obtain,

$$\epsilon h \|\, \chi u \,\| + \|\, u_- \,\| \leq \mathcal{O}(1) \left(\|\, v \,\| + \epsilon h \|\, v_+ \,\| \right) + \mathcal{O}(h^\infty) \left(\|\, u \,\| + \|\, u_- \,\| \right). \tag{6.37}$$

Combining (6.34) and (6.37), we get (6.33). We have thus also proved that the operator

$$\mathcal{P} = \begin{pmatrix} P_\epsilon - z & R_- \\ R_+ & 0 \end{pmatrix} : H(\widehat{\Lambda}_\epsilon) \times \mathbf{C} \to H(\widehat{\Lambda}_\epsilon) \times \mathbf{C} \tag{6.38}$$

is injective, for $z \in \Omega_k(h)$. Now $\mathcal{P}$ is a finite rank perturbation of

$$\begin{pmatrix} P_\epsilon - z & 0 \\ 0 & 0 \end{pmatrix},$$

which is a Fredholm operator of index zero. It follows that $\mathcal{P}$ is also Fredholm of index 0 and hence bijective, since we already know that it is injective. The inverse of $\mathcal{P}$ has the form

$$\mathcal{E} = \begin{pmatrix} E(z) & E_+ \\ E_- & E_{-+}(z) \end{pmatrix}, \tag{6.39}$$

and we recall that the spectrum of P_ϵ in $\Omega_k(h)$ will be the set of values z for which $E_{-+}(z) = 0$.

We finally claim that the components E_+ and $E_{-+}(z)$ in (6.39) are given by $E_+ = V\widehat{E}_+$, and $E_{-+}(z) = \widehat{E}_{-+}(z) = z - \widehat{P}\left(h(k-\theta), \epsilon, \frac{h}{\epsilon}; h\right)$, modulo terms that are $\mathcal{O}(h^\infty)$. Indeed, we need only to check that

$$R_+ V\widehat{E}_+ \equiv 1, \quad (P_\epsilon - z)V\widehat{E}_+ + R_-\widehat{E}_{-+} \equiv 0, \tag{6.40}$$

modulo $\mathcal{O}(h^\infty)$, and at this stage the verification of (6.40) is identical to the corresponding computation from Section 6 of [20]. In particular, we get

$$E_{-+}(z) = z - \widehat{P}\left(h(k-\theta), \epsilon, \frac{h}{\epsilon}; h\right) + \mathcal{O}(h^\infty), \tag{6.41}$$

and we have now proved the first of our two main results.

Theorem 6.2 *Let F_0 be a regular value of $\mathrm{Re}\,\langle q\rangle$ viewed as a function on $p^{-1}(0) \cap \mathbf{R}^4$. Assume that the Lagrangian manifold*

$$\Lambda_{0,F_0} : p = 0, \mathrm{Re}\,\langle q\rangle = F_0$$

is connected, and that $T(0)$ is the minimal period of every closed H_p-trajectory in Λ_{0,F_0}. When α_1 and α_2 are the fundamental cycles in Λ_{0,F_0} with α_1 corresponding to a closed H_p-trajectory of minimal period, we write $S = (S_1, S_2)$ and $k_0 = (k_0(\alpha_1), k_0(\alpha_2))$ for the actions and Maslov indices of the cycles, respectively. Assume furthermore that $\epsilon = \mathcal{O}(h^\delta)$, $\delta > 0$, is such that $h/\epsilon \ll 1$. Let $C > 0$ be sufficiently large. Then the eigenvalues of P_ϵ in the rectangle

$$|\mathrm{Re}\, z| < \frac{1}{C}, \quad |\mathrm{Im}\, z - \epsilon F_0| < \frac{\epsilon}{C} \tag{6.42}$$

are given by

$$z_k = \widehat{P}\left(h\left(k - \frac{k_0}{4}\right) - \frac{S}{2\pi}, \epsilon, \frac{h}{\epsilon}; h\right), \quad k \in \mathbf{Z}^2,$$

modulo $\mathcal{O}(h^\infty)$. Here $\widehat{P}\left(\xi, \epsilon, \frac{h}{\epsilon}; h\right)$ is holomorphic in $\xi \in \mathrm{neigh}(0, \mathbf{C}^2)$, smooth in $\epsilon, \frac{h}{\epsilon} \in \mathrm{neigh}(0, \mathbf{R})$ and has an asymptotic expansion in the space of such functions,

$$\widehat{P}\left(\xi, \epsilon, \frac{h}{\epsilon}; h\right) \sim p(\xi_1) + \epsilon\left(r_0\left(\xi, \epsilon, \frac{h}{\epsilon}\right) + hr_1\left(\xi, \epsilon, \frac{h}{\epsilon}\right) + \cdots\right), \quad h \to 0,$$

with

$$r_0 = i\langle q\rangle + \mathcal{O}(\epsilon + h/\epsilon),$$

and $r_\nu = \mathcal{O}(\epsilon + h/\epsilon)$, $\nu \geq 1$. *We have exactly one eigenvalue for each* $k \in \mathbf{Z}^2$ *such that the corresponding* z_k *falls into the region* (6.42).

Keeping all the general assumptions of the torus case and still taking $F_0 = 0$, we shall next consider the case when the subprincipal symbol of the unperturbed operator $P_{\epsilon=0}$ vanishes. It follows then from the previous arguments, now making use of the full strength of Theorem 2.4, that in this case, microlocally near $\Lambda_{0,0}$,

$$P_\epsilon : H(\Lambda_{\epsilon G}) \to H(\Lambda_{\epsilon G}) \tag{6.43}$$

is equivalent to an operator of the form

$$\widetilde{P}_\epsilon(x_2, \xi, \epsilon; h) \sim \sum_{\nu=0}^{\infty} h^\nu \widetilde{p}_\nu(x_2, \xi, \epsilon), \tag{6.44}$$

acting on $L^2_\theta(\mathbf{T}^2)$, with

$$\widetilde{p}_0(x_2, \xi, \epsilon) = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2), \quad \widetilde{p}_1(x_2, \xi, \epsilon) = \epsilon q_1(x_2, \xi, \epsilon).$$

In what follows we shall discuss the range

$$Mh^2 < \epsilon = \mathcal{O}(h^\delta) \quad M \gg 1,\ \delta > 0. \tag{6.45}$$

Recalling the operators $e^{\epsilon G(x, hD_x, \epsilon)/h}$ and U from (6.4) and (6.10), respectively, we see, as in the general case, that the symbol of $\operatorname{Im} P_\epsilon$ on $H(\Lambda_{\epsilon G})$ is $\mathcal{O}(\epsilon)$, and away from any fixed neighborhood of $\Lambda_{0,0}$ in $\Lambda_{\epsilon G}$, we have $|\operatorname{Im} P_\epsilon(\rho, h)| \sim \epsilon$, if $|\operatorname{Re} P_\epsilon(\rho, h)| < 1/|\mathcal{O}(1)|$.

We write, as in Section 4,

$$\widetilde{P}(x_2, \xi, \epsilon, h) = p(\xi_1) + \epsilon\left(r_0\left(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}\right) + hr_1\left(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}\right) + \cdots\right),$$

where

$$r_0\left(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}\right) = i\langle q\rangle + \mathcal{O}(\epsilon) + \frac{h^2}{\epsilon}\widetilde{p}_2(x_2, \xi, \epsilon),$$

$$r_1(x_2, \xi, \epsilon) = q_1(x_2, \xi, \epsilon) + \frac{h^2}{\epsilon}\widetilde{p}_3(x_2, \xi, \epsilon), \quad r_j(x_2, \xi, \epsilon) = \mathcal{O}\left(\frac{h^2}{\epsilon}\right),\ j \geq 2.$$

Using the canonical transformation κ, generated by the function

$$\psi\left(x, \eta, \epsilon, \frac{h^2}{\epsilon}\right) = x \cdot \eta + \phi_{\mathrm{per}}\left(x_2, \eta, \epsilon, \frac{h^2}{\epsilon}\right),$$

with $\phi_{\rm per} = \mathcal{O}(\epsilon + \frac{h^2}{\epsilon})$, constructed in Section 4, we then argue similarly to the general torus case. We thus introduce an IR-manifold $\widetilde{\Lambda} \subset \widetilde{\mathbf{T}^2} \times \mathbf{C}^2$ which is an $(\epsilon + h^2/\epsilon)$-perturbation of $T^*\mathbf{T}^2$, which agrees with $\kappa(T^*\mathbf{T}^2)$ near $\xi = 0$, and further away from this set coincides with $T^*\mathbf{T}^2$. When constructing $\widetilde{\Lambda}$, we first notice that $\kappa(T^*\mathbf{T}^2)$ has the form

$$\operatorname{Im} x = G'_\xi(\operatorname{Re}(x,\xi)), \quad \operatorname{Im} \xi = -G'_x(\operatorname{Re}(x,\xi)),$$

where $G = G(x_2, \xi, \epsilon, \frac{h^2}{\epsilon})$ is such that

$$\partial_\xi G,\, \partial_{x_2} G = \mathcal{O}\left(\epsilon + \frac{h^2}{\epsilon}\right).$$

As was observed in Section 4, the transformation κ conserves actions, and therefore the smooth function G is single-valued. We may assume that

$$G = \mathcal{O}\left(\epsilon + \frac{h^2}{\epsilon}\right).$$

If we let $\chi(\xi) \in C_0^\infty(\mathbf{R}^2; [0,1])$ be a cutoff function with a small support and such that $\chi = 1$ in a small neighborhood of 0, we define $\widetilde{\Lambda}$ by

$$\operatorname{Im} x = \widetilde{G}'_\xi(\operatorname{Re}(x,\xi)),\ \operatorname{Im} \xi = -\widetilde{G}'_x(\operatorname{Re}(x,\xi)),\ \widetilde{G}(\operatorname{Re}(x,\xi)) = \chi(\operatorname{Re}\xi) G(\operatorname{Re}(x,\xi)).$$

We then obtain the desired globally defined IR-manifold $\widetilde{\Lambda}$ such that $\operatorname{Im} \xi_1 = 0$ on $\widetilde{\Lambda}$. When acting on $H(\widetilde{\Lambda})$, $\widetilde{P_\epsilon}$ is microlocally near $\xi = 0$ unitarily equivalent to an operator on $L^2(\mathbf{T}^2)$, which has the form

$$p(\xi_1) + \epsilon\left(r_0\left(\xi, \epsilon, \frac{h^2}{\epsilon}\right) + h r_1\left(x_2, \xi, \epsilon, \frac{h^2}{\epsilon}\right) + \cdots\right),$$

where

$$r_0\left(\xi, \epsilon, \frac{h^2}{\epsilon}\right) = i\langle q\rangle + \mathcal{O}\left(\epsilon + \frac{h^2}{\epsilon}\right)$$

is independent of x.

It follows, as in the general torus case, that on the Bargmann transform side, $\widetilde{\Lambda}$ can be described by an FBI-weight $\Phi = \Phi(\operatorname{Im} x_1, x_2)$ which does not depend on $\operatorname{Re} x_1$. Repeating the previous arguments, we obtain therefore a new globally defined Hilbert space $H(\widehat{\Lambda})$, associated to an IR-manifold $\widehat{\Lambda} \subset \mathbf{C}^4$, and a Lagrangian torus $\widehat{\Lambda}_{0,0} \subset \widehat{\Lambda}$ such that microlocally near $\widehat{\Lambda}_{0,0}$, $P_\epsilon : H(\widehat{\Lambda}) \to H(\widehat{\Lambda})$ is equivalent to an operator on $L^2_\theta(\mathbf{T}^2)$, described in (4.27), (4.28).

Proposition 6.3 *Assume that the subprincipal symbol of $P_{\epsilon=0}$ vanishes, and consider the range $Mh^2 < \epsilon = \mathcal{O}(h^\delta)$ for $M \gg 1$, $\delta > 0$. There exists an IR-manifold $\widehat{\Lambda} \subset \mathbf{C}^4$ and a smooth Lagrangian torus $\widehat{\Lambda}_{0,0} \subset \widehat{\Lambda}$ such that when $\rho \in \widehat{\Lambda}$ is away* *from a small neighborhood of* $\widehat{\Lambda}_{0,0}$ *in* $\widehat{\Lambda}$ *and* $|\mathrm{Re}\, P_\epsilon(\rho, h)| < 1/C$, *for a sufficiently large* $C > 0$, *it is true that*

$$|\mathrm{Im}\, P_\epsilon(\rho, h)| \sim \epsilon.$$

The manifold $\widehat{\Lambda}$ *is* $(\epsilon + h^2/\epsilon)$*-close to* $\mathbf{R}^4$ *and it coincides with* $\mathbf{R}^4$ *outside a neighborhood of* $p^{-1}(0) \cap \mathbf{R}^4$. *There exists a canonical transformation*

$$\kappa_\epsilon : \mathrm{neigh}(\widehat{\Lambda}_{0,0}, \widehat{\Lambda}) \to \mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2),$$

mapping $\widehat{\Lambda}_{0,0}$ *onto* $\mathbf{T}^2$, *and an elliptic Fourier integral operator* $U : H(\widehat{\Lambda}) \to L^2_\theta(\mathbf{T}^2)$ *associated to* κ_ϵ, *such that, microlocally near* $\widehat{\Lambda}_{0,0}$, $UP_\epsilon = \widehat{P}U$. *Here*

$$\widehat{P} = \widehat{P}(hD_x, \epsilon, \frac{h^2}{\epsilon}; h)$$

has the Weyl symbol, depending smoothly on ϵ, $h^2/\epsilon \in \mathrm{neigh}(0, \mathbf{R})$,

$$\widehat{P}\left(\xi, \epsilon, \frac{h^2}{\epsilon}; h\right) \sim p(\xi_1) + \epsilon \sum_{j=0}^{\infty} h^j r_j \left(\xi, \epsilon, \frac{h^2}{\epsilon}\right).$$

We have

$$r_0 = i\langle q\rangle(\xi) + \mathcal{O}(1)(\epsilon + h^2/\epsilon), \quad r_j = \mathcal{O}(1),\ j \geq 1.$$

Repeating the arguments, leading to Theorem 6.2, and using Proposition 6.3 instead of Proposition 6.1, we then find first that the spectrum of P_ϵ in a region of the form (6.19) is contained in the union of disjoint discs of radii $\epsilon h/|\mathcal{O}(1)|$ around the quasi-eigenvalues $\widehat{P}\left(h(k - \theta), \epsilon, h^2/\epsilon; h\right)$. Furthermore, when z varies in such a disc corresponding to $k \in \mathbf{Z}^2$, such that the corresponding quasi-eigenvalue falls into the region (6.19), an inspection of the previous arguments shows that the Grushin operator

$$\begin{pmatrix} P_\epsilon - z & R_- \\ R_+ & 0 \end{pmatrix} : H(\widehat{\Lambda}) \times \mathbf{C} \to H(\widehat{\Lambda}) \times \mathbf{C}$$

is bijective with the inverse of the norm $\mathcal{O}((\epsilon h)^{-1})$ – see (6.33) for the precise a priori estimate. Here $R_- : \mathbf{C} \to H(\widehat{\Lambda})$ and $R_+ : H(\widehat{\Lambda}) \to \mathbf{C}$ are defined as in (6.30). This leads to the following result.

Theorem 6.4 *Keep all the assumptions and notation of Theorem* 6.2, *and in addition assume that the subprincipal symbol of* $P_{\epsilon=0}$ *vanishes. Let* $\epsilon = \mathcal{O}(1)h^\delta$ *for some fixed* $\delta > 0$ *be such that* $h^2 \ll \epsilon$. *Then the eigenvalues of* P_ϵ *in the rectangle*

$$\left(-\frac{1}{C}, \frac{1}{C}\right) + i\epsilon\left(F_0 - \frac{1}{C}, F_0 + \frac{1}{C}\right)$$

are given by

$$z_k = \widehat{P}\left(h\left(k - \frac{k_0}{4}\right) - \frac{S}{2\pi}, \epsilon, \frac{h^2}{\epsilon}; h\right) + \mathcal{O}(h^\infty),\ k \in \mathbf{Z}^2.$$

Here $C > 0$ is large enough, $\widehat{P}(\xi, \epsilon, h^2/\epsilon; h)$ is holomorphic in $\xi \in \mathrm{neigh}(0, \mathbf{C}^2)$, smooth in ϵ and $h^2/\epsilon \in \mathrm{neigh}(0, \mathbf{R})$, and as $h \to 0$, there is an asymptotic expansion

$$\widehat{P}\left(\xi, \epsilon, \frac{h^2}{\epsilon}; h\right) \sim p(\xi_1) + \epsilon\left(r_0\left(\xi, \epsilon, \frac{h^2}{\epsilon}\right) + hr_1\left(\xi, \epsilon, \frac{h^2}{\epsilon}\right) + \cdots\right).$$

We have

$$r_0\left(\xi, \epsilon, \frac{h^2}{\epsilon}\right) = i\langle q\rangle(\xi) + \mathcal{O}\left(\epsilon + \frac{h^2}{\epsilon}\right), \quad r_j\left(\xi, \epsilon, \frac{h^2}{\epsilon}\right) = \mathcal{O}(1),\ j \geq 1.$$

We shall now turn to the case II from the introduction. Let us recall from Section 1, that if $z \in \mathrm{Spec}\, P_\epsilon$ is such that $|\mathrm{Re}\, z| \leq \delta \to 0$, then

$$\mathrm{Im}\, z \in \epsilon\big[\inf_\Sigma \mathrm{Re}\, \langle q\rangle - o(1), \sup_\Sigma \mathrm{Re}\, \langle q\rangle + o(1)\big],\ h \to 0. \tag{6.46}$$

Here, as in Section 1, we write $\Sigma = p^{-1}(0) \cap \mathbf{R}^4/\exp(\mathbf{R}H_p)$. Our purpose here is to show that the quasi-eigenvalues of Propositions 5.4 and 5.5 give, up to $\mathcal{O}(h^\infty)$, the actual eigenvalues in a set of the form

$$|\mathrm{Re}\, z| \leq \frac{1}{|\mathcal{O}(1)|}, \quad |\mathrm{Im}\, z - \epsilon F_0| \leq \frac{\epsilon}{|\mathcal{O}(1)|},$$

when $F_0 \in \{\inf_\Sigma \mathrm{Re}\, \langle q\rangle, \sup_\Sigma \mathrm{Re}\, \langle q\rangle\}$. As we shall see, the analysis here will be parallel to the torus case just treated, so that in what follows we shall concentrate on the new features of the problem, and some of the computations that are essentially identical to the ones already performed, will not be repeated.

In order to fix the ideas, we shall discuss the case when

$$F_0 = \inf_\Sigma \mathrm{Re}\, \langle q\rangle,$$

and we shall take $F_0 = 0$.

Recall from the beginning of this section that the original operator P_ϵ acting on $H(\Lambda_{\epsilon G})$, is microlocally unitarily equivalent to the operator

$$P_\epsilon \sim \sum_{j=0}^\infty h^j p_j(x, \xi, \epsilon), \tag{6.47}$$

acting on L^2 and defined microlocally near $p^{-1}(0) \cap \mathbf{R}^4$, with

$$p_0 = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2),$$

and the functions $\langle q\rangle$ and $\mathcal{O}(\epsilon^2)$-term are in involution with p. Let $\gamma_1,\dots,\gamma_N\subset p^{-1}(0)\cap\mathbf{R}^4$ be the finitely many trajectories such that $\operatorname{Re}\langle q\rangle=0$ along γ_j, $1\le j\le N$. We know that $T(0)$ is the minimal period of each γ_j, and if $\rho_j\in\Sigma$ is the corresponding point, then the Hessian of $\operatorname{Re}\langle q\rangle$ at ρ_j is positive definite, $1\le j\le N$. Associated to γ_j, we have the quantities $S=S(\gamma_j)$ and $k_0=k_0(\gamma_j)$, the action along γ_j and the Maslov index, respectively, defined as in Section 2, and we recall from [11] that these quantities do not depend on j.

In what follows we shall work microlocally near a fixed critical trajectory, say γ_1. We let $L^2_S(S^1\times\mathbf{R})$ be the space of locally square integrable functions $u(t,x)$ on $\mathbf{R}\times\mathbf{R}$ such that

$$\iint_0^{2\pi}|u(t,x)|^2\,dx\,dt<\infty.$$

and

$$u(t-2\pi,x)=e^{iS/h+ik_0\pi/2}u(t,x).$$

Applying Theorem 2.4 to the canonical transformation κ of Proposition 3.1, we see that there exists an analytic microlocally unitary Fourier integral operator

$$U_0:L^2_S(S^1\times\mathbf{R})\to L^2(\mathbf{R}^2),$$

associated to κ, and defined microlocally from a neighborhood of $\{\tau=x=\xi=0\}$ in $T^*\left(S^1\times\mathbf{R}\right)$ to a neighborhood of γ_1 in $\mathbf{R}^4$, so that we have the two-term Egorov property (2.3). Combining $\exp(i\epsilon H_G)$ with κ, we get a smooth canonical transformation

$$\kappa_\epsilon:\operatorname{neigh}\left(\tau=x=\xi=0,T^*\left(S^1\times\mathbf{R}\right)\right)\to\operatorname{neigh}(\gamma_1,\Lambda_{\epsilon G}),\tag{6.48}$$

where abusing the notation slightly, we write here $\gamma_1\subset\Lambda_{\epsilon G}$ also for the image of γ_1 under the complex canonical transformation $\exp(i\epsilon H_G)$. The operator $e^{\epsilon G(x,hD_x,\epsilon)/h}\circ U_0$ is then associated with κ_ϵ in (6.48), and an application of Egorov's theorem shows that, microlocally near γ_1, we get a unitary equivalence between the operator P_ϵ acting on $H(\Lambda_{\epsilon G})$ and an h-pseudodifferential operator microlocally defined near $\tau=x=\xi=0$ in $T^*\left(S^1\times\mathbf{R}\right)$, with the leading symbol

$$\widetilde{p}_0(\tau,x,\xi,\epsilon)=f(\tau)+i\epsilon\langle q\rangle(\tau,x,\xi)+\mathcal{O}(\epsilon^2),$$

independent of t. Taking into account an additional conjugation by the elliptic operator $e^{iA/h}$, acting on $L^2_S(S^1\times\mathbf{R})$, with

$$A\sim\sum_{k=1}^\infty a_k(t,\tau,x,\xi,\epsilon)h^k,$$

constructed as a formal power series in ϵ, h in Proposition 3.2, we see that microlocally near γ_1, the operator $P_\epsilon:H(\Lambda_{\epsilon G})\to H(\Lambda_{\epsilon G})$ is equivalent to an operator of the form

$$\widetilde{P}_\epsilon(\tau,x,\xi,\epsilon)\sim\sum_{k=0}^\infty h^k\widetilde{p}_k(\tau,x,\xi,\epsilon),\tag{6.49}$$

acting on $L^2_S(S^1 \times \mathbf{R})$, whose full symbol is independent of t. We have

$$\widetilde{p}_0 = f(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2), \tag{6.50}$$

and

$$\mathrm{Re}\,\langle q\rangle(0, x, \xi) \sim x^2 + \xi^2$$

on the real domain.

We shall first consider the general case when the subprincipal symbol of the unperturbed operator $P_{\epsilon=0}$ does not necessarily vanish, and in doing so, it will be assumed that

$$h \ll \epsilon = \mathcal{O}(1)h^\delta, \quad \delta > 0. \tag{6.51}$$

As in Section 5, we write

$$\widetilde{P}_\epsilon = f(\tau) + \epsilon\left(i\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon) + \frac{h}{\epsilon}\widetilde{p}_1 + h\frac{h}{\epsilon}\widetilde{p}_2 + \cdots\right).$$

According to Proposition 5.3, there exists a holomorphic canonical transformation

$$\kappa_{\sigma,\epsilon,\frac{h}{\epsilon}} : \mathrm{neigh}(0, \mathbf{C}^2) \to \mathrm{neigh}(0, \mathbf{C}^2),$$

depending analytically on $\sigma \in \mathrm{neigh}(0, \mathbf{C})$ and smoothly on $\epsilon, \frac{h}{\epsilon} \in \mathrm{neigh}(0, \mathbf{R})$, such that

$$\mathrm{Im}\,\kappa_{\sigma,\epsilon,\frac{h}{\epsilon}}(y, \eta) = \mathcal{O}\left(\epsilon + \frac{h}{\epsilon}\right),$$

when σ, y, η are real, and such that

$$\left(\widetilde{p}_0 + \epsilon\frac{h}{\epsilon}\widetilde{p}_1\right)\left(\sigma, \kappa_{\sigma,\epsilon,\frac{h}{\epsilon}}(y, \eta)\right) = f(\sigma) + i\epsilon g_{\epsilon,\frac{h}{\epsilon}}\left(\sigma, \frac{y^2 + \eta^2}{2}\right).$$

Here $g_{\epsilon,\frac{h}{\epsilon}}(\sigma, q)$ is an analytic function, depending smoothly on $\epsilon, h/\epsilon$, for which

$$\frac{\partial}{\partial q}\mathrm{Re}\,g_{\epsilon,0}(0, 0) > 0.$$

We now lift the family of locally defined canonical transformations $\kappa_{\sigma,\epsilon,\frac{h}{\epsilon}}$ to a canonical transformation

$$\begin{aligned} &\Xi_{\epsilon,\frac{h}{\epsilon}} : \mathrm{neigh}\left(\mathrm{Im}\, s = 0, \sigma = y = \eta = 0, T^*\left(\widetilde{S^1} \times \mathbf{C}\right)\right) \ni (s, \sigma; y, \eta) \\ \mapsto \quad &(t, \tau; x, \xi) \in \mathrm{neigh}\left(\mathrm{Im}\, t = 0, \tau = x = \xi = 0, T^*\left(\widetilde{S^1} \times \mathbf{C}\right)\right) \end{aligned}$$

given by

$$\Xi_{\epsilon,\frac{h}{\epsilon}}(s, \sigma; y, \eta) = (t, \tau; x, \xi) = (s + h(y, \sigma, \eta), \sigma; \kappa_{\sigma,\epsilon,\frac{h}{\epsilon}}(y, \eta)). \tag{6.52}$$

Here $h(y,\sigma,\eta)$ is uniquely determined up to a function $g = g(\sigma)$, and if $\varphi_{\sigma,\epsilon,\frac{h}{\epsilon}}(x,y,\theta)$ is an analytic family of non-degenerate phase functions (in the sense of Hörmander) locally generating the family $\kappa_{\sigma,\epsilon,\frac{h}{\epsilon}}$, then

$$\Phi_{\epsilon,\frac{h}{\epsilon}}(t,x,s,y,\theta,\sigma) := \varphi_{\sigma,\epsilon,\frac{h}{\epsilon}}(x,y,\theta) + (t-s)\sigma$$

is a non-degenerate phase function with θ,σ as fiber variables, such that $\Phi_{\epsilon,\frac{h}{\epsilon}}$ generates the graph of $\Xi_{\epsilon,\frac{h}{\epsilon}}$.

Associated to $\Xi_{\epsilon,\frac{h}{\epsilon}}$, we introduce an IR-manifold $\widetilde{\Lambda} \subset T^*\left(\widetilde{S^1} \times \mathbf{C}\right)$, which in a complex neighborhood of $\tau = x = \xi = 0$, is equal to $\Xi_{\epsilon,\frac{h}{\epsilon}}\left(T^*\left(S^1 \times \mathbf{R}\right)\right)$, and further away from this set agrees with $T^*\left(S^1 \times \mathbf{R}\right)$. In the intermediate region, we construct $\widetilde{\Lambda}$ in such a way that it remains an $(\epsilon + \frac{h}{\epsilon})$-perturbation of $T^*\left(S^1 \times \mathbf{R}\right)$, and so that everywhere on $\widetilde{\Lambda}$, it is true that

$$(t,\tau;x,\xi) \in \widetilde{\Lambda} \Longrightarrow \tau \in \mathbf{R}. \tag{6.53}$$

If we now use the standard FBI-Bargmann transformation, viewed as a mapping on $L^2_S(S^1 \times \mathbf{R})$, so that under the associated canonical transformation, $T^*(S^1 \times \mathbf{R})$ is mapped to $\{(t,\tau;x,\xi) \in T^*(\widetilde{S^1} \times \mathbf{C}); (\tau,\xi) = -\mathrm{Im}\,(t,x)\}$, then as before we see that after an application of such a transformation, the manifold $\widetilde{\Lambda}$ is described by a weight function $\Phi = \Phi(\mathrm{Im}\,t,x)$ which does not depend on $\mathrm{Re}\,t$. At this stage, the situation is similar to the previously analyzed torus case, and, in particular, we see again that the form of the weight $\Phi(\mathrm{Im}\,t,x)$ implies that the term $f(\tau)$ in (6.50) gives no contribution to the imaginary part of the operator. Summing up the discussion so far, we arrive to the following result.

Proposition 6.5 *Make the assumptions of case* II *in the introduction, and assume that*

$$F_0 = \inf_{\Sigma} \mathrm{Re}\,\langle q\rangle = 0.$$

Assume that $\epsilon = \mathcal{O}(h^\delta)$, *for some* $\delta > 0$, *is such that* $h \ll \epsilon$. *There exists a closed IR-manifold* $\Lambda \subset \mathbf{C}^4$ *and finitely many simple closed disjoint curves* $\gamma_1, \ldots, \gamma_N \subset \Lambda$, *which are* $(\epsilon + h/\epsilon)$*-close to the closed* H_p*-trajectories* $\subset p^{-1}(0) \cap \mathbf{R}^4$, *along which* $\mathrm{Re}\,\langle q\rangle = 0$, *such that when* ρ *is outside a small neighborhood of* $\cup_{j=1}^N \gamma_j$ *in* Λ, *then*

$$|\mathrm{Re}\,P_\epsilon(\rho,h)| \geq \frac{1}{|\mathcal{O}(1)|} \quad \text{or} \quad |\mathrm{Im}\,P_\epsilon(\rho,h)| \geq \frac{\epsilon}{|\mathcal{O}(1)|}. \tag{6.54}$$

This estimate is true away from an arbitrarily small neighborhood of $\cup_{j=1}^N \gamma_j$, *provided that the implicit constant in* (6.54) *is chosen sufficiently large. The manifold* Λ *coincides with* $\mathbf{R}^4$ *away from a neighborhood of* $p^{-1}(0) \cap \mathbf{R}^4$ *and is* $(\epsilon + h/\epsilon)$*-close to* $\mathbf{R}^4$ *everywhere. For each* j *with* $1 \leq j \leq N$, *there exists a canonical transformation*

$$\kappa_{\epsilon,j} : \mathrm{neigh}\,(\gamma_j, \Lambda) \to \mathrm{neigh}\left(\tau = x = \xi = 0, T^*(S^1 \times \mathbf{R})\right),$$

whose domain of definition does not intersect the closure of the union of the domains of the $\kappa_{\epsilon,k}$ for $k \neq j$, and an elliptic Fourier integral operator

$$U_j = \mathcal{O}(1) : H(\Lambda) \to L^2_S(S^1 \times \mathbf{R}),$$

associated to $\kappa_{\epsilon,j}$, such that, microlocally near γ_j,

$$U_j P_\epsilon = \widehat{P}_j U_j.$$

Here $\widehat{P}_j = \widehat{P}_j(hD_t, (1/2)(x^2 + (hD_x)^2), \epsilon, \frac{h}{\epsilon}; h)$ has the Weyl symbol

$$\widehat{P}_j\left(\tau, x, \xi, \epsilon, \frac{h}{\epsilon}; h\right) = f(\tau) + i\epsilon G_j\left(\tau, \frac{x^2+\xi^2}{2}, \epsilon, \frac{h}{\epsilon}; h\right),$$

with

$$G_j\left(\tau, q, \epsilon, \frac{h}{\epsilon}; h\right) \sim \sum_{l=0}^{\infty} h^l G_{j,l}\left(\tau, q, \epsilon, \frac{h}{\epsilon}\right), \quad h \to 0,$$

and $G_{j,l}$ holomorphic in $(\tau, q) \in \mathrm{neigh}(0, \mathbf{C}^2)$, smooth in $\epsilon, h/\epsilon \in \mathrm{neigh}(0, \mathbf{R})$. Furthermore, $\mathrm{Re}\, G_{j,0}(0,0,0,0) = 0$ and

$$\frac{\partial}{\partial q} \mathrm{Re}\, G_{j,0}(0,0,0,0) > 0.$$

Take now small open sets $\Omega_j \subset \Lambda$, $1 \leq j \leq N$, such that $\gamma_j \subset \Omega_j$ and

$$\overline{\Omega_j} \cap \overline{\Omega_k} = \emptyset, \quad j \neq k.$$

Let $\chi_j \in C_0^\infty(\Omega_j)$, $0 \leq \chi_j \leq 1$, be such that $\chi_j = 1$ near γ_j, $1 \leq j \leq N$. When $z \in \mathbf{C}$ satisfies

$$|\mathrm{Re}\, z| \leq \frac{1}{C}, \quad |\mathrm{Im}\, z| \leq \frac{\epsilon}{C}, \tag{6.55}$$

and $(P_\epsilon - z)u = v$, it follows from (6.54) by repeating the arguments of the torus case, that

$$||\left(1 - \sum_{j=1}^{N} \chi_j\right) u\,|| \leq \mathcal{O}\left(\frac{1}{\epsilon}\right) ||\, v\,|| + \mathcal{O}(h^\infty) ||\, u\,||. \tag{6.56}$$

We shall now discuss the setup of the global Grushin problem. Associated with each normal form $\widehat{P_j}$, $1 \leq j \leq N$, we have the quasi-eigenvalues given in Proposition 5.4,

$$z(j,k) := f\left(h(k_1 - \frac{k_0}{4}) - \frac{S}{2\pi}\right) + i\epsilon G_j\left(h(k_1 - \frac{k_0}{4}) - \frac{S}{2\pi}, h\left(k_2 + \frac{1}{2}\right), \epsilon, \frac{h}{\epsilon}; h\right),$$

when $1 \leq j \leq N$ and $k = (k_1, k_2) \in \mathbf{Z} \times \mathbf{N}$. We also introduce an ON system of eigenfunctions of the (formally) commuting operators $\widehat{P}_j$,

$$e_k(t,x) = \frac{1}{\sqrt{2\pi}} e^{\frac{i}{h}\left(h(k_1 - \frac{k_0}{4}) - \frac{S}{2\pi}\right)t} e_{k_2}(x), \quad k = (k_1, k_2) \in \mathbf{Z} \times \mathbf{N},$$

which forms an ON basis in $L^2_S(S^1 \times \mathbf{R})$. Here $e_{k_2}(x)$, $k_2 \in \mathbf{N}$, are the normalized eigenfunctions of $1/2(x^2 + (hD_x)^2)$ with eigenvalues $(k_2 + 1/2)h$.

When $1 \leq j \leq N$, let

$$M_j = \# \left\{ z(j,k), |\mathrm{Re}\, z(j,k)| < \frac{1}{|\mathcal{O}(1)|}, |\mathrm{Im}\, z(j,k)| < \frac{\epsilon}{|\mathcal{O}(1)|} \right\}.$$

Then $M_j = \mathcal{O}(h^{-2})$ and we let $k(j,1), \ldots, k(j, M_j) \in \mathbf{Z} \times \mathbf{N}$ be the corresponding half-lattice points. We introduce the auxiliary operator

$$R_+ : H(\Lambda) \to \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N},$$

given by

$$R_+ u(j)(l) = (U_j \chi_j u | e_{k(j,l)}), \quad 1 \leq j \leq N,\ 1 \leq l \leq M_j.$$

Here the inner product in the right-hand side is taken in $L^2_S(S^1 \times \mathbf{R})$. Define also

$$R_- : \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N} \to H(\Lambda),$$

by

$$R_- u_- = \sum_{j=1}^{N} \sum_{l=1}^{M_j} u_-(j)(l) V_j e_{k(j,l)}.$$

Here V_j is a microlocal inverse of U_j. We then claim that for $z \in \mathbf{C}$ satisfying (6.55), with a sufficiently large $C > 0$, the Grushin operator

$$\mathcal{P} = \begin{pmatrix} P_\epsilon - z & R_- \\ R_+ & 0 \end{pmatrix} : H(\Lambda) \times \left(\mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N} \right) \to H(\Lambda) \times \left(\mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N} \right) \quad (6.57)$$

is bijective. Indeed, when $v \in H(\Lambda)$ and $v_+ \in \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N}$, let us consider

$$\begin{cases} (P_\epsilon - z)u + R_- u_- = v, \\ R_+ u = v_+. \end{cases} \quad (6.58)$$

As in (6.56), we get

$$\| \left(1 - \sum_{j=1}^{N} \chi_j \right) u \| \leq \mathcal{O}\left(\frac{1}{\epsilon} \right) \| v \| + \mathcal{O}(h^\infty) \left(\| u \| + \| u_- \| \right).$$

Applying χ_j and then U_j, $1 \leq j \leq N$, to the first equation in (6.58), we get

$$\begin{cases} (\widehat{P}_j - z) U_j \chi_j u + \sum_{l=1}^{M_j} u_-(j)(l) e_{k(j,l)} = \\ \qquad U_j \left(\chi_j v + [P_\epsilon, \chi_j] u \right) + R_\infty u + R_{-,\infty}(j) u_-, \\ (U_j \chi_j u | e_{k(j,l)}) = v_+(j)(l), \quad 1 \leq l \leq M_j, \end{cases} \quad (6.59)$$

and here $R_\infty = R_\infty(j) = \mathcal{O}(h^\infty)$ and $R_{-,\infty}(j) = \mathcal{O}(h^\infty)$ in the corresponding operator norms. For each j, $1 \leq j \leq N$, we get a microlocally well-posed Grushin problem for $\widehat{P}_j - z$ in $L^2_S(S^1 \times \mathbf{R})$, with inverse of the norm $\mathcal{O}(1/\epsilon)$, and the global well-posedness of (6.58) follows. The inverse $\mathcal{E}$ of $\mathcal{P}$ in (6.57) has the form

$$\mathcal{E} = \begin{pmatrix} E(z) & E_+ \\ E_- & E_{-+}(z) \end{pmatrix}, \tag{6.60}$$

and a straightforward computation shows that

$$E_+ : \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N} \rightarrow H(\Lambda)$$

modulo $\mathcal{O}(h^\infty)$, is given by

$$E_+ v_+ \equiv \sum_{j=1}^{N} \sum_{l=1}^{M_j} v_+(j)(l) V_j e_{k(j,l)} = R_- v_+,$$

and $E_{-+}(z) \in \mathcal{L}\left(\mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N}, \mathbf{C}^{M_1} \times \cdots \times \mathbf{C}^{M_N}\right)$ is a block diagonal matrix with the blocks $E_{-+}(z)(j) \in \mathcal{L}(\mathbf{C}^{M_j}, \mathbf{C}^{M_j})$, $1 \leq j \leq N$, of the form

$$E_{-+}(z)(j)(m,n) \equiv (z - z(j, k(j,m)))\, \delta_{mn}, \quad 1 \leq m \leq n \leq M_j,$$

modulo $\mathcal{O}(h^\infty)$. The computation of eigenvalues near the boundary of the band has therefore been justified, and we get the second of our two main results.

Theorem 6.6 *Assume that*

$$F_0 = \inf_{\Sigma} \operatorname{Re} \langle q \rangle$$

is achieved along finitely many closed H_p-trajectories $\gamma_1, \ldots, \gamma_N \subset p^{-1}(0) \cap \mathbf{R}^4$ of minimal period $T(0)$, and that the Hessian of $\operatorname{Re}\langle q \rangle$ at the corresponding points $\rho_j \in \Sigma$, $j = 1, \ldots, N$, is positive definite. Let us write S and k_0 to denote the common values of the action and the Maslov index of γ_j, $j = 1, \ldots, N$, respectively. Assume that $\epsilon = \mathcal{O}(h^\delta)$ for a fixed $\delta > 0$, is such that $h \ll \epsilon$. Let $C > 0$ be sufficiently large. Then the eigenvalues of P_ϵ in the set

$$\left(-\frac{1}{C}, \frac{1}{C}\right) + i\epsilon \left(F_0 - \frac{1}{C}, F_0 + \frac{1}{C}\right) \tag{6.61}$$

are given by

$$\begin{aligned} z(j,k) = f &\left(h\left(k_1 - \frac{k_0}{4}\right) - \frac{S}{2\pi}\right) \\ &+ i\epsilon G_j \left(h\left(k_1 - \frac{k_0}{4}\right) - \frac{S}{2\pi}, h\left(\frac{1}{2} + k_2\right), \epsilon, \frac{h}{\epsilon}; h\right), \end{aligned}$$

modulo $\mathcal{O}(h^\infty)$*, when* $1 \le j \le N$ *and* $(k_1, k_2) \in \mathbf{Z}\times\mathbf{N}$*. Here* $f(\tau)$ *is real-valued with* $f(0) = 0$ *and* $f'(0) > 0$*. The function* $G_j(\tau, q, \epsilon, h/\epsilon; h)$*,* $1 \le j \le N$*, is analytic in* τ *and* q *in a neighborhood of* $(0, 0) \in \mathbf{C}^2$*, and smooth in* $\epsilon, h/\epsilon \in \mathrm{neigh}(0, \mathbf{R})$*, and has an asymptotic expansion in the space of such functions, as* $h \to 0$*,*

$$G_j\left(\tau, q, \epsilon, \frac{h}{\epsilon}; h\right) \sim \sum_{l=0}^{\infty} G_{j,l}\left(\tau, q, \epsilon, \frac{h}{\epsilon},\right) h^l.$$

We have $\mathrm{Re}\, G_{j,0}(0, 0, 0, 0) = F_0$ *and*

$$\frac{\partial}{\partial q} \mathrm{Re}\, G_{j,0}(0, 0, 0, 0) > 0, \quad 1 \le j \le N.$$

Remark. With obvious modifications, Theorem 6.6 describes the eigenvalues in the region (6.61), when $F_0 = \sup_\Sigma \mathrm{Re}\,\langle q\rangle$, if we assume that F_0 is attained along finitely many trajectories of minimal period $T(0)$, such that the transversal Hessian of $\mathrm{Re}\,\langle q\rangle$ along the trajectories is negative definite.

The treatment of the remaining case of the eigenvalues near the boundary of the band (6.61), when the subprincipal symbol of $P_{\epsilon=0}$ vanishes proceeds in full analogy with the previously analyzed torus case. Thus, restricting attention to the region

$$Mh^2 < \epsilon = \mathcal{O}(h^\delta), \quad M \gg 1,$$

we find that the symbol of $\mathrm{Im}\, P_\epsilon$, acting on $H(\Lambda_{\epsilon G})$ is $\mathcal{O}(\epsilon)$, and away from an arbitrarily small but fixed neighborhood of $\cup_{j=1}^N \gamma_j$ we have that $|\mathrm{Im}\, P_\epsilon(\rho)| \ge \epsilon/C$ when we restrict the attention to the region $|\mathrm{Re}\, P_\epsilon(\rho)| \le 1/C$.

When working microlocally near $\tau = x = \xi = 0$ in $T^*(S^1 \times \mathbf{R})$ and simplifying the operator (6.49) further, we use Proposition 5.3 to find a holomorphic canonical transformation

$$\kappa_{\sigma,\epsilon,\frac{h^2}{\epsilon}} : \mathrm{neigh}(0, \mathbf{C}^2) \to \mathrm{neigh}(0, \mathbf{C}^2)$$

depending analytically on $\sigma \in \mathrm{neigh}(0, \mathbf{C})$ and smoothly on $\epsilon, h^2/\epsilon \in \mathrm{neigh}(0, \mathbf{R})$, such that

$$\left(\widetilde{p}_0 + \epsilon\frac{h^2}{\epsilon}\widetilde{p}_2\right)\left(\sigma, \kappa_{\sigma,\epsilon,\frac{h^2}{\epsilon}}(y, \eta)\right) = f(\sigma) + i\epsilon g_{\epsilon,\frac{h^2}{\epsilon}}\left(\sigma, \frac{y^2 + \eta^2}{2}\right).$$

As before, associated to $\kappa_{\sigma,\epsilon,\frac{h^2}{\epsilon}}$, we construct an IR-submanifold of $T^*(\widetilde{S^1 \times \mathbf{C}})$ which is $(\epsilon + h^2/\epsilon)$-close to $T^*(S^1 \times \mathbf{R})$, and which has the property that τ is real along this submanifold. This leads to a new IR-manifold $\Lambda \subset \mathbf{C}^4$ such that on Λ, $\mathrm{Im}\, P_\epsilon$ has a symbol of modulus $\sim \epsilon$ in the region $|\mathrm{Re}\, P_\epsilon| < 1/C$, when away from the union of small neighborhoods Ω_j of $\gamma_j \subset \Lambda$, $1 \le j \le N$. In Ω_j, P_ϵ is equivalent to an operator constructed in Section 5, which has the form

$$f(hD_t) + i\epsilon G_j\left(hD_t, \frac{x^2 + (hD_x)^2}{2}, \epsilon, \frac{h^2}{\epsilon}; h\right),$$

with

$$G_j\left(\tau, q, \epsilon, \frac{h^2}{\epsilon}; h\right) \sim \sum_{l=1}^{\infty} G_{j,l}\left(\tau, q, \epsilon, \frac{h^2}{\epsilon}\right) h^l.$$

Again we see that we have a globally well-posed Grushin problem for $P_\epsilon - z$ in the h-dependent Hilbert space $H(\Lambda)$. The following result complements Theorem 6.6.

Theorem 6.7 *Make the assumptions of Theorem* 6.6*, and assume in addition that the subprincipal symbol of* $P_{\epsilon=0}$ *vanishes. Then for* ϵ *in the range*

$$h^2 \ll \epsilon < h^\delta, \quad \delta > 0,$$

the eigenvalues of P_ϵ *in the set of the form*

$$\left(-\frac{1}{C}, \frac{1}{C}\right) + i\epsilon\left(F_0 - \frac{1}{C}, F_0 + \frac{1}{C}\right), \quad C \gg 1,$$

are given by

$$f\left(h\left(k_1 - \frac{k_0}{4}\right) - \frac{S}{2\pi}\right) + i\epsilon G_j\left(h\left(k_1 - \frac{k_0}{4}\right) - \frac{S}{2\pi}, h\left(\frac{1}{2} + k_2\right), \epsilon, \frac{h^2}{\epsilon}; h\right),$$

modulo $\mathcal{O}(h^\infty)$*, when* $1 \le j \le N$ *and* $(k_1, k_2) \in \mathbf{Z}\times\mathbf{N}$*. Here* $f(\tau)$ *is real-valued with* $f(0) = 0$ *and* $f'(0) > 0$*. The function* $G_j(\tau, q, \epsilon, h^2/\epsilon; h)$ *for* $1 \le j \le N$*, is analytic in* τ *and* q *in a neighborhood of* $(0,0) \in \mathbf{C}^2$*, and smooth in* $\epsilon, h^2/\epsilon \in \mathrm{neigh}(0, \mathbf{R})$*, and has an asymptotic expansion in the space of such functions, as* $h \to 0$*,*

$$G_j\left(\tau, q, \epsilon, \frac{h^2}{\epsilon}; h\right) \sim \sum_{l=0}^{\infty} G_{j,l}\left(\tau, q, \epsilon, \frac{h^2}{\epsilon}\right) h^l,$$

where $\mathrm{Re}\, G_{j,0}(0,0,0,0) = F_0$ *and*

$$\frac{\partial}{\partial q}\mathrm{Re}\, G_{j,0}(0,0,0,0) > 0.$$

7 Barrier top resonances in the resonant case

Consider

$$P = -h^2\Delta + V(x), \quad p(x,\xi) = \xi^2 + V(x),\ x, \xi \in \mathbf{R}^2, \tag{7.1}$$

and let us assume that $V(x)$ is real-valued, and that it extends holomorphically to a set $\{x \in \mathbf{C}^2; |\mathrm{Im}\, x| \le \langle \mathrm{Re}\, x\rangle / C\}$, for some $C > 0$, and tends to 0 when $x \to \infty$ in that set. The resonances of P can be defined in an angle $\{z \in \mathbf{C}; -2\theta_0 < \arg z < 0\}$ for some fixed small $\theta_0 > 0$, as the eigenvalues of $P\Big|_{e^{i\theta_0}\mathbf{R}^2}$ in the same region.

We shall assume that $V(0) = E_0 > 0$, $\nabla V(0) = 0$, and $V''(0)$ is a negative definite quadratic form. Assume also that the union of trapped H_p-trajectories in $p^{-1}(E_0) \cap \mathbf{R}^4$ is reduced to $(0,0) \in \mathbf{R}^4$. (We recall that a trapped trajectory is a maximal integral curve of the Hamilton vector field H_p, contained in a bounded set.) We are then interested in resonances of P near E_0, created by the critical point of V. After a linear symplectic change of coordinates, and a conjugation of P by means of the corresponding metaplectic operator, we may assume that as $(x, \xi) \to 0$,

$$p(x,\xi) - E_0 = \sum_{j=1}^{2} \frac{\lambda_j}{2}\left(\xi_j^2 - x_j^2\right) + p_3(x) + p_4(x) + \cdots, \quad \lambda_j > 0. \tag{7.2}$$

Here $p_j(x)$ is a homogeneous polynomial of degree $j \geq 3$.

For future reference we recall that according to the theory of resonances developed in [12], the resonances of P in a fixed h-independent neighborhood of E_0 can also be viewed as the eigenvalues of $P : H(\Lambda_{tG}, 1) \to H(\Lambda_{tG}, 1)$, equipped with the domain $H(\Lambda_{tG}, \langle \xi \rangle^2)$. Here $G \in C^\infty(\mathbf{R}^2; \mathbf{R})$ is an escape function in the sense of [12], $t > 0$ is sufficiently small and fixed, and Λ_{tG} is a suitable IR-deformation of $\mathbf{R}^4$, associated with the function G. The Hilbert space $H(\Lambda_{tG}, 1)$ consists of all tempered distributions u such that a suitable FBI transform Tu belongs to a certain exponentially weighted L^2-space. We refer to [12] for the original presentation of the microlocal theory of resonances, and to [18] for a simplified version of the theory, which is applicable in the present setting of operators with globally analytic coefficients, converging to the Laplacian at infinity. Here we shall only remark that as in [17], the escape function G can be chosen such that $G = x \cdot \xi$ in a neighborhood of $(0,0)$, and such that $H_p G > 0$ on $p^{-1}(E_0) \setminus \{(0,0)\}$.

Under the assumptions above, but without any restriction on the dimension and without any assumption on the signature of $V''(0)$, all resonances in a disc around E_0 of radius Ch were determined in [23]. Here $C > 0$ is arbitrarily large and fixed. (See also [7].) Specializing the result of [23] to the present barrier top case, we may recall that in this disc, the resonances are of the form

$$E_0 - i\left(k_1 + \frac{1}{2}\right)\lambda_1 h - i\left(k_2 + \frac{1}{2}\right)\lambda_2 h + \mathcal{O}(h^{3/2}), \quad h \to 0, \quad k = (k_1, k_2) \in \mathbf{N}^2. \tag{7.3}$$

Furthermore, in the non-resonant case, i.e., when

$$\lambda \cdot k \neq 0, \quad 0 \neq k \in \mathbf{Z}^2, \tag{7.4}$$

a result of Kaidi and Kerdelhué [17] extended [23] to obtain all resonances in a disc around E_0 of radius h^δ, for each fixed $\delta > 0$ and $h > 0$ small enough depending on δ. In this case, the resonances are given by asymptotic expansions in integer powers of h, with the leading term as in (7.3).

Throughout this section we shall work under the following resonant assumption,

$$\lambda \cdot k = 0, \text{ for some } 0 \neq k \in \mathbf{Z}^2. \tag{7.5}$$

In this case we shall show how to obtain a description of all the resonances in an energy shell of the form

$$h^{4/5} \ll |E - E_0| < \mathcal{O}(1)h^{\delta}, \quad \delta > 0,$$

provided that we avoid an arbitrarily small half-cubic neighborhood of $E_0 - i[0, \infty)$. The starting point is a reduction to an eigenvalue problem for a scaled operator, as in [17], [20], [24]. In these works it was shown how to adapt the theory of [12] so that P can be realized as an operator acting on a suitable $H(\Lambda)$-space, where $\Lambda \subset \mathbf{C}^4$ is an IR-manifold which coincides with $T^*\left(e^{i\pi/4}\mathbf{R}^2\right)$ near $(0,0)$, and further away from a neighborhood of this point, it agrees with Λ_{tG}. Furthermore, Λ has the property that on this manifold, $p - E_0$ is elliptic away from a small neighborhood of $(0,0)$, and this neighborhood can be chosen arbitrarily small, provided that the constant in the elliptic estimate is taken sufficiently large. Using a Grushin reduction exactly as in [20], we may and will therefore reduce the study of resonances of P near E_0 to an eigenvalue problem for P after the complex scaling, which near $(0,0)$ is given by $x = e^{i\pi/4}\widetilde{x}$, $\xi = e^{-i\pi/4}\widetilde{\xi}$, $\widetilde{x}, \widetilde{\xi} \in \mathbf{R}$.

Using (7.2) and dropping the tildes from the notation, we see that the principal symbol of the scaled operator has the form

$$E_0 - i\left(p_2(x,\xi) + ie^{3\pi i/4}p_3(x) + ie^{4i\pi/4}p_4(x) + \cdots\right), \quad (x,\xi) \to 0, \tag{7.6}$$

where

$$p_2(x,\xi) = \sum_{j=1}^{2} \frac{\lambda_j}{2}\left(\xi_j^2 + x_j^2\right) \tag{7.7}$$

is the harmonic oscillator. In what follows we shall therefore consider an h-pseudo-differential operator P on $\mathbf{R}^2$, microlocally defined near $(0,0)$, with the leading symbol

$$p(x,\xi) = p_2(x,\xi) + ie^{3\pi i/4}p_3(x) + \cdots, \quad (x,\xi) \to 0, \tag{7.8}$$

and with the vanishing subprincipal symbol. We extend P to be globally defined as a symbol of class $S^0(\mathbf{R}^4) = C_{\mathrm{b}}^{\infty}(\mathbf{R}^4)$, with the asymptotic expansion

$$P(x,\xi;h) \sim p(x,\xi) + h^2 p^{(2)}(x,\xi) + \cdots,$$

in this space, and so that

$$|p(x,\xi)| \geq \frac{1}{C}, \quad C > 0,$$

outside a small neighborhood of $(0,0)$.

We shall be interested in eigenvalues E of P with $|E| \sim \epsilon^2$, $0 < \epsilon \ll 1$. It follows from [26] that the corresponding eigenfunctions are concentrated in a region where $|(x,\xi)| \sim \epsilon$, and so we introduce the change of variables $x = \epsilon y$, $h^\delta \leq \epsilon \leq 1$, $0 < \delta < 1/2$. Then

$$\frac{1}{\epsilon^2}P(x, hD_x; h) = \frac{1}{\epsilon^2}P(\epsilon(y, \widetilde{h}D_y); h), \quad \widetilde{h} = \frac{h}{\epsilon^2} \ll 1.$$

The corresponding new symbol is

$$\frac{1}{\epsilon^2}P(\epsilon(y,\eta); h) \sim \frac{1}{\epsilon^2}p(\epsilon(y,\eta)) + \epsilon^2\widetilde{h}^2 p^{(2)}(\epsilon(y,\eta)) + \cdots,$$

to be considered in the region where $|(y,\eta)| \sim 1$. The leading symbol becomes

$$\frac{1}{\epsilon^2}p(\epsilon(y,\eta)) = p_2(y,\eta) + i\epsilon e^{3\pi i/4}p_3(y) + \mathcal{O}(\epsilon^2),$$

for (y,η) in a fixed neighborhood of $(0,0)$.

Now the resonant assumption (7.5) implies that the H_{p_2}-flow is periodic on $p_2^{-1}(E)$, for $E \in \text{neigh}(1, \mathbf{R})$, with period $T > 0$ which does not depend on E. For $z \in \text{neigh}(1, \mathbf{C})$, we shall then discuss the invertibility of

$$1/\epsilon^2 P(x, hD_x; h) - z$$

in the range of ϵ, dictated by Theorem 6.4, and using $\widetilde{h}$ as the new semiclassical parameter. Indeed, all the assumptions of that theorem are satisfied in a fixed neighborhood of $(0,0)$, and outside such a neighborhood, we have ellipticity which guarantees the invertibility there.

Proposition 7.1 *Assume that* (7.5) *holds. When p_3 is a homogeneous polynomial of degree* 3 *on $\mathbf{R}^2$, we let $\langle p_3\rangle$ denote the average of p_3 along the trajectories of the Hamilton vector field of p_2 in* (7.7)*, and assume that $\langle p_3\rangle$ is not identically zero. Let $F_0 \in \mathbf{R}$ be a regular value of* $\cos(3\pi/4)\langle p_3\rangle$ *restricted to* $p_2^{-1}(1)$*, and assume that T is the minimal period of the H_{p_2}-trajectories in the manifold Λ_{1,F_0} given by*

$$\Lambda_{1,F_0} : p_2 = 1, \cos\left(\frac{3\pi}{4}\right)\langle p_3\rangle = F_0.$$

Assume that Λ_{1,F_0} is connected. Let ϵ satisfy

$$h^{2/5} \ll \epsilon = \mathcal{O}(1)h^\delta, \quad \delta > 0. \tag{7.9}$$

Then for z in the set

$$\left[1 - \frac{1}{|\mathcal{O}(1)|}, 1 + \frac{1}{|\mathcal{O}(1)|}\right] + i\epsilon\left[F_0 - \frac{1}{|\mathcal{O}(1)|}, F_0 + \frac{1}{|\mathcal{O}(1)|}\right], \tag{7.10}$$

the operator $\epsilon^{-2}P(x, hD_x; h) - z : L^2 \to L^2$ *is non-invertible precisely when* $z = z_k$ *for some* $k \in \mathbf{Z}^2$*, where the numbers* z_k *satisfy*

$$z_k = \widehat{P}\left(\widetilde{h}(k - \frac{\alpha}{4}) - \frac{S}{2\pi}, \epsilon, \frac{\widetilde{h}^2}{\epsilon}; \widetilde{h}\right) + \mathcal{O}(h^\infty), \quad \widetilde{h} = \frac{h}{\epsilon^2}.$$

Here $\widehat{P}\left(\xi, \epsilon, \frac{\widetilde{h}^2}{\epsilon}; \widetilde{h}\right)$ *has an expansion as* $\widetilde{h} \to 0$,

$$\widehat{P}\left(\xi, \epsilon, \frac{\widetilde{h}^2}{\epsilon}; \widetilde{h}\right) \sim p_2(\xi_1) + \epsilon \sum_{j=0}^{\infty} \widetilde{h}^j r_j \left(\xi, \epsilon, \frac{\widetilde{h}^2}{\epsilon}\right),$$

where

$$r_0 = ie^{3\pi i/4}\langle p_3\rangle(\xi) + \mathcal{O}\left(\epsilon + \frac{\widetilde{h}^2}{\epsilon}\right).$$

The coordinates $\xi_1 = \xi_1(E)$ *and* $\xi_2 = \xi_2(E, F)$ *are the normalized actions of*

$$\Lambda_{E,F} : p_2 = E, \cos\left(\frac{3\pi}{4}\right)\langle p_3\rangle = F,$$

for $E \in \mathrm{neigh}(1, \mathbf{R})$, $F \in \mathrm{neigh}(F_0, \mathbf{R})$, *given by*

$$\xi_j = \frac{1}{2\pi}\left(\int_{\gamma_j(E,F)} \eta\, dy - \int_{\gamma_j(1,F_0)} \eta\, dy\right), \quad j = 1, 2, \tag{7.11}$$

with $\gamma_j(E, F)$ *being fundamental cycles in* $\Lambda_{E,F}$*, such that* $\gamma_1(E, F)$ *corresponds to a closed* H_{p_2}*-trajectory of minimal period* T*. Furthermore,*

$$S_j = \int_{\gamma_j(1,F_0)} \eta\, dy, \quad j = 1, 2, \;\; S = (S_1, S_2), \tag{7.12}$$

and $\alpha \in \mathbf{Z}^2$ *is fixed.*

Remark. In the case when the compact manifold Λ_{1,F_0} has finitely many connected components Λ_j, $1 \leq j \leq M$, with each Λ_j being diffeomorphic to a torus, the set of z in (7.10) for which the operator $\epsilon^{-2}P(x, hD_x; h) - z$ is non-invertible agrees with the union of the quasi-eigenvalues constructed for each component, up to an error which is $\mathcal{O}(h^\infty)$. In the following discussion, for simplicity it will be tacitly assumed that Λ_{1,F_0} is connected.

The reduction by complex scaling together with the scaling argument above and Proposition 7.1 allows us to describe the resonances E of the operator (7.1) in the set

$$h^{4/5} \ll |E - E_0| = \mathcal{O}(1)h^\delta, \quad \delta > 0, \tag{7.13}$$

by

$$E = E_0 - i\epsilon^2 \widehat{P}\left(\widetilde{h}\left(k - \frac{\alpha}{4}\right) - \frac{S}{2\pi}, \epsilon, \frac{\widetilde{h}^2}{\epsilon}; \widetilde{h}\right) + \mathcal{O}(h^\infty), \tag{7.14}$$

where we choose $\epsilon > 0$ with $|E - E_0|/\epsilon^2 \sim 1$. The description (7.14) is valid provided that we exclude sets of the form

$$E \in \mathbf{C},\ \left|\operatorname{Re} E - E_0 - F_0 |\operatorname{Im} E|^{3/2}\right| < \frac{1}{|\mathcal{O}(1)|} |\operatorname{Im} E|^{3/2}, \tag{7.15}$$

from the domain (7.13). Here F_0 varies over the set of critical values of $\cos(3\pi/4)$ $\langle p_3 \rangle$ restricted to $p_2^{-1}(1)$. Indeed, writing $E = E_0 - i\epsilon^2 z$, we see that the set (7.15) in the E-plane corresponds to the set $|\operatorname{Im} z - \epsilon F_0| < \epsilon/|\mathcal{O}(1)|$ in the z-plane. It is also clear that when $F_0 \in \{\inf_{p_2^{-1}(1)} \cos(3\pi/4)\langle p_3\rangle, \sup_{p_2^{-1}(1)} \cos(3\pi/4)\langle p_3\rangle\}$, an application of Theorem 6.7 will allow us to extend a description of the resonances to a set of the form (7.15), provided that the assumptions of that theorem are satisfied. In what follows, we shall content ourselves by discussing an explicit example.

Our starting point will be deriving an expression for $\langle p_3 \rangle$. Consider

$$p_2(x, \xi) = \sum_{j=1}^{2} \frac{\lambda_j}{2}(x_j + \xi_j^2), \quad \lambda_j > 0,$$

where the λ_j satisfy (7.5). In order to describe the H_{p_2}-flow, it is convenient to introduce the action-angle variables $I_j \geq 0$, $\tau_j \in \mathbf{R}/2\pi\mathbf{Z}$ for p_2, given by

$$x_j = \sqrt{2I_j} \cos \tau_j,\ \xi_j = -\sqrt{2I_j} \sin \tau_j. \tag{7.16}$$

Then $p_2 = \sum \lambda_j I_j$ and the Hamilton flow is given by $\mathbf{R} \ni t \mapsto (I(t), \tau(t))$, with $I(t) = I(0)$, $\tau(t) = \tau(0) + t\lambda$, $\lambda = (\lambda_1, \lambda_2)$. In the original coordinates, this gives

$$\begin{cases} x_j(t) = \sqrt{2I_j(0)} \cos(\tau_j(0) + \lambda_j t) \\ \xi_j(t) = -\sqrt{2I_j(0)} \sin(\tau_j(0) + \lambda_j t), \end{cases} \tag{7.17}$$

and we get a combination of two rotations in (x_j, ξ_j), $j = 1, 2$, with minimal periods $2\pi/\lambda_j$ (except in the degenerate cases when one of the (x_j, ξ_j) vanishes). Avoiding the totally degenerate case when $I = 0$, we get trajectories with

- minimal period $2\pi/\lambda_2$ when $I_1(0) = 0$,
- minimal period $2\pi/\lambda_1$ when $I_2(0) = 0$,
- minimal period $T = -k_2^0 2\pi/\lambda_1 = k_1^0 2\pi/\lambda_2$, when both $I_1(0)$ and $I_2(0)$ are $\neq 0$.

Here we let $k^0 = (k_1^0, k_2^0)$ be the point satisfying (7.5), which has minimal norm and positive first component. The integers k in (7.5) are equally spaced on the straight line $\lambda^\perp$, and it will be convenient to represent them in the form nk^0, $n \in \mathbf{Z} \setminus \{0\}$.

We shall now compute the averages $\langle x^\alpha \rangle$ along the H_{p_2}-trajectories of a monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$. Using (7.17), we get

$$\langle x^\alpha \rangle = I(0)^{\frac{\alpha}{2}} 2^{\frac{|\alpha|}{2}} \frac{1}{T} \int_0^T (\cos(\tau_1(0) + \lambda_1 t))^{\alpha_1} (\cos(\tau_2(0) + \lambda_2 t))^{\alpha_2} dt \tag{7.18}$$

$$= \frac{I(0)^{\frac{\alpha}{2}}}{2^{\frac{|\alpha|}{2}}} \frac{1}{T} \int_0^T (e^{i(\tau_1(0)+\lambda_1 t)} + e^{-i(\tau_1(0)+\lambda_1 t)})^{\alpha_1} (e^{i(\tau_2(0)+\lambda_2 t)} + e^{-i(\tau_2(0)+\lambda_2 t)})^{\alpha_2} dt.$$

Here the integrand can be developed with the binomial theorem,

$$\sum_{k_1=0}^{\alpha_1} \sum_{k_2=0}^{\alpha_2} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} e^{i((2k_1-\alpha_1)\tau_1(0)+(2k_2-\alpha_2)\tau_2(0))} e^{i((2k_1-\alpha_1)\lambda_1+(2k_2-\alpha_2)\lambda_2)t},$$

and only the terms with $(2k_1 - \alpha_1)\lambda_1 + (2k_2 - \alpha_2)\lambda_2 = 0$ can give a non-vanishing contribution to the integral. This means that $2k - \alpha = nk^0$ for some $n \in \mathbf{Z}$, i.e., $\alpha + nk^0 = 2k$ with $0 \le k \le \alpha$ componentwise. We get

$$\langle x^\alpha \rangle = \frac{I(0)^{\alpha/2}}{2^{|\alpha|/2}} \sum_{\substack{\alpha+nk_0=2k \\ 0 \le k \le \alpha}} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} \cos((2k_1 - \alpha_1)\tau_1(0) + (2k_2 - \alpha_2)\tau_2(0)), \tag{7.19}$$

where it is understood that $n \in \mathbf{Z}$, $k \in \mathbf{N}^2$, and where we notice that if $\alpha + nk_0 = 2k$, $0 \le k \le \alpha$, then $\widetilde{k} := \alpha - k$ also participates in the sum, since $0 \le \widetilde{k} \le \alpha$ and $\alpha - nk_0 = 2\widetilde{k}$. Also notice that the cosine in (7.19) can be written in the form $\cos(nk_0 \cdot \tau(0))$. In order to find the non-vanishing terms in (7.19), we consider the "line" $\mathbf{Z} \ni n \mapsto \alpha + nk_0 \in \mathbf{Z}^2$. The points on this line in the rectangle $([0, 2\alpha_1] \times [0, 2\alpha_2]) \cap \mathbf{N}^2$ with even coordinates correspond to the terms in (7.19).

Example 1. Let $k^0 = (1, -1)$, corresponding for instance to $\lambda = (1, 1)$. In this case the two components of α must have the same parity.
For $\alpha = (2, 0)$ we have only one term with $n = 0$, $k = (1, 0)$, and $\langle x_1^2 \rangle = I_1(0)$.
For $\alpha = (0, 2)$ we get similarly $\langle x_2^2 \rangle = I_2(0)$.
For $\alpha = (1, 1)$ we get two terms with $n = 1, k = (1, 0)$ and $n = -1, k = (0, 1)$ respectively, and $\langle x_1 x_2 \rangle = \sqrt{I_1(0) I_2(0)} \cos(\tau_1(0) - \tau_2(0))$.
For $|\alpha| = 3$ we get no non-vanishing terms.
For $\alpha = (4, 0)$ we have one term with $n = 0$, $k = (2, 0)$ and we get $\langle x_1^4 \rangle = \frac{3}{2} I_1(0)^2$.
For $\alpha = (0, 4)$ we get similarly, $\langle x_2^4 \rangle = \frac{3}{2} I_2(0)^2$.
For $\alpha = (2, 2)$ we get one term with $n = 2, k = (2, 0)$ and one with $n = -2, k = (0, 2)$, We also have a term with $n = 0, k = (1, 1)$, and this leads to $\langle x_1^2 x_2^2 \rangle = I_1(0) I_2(0)(1 + \frac{1}{2} \cos 2(\tau_1(0) - \tau_2(0)))$.

It follows from Example 1 that Proposition 7.1 does not apply when $\lambda = \text{Const.} (1,1)$, since in this case $\langle p_3 \rangle \equiv 0$. We shall therefore consider a different choice of the resonant frequencies.

Example 2. Let us take $k^0 = (2,-1)$, corresponding for instance to $\lambda = (1,2)$, and let $|\alpha| = 3$. For $\alpha = (3,0),(0,3),(1,2)$ it follows from (7.19) that $\langle x^\alpha \rangle = 0$. For $\alpha = (2,1)$ we get two terms, one with $n = 1, k = (2,0)$ and one with $n = -1, k = (0,1)$. It follows that

$$\langle x_1^2 x_2 \rangle = 2^{-1/2} I_1(0) I_2(0)^{1/2} \cos(2\tau_1(0) - \tau_2(0)). \tag{7.20}$$

For future reference, we shall also describe how the averages $\langle x^\alpha \rangle$ can be computed after a suitable complex linear change of symplectic coordinates. Introduce

$$\begin{cases} y = \frac{1}{\sqrt{2}}(x - i\xi) \\ \eta = \frac{1}{i\sqrt{2}}(x + i\xi) \end{cases}, \quad \begin{cases} x = \frac{1}{\sqrt{2}}(y + i\eta) \\ \xi = \frac{i}{\sqrt{2}}(y - i\eta) \end{cases}.$$

In these coordinates $p = \sum_{j=1}^2 i\lambda_j y_j \eta_j$, and

$$\exp(tH_p)(y,\eta) = (e^{it\lambda_1} y_1, e^{it\lambda_2} y_2, e^{-it\lambda_1}\eta_1, e^{-it\lambda_2}\eta_2),$$

so that

$$\langle y^\alpha \eta^\beta \rangle = \frac{1}{T}\int_0^T e^{i\lambda\cdot(\alpha-\beta)t} dt y^\alpha \eta^\beta = \begin{cases} y^\alpha \eta^\beta \text{ if } \lambda\cdot(\alpha-\beta) = 0, \\ 0 \text{ otherwise.} \end{cases}$$

We apply this to

$$x^\alpha = \frac{1}{2^{|\alpha|/2}} \sum_{0\le k\le \alpha} \binom{\alpha}{k} y^k (i\eta)^{\alpha-k},$$

and get

$$\langle x^\alpha \rangle = 2^{-|\alpha|} \sum_{\substack{\alpha + nk_0 = 2k \\ 0 \le k \le \alpha}} \binom{\alpha}{k} (x - i\xi)^k (x + i\xi)^{\alpha-k}. \tag{7.21}$$

As before we check that for each term present there is also the complex conjugate.

The computations of Examples 1 and 2 can be written like (7.21). We shall only do it for the last example with $k^0 = (2,-1), \alpha = (2,1)$:

$$\langle x_1^2 x_2 \rangle = \frac{1}{4}\mathrm{Re}\,((x_1 + i\xi_1)^2 (x_2 - i\xi_2)) = \frac{1}{4}(x_1^2 x_2 + 2x_1\xi_1\xi_2 - x_2\xi_1^2). \tag{7.22}$$

We may assume that $\lambda = (1,2)$, so that

$$p_2 = \frac{1}{2}(x_1^2 + \xi_1^2) + (x_2^2 + \xi_2^2), \tag{7.23}$$

and we may then check directly that $H_{p_2}\langle x_1^2 x_2 \rangle = 0$.

From (7.22) and (7.23) it is clear that dp_2 and $d\langle x_1^2x_2\rangle$ are linearly independent except on some set of measure 0. When computing the critical points of $\langle x_1^2x_2\rangle$ on $p_2^{-1}(1)$, we shall first make use of the (I,τ)-coordinates. From (7.20) we recall that

$$p_2 = I_1 + 2I_2,\ \sqrt{2}\langle x_1^2x_2\rangle = I_1 I_2^{\frac{1}{2}}\cos(2\tau_1-\tau_2). \tag{7.24}$$

It follows from the Hamilton equations that $\theta := 2\tau_1-\tau_2$ is invariant under the H_{p_2}-flow, and we can therefore work in the coordinates I_1, I_2, θ. We have

$$dp_2 = dI_1 + 2dI_2, \sqrt{2}d\langle x_1^2x_2\rangle = (I_2^{\frac{1}{2}}\cos\theta)dI_1 + \frac{1}{2}I_1I_2^{-\frac{1}{2}}(\cos\theta)dI_2 - I_1I_2^{\frac{1}{2}}(\sin\theta)d\theta. \tag{7.25}$$

If $\theta\notin\pi\mathbf{Z}$, $I_1, I_2\neq 0$, we have $\partial_\theta\langle x_1^2x_2\rangle\neq 0$, and hence the differentials are linearly independent. Still with $I_1, I_2\neq 0$, let $\theta\in\pi\mathbf{Z}$, so that $\cos\theta=\pm 1$. Then the differentials are linearly dependent iff

$$0=\det\begin{pmatrix} 1 & 2\\ I_2^{1/2} & \frac{1}{2}I_1I_2^{-\frac{1}{2}}\end{pmatrix},\qquad \text{i.e., iff}\qquad I_1 = 4I_2.$$

This gives two closed trajectories inside the energy surface $p_2=1$ and the corresponding values for $\langle x_1^2x_2\rangle$:

$$I_1=\frac{2}{3},\ I_2=\frac{1}{6},\ 2\tau_1-\tau_2=0;\ \ \langle x_1^2x_2\rangle=\frac{1}{3\sqrt{3}}, \tag{7.26}$$

and

$$I_1=\frac{2}{3},\ I_2=\frac{1}{6},\ 2\tau_1-\tau_2=\pi;\ \ \langle x_1^2x_2\rangle=\frac{-1}{3\sqrt{3}}. \tag{7.27}$$

When $I_1=0$ or $I_2=0$, the question of linear independence of the differentials should be analyzed directly in the (x,ξ)-coordinates (or (y,η)-coordinates), and here we shall use (7.22). On the plane $I_1=0$, corresponding to $x_1=\xi_1=0$, we have $d\langle x_1^2x_2\rangle=0$, so here we have linear dependence, with the corresponding critical value $\langle x_1^2x_2\rangle=0$. On the plane $I_2=0$, corresponding to $x_2=\xi_2=0$, we have

$$\begin{cases} d\langle x_1^2x_2\rangle = \frac{1}{4}(x_1^2-\xi_1^2)dx_2+\frac{1}{2}x_1\xi_1 d\xi_2,\\ dp_2 = x_1dx_1+\xi_1d\xi_1,\end{cases}$$

and these differentials are independent, since we avoid the point $x=\xi=0$.

We shall now look at the nature of the critical points of $\langle x_1^2x_2\rangle$, when viewed as a function on $\Sigma := p_2^{-1}(1)/\exp(\mathbf{R}H_p)$. For the trajectories found in (7.26) and (7.27), we use θ and I_2 as local coordinates on Σ, and using (7.24) together with $I_1=1-2I_2$, we get for $\theta=k\pi$, $k=0,1$, $I_2=1/6$ and $f=\sqrt{2}\langle x_1^2x_2\rangle$,

$$\partial_\theta\partial_{I_2}f=0,\ \partial_\theta^2 f = -(1-2I_2)I_2^{\frac{1}{2}}(-1)^k,\ \partial_{I_2}^2 f = -(-1)^k\left(\frac{1}{4}I_2^{-\frac{3}{2}}+\frac{3}{2}I_2^{-\frac{1}{2}}\right).$$

For $k=0$ we therefore have a non-degenerate maximum and for $k=1$ we get a non-degenerate minimum.

For the third trajectory, given by

$$x_1 = \xi_1 = 0, \quad x_2^2 + \xi_2^2 = 1, \tag{7.28}$$

we use that $\langle x_1^2 x_1 \rangle$ vanishes to the second order there, and hence that the transversal Hessian in $p_2^{-1}(1)$ can be identified with the free Hessian with respect to x_1, ξ_1, which is given by the matrix

$$\frac{1}{2}\begin{pmatrix} x_2 & \xi_2 \\ \xi_2 & -x_2 \end{pmatrix}.$$

The eigenvalues are $\frac{1}{2}$ and $-\frac{1}{2}$. Thus we have a non-degenerate saddle point.

We summarize the discussion above in the following proposition.

Proposition 7.2 *Let*

$$p_2(x,\xi) = \frac{1}{2}\left(x_1^2 + \xi_1^2\right) + (x_2^2 + \xi_2^2).$$

Then the H_{p_2}-flow is periodic in $p_2^{-1}(E)$, for $E \in \mathrm{neigh}(1, \mathbf{R})$, with period $T = 2\pi$. If

$$p_3(x) = a_{3,0}x_1^3 + a_{1,2}x_1x_2^2 + x_1^2x_2 + a_{0,3}x_2^3,$$

then we have

$$\langle p_3\rangle(x,\xi) = \frac{1}{4}\left(x_1^2x_2 + 2x_1\xi_1\xi_2 - x_2\xi_1^2\right).$$

The differential of $\langle p_3\rangle$, restricted to $p_2^{-1}(1)$, vanishes along three closed H_{p_2}-trajectories, given by (7.26), (7.27), *and* (7.28). *These critical trajectories are non-degenerate in the sense that the transversal Hessian of $\langle p_3\rangle$ is non-degenerate. The set of the critical values of $\langle p_3\rangle$ is $\{\pm(3\sqrt{3})^{-1}, 0\}$, and the maximum and the minimum of $\langle p_3\rangle$ are attained along the trajectories* (7.26) *and* (7.27)*, respectively. The transversal Hessian of $\langle p_3\rangle$ along* (7.28) *has the signature* $(1,-1)$. *The minimal period of the trajectories in* (7.26) *and* (7.27) *is equal to $T = 2\pi$, and the minimal period in* (7.28) *is π. Let finally F_0 be a regular value of $\langle p_3\rangle$ restricted to $p_2^{-1}(1)$. Then the minimal period of every closed H_{p_2}-trajectory in the Lagrangian manifold*

$$\Lambda_{1,F_0} : p_2 = 1, \langle p_3\rangle = F_0$$

is equal to $T = 2\pi$.

We now return to the operator P with principal symbol p in (7.1). Under the general assumptions from the beginning of this section, we shall assume that as $(x,\xi) \to 0$, we have

$$p(x,\xi) - E_0 = \frac{1}{2}(\xi_1^2 - x_1^2) + (\xi_2^2 - x_2^2) + p_3(x) + \mathcal{O}(x^4),$$

where

$$p_3(x) = a_{3,0}x_1^3 + a_{1,2}x_1x_2^2 + x_1^2x_2 + a_{0,3}x_2^3.$$

Let us write $A_1 = -(3\sqrt{6})^{-1}$, $A_2 = (3\sqrt{6})^{-1}$, and $A_3 = 0$.

Proposition 7.3 *The resonances of P in the domain*

$$\left\{z \in \mathbf{C};\ h^{4/5} \ll |z-E_0| = \mathcal{O}(1)h^{\delta}\right\} \setminus \bigcup_{j=1}^{3}\{z; \left|\operatorname{Re} z - E_0 - A_j \left|\operatorname{Im} z\right|^{3/2}\right| < \eta \left|\operatorname{Im} z\right|^{3/2}\}, \quad (7.29)$$

where $\delta,\ \eta > 0$ are arbitrary but fixed, are given by

$$\sim E_0 - i\left(h(k_1 - \alpha_1/4) + \epsilon^3 \sum_{j=0}^{\infty} h^j \epsilon^{-2j} r_j\left(\frac{h}{\epsilon^2}\left(k - \frac{\alpha}{4}\right) - \frac{S}{2\pi}, \epsilon, \frac{h^2}{\epsilon^5}\right)\right), \quad (7.30)$$

with

$$r_0\left(\xi, \epsilon, \frac{h^2}{\epsilon^5}\right) = ie^{3\pi i/4}\langle p_3\rangle(\xi) + \mathcal{O}\left(\epsilon + \frac{h^2}{\epsilon^5}\right),$$

$$r_j\left(\xi, \epsilon, \frac{h^2}{\epsilon^5}\right) = \mathcal{O}\left(\epsilon + \frac{h^2}{\epsilon^5}\right), \quad j \geq 1$$

analytic in $\xi \in \operatorname{neigh}(0, \mathbf{C}^2)$, and smooth in $\epsilon, h^2/\epsilon \in \operatorname{neigh}(0, \mathbf{R})$. We have $k = (k_1, k_2) \in \mathbf{Z}^2$, $S = (S_1, S_2)$ with $S_1 = 2\pi$, and $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}^2$ is fixed, and we choose $\epsilon > 0$ with $|E - E_0| \sim \epsilon^2$. The resonances in the set

$$\left\{z \in \mathbf{C},\ \left|\operatorname{Re} z - E_0 - A_1 \left|\operatorname{Im} z\right|^{3/2}\right| < \eta \left|\operatorname{Im} z\right|^{3/2}\right\} \text{ and } h^{4/5} \ll |z - E_0| = \mathcal{O}(1)h^{\delta}, \quad (7.31)$$

are given by E_0 plus

$$\frac{1}{i}\left(h\left(k_1 - \frac{\alpha_1}{4}\right) + i\epsilon^3 \sum_{j=0}^{\infty} h^j \epsilon^{-2j} G_j\left(\frac{h}{\epsilon^2}\left(k_1 - \frac{\alpha_1}{4}\right) - 1, \frac{h}{\epsilon^2}\left(k_2 + \frac{1}{2}\right), \epsilon, \frac{h^2}{\epsilon^5}\right)\right), \quad (7.32)$$

with $(k_1, k_2) \in \mathbf{Z} \times \mathbf{N}$, $\alpha_1 \in \mathbf{Z}$, and $|E - E_0| \sim \epsilon^2$. The function $G_0(\tau, q, \epsilon, h^2/\epsilon^5)$ is such that $\operatorname{Re} G(0,0,0,0) = A_1$ and $\frac{\partial}{\partial q}\operatorname{Re} G_0(0,0,0,0) > 0$. An analogous description of resonances is valid in the domain (7.31) with A_1 replaced by A_2.

Here in (7.30) we have also used that when expressed in terms of the action coordinates from (7.11), it is true that $p_2(\xi_1) = \xi_1 + 1$.

Remark. If we replace $r_j(\xi, \epsilon, h^2/\epsilon^5)$ in (7.30) by $r_j(\xi + S/2\pi, \epsilon, h^2/\epsilon^5)$, then we get

$$\sim E_0 - i\left(h(k_1 - \alpha_1/4) + \epsilon^3 \sum_{j=0}^{\infty} h^j \epsilon^{-2j} r_j\left(\frac{h}{\epsilon^2}\left(k - \frac{\alpha}{4}\right), \epsilon, \frac{h^2}{\epsilon^5}\right)\right).$$

Now let us notice that the choice of ϵ is not unique, and replacing ϵ by $\lambda\epsilon$, with $\lambda \sim 1$, does not affect the resonances. It follows therefore that

$$r_j(\xi, \epsilon, \tau) = \lambda^{3-2j} r_j\left(\frac{\xi}{\lambda^2}, \lambda\epsilon, \frac{\tau}{\lambda^5}\right). \quad (7.33)$$

Using this, we define

$$r_j(\xi, 1, \tau) = \epsilon^{3-2j} r_j\left(\frac{\xi}{\epsilon^2}, \epsilon, \frac{\tau}{\epsilon^5}\right),$$

when $|\xi| \sim \epsilon^2$ and $|\tau| \leq \mathcal{O}(\epsilon^5)$. Then (7.30) becomes

$$\sim E_0 - i\left(h(k_1 - \alpha_1/4) + \sum_{j=0}^{\infty} h^j r_j\left(h\left(k - \frac{\alpha}{4}\right), 1, h^2\right)\right).$$

A Function spaces and FBI-transforms on manifolds

Let X be a compact analytic manifold of dimension n. In this section we first review some parts of Section 1 in [27] about how to define global FBI-transforms on X, and function spaces associated to certain IR-deformations of the real cotangent space. After that we shall perform Bargmann type transforms which allow us to view the above-mentioned function spaces, microlocally in a bounded frequency region, as weighted spaces of holomorphic functions. The theory in [27] is an adaptation to the case of compact manifolds of the one in [12] and this as well as the Bargmann transform below are closely related to similar ideas and techniques, developed in [6], [4], [28], [32], [10].

We equip X with some analytic Riemannian metric so that we have a distance d and a volume density dy. Let $\phi(\alpha, y)$ be an analytic function on $\{(\alpha, y) \in T^*X \times X;\, d(\alpha_x, y) < 1/C\}$ (using the notation $\alpha = (\alpha_x, \alpha_\xi)$, $\alpha_x \in X$, $\alpha_\xi \in T^*_{\alpha_x}X$) with the following two properties (A) and (B):

(A) ϕ has a holomorphic extension to a domain of the form

$$\{(\alpha, y) \in T^*\widetilde{X} \times \widetilde{X};\, |\operatorname{Im}\alpha_x|, |\operatorname{Im} y| < \frac{1}{C}, |\operatorname{Re}\alpha_x - \operatorname{Re} y| < \frac{1}{C}, |\operatorname{Im}\alpha_\xi| < \frac{1}{C}|\langle\alpha_\xi\rangle|\} \tag{A.1}$$

and satisfies $|\phi| \leq \mathcal{O}(1)|\langle\alpha_\xi\rangle|$ there.

Here $\widetilde{X}$ is some complexification of X and $T^*\widetilde{X}$ denotes the cotangent space in the sense of complex manifolds with pointwise fiber spanned by the pointwise (1,0)-forms. We write $\langle\alpha_\xi\rangle = \sqrt{1 + \alpha_\xi^2}$ with α_ξ^2 defined by means of the dual metric, and as below, we shall often give statements in local coordinates whenever convenient and leave to the reader to check that the statements make sense globally. Notice that by the Cauchy inequalities,

$$\partial_{\alpha_x}^k \partial_{\alpha_\xi}^\ell \partial_y^m \phi = \mathcal{O}_{k,\ell,m}(1)|\langle\alpha_\xi\rangle|^{1-|\ell|}, \tag{A.2}$$

in a set of the form (A.1), with a slightly increased constant C.

The second assumption is

(B) $\phi(\alpha, \alpha_x) = 0$, $(\partial_y \phi)(\alpha, \alpha_x) = -\alpha_\xi$, $\mathrm{Im}\,(\partial_y^2 \phi)(\alpha, \alpha_x) \sim |\langle \mathrm{Re}\,\alpha_\xi \rangle| I$.

By Taylor's formula, we have

$$\phi(\alpha, y) = \alpha_\xi \cdot (\alpha_x - y) + \mathcal{O}(1)\langle \alpha_\xi \rangle |\alpha_x - y|^2, \tag{A.3}$$

and on the real domain, for $d(\alpha_x, y) \leq 1/C$, with C sufficiently large, we have:

$$\mathrm{Im}\,\phi(\alpha, y) \sim \langle \alpha_\xi \rangle (\alpha_x - y)^2. \tag{A.4}$$

The following example was found in a joint discussion with M. Zworski: Let $\exp_x : T_x X \to X$ be the geodesic exponential map. Then we can take

$$\phi(\alpha, y) = -\alpha_\xi \cdot \exp_{\alpha_x}^{-1}(y) + \frac{i}{2}\langle \alpha_\xi \rangle d(\alpha_x, y)^2. \tag{A.5}$$

Let $\Lambda \subset T^*\widetilde{X}$ be a closed I-Lagrangian manifold which is close to T^*X in the C^∞-sense and which coincides with this set outside a compact set. Recall that "I-Lagrangian" means Lagrangian for the real symplectic form $-\mathrm{Im}\,\sigma$, where $\sigma = \sum d\alpha_{\xi_j} \wedge d\alpha_{x_j}$ is the standard complex symplectic form. This means that if we choose (analytic) coordinates y in X and let (y, η) be the corresponding canonical coordinates on T^*X and $T^*\widetilde{X}$, then Λ is of the form $\{(y, \eta) + iH_G(y, \eta);\ (y, \eta) \in T^*X\}$ for some real-valued smooth function $G(y, \eta)$ which is close to 0 in the C^∞-sense and has compact support in η. Here H_G denotes the Hamilton field of G. Since Λ is close to T^*X, it is also R-symplectic in the sense that the restriction to Λ of $\mathrm{Re}\,\sigma$ is non-degenerate. (We say that Λ is an IR-manifold.) It follows that

$$d\alpha_{|\Lambda} = d\alpha_{x_1} \wedge \cdots \wedge d\alpha_{x_n} \wedge d\alpha_{\xi_1} \wedge \cdots \wedge d\alpha_{\xi_n}{}_{|\Lambda} = \frac{1}{n!}\sigma^n{}_{|\Lambda}$$

is a real non-vanishing $2n$-form on Λ, that we view as a positive density.

We also need some symbol classes. A smooth function $a(x, \xi; h)$, defined on Λ or on a suitable neighborhood of T^*X in $T^*\widetilde{X}$ is said to be of class $S^{m,k}$, if

$$\partial_x^p \partial_\xi^q a = \mathcal{O}(1) h^{-m} \langle \xi \rangle^{k-q}. \tag{A.6}$$

A formal classical symbol $a \in S^{m,k}_{\mathrm{cl}}$ is of the form $a \sim h^{-m}(a_0 + h a_1 + \cdots)$ where $a_j \in S^{0,k-j}$ is independent of h. Here and in the following, we let $0 < h \leq h_0$ for some sufficiently small $h_0 > 0$. When the domain of definition is real or equal to Λ, we can find a realization of a in $S^{m,k}$ (denoted by the same letter a) so that

$$a - h^{-m} \sum_0^N h^j a_j \in S^{-(N+1)+m, k-(N+1)}.$$

When the domain of definition is a complex domain, we say that $a \in S^{m,k}_{\mathrm{cl}}$ is a formal classical analytic symbol ($a \in S^{m,k}_{\mathrm{cla}}$) if a_j are holomorphic and satisfy

$$|a_j| \le C_0 C^j (j!) |\langle \xi \rangle|^{k-j}. \tag{A.7}$$

It is then standard, that we can find a realization $a \in S^{m,k}$ (denoted by the same letter a) such that

$$\begin{aligned} &\partial_x^k \partial_\xi^\ell \overline{\partial}_{x,\xi} a = \mathcal{O}_{k,\ell}(1) e^{-|\langle \xi \rangle|/Ch}, \\ &|a - h^{-m} \sum_{0 \le j \le |\langle \xi \rangle|/C_0 h} h^j a_j| \le \mathcal{O}(1) e^{-|\langle \xi| \rangle/C_1 h}, \end{aligned} \tag{A.8}$$

where in the last estimate $C_0 > 0$ is sufficiently large and $C, C_1 > 0$ depend on C_0. We will denote by $S^{m,k}_{\mathrm{cl}}$ and $S^{m,k}_{\mathrm{cla}}$ also the classes of realizations of classical symbols. We say that a classical (analytic) symbol $a \sim h^{-m}(a_0 + h a_1 + \cdots)$ is elliptic, if a_0 is elliptic, so that $a_0^{-1} \in S^{0,-k}$. Take such an elliptic $a(\alpha, y; h) \in S^{\frac{3n}{4}, \frac{n}{4}}_{\mathrm{cla}}$ and put

$$Tu(\alpha; h) = \int e^{\frac{i}{h}\phi(\alpha, y)} a(\alpha, y; h) \chi(\alpha_x, y) u(y) dy, \tag{A.9}$$

where χ is smooth with support close to the diagonal and equal to 1 in a neighborhood of the same set.

According to [27] there exists $b(\alpha, x; h) \in S^{\frac{3n}{4}, \frac{n}{4}}_{\mathrm{cla}}$, such that if

$$Sv(x) = \int_{T^*X} e^{-\frac{i}{h}\phi^*(x,\alpha)} b(\alpha, x; h) \chi(\alpha_x, x) v(\alpha) d\alpha, \tag{A.10}$$

then

$$STu = u + Ru, \tag{A.11}$$

where R has a distribution kernel $R(x, y; h)$ satisfying

$$|\partial_x^\alpha \partial_y^\ell R| \le C_{k,\ell} e^{-\frac{1}{C_0 h}}. \tag{A.12}$$

Here we denote in general by f^*, the holomorphic extension of the complex conjugate of f.

With Λ as above, we put

$$T_\Lambda u = Tu_{|_\Lambda}, \tag{A.13}$$

and define $S_\Lambda v$ by (A.10), but with T^*X replaced by Λ. Then,

$$S_\Lambda T_\Lambda u = u + R_\Lambda u, \tag{A.14}$$

where R_Λ satisfies (A.12) (with a slightly larger C_0 and under the assumption that Λ is sufficiently close to T^*X). In fact, using Stokes' formula and the exponential decrease of $\overline{\partial}$ of the symbols involved, we see that $S_\Lambda T_\Lambda$ coincides up to an exponentially small error with ST.

Since Λ is I-Lagrangian, we can find locally a real-valued smooth function $H(\alpha)$ on Λ, such that

$$dH = -\mathrm{Im}\,(\alpha_\xi \cdot d\alpha_x)_{|\Lambda}. \tag{A.15}$$

Indeed, $-\mathrm{Im}\,(\alpha_\xi \cdot d\alpha_x)$ is a primitive of $-\mathrm{Im}\,\sigma$ and the latter vanishes on Λ, so the right-hand side of (A.15) is closed.

We assume:

$$\text{The equation (A.15) has a global solution } H \in C^\infty(\Lambda; \mathbf{R}). \tag{A.16}$$

Notice that this property is equivalent to

$$\mathrm{Im} \int_\gamma (\alpha_\xi \cdot d\alpha_x) = 0, \text{ for all closed curves } \gamma \subset \Lambda. \tag{A.17}$$

When (A.16) is fulfilled, H is well defined up to a constant, and we shall always choose H to be zero for large α_ξ.

As in [27] we notice that (A.16) is fulfilled in the case of IR-manifolds generated by a weight $G \in C^\infty(T^*\widetilde{X}; \mathbf{R})$ in the following way: Let $H_G = H_G^{\mathrm{Im}\,\sigma}$ be the Hamilton field of G with respect to $\mathrm{Im}\,\sigma$, and assume that $G = 0$ in the region where $|\alpha_\xi|$ is large. Then for t real with $|t|$ small enough, we can consider the IR-manifold $\Lambda_t = \exp(tH_G)(\Lambda_0)$, where $\Lambda_0 = T^*X$. Then we get (A.16) with $H = H_t$ given by

$$H_t = \int_0^t (\exp(s-t)H_G)^*(G + \langle H_G, \omega \rangle)ds, \tag{A.18}$$

where $\omega = -\mathrm{Im}\,(\alpha_\xi \cdot d\alpha_x)$

The function H appears naturally in connection with T_Λ. We have $d_\alpha \phi = \alpha_\xi \cdot d\alpha_x + \mathcal{O}(|\alpha_x - y|)$, so $(d_\alpha\phi)(\alpha, \alpha_x) = \alpha_\xi \cdot d\alpha_x$ and

$$-\mathrm{Im}\,(d_\alpha\phi)(\alpha, \alpha_x)_{|\Lambda} = d_\alpha H. \tag{A.19}$$

Definition. For $m \in \mathbf{R}$, put

$$H(\Lambda; \langle\alpha_\xi\rangle^m) = \{u \in \mathcal{D}'(X);\, T_\Lambda u \in L^2(\Lambda; e^{-2H/h}|\langle\alpha_\xi\rangle|^{2m} d\alpha)\}. \tag{A.20}$$

When $\Lambda = T^*X$ we get the usual h-Sobolev spaces, and in particular the case $m = 0$ just gives $L^2(X)$. For general Λ we get the same spaces, but the equivalence of the norm

$$\|u\|_{H(\Lambda, \langle\alpha_\xi\rangle^m)} = \|T_\Lambda u\|_{L^2(\Lambda; e^{-2H/h}|\langle\alpha_\xi\rangle|^{2m} d\alpha)} \tag{A.21}$$

with the h-m-Sobolev norm $\|u\|_{H(T^*X, \langle\alpha_\xi\rangle^m)}$ is no longer uniform with respect to h, in general.

Recall from [27] that if we choose another FBI-transform $\widetilde{T}$ of the same type as T but with different phase $\widetilde{\phi}$ and amplitude $\widetilde{a}$, then for Λ close enough to T^*X, the definition (A.20) does not change if we replace T by $\widetilde{T}$, and we get a new norm which is equivalent to the previous one, uniformly with respect to h. This follows from a fairly explicit description of $\widetilde{T}_\Lambda T_\Lambda^{-1}$.

We also know that $Tu = T_{T^*X}u$ and $T_\Lambda u$ satisfy compatibility conditions similar to the Cauchy-Riemann equations for holomorphic functions. For the analysis in the most interesting region where ξ is bounded, it will be convenient to work with transforms which are holomorphic up to exponentially small errors, and for that we make a different choice of T, and take an FBI-transform as in [28], now with a global choice of phase (cf [4], [10], [32]).

The function $d(x,y)^2$ is analytic in a neighborhood of the diagonal in $X \times X$, so we can consider it as a holomorphic function in a region

$$\{(x,y) \in \widetilde{X} \times \widetilde{X};\, \mathrm{dist}\,(x,y) < \frac{1}{C},\, |\mathrm{Im}\, x|, |\mathrm{Im}\, y| < \frac{1}{C}\}.$$

Put

$$\phi(x,y) = i\lambda d(x,y)^2, \tag{A.22}$$

where $\lambda > 0$ is a constant that we choose large enough, depending on the size of the neighborhood of the zero section in T^*X, that we wish to cover.

For $x \in \widetilde{X}$, $|\mathrm{Im}\, x| < 1/C$, put

$$\mathcal{T}u(x;h) = h^{-\frac{3n}{4}} \int e^{\frac{i}{h}\phi(x,y)} \chi(x,y) u(y) dy,\ u \in \mathcal{D}'(X), \tag{A.23}$$

where χ is a smooth cut-off function with support in $\{(x,y) \in \widetilde{X} \times X;\, |\mathrm{Im}\, x| < 1/C,\, d(y, y(x)) < 1/C\}$. Here $y(x) \in X$ is the point close to x, where $X \ni y \mapsto -\mathrm{Im}\, \phi(x,y)$ attains its non-degenerate maximum. We have the following facts ([28]):

The function $\Phi_0(x) = -\mathrm{Im}\, \phi(x, y(x))$, $x \in \widetilde{X}$, $|\mathrm{Im}\, x| < 1/C$, is strictly plurisubharmonic and is of the order of magnitude $\sim |\mathrm{Im}\, x|^2$.

$\Lambda_{\Phi_0} := \{(x, \frac{2}{i}\partial\Phi_0) \in T^*\widetilde{X}\}$ is an IR-manifold given by $\Lambda_{\Phi_0} = \kappa_{\mathcal{T}}(T^*X)$, where $\kappa_{\mathcal{T}}$ is the complex canonical transform associated to $\mathcal{T}$, given by $(y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y))$. Here and in the following, we identify $\widetilde{X}$ with its intersection with a tubular neighborhood of X which is independent of the choice of λ in (A.22).

If $L^2_{\Phi_0} = L^2(\widetilde{X}; e^{-2\Phi_0/h}L(dx))$, for $L(dx)$ denoting a choice of Lebesgue measure (up to a non-vanishing continuous factor), then $\mathcal{T} = \mathcal{O}(1) : L^2(X) \to L^2_{\Phi_0}$, $\overline{\partial}_x \mathcal{T} = \mathcal{O}(e^{-1/Ch}) : L^2(X) \to L^2_{\Phi_0}$. This means that up to an exponentially small error $\mathcal{T}u$ is holomorphic for $u \in L^2(X)$ (and even for $u \in \mathcal{D}'(X)$). A natural choice of Lebesgue measure might be $(n!)^{-1}|\pi_*(\sigma_{|\Lambda_{\Phi_0}})^n|$, where $\pi : \Lambda_{\Phi_0} \to \widetilde{X}$ is the natural projection.

Let $H_{\Phi_0}(\widetilde{X}) \subset L^2_{\Phi_0}(\widetilde{X})$ be the subspace of holomorphic functions. Assuming, as we may, that $\widetilde{X}$ is a Stein ("pseudoconvex") domain, we can apply the well-known L^2 results of Hörmander for the $\overline{\partial}$-operator and replace $\mathcal{T}$ by $\widetilde{\mathcal{T}} = \mathcal{T} + K$, where $K = \mathcal{O}(e^{-1/(Ch)}) : L^2(X) \to L^2_{\Phi_0}(\widetilde{X})$, so that $\widetilde{\mathcal{T}} : L^2(X) \to H_{\Phi_0}(\widetilde{X})$. In the main text we do not distinguish between $\mathcal{T}$ and $\widetilde{\mathcal{T}}$.

Unitarity: Modulo exponentially small errors and microlocally, $\mathcal{T}$ is unitary $L^2(X) \to L^2(\widetilde{X}; a_0 e^{-2\Phi_0/h} L(dx))$, where $L(dx)$ is chosen as indicated above, and $a_0(x;h)$ is a positive elliptic analytic symbol of order 0.

Let $\Lambda \subset T^*\widetilde{X}$ be an IR-manifold as before, satisfying (A.16) (or the equivalent condition (A.17)). Then $\kappa_{\mathcal{T}}(\Lambda) = \Lambda_\Phi$, where $\Phi = \Phi_\Lambda$, can be normalized by the requirement that $\Phi = \Phi_0$ near the boundary of $\widetilde{X}$. (Here is where we have to choose λ large enough, depending on Λ. In the applications, for a given elliptic operator, Λ and T^*X will coincide outside a fixed compact neighborhood of the zero section, and the whole study will be carried out with a fixed λ.)

Let $\Omega \subset T^*X$ be the open neighborhood of the 0-section, given by $\pi_x \kappa_{\mathcal{T}} \Omega = \widetilde{X}$ and view also Ω as a subset of Λ in the natural sense, assuming that T^*X and Λ coincide in a neighborhood of the closure of the complement of Ω. If $\chi \in C_0^\infty(\Omega)$, then the norm $\|u\|_{H(\Lambda,\langle\alpha_\xi\rangle^m)}$ is equivalent to the norm

$$\|\mathcal{T}u\|_{L^2_\Phi} + \|(1-\chi)T_\Lambda u\|_{L^2(\Lambda; e^{-2H/h}|\langle\alpha_\xi\rangle|^{2m} d\alpha)}$$

uniformly with respect to h.

Acknowledgments

We would like to thank Anders Melin and Maciej Zworski for useful discussions. The first author gratefully acknowledges the support of the Swedish Foundation for International Cooperation in Research and Higher Education (STINT) as well as of the MSRI postdoctoral fellowship.

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Michael Hitrik
Department of Mathematics
University of California
Los Angeles, CA 90095–1555
USA
email: hitrik@math.ucla.edu

Johannes Sjöstrand
Centre de Mathématiques
UMR 7640 CNRS
Ecole Polytechnique
F-91128 Palaiseau
France
email: johannes@math.polytechnique.fr

Communicated by Bernard Helffer
submitted 13/03/03, accepted 06/10/03