Calc. Var.
DOI 10.1007/s00526-008-0187-0 **Calculus of Variations**

Non-simple blow-up solutions for the Neumann two-dimensional sinh-Gordon equation

Pierpaolo Esposito · Juncheng Wei

Received: 31 January 2008 / Accepted: 5 May 2008

Abstract For the Neumann sinh-Gordon equation on the unit ball $B \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = \lambda^+ \left(\frac{e^u}{\int_B e^u} - \frac{1}{\pi} \right) - \lambda^- \left(\frac{e^{-u}}{\int_B e^{-u}} - \frac{1}{\pi} \right) & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

we construct sequence of solutions which exhibit a multiple blow up at the origin, where $\lambda^\pm$ are positive parameters. It answers partially an open problem formulated in Jost et al. [Calc Var Partial Diff Equ 31(2):263–276].

Mathematics Subject Classification (2000) 35J60 · 35B33 · 35J25 · 35J20 · 35B40

1 Introduction and statement of main results

In this paper, we consider the Neumann sinh-Gordon equation

$$\begin{cases} -\Delta u = \lambda^+ \left(\frac{e^u}{\int_\Omega e^u} - \frac{1}{|\Omega|} \right) - \lambda^- \left(\frac{e^{-u}}{\int_\Omega e^{-u}} - \frac{1}{|\Omega|} \right) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

The research of the first named author is supported by M. U. R. S. T., project "Variational methods and nonlinear differential equations". The research of the second named author is supported by an Earmarked grant from RGC of Hong Kong.

P. Esposito (✉)
Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. Leonardo Murialdo, 1-00146 Rome, Italy
e-mail: esposito@mat.uniroma3.it

J. Wei
Department of Mathematics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong, China
e-mail: wei@math.cuhk.edu.hk

on a smooth domain $\Omega \subset \mathbb{R}^2$, where ν denotes the unit outward normal to $\partial\Omega$ and $\lambda^{\pm}$ are positive parameters.

The analysis of non compact solutions to (1.1) has recently attracted a lot of interest. Let u_n be a sequence of solutions to (1.1) with uniformly bounded parameters $\lambda_n^{\pm}$. We define the positive/negative blow-up set of $\{u_n\}$ as

$$S_{\pm} = \left\{ x \in \Omega : \exists\, x_n \to \Omega \text{ s.t. } \ln \lambda_n^{\pm} \pm u_n(x_n) - \ln \int_{\Omega} e^{\pm u_n} \to +\infty \text{ as } n \to +\infty \right\}$$

and we can associate (up to a subsequence) to every $p \in S_{\pm}$ its positive/negative limiting mass

$$m_{\pm}(p) = \lim_{r \to 0} \lim_{n \to +\infty} \frac{\lambda_n^{\pm} \int_{B_r(p)} e^{\pm u_n}}{\int_{\Omega} e^{\pm u_n}}.$$

In particular, $S_{\pm}$ is a finite set and

$$\lambda_n^{\pm} \frac{e^{\pm u_n}}{\int_B e^{\pm u_n}} \rightharpoonup \sum_{p \in S_{\pm}} m_{\pm}(p) \delta_p$$

weakly in the sense of measures, as $n \to +\infty$. In a recent paper [8], Jost, Wang, Ye and Zhou proved that a quantization of the limiting masses holds: $m_{\pm}(p)$ are multiples of 8π. It is the analogue of a result by Li and Shafrir [9] for the mean field equation.
In view of a relationship in [10]

$$(m_+(p) - m_-(p))^2 = 8\pi \left(m_+(p) + m_-(p) \right),$$

it follows that for any $p \in S_+ \cap S_-$ the couple $(m_+(p), m_-(p))$, up to the order, takes the value

$$8\pi \left(\frac{k(k-1)}{2}, \frac{k(k+1)}{2} \right), \quad k \in \mathbb{N} \setminus \{0\}.$$

An open problem raised in [8] concerns whether or not in general k must be 1. (See Problem 1 of [8].) Let us stress that $k = 1$ corresponds to a simple blow up in p while $k > 1$ gives rise to a non-simple (multiple) blow up.

In this paper, we will give a negative answer to this question. We consider the following problem on the unit ball B:

$$\begin{cases} -\Delta u = \rho^2 \left(e^u - \frac{1}{\pi} \int_B e^u \right) - \rho^2 \left(e^{-u} - \frac{1}{\pi} \int_B e^{-u} \right) & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B. \end{cases} \tag{1.2}$$

The result we have is:

Theorem 1.1 *There exists $\rho_0 > 0$ small such that for any $0 < \rho \le \rho_0$ problem (1.2) has a solution u_ρ such that as $\rho \to 0$*

$$\rho^2 e^{u_\rho} \rightharpoonup 8\pi \delta_0, \quad \rho^2 e^{-u_\rho} \rightharpoonup 24\pi \delta_0 \tag{1.3}$$

weakly in the sense of measure in $\overline{B}$.

The solution u_ρ is constructed by superposing a positive bubble centered at the origin and 3 negative bubbles centered at la_j, where $a_j = e^{\frac{2\pi i j}{3}}$, $j = 0, 1, 2$, are the 3-roots of unity and $l = l(\rho) \to 0$ as $\rho \to 0$. Setting $\lambda_\rho^\pm = \rho^2 \int_B e^{\pm u_\rho}$, by (1.3) we have that

$$\lambda_\rho^+ \frac{e^{u_\rho}}{\int_B e^{u_\rho}} \rightharpoonup 8\pi\delta_0, \quad \lambda_\rho^- \frac{e^{-u_\rho}}{\int_B e^{-u_\rho}} \rightharpoonup 24\pi\delta_0$$

weakly in the sense of measure in $\overline{B}$, as $\rho \to 0$. In this way, u_ρ is a sequence of solutions to (1.1) with parameters $\lambda_\rho^\pm$ for which $0 \in S_+ \cap S_-$ and the limiting masses satisfy $m_+(0) = 8\pi$, $m_-(0) = 24\pi$. Hence, in general $k = 1$ does not hold.

We can recover an example of non simple blow up for the Dirichlet sinh-Gordon equation too (see also [2] for the case of simple blow up points). Let u_ρ^0 be the solution of

$$\begin{cases} \Delta u_\rho^0 = \frac{\rho^2}{\pi} \int_B (e^{u_\rho} - e^{-u_\rho}) & \text{in } B \\ u_\rho^0 = u_\rho & \text{on } \partial B. \end{cases}$$

The function $v_\rho = u_\rho - u_\rho^0$ satisfies

$$\begin{cases} -\Delta v = \rho^2 (V_\rho^+ e^v - V_\rho^- e^{-v}) & \text{in } B \\ v_\rho = 0 & \text{on } \partial B, \end{cases} \tag{1.4}$$

where the potentials $V_\rho^\pm = e^{\pm u_\rho^0} \to V^\pm$ uniformly as $\rho \to 0$ for some explicit functions $V^\pm$. In fact, V^+ has a local minimum at the origin while V^- has a local maximum at the origin. This suggests that the existence of non-simple blow-up solutions depends very much on the local structure of $V^\pm$. Our computations also suggest that when $V_\rho^\pm = 1$, problem (1.4) has only simple blow-ups.

For ϵ, δ and $l > 0$, let us define

$$U^+(x) = \ln \frac{8\delta^2}{(\delta^2\rho^2 + |x|^2)^2}, \quad U_j^- = \ln \frac{8\epsilon^2}{(\epsilon^2\rho^2 + |x - la_j|^2)^2}, \quad j = 0, 1, 2,$$

which are solutions of $-\Delta U = \rho^2 e^U$ in $\mathbb{R}^2$. Let us introduce the projection operator $P : C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$: given $u \in C^{2,\alpha}(\overline{\Omega})$, let Pu be the solution of

$$\begin{cases} \Delta Pu = \Delta u - \frac{1}{\pi} \int_B \Delta u & \text{in } B \\ \frac{\partial Pu}{\partial \nu} = 0 & \text{on } \partial B \\ \int_B Pu = 0. \end{cases}$$

For a suitable choice of ϵ, δ and l, $PU := PU^+ - PU^-$ is a good approximating solution to (1.2), where $U^- = \sum_{j=0}^2 U_j^-$. Our solution u_ρ will be in the form $PU + \phi$, where ϕ is a remainder term small in $L^\infty(\Omega)$−norm and $l = l(\rho)$ satisfies $l(\rho) \to 0$ as $\rho \to 0$. The existence of $l(\rho)$ will follow by means of a Lyapunov–Schmidt finite dimensional reduction and crucial will be the property that 0 is a critical point of the related Green's function. This procedure has been used in many other papers. See [1,2,4–7,11] and the references therein. The main difficulties here are the estimates of the distance between bubbles.

Theorem 1.1 is the first nontrivial example of non-simple blow up solutions for sinh-Gordon equations. Previous known examples of non-simple blow up solutions are for Liouville equation on a disk in [3] (without boundary condition) or Liouville equation with anisotropic coefficients in [11].

The paper is organized as follows. In Sect. 2 we describe exactly the ansatz for the solution we are looking for and we rewrite the problem in term of a linear operator L (for which a solvability theory is performed in Appendix B). In Sect. 3 we solve an auxiliary non linear problem and reduce (1.2) to find critical points of a function $\widetilde{E}_\rho(l)$. In Sect. 4 we prove Theorem 1.1 and an aymptotic expansion of $\widetilde{E}_\rho(l)$ for l small has to be performed. A coefficient in the expansion is given in integral form and its sign is crucial to have critical points of $\widetilde{E}_\rho(l)$ for l small: Appendix A is devoted to the exact computation of such an integral.

2 Approximating solutions

First of all, let us introduce the Neumann Green's function $G(x, y)$ on B, i.e. the solution of the problem

$$\begin{cases} -\Delta_x G(x,y) = \delta_y - \frac{1}{\pi} & \text{in } B \\ \frac{\partial G}{\partial \nu}(x,y) = \nabla_x G(x,y)\cdot x = 0 & \text{on } \partial B \\ \int_B G(x,y)dx = 0. \end{cases}$$

On B the regular part $H(x, y)$ of $G(x, y)$, defined as $H(x, y) = G(x, y) + \frac{1}{2\pi}\ln|x-y|$, turns out to be:

$$H(x,y) = -\frac{1}{4\pi}\ln\left(|x|^2|y|^2 - 2x\cdot y + 1\right) + \frac{1}{4\pi}|x|^2 + c(y),$$

where $c(y)$ is chosen to have $\int_B G(x, y)dx = 0$. Here and in the sequel, the expression $x \cdot y$ will denote both the inner product in $\mathbb{R}^2$, $x\cdot y = \sum_{j=1}^2 x^j y^j$ and the inner product in $\mathbb{C}$, $x\cdot y = \text{Re}\,(x\bar{y})$, depending on whether x, y are considered as points in $\mathbb{R}^2$ or $\mathbb{C}$.

For $y = 0$ it is easy to compute $c(0) = -\frac{3}{8\pi}$. Since $G(x, y)$ is a symmetric function, we can deduce that

$$c(y) = H(0,y) = H(y,0) = \frac{|y|^2}{4\pi} + c(0) = \frac{|y|^2}{4\pi} - \frac{3}{8\pi}.$$

Hence, the expression of $H(x, y)$ becomes

$$H(x,y) = -\frac{1}{4\pi}\ln\left(|x|^2|y|^2 - 2x\cdot y + 1\right) + \frac{|x|^2+|y|^2}{4\pi} - \frac{3}{8\pi}.$$

Given $a_j = e^{\frac{2\pi i j}{3}}$, $j = 0, 1, 2$, the 3-roots of unity, define

$$\delta = \frac{1}{\sqrt{8}}e^{4\pi H(0,0) - 4\pi\sum_{j=0}^2 G(0,la_j)}$$

$$\epsilon_j = \frac{1}{\sqrt{8}}e^{4\pi H(la_j,la_j) + 4\pi\sum_{m\neq j} G(la_m,la_j) - 4\pi G(la_j,0)}, \quad j = 0,1,2.$$

Since for symmetry ϵ_j does not depend on $j = 0, 1, 2$, we will refer to it simply as ϵ. Since $a_j \cdot a_m = -\frac{1}{2}$ for $j \neq m$, we get that

$$\delta = \frac{1}{\sqrt{8}}e^{3(1-l^2)}l^6, \quad \epsilon = \frac{e^{5l^2-3}}{9\sqrt{8}}(1-l^6)^{-2}l^{-2}.$$

We describe asymptotically the action of P on $U_\pm$ in the following Lemma:

Lemma 2.1 *Let $j = 0, 1, 2$. There hold*

$$PU^+ = U^+ - \ln(8\delta^2) + 8\pi H(x,0) + O(\delta^2\rho^2|\ln\,\delta\rho|)$$
$$PU_j^- = U_j^- - \ln(8\epsilon^2) + 8\pi H(x,la_j) + O(\epsilon^2\rho^2|\ln\,\epsilon\rho|)$$

uniformly in Ω, *as* $\delta\rho, \epsilon\rho \to 0$. *In particular, there hold*

$$PU^+ = 8\pi G(x,0) + O(\delta^2\rho^2|\ln\ \delta\rho| + \delta^2\rho^2|x|^{-2})$$
$$PU_j^- = 8\pi G(x, la_j) + O(\epsilon^2\rho^2|\ln\ \epsilon\rho| + \epsilon^2\rho^2|x - la_j|^{-2}).$$

Proof First, let us observe that

$$-\int_B \Delta U^+ = \rho^2 \int_B e^{U^+} = \int_{|x|\le 1/\delta\rho} \frac{8dx}{(1+|x|^2)^2} = 8\pi + O(\delta^2\rho^2) \tag{2.1}$$

$$-\int_B \Delta U_j^- = \rho^2 \int_B e^{U_j^-} = \rho^2 \int_{|x-la_j|\le 1/2} e^{U_j^-} + O(\epsilon^2\rho^2) \tag{2.2}$$
$$= 8\pi + O(\epsilon^2\rho^2).$$

Let us justify the validity of the expansion for PU^+. Since

$$\frac{\partial U^+}{\partial \nu} = -\frac{4}{\delta^2\rho^2+1} = 8\pi \frac{\partial}{\partial \nu}\left(-\frac{1}{2\pi}\ln|x|\right) + O(\delta^2\rho^2) \quad \text{on } \partial B,$$

the function $\varphi = PU^+ - U^+ + \ln(8\delta^2) - 8\pi H(x,0)$ satisfies

$$\Delta\varphi = O(\delta^2\rho^2) \text{ in } B, \quad \frac{\partial\varphi}{\partial\nu} = -8\pi\frac{\partial G(x,0)}{\partial\nu} + O(\delta^2\rho^2) = O(\delta^2\rho^2) \text{ on } \partial B$$

in view of (2.1). Since $\int_B \ln(\frac{\delta^2\rho^2}{|x|^2}+1) = O(\delta^2\rho^2|\ln\ \delta\rho|)$, we easily get that

$$\int_B \varphi = \int_B (PU^+ - 8\pi G(x,0)) + 2\int_B \ln\left(\frac{\delta^2\rho^2}{|x|^2}+1\right) = O(\delta^2\rho^2|\ln\ \delta\rho|).$$

By the representation formula

$$\varphi(x) = \frac{1}{\pi}\int_B \varphi - \int_B G(x,y)\Delta\varphi(y)dy + \int_{\partial B} G(x,y)\frac{\partial\varphi}{\partial\nu}(y)d\sigma(y)$$

for every $x \in B$, finally we get that $\varphi = O(\delta^2\rho^2|\ln\ \delta\rho|)$ uniformly in Ω, as $\delta\rho \to 0$. Similarly, the expansion of PU^- follows and the proof is done. □

In order to find solutions we will need a-posteriori that l^4 has to behave like ρ, as $\rho \to 0$. In order to simplify the estimates and make the argument more clear, in the sequel we will assume that

$$\exists\, C > 1 : \quad C^{-1}\rho \le l^4 \le C\rho. \tag{2.3}$$

Let

$$W(x) = \left(\frac{(\delta\rho)^{\frac14}}{(\delta^2\rho^2+|x|^2)^{\frac98}} + \sum_{j=0}^{2}\frac{(\epsilon\rho)^{\frac14}}{(\epsilon^2\rho^2+|x-la_j|^2)^{\frac98}}\right)^{-1}.$$

For any $h \in L^\infty(\Omega)$, introduce the weighted norm

$$\|h\|_* = \sup_{x\in\Omega}\left|W(x)h(x)\right|.$$

Let us stress that there are many choices for the exponents in the weight function $W(x)$ and ours turns out to be satisfactory.

With Lemma 2.1 in hands, we can evaluate how good is the approximating solution PU in $\|\cdot\|_*$:

Proposition 2.2 *Assume* (2.3). *There holds*

$$\|\Delta PU+\rho^2\left(e^{PU}-\frac{1}{\pi}\int\limits_B e^{PU}\right)-\rho^2\left(e^{-PU}-\frac{1}{\pi}\int\limits_B e^{-PU}\right)\|_*=O(l^{\frac{3}{2}}|\ln l|)$$

as $\rho, l\to 0$.

Proof We have that

$$\begin{aligned}R:=\Delta PU+\rho^2\left(e^{PU}-\frac{1}{\pi}\int\limits_B e^{PU}\right)-\rho^2\left(e^{-PU}-\frac{1}{\pi}\int\limits_B e^{-PU}\right)\\ =\rho^2(e^{PU}-e^{U^+})-\rho^2\left(e^{-PU}-\sum_{j=0}^{2}e^{U_j^-}\right)\\ -\frac{\rho^2}{\pi}\int\limits_B(e^{PU}-e^{U^+})+\frac{\rho^2}{\pi}\int\limits_B\left(e^{-PU}-\sum_{j=0}^{2}e^{U_j^-}\right).\end{aligned}$$

Let $R^+=\rho^2(e^{PU}-e^{U^+})$ and $R^-=\rho^2(e^{-PU}-\sum_{j=0}^2 e^{U_j^-})$ in order to get $R=R^+-R^--\frac{1}{\pi}\int_B(R^+-R^-)$.

Estimate on R^+. By the choice of δ and Lemma 2.1 we get that

$$\begin{aligned}PU-U^+&=(PU^+-U^+)-PU^-\\ &=8\pi\,(H(x,0)-H(0,0))-8\pi\sum_{j=0}^{2}(H(x,la_j)-H(0,la_j))\\ &\quad+2\sum_{j=0}^{2}\ln(\epsilon^2l^{-2}\rho^2+|l^{-1}x-a_j|^2)+O(\epsilon^2\rho^2|\ln\,\epsilon\rho|)\end{aligned}$$

uniformly in Ω. By $\sum_{j=0}^2 a_j=0$ note that the expansions

$$\begin{aligned}\sum_{j=0}^{2}\ln(\epsilon^2l^{-2}\rho^2+|l^{-1}x-a_j|^2)=2\sum_{j=0}^{2}\ln|l^{-1}x-a_j|+O\left(\frac{l^2}{|l^{-1}x-a_j|^2}\right)\\ =-2\left(\sum_{j=0}^{2}a_j\right)\cdot\frac{x}{l}+O\left(\frac{|x|^2}{l^2}+l^2\right)=O\left(\frac{|x|^2}{l^2}+l^2\right),\end{aligned}\tag{2.4}$$

in $B_{l/2}(0)$, and

$$\begin{aligned} &H(x,0)-H(0,0)-\sum_{j=0}^{2}\big(H(x,la_j)-H(0,la_j)\big)\\ &=-\frac{|x|^2}{2\pi}+\frac{1}{4\pi}\sum_{j=0}^{2}\ln(l^2|x|^2-2lx\cdot a_j+1)\\ &=-\frac{|x|^2}{2\pi}-\frac{l}{2\pi}\left(\sum_{j=0}^{2}a_j\right)\cdot x+O(l^2|x|^2)=O(|x|^2) \end{aligned} \tag{2.5}$$

in Ω hold. Hence, we get that

$$\rho^2e^{PU}=\rho^2\prod_{j=0}^{2}(\epsilon^2l^{-2}\rho^2+|l^{-1}x-a_j|^2)^2e^{U^+}\left(1+O(|x|^2+l^4|\ln l|)\right) \tag{2.6}$$

uniformly in Ω and in particular, by (2.4) in $B_{l/2}(0)$ there holds

$$\rho^2e^{PU}=\rho^2e^{U^+}(1+O(l^{-2}|x|^2+l^2)). \tag{2.7}$$

Then, there holds $\int_{B_{l/2}(0)}|R^+|=O(l^2)$ and

$$\begin{aligned} |W(x)R^+(x)|&\le\frac{(\delta^2\rho^2+|x|^2)^{\frac{9}{8}}}{(\delta\rho)^{\frac14}}|R^+(x)|\\ &\le C\left(\delta^2l^{-2}\rho^2\frac{|y|^2}{(1+|y|^2)^{\frac78}}+\frac{l^2}{(1+|y|^2)^{\frac78}}\right)=O(l^2) \end{aligned}$$

in $B_{l/2}(0)$, where $y=\frac{x}{\delta\rho}\in B_{l/2\delta\rho}(0)$. Outside $B_{l/2}(0)$, firstly we have that

$$\rho^2We^{U^+}\le\frac{(\delta\rho)^{\frac74}}{(\delta^2\rho^2+|x|^2)^{\frac78}}=O(\delta^{\frac74}l^{-\frac74}\rho^{\frac74})=O(l^{\frac{63}{4}}) \tag{2.8}$$

in $B\backslash B_{l/2}(0)$. Secondly, by (2.6) we deduce that

$$\begin{aligned} e^{PU}&=O\left(\frac{\prod_{j=0}^{2}(\epsilon^2\rho^2+|x-la_j|^2)^2}{(\delta^2\rho^2+|x|^2)^2}\right)\\ &=O\left(\frac{(\epsilon^2\rho^2+|x|^2+l^2)^2}{(\delta^2\rho^2+|x|^2)^2}\right)=O((\epsilon^2l^{-2}\rho^2+1)^2)=O(1) \end{aligned} \tag{2.9}$$

in $B\backslash B_{l/2}(0)$ and then

$$\rho^2We^{PU}=O\left(\rho^2\frac{(\epsilon^2\rho^2+|x-la_0|^2)^{\frac98}}{(\epsilon\rho)^{\frac14}}\right)=O\left(\frac{\rho^{\frac74}}{\epsilon^{\frac14}}\right)=O\left(l^{\frac{15}{2}}\right) \tag{2.10}$$

in $B\backslash B_{l/2}(0)$. Hence, by (2.8) and (2.10) we get that $|WR^+|=O(l^{\frac{15}{2}})$ in $B\backslash B_{l/2}(0)$. By (2.9) it is easily seen that

$$\int_{B\backslash B_{l/2}(0)}|R^+|\le\rho^2\left(\int_{B\backslash B_{l/2}(0)}e^{PU}+\int_{B\backslash B_{l/2}(0)}e^{U^+}\right)=O(l^8).$$

Finally, combining the estimates in $B_{l/2}(0)$ and in $B \setminus B_{l/2}(0)$ we get that

$$\|R^+\|_* + \int_B |R^+| = O(l^2). \tag{2.11}$$

Estimate on R^-. Fix $j = 0, 1, 2$. On $B_{l/2}(la_j)$ we have that

$$R^- = \rho^2 \left(e^{-PU} - \sum_{m=0}^{2} e^{U_m^-} \right) = \left(\rho^2 e^{-PU} - \rho^2 e^{U_j^-} \right) - \sum_{m \neq j} \frac{8\epsilon^2 \rho^2}{(\epsilon^2 \rho^2 + |x - la_m|^2)^2}.$$

As for R^+, we can write in Ω:

$$\begin{aligned} -PU - U_j^- &= (PU_j^- - U_j^-) + \sum_{m \neq j} PU_m^- - PU^+ = 8\pi \sum_{m=0}^{2} \left(H(x, la_m) - H(la_j, la_m) \right) \\ &\quad - 8\pi \left(H(x,0) - H(la_j, 0) \right) + 2 \ln(\delta^2 l^{-2} \rho^2 + |l^{-1} x|^2) \\ &\quad - 2 \sum_{m \neq j} \ln \frac{\epsilon^2 l^{-2} \rho^2 + |l^{-1} x - a_m|^2}{|a_j - a_m|^2} + O(\epsilon^2 \rho^2 |\ln \epsilon\rho|), \end{aligned}$$

by means of by the choice of ϵ and Lemma 2.1. We compute now the Taylor expansion of

$$\begin{aligned} &\sum_{m=0}^{2} \left(H(x, la_m) - H(la_j, la_m) \right) - \left(H(x,0) - H(la_j, 0) \right) \\ &\quad = \frac{|x|^2 - l^2}{2\pi} + O(l|x - la_j|) = O(l|x - la_j| + |x - la_j|^2). \end{aligned} \tag{2.12}$$

Hence, we get that

$$\begin{aligned} \rho^2 e^{-PU} &= \rho^2 e^{U_j^-} (\delta^2 l^{-2} \rho^2 + |l^{-1} x|^2)^2 \prod_{m \neq j} \frac{|a_j - a_m|^4}{(\epsilon^2 l^{-2} \rho^2 + |l^{-1} x - a_m|^2)^2} \\ &\quad \times \left(1 + O(l^4 |\ln l| + l|x - la_j| + |x - la_j|^2) \right) \end{aligned} \tag{2.13}$$

uniformly in Ω, for any $j = 0, 1, 2$. Note that on $B_{l/2}(la_j)$

$$\begin{aligned} &\ln(\delta^2 l^{-2} \rho^2 + |l^{-1} x|^2) - \sum_{m \neq j} \ln \frac{\epsilon^2 l^{-2} \rho^2 + |l^{-1} x - a_m|^2}{|a_j - a_m|^2} \\ &= 2 \ln |l^{-1} x| - 2 \sum_{m \neq j} \ln \frac{|l^{-1} x - a_m|}{|a_j - a_m|} + O(l^2) \\ &= 2 \frac{a_j}{l} \cdot (x - la_j) - 2 \sum_{m \neq j} \frac{a_j - a_m}{3l} \cdot (x - la_j) + O(l^2 + l^{-2} |x - la_j|^2) \\ &= O(l^2 + l^{-2} |x - la_j|^2) \end{aligned}$$

because

$$\sum_{m \neq j} \frac{a_j - a_m}{3} = \frac{2}{3} a_j - \frac{1}{3} \sum_{m \neq j} a_m = a_j.$$

Hence, we deduce that

$$\rho^2 e^{-PU} = \rho^2 e^{U_j^-}\left(1 + O(l^2 + l|x - la_j| + l^{-2}|x - la_j|^2)\right) \tag{2.14}$$

in $B_{l/2}(la_j)$, $j = 0, 1, 2$, and then

$$|R^-| \le C\rho^2 e^{U_j^-}\left(l^{-2}|x - la_j|^2 + l|x - la_j| + l^2\right) + O\left(\sum_{m\ne j} \frac{\epsilon^2\rho^2}{(\epsilon^2\rho^2 + |x - la_m|^2)^2}\right).$$

In turn, we get that $\int_{B_{l/2}(la_j)} |R^-| = O(l^2|\ln l|)$ and the estimate

$$\begin{aligned}|W(x)R^-(x)| \le& \frac{1}{(1 + |y|^2)^{\frac{7}{8}}}\left(\epsilon^2 l^{-2}\rho^2|y|^2 + \epsilon l\rho|y| + l^2\right)\\ &+ C\sum_{m\ne j}\frac{(\epsilon\rho)^{\frac{7}{4}}}{(\epsilon^2\rho^2 + |x - la_m|^2)^{\frac{7}{8}}} = O(l^{\frac{7}{4}})\end{aligned}$$

does hold in $B_{l/2}(la_j)$, where $y = \frac{x - la_j}{\epsilon\rho} \in B_{l/2\epsilon\rho}(0)$.
Setting $\tilde{B} := B\setminus \bigcup_{j=0}^{2} B_{l/2}(la_j)$, we have that

$$\rho^2 W e^{U_j^-} \le \frac{(\epsilon\rho)^{\frac{7}{4}}}{(\epsilon^2\rho^2 + |x - la_j|^2)^{\frac{7}{8}}} = O(\epsilon^{\frac{7}{4}} l^{-\frac{7}{4}}\rho^{\frac{7}{4}}) = O(l^{\frac{7}{4}}) \tag{2.15}$$

in $\tilde{B}$. Since by Lemma 2.1

$$\begin{aligned}e^{-PU} &= \frac{(\delta^2\rho^2 + |x|^2)^2}{\prod_{j=0}^{2}(\epsilon^2\rho^2 + |x - la_j|^2)^2} e^{-8\pi H(x,0) + 8\pi\sum_{j=0}^{2} H(x,la_j)}\left(1 + O(l^4|\ln l|)\right)\\ &= O\left(\frac{(\delta^2\rho^2 + |x|^2)^2}{\prod_{j=0}^{2}(\epsilon^2\rho^2 + |x - la_j|^2)^2}\right),\end{aligned} \tag{2.16}$$

we get that in $\tilde{B}$

$$\begin{aligned}\rho^2 W e^{-PU} &\le C\frac{\rho^2}{(\epsilon\rho)^{\frac{1}{4}}}\frac{(\delta^2\rho^2 + |x - la_1|^2 + l^2)^2}{(\epsilon^2\rho^2 + |x - la_0|^2)^{\frac{7}{8}}\prod_{j=1}^{2}(\epsilon^2\rho^2 + |x - la_j|^2)^2}\\ &\le C'\frac{\rho^2}{(\epsilon\rho)^{\frac{1}{4}}} l^{-\frac{23}{4}}\left(1 + \frac{l^4}{(\epsilon^2\rho^2 + |x - la_1|^2)^2}\right) = O\left(l^{\frac{7}{4}}\right)\end{aligned} \tag{2.17}$$

in view of $\delta \le \epsilon$. Then, by (2.15) and (2.17) we get that $|WR^-| = O(l^{\frac{7}{4}})$ in $\tilde{B}$ and by (2.16) it follows easily that

$$\begin{aligned}\int_{\tilde{B}} |R^-| &= O\left(\rho^2\int_{\tilde{B}}\prod_{j\ne 0}(\epsilon^2\rho^2 + |x - la_j|^2)^{-2} + l^2\right) = O\left(l^4\int_{\tilde{B}} |x - la_2|^{-4} + l^2\right)\\ &= O\left(l^2\int_{l^{-1}\tilde{B}} |y - a_2|^{-4} + l^2\right) = O(l^2).\end{aligned}$$

The estimates on each $B_{l/2}(la_j)$ and in $\tilde{B}$ yield to

$$\|R^-\|_* = O(l^{\frac{7}{4}}), \quad \int_B |R^-| = O(l^2|\ln l|). \tag{2.18}$$

Finally, by (2.11) and (2.18) we get that

$$\|R\|_* \le \|R^+\|_* + \|R^-\|_* + \frac{1}{\pi}\left(\int_B |R^+| + |\int_B R^-|\right)(\sup_B W) = O(l^{\frac{3}{2}}|\ln l|)$$

because

$$\sup_B W \le \frac{C}{(\epsilon\rho)^{\frac{1}{4}}} = O(l^{-\frac{1}{2}}). \tag{2.19}$$

□

Remark 2.3 Let us observe that (2.7) implies $\rho^2 e^{PU} \le C\rho^2 e^{U^+}$ in $B_{l/2}(0)$ and (2.9) yields to

$$\rho^2 e^{PU} \le C' \rho^2 \le C\frac{\rho^2}{(\epsilon^2\rho^2 + |x - la_2|^2)^2} \le C\frac{\epsilon^2\rho^2}{(\epsilon^2\rho^2 + |x - la_2|^2)^2}$$

in $B \setminus B_{l/2}(0)$. Similarly, (2.14) gives $\rho^2 e^{-PU} \le C\rho^2 e^{U_j^-}$ in $B_{l/2}(la_j)$ and by (2.16) we deduce that in $\tilde{B}$ there holds

$$\begin{aligned}\rho^2 e^{-PU} &\le C''\rho^2\frac{(\delta^2\rho^2 + |x - la_1|^2 + l^2)^2}{\prod_{j=0}^2(\epsilon^2\rho^2 + |x - la_j|^2)^2} \le C'\frac{\rho^2 l^{-4}}{(\epsilon^2\rho^2 + |x - la_2|^2)^2}\\ &\le C\frac{\epsilon^2\rho^2}{(\epsilon^2\rho^2 + |x - la_2|^2)^2}.\end{aligned}$$

In conclusion, the global estimate

$$\rho^2(e^{PU} + e^{-PU}) \le D_0\left(e^{U^+} + \sum_{j=0}^2 e^{U_j^-}\right) \tag{2.20}$$

does hold in B, for some constant $D_0 > 0$. Moreover, (2.7) and (2.14) give that

$$\begin{aligned}\delta^2\rho^4(e^{PU} + e^{-PU})(\delta\rho y) &\to \tfrac{8}{(1+|y|^2)^2}\\ \epsilon^2\rho^4(e^{PU} + e^{-PU})(\epsilon\rho y + la_j) &\to \tfrac{8}{(1+|y|^2)^2}\end{aligned} \tag{2.21}$$

uniformly on compact set of $\mathbb{R}^2$ as $l \to 0$.

We will look for a solution u of problem (1.2) in the form $u = PU + \phi$, with ϕ a remainder term small in $\|\cdot\|_*$, which is $\frac{2\pi}{3}$−periodic (in the angular variable) and even in the second variable. Identifying $x = (x_1, x_2) \in \mathbb{R}^2$ and $x_1 + ix_2 \in \mathbb{C}$, let us introduce

$$\mathcal{S} = \{u \in L^1(B) : \ u(e^{\frac{2\pi i}{3}}x) = u(x), \quad u(\bar{x}) = u(x) \quad a.e. \text{ in } B\}$$

as the space of $\frac{2\pi}{3}$−periodic functions on B which are even in x_2. We have that $U^\pm$ and $\sum_{j=0}^2 e^{U_j^-}$ are in $\mathcal{S}$. Then

$$-\Delta PU = \rho^2\left(e^{U^+} - \frac{1}{\pi}\int_B e^{U^+}\right) - \left(\sum_{j=0}^2 e^{U_j^-} - \frac{1}{\pi}\sum_{j=0}^2\int_B e^{U_j^-}\right)$$

is invariant under $\frac{2\pi}{3}$−rotation and conjugation. Since $G(e^{\frac{2\pi i}{3}}x, y) = G(x, e^{-\frac{2\pi i}{3}}y)$ and $G(\bar{x}, y) = G(x, \bar{y})$, by the representation formula for PU:

$$PU(x) = \int_B G(x, y)(-\Delta PU)(y)dy, \quad \forall\, x \in B,$$

simple changes of variable yield to $PU \in \mathcal{S}$.

We take the remainder term ϕ in $W^{2,2}(B) \cap \mathcal{S}$ with $\int_B \phi = 0$. In terms of ϕ, equation (1.2) becomes

$$\begin{cases} L(\phi) = -[R + N(\phi)] & \text{in } B, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial B, \end{cases}$$

where

$$L(\phi) = \Delta\phi + \rho^2(e^{PU} + e^{-PU})\phi - \frac{\rho^2}{\pi}\int_B (e^{PU} + e^{-PU})\phi,$$

$$N(\phi) = \rho^2 e^{PU}(e^{\phi} - 1 - \phi) - \rho^2 e^{-PU}(e^{-\phi} - 1 + \phi)$$
$$- \frac{\rho^2}{\pi}\int_B e^{PU}(e^{\phi} - 1 - \phi) + \frac{\rho^2}{\pi}\int_B e^{-PU}(e^{-\phi} - 1 + \phi).$$

Recall that

$$R = \Delta PU + \rho^2(e^{PU} - e^{-PU}) - \frac{\rho^2}{\pi}\int_B (e^{PU} - e^{-PU}).$$

Let us stress that R, $L(\phi)$ and $N(\phi)$ are in $\mathcal{S}$ and there holds:

$$\int_B R = \int_B L(\phi) = \int_B N(\phi) = 0.$$

3 The finite dimensional reduction

Let us introduce the functions

$$Y_0 = 2\frac{|x|^2 - \delta^2\rho^2}{\delta^2\rho^2 + |x|^2}, \quad Z_{0,j} = 2\frac{|x - la_j|^2 - \epsilon^2\rho^2}{\epsilon^2\rho^2 + |x - la_j|^2} \quad j = 0, 1, 2$$

and

$$Y = 4\frac{\delta\rho x}{\delta^2\rho^2 + |x|^2}, \quad Z_j = 4\frac{\epsilon\rho(x - la_j)}{\epsilon^2\rho^2 + |x - la_j|^2} \quad j = 0, 1, 2.$$

Define

$$Z = \sum_{j=0}^{2} Z_j \cdot a_j = \sum_{j=0}^{2} 4\frac{\epsilon\rho(x - la_j)\cdot a_j}{\epsilon^2\rho^2 + |x - la_j|^2}$$

and observe that $Z \in \mathcal{S}$. Setting $\mathcal{S}_0 = \mathcal{S} \cap \{\int_B u = 0\}$, we are interested in solving the following linear problem associated to L: given $h \in L^\infty(B) \cap \mathcal{S}_0$, find a function $\phi \in W^{2,2}(B) \cap \mathcal{S}_0$

such that

$$\begin{cases} L(\phi) = h + c\Delta PZ & \text{in } B \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial B \\ \int_B \Delta PZ\phi = 0, \end{cases} \tag{3.1}$$

for some coefficient $c \in \mathbb{R}$.

We will follow the approach in [4] as re-formulated in [6,7], developed there for a Dirichlet linear problem (see also [5]). Asymptotically the kernel of L is composed by linear combinations of Y_0, $Z_{0,j}$, Y_k, $(Z_j)_k$ for $j = 0, 1, 2$ and $k = 1, 2$. The elements $\frac{2\pi}{3}$−periodic in the kernel of L are forced to be linear combinations of Y_0, $\sum_{j=0}^2 Z_{0,j}$, Re $\left(\sum_{j=0}^2 Z_j a_j^2\right)$ and Im $\left(\sum_{j=0}^2 Z_j a_j^2\right)$, where a_j^2 is the complex square. Note that

$$\left(\sum_{j=0}^{2} Z_j a_j^2\right)(\bar{x}) = \overline{\left(\sum_{j=0}^{2} Z_j a_j^2\right)(x)},$$

and then the kernel of L in $\mathcal{S}$ is spanned by Y_0, $\sum_{j=0}^2 Z_{0,j}$ and

$$Z = \text{Re}\left(\sum_{j=0}^{2} Z_j a_j^2\right) = \sum_{j=0}^{2} Z_j \cdot a_j.$$

Among them, only Z has "asymptotically null average on B", and then, we expect that asymptotically the kernel of L in $\mathcal{S}_0$ should be generated simply by Z. In Appendix B we will show that the picture above is correct:

Proposition 3.1 *Assume* (2.3)*. There exist $l_0 > 0$ and $C > 0$ such that, for any $h \in L^\infty(B) \cap \mathcal{S}_0$ and $0 < l \le l_0$, there is a unique solution $\phi \in W^{2,2}(B) \cap \mathcal{S}_0$ to* (3.1) *with*

$$\|\phi\|_\infty \le C|\ln l|\|h\|_*, \quad \|\phi\|_{H_0^1(B)} \le C\left(\|\phi\|_\infty + \|h\|_*\right). \tag{3.2}$$

Based on it, we solve now the following nonlinear auxiliary problem:

$$\begin{cases} -\Delta(PU + \phi) = \rho^2(e^{PU+\phi} - e^{-PU-\phi}) & \text{in } B \\ -\frac{\rho^2}{\pi}\int_B (e^{PU+\phi} - e^{-PU-\phi}) + c\Delta PZ & \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial B \\ \int_B \Delta PZ\phi = 0, \end{cases} \tag{3.3}$$

for some $\phi \in W^{2,2}(B) \cap \mathcal{S}_0$ and a coefficient $c \in \mathbb{R}$. The following result holds:

Proposition 3.2 *Assume* (2.3)*. There exist $C > 0$ and $l_0 > 0$ such that for any $0 < l \le l_0$ problem* (3.3) *has a unique solution $\phi_\rho(l) \in W^{2,2}(B) \cap \mathcal{S}_0$ which satisfies $\|\phi_\rho(l)\|_\infty \le C l^{\frac{3}{2}} \ln^2 l$. Furthermore, the function $l \to \phi_\rho(l)$ is a C^1 function in $L^\infty(B)$ and in $H^1(B)$.*

Proof We can rewrite (3.3) in the following way

$$L(\phi) = -(R + N(\phi)) - c\Delta PZ.$$

Let us denote by $\mathcal{L}_0^*$ the function space $\mathcal{L}_0 := L^\infty(B) \cap \mathcal{S}_0$ endowed with the norm $\|\cdot\|_*$ instead of $\|\cdot\|_\infty$. Proposition 3.1 ensures that the unique solution $\phi = T(h)$ of (3.1) defines a continuous linear map from the Banach space $\mathcal{L}_0^*$ into $\mathcal{L}_0$, with a norm bounded by a multiple of $|\ln l|$. Then, problem (3.3) becomes

$$\phi = \mathcal{A}(\phi) := -T\left[R + N(\phi)\right].$$

Let $\mathcal{B}_r := \left\{\phi \in \mathcal{L}_0 :\ \|\phi\|_\infty \leq r l^{\frac{3}{2}} \ln^2 l\right\}$, for some $r > 0$. Since

$$\begin{aligned}|\rho^2 e^{PU}(e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2)| &= |(\rho^2 e^{U^+} + R^+)(e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2)| \\ &\leq C^{'}(\max_{i=1,2} \|\phi_i\|_\infty)\|\phi_1 - \phi_2\|_\infty(\rho^2 e^{U^+} + |R^+|),\end{aligned}$$

by (2.11) we get that

$$\|\rho^2 e^{PU}(e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2)\|_* \leq C(\max_{i=1,2} \|\phi_i\|_\infty)\|\phi_1 - \phi_2\|_\infty$$

and

$$\left\|\frac{\rho^2}{\pi}\int_B e^{PU}(e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2)\right\|_* \leq C l^{-\frac{1}{2}}(\max_{i=1,2} \|\phi_i\|_\infty)\|\phi_1 - \phi_2\|_\infty,$$

in view of (2.19). Combining with the similar estimates for $\rho^2 e^{-PU}(e^{-\phi_1} - e^{-\phi_2} + \phi_1 - \phi_2)$, we get that

$$\|N(\phi_1) - N(\phi_2)\|_* \leq C l^{-\frac{1}{2}}(\max_{i=1,2} \|\phi_i\|_\infty)\|\phi_1 - \phi_2\|_\infty.$$

Since $N(0) = 0$, in particular we have that

$$\|N(\phi)\|_* \leq C l^{-\frac{1}{2}} \|\phi\|_\infty^2. \tag{3.4}$$

Hence, by Propositions 2.2 and 3.1 we get that

$$\begin{aligned}&\|\mathcal{A}(\phi)\|_\infty \leq C|\ln l|(\|R\|_* + \|N(\phi)\|_*) \leq C^{'} l^{\frac{3}{2}} \ln^2 l + C^{''} l^{\frac{5}{2}} |\ln^5 l| \leq r l^{\frac{3}{2}} \ln^2 l \\ &\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_\infty \leq C|\ln l| \|N(\phi_1) - N(\phi_2)\|_* \leq l|\ln^3 l| \|\phi_1 - \phi_2\|_\infty\end{aligned}$$

for all $\phi, \phi_1, \phi_2 \in \mathcal{B}_r$, with $r = 2C^{'}$ and l small enough. Since $\mathcal{A}$ is a contraction mapping of $\mathcal{B}_r$, a unique fixed point of $\mathcal{A}$ exists in $\mathcal{B}_r$. The regularity of the map $l \to \phi_\rho(l)$ follows using standard arguments (see for example [6]). □

After problem (3.3) has been solved, we find a solution to problem (1.2), if we are able to find $l > 0$ small such that the coefficients $c(l)$ in (3.3) vanish. Let us introduce the energy functional $E_\rho : H_0 \to \mathbb{R}$ given by

$$E_\rho(u) := \frac{1}{2}\int_B |\nabla u|^2 - \rho^2 \int_B (e^u + e^{-u}),$$

where $H_0 = H^1(B) \cap \mathcal{S}_0$. A critical point u of E_ρ on H_0 yields to a $\frac{2\pi}{3}$−periodic and x_2−even solution of

$$\begin{cases} -\Delta u = \rho^2(e^u - e^{-u}) - \lambda & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B,\end{cases}$$

for some Lagrange multiplier λ. Integrating the equation on B, we get that $\lambda = \frac{1}{\pi}\int_B (e^u - e^{-u})$ and we recover a solution to (1.2).

We introduce the finite dimensional restriction $\widetilde{E}_\rho : (0, l_0) \to \mathbb{R}$ given by

$$\widetilde{E}_\rho(l) := E_\rho\left(PU + \phi_\rho(l)\right). \tag{3.5}$$

Since the map $l \to \phi_\rho(l)$ is a C^1 function in $H^1(B)$, we have that $\widetilde{E}_\rho(l)$ is a C^1-function and the following result is standard:

Lemma 3.3 *Assume* (2.3). *Let l be a critical point of $\widetilde{E}_\rho$. If l is small, then $PU + \phi_\rho(l)$ is a critical point of E_ρ in H_0, namely a solution to problem* (1.2).

Proof If $l > 0$ is a critical point of $\widetilde{E}_\rho$, we have that

$$\int_B \nabla(PU + \phi_\rho)\nabla(\partial_l PU + \partial_l \phi_\rho) - \rho^2 \int_B (e^{PU+\phi_\rho} - e^{-PU-\phi_\rho})(\partial_l PU + \partial_l \phi_\rho) = 0.$$

Since $\partial_l PU$ and $\partial_l \phi_\rho$ have zero average on B, by (3.3) we can rewrite this condition as

$$c(l)\int_B \Delta PZ(\partial_l PU + \partial_l \phi_\rho) = c(l)\int_B \Delta Z(\partial_l PU + \partial_l \phi_\rho) = 0.$$

Differentiating $\int_B \Delta PZ\phi_\rho = \int_B \Delta Z\phi_\rho = 0$ in l, we get that

$$\int_B \Delta Z\partial_l\phi_\rho = -\int_B \partial_l(\Delta Z)\phi_\rho = \rho^2\sum_{j=0}^{2}\int_B e^{U_j^-}\left(Z_j\partial_l U_j^- + \partial_l Z_j\right)\cdot a_j\phi_\rho.$$

Since

$$\partial_l U^+ = Y_0\frac{\partial_l\delta}{\delta}, \quad \partial_l U_i^- = Z_{0,i}\frac{\partial_l\epsilon}{\epsilon} + \frac{1}{\epsilon\rho}Z_i\cdot a_i, \tag{3.6}$$

we get easily that

$$Z_j\partial_l U_j^- + \partial_l Z_j = O(\frac{1}{\epsilon\rho}).$$

Hence, by Proposition 3.2 we have that

$$\left|\int_B \Delta PZ\partial_l\phi_\rho\right| = O\left(\frac{\|\phi_\rho\|_\infty}{\epsilon\rho}\right) = O\left(\frac{l^{\frac{3}{2}}\ln^2 l}{\epsilon\rho}\right). \tag{3.7}$$

By (3.6) we deduce the expression for $\partial_l U$:

$$\partial_l U = Y_0\frac{\partial_l\delta}{\delta} - \sum_{j=0}^{2} Z_{0,j}\frac{\partial_l\epsilon}{\epsilon} - \frac{1}{\epsilon\rho}Z.$$

Arguing as in Lemma 2.1, it is easy to establish the following expansions:

$$PY_0 = Y_0 + 2 + O(\delta\rho), \quad PZ_{0,j} = Z_{0,j} + 2 + O(\epsilon\rho), \quad PZ = Z + O(\epsilon\rho l) \tag{3.8}$$

uniformly in Ω as $l \to 0$. As far as (3.8), let us simply observe that

$$\frac{\partial Z}{\partial\nu} = \epsilon\rho\frac{\partial}{\nu}\left(x\cdot\sum_{j=0}^{2} a_j\right) + O(\epsilon\rho l) = O(\epsilon\rho l) \quad \text{on } \partial B$$

$$\int_B Z = 3\epsilon\rho\int_B \frac{x - la_0}{|x - la_0|^2}\cdot a_0 + O(\epsilon\rho l) = O(\epsilon\rho l)$$

because $\sum_{j=0}^{2} a_j = 0$. Then, we get that

$$\partial_l PU = P(\partial_l U) = -\frac{1}{\epsilon\rho} Z + O\left(\frac{1}{l}\right)$$

uniformly in Ω as $l \to 0$. First, let us compute the following expansion:

$$\begin{aligned}
\int_B (\Delta PZ)(PZ) &= \int_B (\Delta Z)(PZ) = \int_B (\Delta Z)Z + O\left(\epsilon\rho l \int_B |\Delta Z|\right) \\
&= -\rho^2 \sum_{j,m=0}^{2} \int_B e^{U_j^-} (Z_j \cdot a_j)(Z_m \cdot a_m) + O(\epsilon\rho l) \\
&= -\sum_{j=0}^{2} \int_{|y|\le 1/\epsilon\rho} \frac{128(y\cdot a_j)^2}{(1+|y|^2)^4} \\
&\quad - \sum_{j\neq m} \int_{|y|\le 1/\epsilon\rho} \frac{128(y\cdot a_j)}{(1+|y|^2)^3} \frac{(y + l\epsilon^{-1}\rho^{-1}(a_j - a_m))\cdot a_m}{1+|y+l\epsilon^{-1}\rho^{-1}(a_j-a_m)|^2} + O(\epsilon\rho l) \\
&= -3 \int_{\mathbb{R}^2} \frac{128 y_1^2}{(1+|y|^2)^4} + o(1)
\end{aligned} \tag{3.9}$$

as $l \to 0$, by means of the Lebesgue's theorem. By the expansion of $\partial_l PU$ and (3.9) we deduce that

$$\begin{aligned}
\int_B (\Delta PZ)(\partial_l PU) &= \int_B (\Delta Z)(\partial_l PU) \\
&= -\frac{1}{\epsilon\rho} \int_B (\Delta Z)Z + O\left(\frac{1}{l}\int_B |\Delta Z|\right) \\
&= \frac{3}{\epsilon\rho}\left(\int_{\mathbb{R}^2} \frac{128 y_1^2}{(1+|y|^2)^4} + o(1)\right)
\end{aligned} \tag{3.10}$$

as $l \to 0$. Combining (3.7) and (3.10), finally we get that

$$0 = c(l) \int_B \Delta PZ(\partial_l PU + \partial_l \phi_\rho) = \frac{3c(l)}{\epsilon\rho}\left(\int_{\mathbb{R}^2} \frac{128 y_1^2}{(1+|y|^2)^4} + o(1)\right)$$

as $l \to 0$. It implies that $c(l) = 0$ for l small enough. □

4 Energy expansion

In view of Lemma 3.3, it is crucial to write down the expansion of $\widetilde{E}_\rho$ as $\rho, l \to 0$. We have that

Theorem 4.1 *Assume* (2.3). *It holds*

$$\widetilde{E}_\rho(l) = -64\pi \ln \rho + D_2 - 96\pi l^2 - 32\pi \epsilon^2 l^{-2} \rho^2 + o(l^2)$$

as $l \to 0$, *where* $D_2 = 96\pi \ln 2 - 16\pi + 48\pi \ln 3$.

Since $\epsilon = \frac{e^{5l^2-3}}{9\sqrt{8}}(1-l^6)^{-2} l^{-2}$, by (2.3) we can further write the expansion of $\widetilde{E}_\rho(l)$ as

$$\widetilde{E}_\rho(l) = -64\pi \ln \rho + D_2 - 96\pi l^2 - \frac{4\pi}{81 e^6} l^{-6} \rho^2 + o(l^2)$$

as $l \to 0$. The non-constant main order term $P_\rho(l) = -96\pi l^2 - \frac{4\pi}{81e^6} l^{-6}\rho^2$ has a strict maximum point at $(648e^6)^{-\frac{1}{8}} \rho^{\frac{1}{4}}$. It is now easy to see that

$$P_\rho((647e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}),\ P((649e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}) < P((648e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}).$$

Since at these points the values of P_ρ are of order $\sqrt{\rho}$ and $o(l^2) = o(\sqrt{\rho})$, we get that for ρ small the above inequalities still hold true for $\widetilde{E}_\rho$:

$$\widetilde{E}_\rho((647e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}),\ \widetilde{E}_\rho((649e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}) < \widetilde{E}_\rho((648e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}).$$

Hence, $\widetilde{E}_\rho$ has a maximum point $l_\rho \in ((647e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}}, (649e^6)^{-\frac{1}{8}}\rho^{\frac{1}{4}})$ (which is consistent with the assumption (2.3) for $C > 0$ large). Lemma 3.3 now yields to the existence part in Theorem 1.1. The verification of (1.3) follows by construction of the approximating solutions PU and (2.3).

Proof of Theorem 4.1 The function $\phi = \phi_\rho(l)$ satisfies

$$L(\phi) = -(R + N(\phi)) - c(l)\Delta PZ$$

as observed in the proof of Proposition 3.2. Multiply it by ϕ and integrate on B in order to get

$$\int_B |\nabla \phi|^2 = \rho^2 \int_B (e^{PU} + e^{-PU})\phi^2 + \int_B (R + N(\phi))\phi.$$

Recall that $\int_B \phi = \int_B \Delta PZ\phi = 0$. By (2.20), (3.4) and Propositions 2.2, 3.2, we get that

$$\int_B |\nabla \phi|^2 \le C\|\phi\|_\infty^2 + (\|R\|_* + \|N(\phi)\|_*)\|\phi\|_\infty \le C' l^3 \ln^4 l. \tag{4.1}$$

Since

$$\int_B \nabla PU \nabla \phi = \int_B \left(-\Delta U + \frac{1}{\pi}\int_B \Delta U\right)\phi = \rho^2 \int_B \left(e^{U^+} - \sum_{j=0}^{2} e^{U_j^-}\right)\phi$$

in view of $\int_B \phi = 0$, we get that

$$\int_B \nabla PU \nabla \phi - \rho^2 \int_B (e^{PU} - e^{-PU})\phi = -\int_B (R^+ - R^-)\phi.$$

In view of (2.20) we can write now $\widetilde{E}_\rho(l)$ in the form:

$$\widetilde{E}_\rho(l) = E(l) - \int_B (R^+ - R^-)\phi + \frac{1}{2}\int_B |\nabla\phi|^2 + O\left(\rho^2 \int_B (e^{PU} + e^{-PU})\phi^2\right)$$
$$= E(l) + O\left(\|\phi\|_\infty \int_B (|R^+| + |R^-|) + \int_B |\nabla\phi|^2 + \|\phi\|_\infty^2\right),$$

where

$$E(l) = \frac{1}{2}\int_B |\nabla PU|^2 - \rho^2 \int_B (e^{PU} + e^{-PU}).$$

By (2.11), (2.18), (4.1) and Proposition 3.2 finally we get:

$$\widetilde{E}_\rho(l) = E(l) + o(l^2) \tag{4.2}$$

as $l \to 0$.

We are led now to expand the functional $E(l)$. First, we consider the gradient term:

$$\int_B |\nabla PU|^2 = \rho^2 \int_B \left(e^{U^+} - \sum_{j=0}^{2} e^{U_j^-}\right) PU$$
$$= \rho^2 \int_B e^{U^+}\left[U^+ + 8\pi\,(H(x,0) - H(0,0)) - 8\pi \sum_{j=0}^{2}\big(H(x,la_j) - H(0,la_j)\big) + O(l^4|\ln l|)\right]$$
$$+ \rho^2 \sum_{j=0}^{2}\int_B e^{U_j^-}\left[U_j^- - 8\pi\,\big(H(x,0) - H(la_j,0)\big) + 8\pi \sum_{m=0}^{2}(H(x,la_m)\right.$$
$$\left. - H(la_j,la_m)) + O(l^4|\ln l|)\right] + 2\rho^2 \sum_{j=0}^{2}\int_B e^{U^+}\ln(\epsilon^2 l^{-2}\rho^2 + |l^{-1}x - a_j|^2)$$
$$- 2\rho^2 \sum_{j=0}^{2}\int_B e^{U_j^-}\left(-2\ln 3 + \sum_{m\neq j}\ln(\epsilon^2 l^{-2}\rho^2 + |l^{-1}x - a_m|^2) - \ln(\delta^2 l^{-2}\rho^2 + |l^{-1}x|^2)\right)$$
$$= I + II + III + IV$$

by means of Lemma 2.1.

As far as I, by (2.5) we get that

$$I = \int_{B_{1/\delta\rho(0)}} \frac{8}{(1+|y|^2)^2}\left(-4\ln\rho - \ln\frac{\delta^2}{8} - 2\ln(1+|y|^2)\right) + O(l^4|\ln l|)$$
$$= -32\pi\ln\rho - 96\pi\ln l + (48\pi\ln 2 - 64\pi) + 48\pi l^2 + O(l^4|\ln l|)$$

in view of $\delta = \frac{1}{\sqrt{8}} e^{3(1-l^2)} l^6$, where

$$\int_{\mathbb{R}^2} \frac{\ln(1+|y|^2)}{(1+|y|^2)^2} = \pi \int_0^{+\infty} \frac{\ln(1+s)}{(1+s)^2} = -\pi\,\frac{\ln(1+s)}{1+s}\Big|_0^{+\infty} + \pi \int_0^{+\infty} \frac{1}{(1+s)^2} = \pi.$$

Similarly, by (2.12) we deduce that

$$
\begin{aligned}
II &= \sum_{j=0}^{2} \int\limits_{B_{1/2}(la_j)} \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x-la_j|^2)^2} \ln \frac{8\epsilon^2}{(\epsilon^2\rho^2+|x-la_j|^2)^2} + O(l^3)\\
&= \sum_{j=0}^{2} \int\limits_{B_{1/2\epsilon\rho}(0)} \frac{8}{(1+|y|^2)^2} \left(-4\ln\rho - \ln\frac{\epsilon^2}{8} - 2\ln(1+|y|^2)\right) + O(l^3)\\
&= -96\pi \ln\rho + 96\pi \ln l + 3(48\pi \ln 2 + 32\pi + 32\pi \ln 3) - 240\pi l^2 + O(l^3)
\end{aligned}
$$

in view of $\epsilon = \frac{e^{5l^2-3}}{9\sqrt{8}}(1-l^6)^{-2}l^{-2}$. As far as III, let us expand the following integrals:

$$
\begin{aligned}
&\int\limits_{B_{l/2}(la_j)} \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2} \ln(\epsilon^2 l^{-2}\rho^2 + |l^{-1}x - a_j|^2)\\
&\quad= \int\limits_{B_{1/2}(a_j)} \frac{8\delta^2 l^{-2}\rho^2}{(\delta^2 l^{-2}\rho^2+|y|^2)^2} \ln(\epsilon^2 l^{-2}\rho^2 + |y-a_j|^2)\\
&\quad= 8\delta^2 l^{-2}\rho^2 \int\limits_{B_{1/2}(a_j)} |y|^{-4} \ln(\epsilon^2 l^{-2}\rho^2 + |y-a_j|^2) + O(\delta^4 l^{-4}\rho^4)\\
&\quad= 8\delta^2 l^{-2}\rho^2 \int\limits_{B_{1/2}(a_j)} |y|^{-4} \ln|y-a_j|^2 + o(\delta^2 l^{-2}\rho^2)\\
&\quad= \int\limits_{B_{l/2}(la_j)} \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2} \ln|l^{-1}x - a_j|^2 + o(\delta^2 l^{-2}\rho^2)
\end{aligned}
$$

because of $2\ln|y-a_j| \le \ln(\epsilon^2 l^{-2}\rho^2 + |y-a_j|^2) \le 0$, $(\delta^2 l^{-2}\rho^2 + |y|^2)^{-2} = |y|^{-4} + O(\delta^2 l^{-2}\rho^2)$ in $B_{1/2}(a_j)$ and the Lebesgue's theorem;

$$
\begin{aligned}
&\int\limits_{B\setminus B_{l/2}(la_j)} \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2} \ln(\epsilon^2 l^{-2}\rho^2 + |l^{-1}x - a_j|^2)\\
&= \int\limits_{B\setminus B_{l/2}(la_j)} \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2} \left(\ln|l^{-1}x-a_j|^2 + \frac{\epsilon^2 l^{-2}\rho^2}{|l^{-1}x-a_j|^2} + O(\epsilon^4 l^{-4}\rho^4)\right)\\
&= \int\limits_{B\setminus B_{l/2}(la_j)} \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2} \ln|l^{-1}x-a_j|^2 + \epsilon^2 l^{-2}\rho^2\\
&\int\limits_{B_{1/\delta\rho}\setminus B_{l/2\delta\rho}(l/\delta\rho\, a_j)} \frac{8}{(1+|y|^2)^2} |\delta l^{-1}\rho y - a_j|^{-2} + O(\epsilon^4 l^{-4}\rho^4)\\
&= \int\limits_{B\setminus B_{l/2}(la_j)} \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2} \ln|l^{-1}x-a_j|^2 + 8\pi\epsilon^2 l^{-2}\rho^2 + o(\epsilon^2 l^{-2}\rho^2)
\end{aligned}
$$

because of $|\delta l^{-1}\rho y - a_j|^{-2} \le 4$ in $B_{1/\delta\rho}\setminus B_{l/2\delta\rho}(l/\delta\rho\, a_j) \to \mathbb{R}^2$ as $\delta l^{-1}\rho \to 0$ and the Lebesgue' theorem.

Summing up the previous expansions, we get that

$$\int_B \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2}\ln(\epsilon^2 l^{-2}\rho^2+|l^{-1}x-a_j|^2)$$
$$=\int_B \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2}\ln|l^{-1}x-a_j|^2+8\pi\epsilon^2 l^{-2}\rho^2+o(\delta^2 l^{-2}\rho^2+\epsilon^2 l^{-2}\rho^2). \quad (4.3)$$

Let us note that (4.3) holds whenever $\delta l^{-1}\rho, \epsilon l^{-1}\rho \to 0$. Then, by (4.3) we get for III and IV:

$$III = 2\rho^2\sum_{j=0}^{2}\int_B e^{U^+}\ln(\epsilon^2 l^{-2}\rho^2+|l^{-1}x-a_j|^2)$$
$$=2\int_B \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2}\ln|l^{-3}x^3-1|^2+48\pi\epsilon^2 l^{-2}\rho^2+o(l^2)$$

and

$$IV = 96\pi\ln 3-2\sum_{j=0}^{2}\int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x-la_j|^2)^2}\left(\ln\frac{\prod_{m\neq j}(\epsilon^2 l^{-2}\rho^2+|l^{-1}x-a_m|^2)}{\delta^2 l^{-2}\rho^2+|l^{-1}x|^2}\right)$$
$$+O(l^4)$$
$$=96\pi\ln 3+2\sum_{j=0}^{2}\int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x|^2)^2}\left(\ln\frac{\delta^2 l^{-2}\rho^2+|l^{-1}x+a_j|^2}{\prod_{m\neq j}(\epsilon^2 l^{-2}\rho^2+|l^{-1}x+a_j-a_m|^2)}\right)$$
$$+O(l^4|\ln l|)$$
$$=96\pi\ln 3+2\int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x|^2)^2}\ln|l^{-3}x^3+1|^2$$
$$-2\sum_{j=0}^{2}\int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x|^2)^2}\ln|l^{-2}x^2+3l^{-1}xa_j+3a_j^2|^2-96\pi\epsilon^2 l^{-2}\rho^2+o(l^2)$$
$$=96\pi\ln 3+2\int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x|^2)^2}\ln|l^{-3}x^3+1|^2$$
$$-2\int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2+|x|^2)^2}\ln|l^{-6}x^6+27|^2-96\pi\epsilon^2 l^{-2}\rho^2+o(l^2),$$

where x^2, $x^3$and x^6 denote powers of a complex number $x\in\mathbb{C}$. By the change of variable $t=l^{-3}r^3$, we compute now

$$2\int_B \frac{8\delta^2\rho^2}{(\delta^2\rho^2+|x|^2)^2}\ln|l^{-3}x^3-1|^2 = 32\int_0^1 \frac{\delta^2\rho^2 r dr}{(\delta^2\rho^2+r^2)^2}\int_0^{2\pi}\ln|l^{-3}r^3e^{3i\theta}-1|d\theta$$
$$=\frac{32\delta^2 l^{-2}\rho^2}{3}\int_0^{1/l^3}\frac{dt}{t^{\frac{1}{3}}(\delta^2 l^{-2}\rho^2+t^{\frac{2}{3}})^2}\int_0^{2\pi}\ln|te^{3i\theta}-1|d\theta$$

$$= \frac{32\delta^2 l^{-2}\rho^2}{3} \int_0^{1/l^3} t^{-\frac{5}{3}} dt \int_0^{2\pi} \ln|te^{i\theta} - 1| d\theta + o(\delta^2 l^{-2}\rho^2)$$

$$= 24\delta^2 l^{-2}\rho^2 \int_0^{1/l^3} \Delta(t^{\frac{-2}{3}}) t dt \int_0^{2\pi} \ln|te^{i\theta} - 1| d\theta + o(\delta^2 l^{-2}\rho^2)$$

$$= 24\delta^2 l^{-2}\rho^2 \int_{B_{1/l^3}} \Delta(|x|^{\frac{-2}{3}}) \ln|x - 1| + o(\delta^2 l^{-2}\rho^2)$$

because $\int_0^{2\pi} \ln|te^{i\theta} - 1| d\theta = O(t)$ as $t \to 0$ and the Lebesgue's theorem. Since

$$\int_\Omega \Delta u_0 \ln|x-1| = \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial \nu} \ln|x-1| - u_0 \frac{x-1}{|x-1|^2} \cdot \nu \right) + 2\pi u_0(1)$$

for any domain Ω containing the singularity 1, we get

$$2 \int_B \frac{8\delta^2\rho^2}{(\delta^2\rho^2 + |x|^2)^2} \ln|l^{-3}x^3 - 1|^2 = 48\pi\delta^2 l^{-2}\rho^2 + o(\delta^2 l^{-2}\rho^2). \tag{4.4}$$

Similarly, it is straightforward to see that

$$\begin{aligned} 2 \int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2 + |x|^2)^2} \ln|l^{-3}x^3 + 1|^2 &= 48\pi\epsilon^2 l^{-2}\rho^2 + o(\epsilon^2 l^{-2}\rho^2) \\ 2 \int_B \frac{8\epsilon^2\rho^2}{(\epsilon^2\rho^2 + |x|^2)^2} \ln|l^{-6}x^6 + 27|^2 &= 96\pi \ln 3 + 32\pi\epsilon^2 l^{-2}\rho^2 + o(\epsilon^2 l^{-2}\rho^2). \end{aligned} \tag{4.5}$$

By (4.4)–(4.5) we get the expansions for III and IV:

$$III = 48\pi\epsilon^2 l^{-2}\rho^2 + o(l^2), \quad IV = -80\pi\epsilon^2 l^{-2}\rho^2 + o(l^2).$$

By the estimates on I, II, III and IV finally we get for the gradient term:

$$\frac{1}{2} \int_B |\nabla PU|^2 = -64\pi \ln\rho + D_1 - 96\pi l^2 - 16\pi\epsilon^2 l^{-2}\rho^2 + o(l^2) \tag{4.6}$$

where $D_1 = 96\pi \ln 2 + 16\pi + 48\pi \ln 3$.

To conclude the asymptotic expansion of $E(l)$, we need to consider the nonlinear term $\rho^2 \int_B (e^{PU} + e^{-PU})$. By (2.6) we can write

$$\begin{aligned} &\rho^2 \int_B e^{PU} \\ &= \rho^2 \int_B \prod_{j=0}^{2} (\epsilon^2 l^{-2}\rho^2 + |l^{-1}x - a_j|^2)^2 e^{U^+} \left(1 + O(|x|^2 + l^4 |\ln l|)\right) \\ &= \rho^2 \int_B \left(|l^{-3}x^3 - 1|^4 + 2\epsilon^2 l^{-2}\rho^2 \sum_{m=0}^{2} |l^{-3}x^3 - 1|^2 |l^{-2}x^2 + a_m l^{-1}x + a_m^2|^2 \right) e^{U^+} \end{aligned}$$

$$+\epsilon^4 l^{-4}\rho^4 O\left(\rho^2\int_B e^{U^+}(1+|\frac{x}{l}|^8)\right)+O\left(\rho^2\int_B e^{U^+}(1+|\frac{x}{l}|^{12})(|x|^2+l^4|\ln l|)\right)$$
$$=\rho^2\int_B\left(1+6\epsilon^2 l^{-2}\rho^2+O(|\frac{x}{l}|^3+|\frac{x}{l}|^{12}+\epsilon^2 l^{-2}\rho^2|\frac{x}{l}|+\epsilon^2 l^{-2}\rho^2|\frac{x}{l}|^{10})\right)e^{U^+}$$
$$+O(l^4|\ln l|)=8\pi+48\pi\epsilon^2 l^{-2}\rho^2+o(l^2). \tag{4.7}$$

Splitting the integral on each $B_{l/2}(la_j)$ and $\tilde{B}$, by (2.13) and (2.16) we can write

$$\rho^2\int_B e^{-PU}=81\sum_{j=0}^{2}\int_{B_{1/2}(a_j)}\frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y-a_j|^2)^2}(\delta^2 l^{-2}\rho^2+|y|^2)^2$$
$$\times\prod_{m\neq j}(\epsilon^2 l^{-2}\rho^2+|y-a_m|^2)^{-2}\left(1+O(l^4|\ln l|+l^2|y-a_j|+l^2|y-a_j|^2)\right)$$
$$+\rho^2\int_{\tilde{B}}\frac{(\delta^2\rho^2+|x|^2)^2}{\prod_{m=0}^2(\epsilon^2\rho^2+|x-la_m|^2)^2}e^{-8\pi H(x,0)+8\pi\sum_{m=0}^2 H(x,la_m)}\left(1+O(l^4|\ln l|)\right)$$
$$=81\sum_{j=0}^{2}\int_{B_{1/2}(a_j)}\frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y-a_j|^2)^2}|y|^4|y^2+a_jy+a_j^2|^{-4}$$
$$-324\epsilon^2 l^{-2}\rho^2\sum_{j=0}^{2}\int_{B_{1/2}(a_j)}\frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y-a_j|^2)^2}|y|^4(|y|^2+a_j\cdot y+1)|y^2+a_jy+a_j^2|^{-6}$$
$$+\rho^2 l^{-6}\int_{l^{-1}\tilde{B}}\frac{(\delta^2 l^{-2}\rho^2+|y|^2)^2}{\prod_{j=0}^2(\epsilon^2 l^{-2}\rho^2+|y-a_j|^2)^2}e^{-8\pi H(ly,0)+8\pi\sum_{m=0}^2 H(ly,la_m)}+O(l^3)$$
$$=81\sum_{j=0}^{2}\int_{B_{1/2}(0)}\frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y|^2)^2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}$$
$$-32\pi\epsilon^2 l^{-2}\rho^2+648\epsilon^2 l^{-2}\rho^2\int_{\tilde{R}}\frac{|y|^4}{|y^3-1|^4}+o(l^2), \tag{4.8}$$

where $\tilde{R}=\mathbb{R}^2\setminus\cup_{j=0}^2 B_{1/2}(a_j)$, because

$$e^{-8\pi H(0,0)+8\pi\sum_{m=0}^2 H(0,la_m)}l^{-4}=648\epsilon^2(1+o(1))$$

as $l\to 0$.

Adding (4.7) and (4.8) we obtain the expansion:

$$\rho^2\int_B(e^{PU}+e^{-PU})=8\pi+16\pi\epsilon^2 l^{-2}\rho^2+648\epsilon^2 l^{-2}\rho^2\int_{\tilde{R}}\frac{|y|^4}{|y^3-1|^4}$$
$$+81\int_{B_{1/2}(0)}\frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y|^2)^2}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}\right)+o(l^2). \tag{4.9}$$

In polar coordinates with respect to 0, letting $\alpha = \epsilon l^{-1}\rho$ the following term rewrites as

$$\begin{aligned}
&\int_{B_{1/2}(0)} \frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y|^2)^2}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}\right)\\
&=\int_0^{1/2\alpha}\frac{8rdr}{(1+r^2)^2}\int_0^{2\pi}\left(\sum_{j=0}^{2}|\alpha re^{i\theta}+a_j|^4|\alpha^2r^2e^{2i\theta}+3\alpha ra_je^{i\theta}+3a_j^2|^{-4}\right)d\theta\\
&=\int_0^{2\pi}\sum_{j=0}^{2}\left[-\frac{4}{1+r^2}|\alpha re^{i\theta}+a_j|^4|\alpha^2r^2e^{2i\theta}+3\alpha ra_je^{i\theta}+3a_j^2|^{-4}\,|_0^{1/2\alpha}\right.\\
&\quad\left.+\int_0^{1/2\alpha}\frac{4\alpha}{1+r^2}f_j(\alpha r,\theta)dr\right]d\theta\\
&=\int_0^{2\pi}\left[-\frac{16\alpha^2}{1+4\alpha^2}\sum_{j=0}^{2}\left(|\frac{1}{2}e^{i\theta}+a_j|^4|\frac{1}{4}e^{2i\theta}+\frac{3}{2}a_je^{i\theta}+3a_j^2|^{-4}\right)+\frac{4}{27}\right.\\
&\quad\left.+\int_0^{1/2}\frac{4\alpha^2}{\alpha^2+r^2}\left(\sum_{j=0}^{2}f_j(r,\theta)\right)dr\right]d\theta,
\end{aligned}$$

where

$$\begin{aligned}
f_j(r,\theta)=&\,4|re^{i\theta}+a_j|^2|r^2e^{2i\theta}+3ra_je^{i\theta}+3a_j^2|^{-4}(re^{i\theta}+a_j)\cdot e^{i\theta}\\
&-4|re^{i\theta}+a_j|^4|r^2e^{2i\theta}+3ra_je^{i\theta}+3a_j^2|^{-6}(r^2e^{2i\theta}+3ra_je^{i\theta}+3a_j^2)\cdot(2re^{2i\theta}+3a_je^{i\theta}).
\end{aligned}$$

Set $f(r,\theta)=\sum_{j=0}^{2}f_j(r,\theta)$. Recalling that $\sum_{j=0}^{2}a_j^2=0$, it is tedious but straightforward to show that

$$f(0,\theta)=\sum_{j=0}^{2}\left(\frac{4}{81}a_j\cdot e^{i\theta}-\frac{4}{81}a_j\cdot e^{i\theta}\right)=0$$

$$\frac{\partial}{\partial r}f(0,\theta)=-\frac{8}{243}\left(\sum_{j=0}^{2}a_j^2\right)\cdot e^{2i\theta}=0.$$

Since $|f(r,\theta)|\le Cr^2$ in $(0,\frac{1}{2})\times[0,2\pi]$, we get that

$$\begin{aligned}
&\int_{B_{1/2}(0)} \frac{8\epsilon^2 l^{-2}\rho^2}{(\epsilon^2 l^{-2}\rho^2+|y|^2)^2}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}\right)\\
&=\frac{8}{27}\pi-16\alpha^2\int_0^{2\pi}\sum_{j=0}^{2}\left(\left|\frac{1}{2}e^{i\theta}+a_j\right|^4\left|\frac{1}{4}e^{2i\theta}+\frac{3}{2}a_je^{i\theta}+3a_j^2\right|^{-4}\right)d\theta\\
&\quad+4\alpha^2\int_0^{2\pi}d\theta\int_0^{1/2}r^{-2}f(r,\theta)dr+o(\alpha^2)
\end{aligned}$$

as $\alpha \to 0$. Since

$$f(r,\theta)=\frac{\partial}{\partial r}\left(\sum_{j=0}^{2}|re^{i\theta}+a_j|^4|r^2e^{2i\theta}+3ra_je^{i\theta}+3a_j^2|^{-4}\right),$$

we can write

$$\begin{aligned}
&\int_0^{2\pi}d\theta\int_0^{1/2}\frac{4}{r^2}f(r,\theta)\\
&=\int_0^{2\pi}\left[\frac{4}{r^2}\left(\sum_{j=0}^{2}|re^{i\theta}+a_j|^4|r^2e^{2i\theta}+3ra_je^{i\theta}+3a_j^2|^{-4}-\frac{1}{27}\right)\Big|_0^{1/2}\right.\\
&\quad\left.+\int_0^{1/2}\frac{8}{r^3}\left(\sum_{j=0}^{2}|re^{i\theta}+a_j|^4|r^2e^{2i\theta}+3ra_je^{i\theta}+3a_j^2|^{-4}-\frac{1}{27}\right)dr\right]d\theta\\
&=16\int_0^{2\pi}\left(\sum_{j=0}^{2}\left|\frac{1}{2}e^{i\theta}+a_j\right|^4\left|\frac{1}{4}e^{2i\theta}+\frac{3}{2}a_je^{i\theta}+3a_j^2\right|^{-4}\right)d\theta-\frac{32\pi}{27}\\
&\quad+\int_{B_{1/2}}\frac{8}{|y|^4}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}-\frac{1}{27}\right).
\end{aligned}$$

So, we get that

$$\begin{aligned}
&\int_{B_{1/2}(0)}\frac{8\epsilon^2l^{-2}\rho^2}{(\epsilon^2l^{-2}\rho^2+|y|^2)^2}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}\right)=\frac{8}{27}\pi-\frac{32\pi}{27}\epsilon^2l^{-2}\rho^2\\
&+\epsilon^2l^{-2}\rho^2\int_{B_{1/2}}\frac{8}{|y|^4}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}-\frac{1}{27}\right)+o(l^2).
\end{aligned}$$

Finally, by (4.9) the following expansion does hold:

$$\begin{aligned}
\rho^2\int_B(e^{PU}+e^{-PU})&=32\pi-80\pi\epsilon^2l^{-2}\rho^2+648\epsilon^2l^{-2}\rho^2\int_{\tilde R}\frac{|y|^4}{|y^3-1|^4}\\
&+81\epsilon^2l^{-2}\rho^2\int_{B_{1/2}}\frac{8}{|y|^4}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}-\frac{1}{27}\right)+o(l^2).
\end{aligned}\tag{4.10}$$

Combining (4.6), (4.10) and the following Lemma

Lemma 4.2 *There holds*

$$\int_{\tilde R}\frac{|y|^4}{|y^3-1|^4}+\int_{B_{1/2}}\frac{1}{|y|^4}\left(\sum_{j=0}^{2}|y+a_j|^4|y^2+3a_jy+3a_j^2|^{-4}-\frac{1}{27}\right)=\frac{4}{27}\pi.$$

we obtain that

$$E(l) = -64\pi \ln \rho + D_2 - 96\pi l^2 - 32\pi \epsilon^2 l^{-2} \rho^2 + o(l^2),$$

where $D_2 = 96\pi \ln 2 - 16\pi + 48\pi \ln 3$. With the aid of (4.2), the proof is done. □

5 Appendix A

In this Appendix we will establish the validity of Lemma 4.2. We need to compute the value of

$$I_0 := \int_{\tilde{R}} \frac{|y|^4}{|y^3 - 1|^4} + \int_{B_{1/2}} \frac{1}{|y|^4} \left(\sum_{j=0}^{2} |y + a_j|^4 |y^2 + 3a_j y + 3a_j^2|^{-4} - \frac{1}{27} \right),$$

where $\tilde{R} = \mathbb{R}^2 \setminus \cup_{j=0}^2 B_{1/2}(a_j)$. Since

$$\begin{aligned}
&\int_{B_{1/2}} \frac{1}{|y|^4} \left(\sum_{j=0}^{2} |y + a_j|^4 |y^2 + 3a_j y + 3a_j^2|^{-4} - \frac{1}{27} \right) \\
&= \sum_{j=0}^{2} \int_{B_{1/2}} \left(|y + a_j|^4 |y|^{-4} |y^2 + 3a_j y + 3a_j^2|^{-4} - \frac{|y|^{-4}}{81} \right) \\
&= \sum_{j=0}^{2} \int_{B_{1/2}(a_j)} \left(|y|^4 |y - a_j|^{-4} |y^2 + a_j y + a_j^2|^{-4} - \frac{|y - a_j|^{-4}}{81} \right) \\
&= \sum_{j=0}^{2} \int_{B_{1/2}(a_j)} \left(\frac{|y|^4}{|y^3 - 1|^4} - \frac{|y - a_j|^{-4}}{81} \right),
\end{aligned}$$

let us rewrite I_0 in a more useful way:

$$\begin{aligned}
I_0 &= \int_{\tilde{R}} \frac{|y|^4}{|y^3 - 1|^4} + \sum_{j=0}^{2} \int_{B_{1/2}(a_j) \setminus C_{\epsilon,j}} \left(\frac{|y|^4}{|y^3 - 1|^4} - \frac{|y - a_j|^{-4}}{81} \right) + o(1) \\
&= \int_{\mathbb{R}^2 \setminus \cup_{j=0}^2 C_{\epsilon,j}} \frac{|y|^4}{|y^3 - 1|^4} - \frac{1}{81} \sum_{j=0}^{2} \int_{B_{1/2}(a_j) \setminus C_{\epsilon,j}} |y - a_j|^{-4} + o(1)
\end{aligned}$$

as $\epsilon \to 0$, where in complex notations $C_{\epsilon,j} = a_j \, (B_\epsilon(1))^{\frac{1}{3}}$ and

$$(B_\epsilon(1))^{\frac{1}{3}} = \{y \in B_{1/2}(1) : \ y^3 \in B_\epsilon(1)\}.$$

Setting $C = \{y = \rho e^{i\theta} \in \mathbb{C} : \rho \geq 0, \theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]\}$, by the change of variable $y \to a_j y$ we get that

$$I_0 = \sum_{j=0}^{2} \int\limits_{a_j(C \setminus B_\epsilon(1)^{\frac{1}{3}})} \frac{|y|^4}{|y^3 - 1|^4} - \frac{1}{81} \sum_{j=0}^{2} \int\limits_{a_j(B_{1/2}(1) \setminus B_\epsilon(1)^{\frac{1}{3}})} |y - a_j|^{-4} + o(1)$$

$$= 3 \int\limits_{C \setminus B_\epsilon(1)^{\frac{1}{3}}} \frac{|y|^4}{|y^3 - 1|^4} - \frac{1}{27} \int\limits_{B_{1/2}(1) \setminus B_\epsilon(1)^{\frac{1}{3}}} |y - 1|^{-4} + o(1).$$

Under the change of variable $z = y^3$, the volume element is $dz = 9|y|^4 dy$ and I_0 becomes

$$I_0 = \frac{1}{3} \int\limits_{\mathbb{R}^2 \setminus B_\epsilon(1)} \frac{dz}{|z - 1|^4} - \frac{1}{27} \int\limits_{B_{1/2}(1) \setminus B_\epsilon(1)^{\frac{1}{3}}} |y - 1|^{-4} + o(1)$$

as $\epsilon \to 0$.

It is crucial now to understand the asymptotic shape of $B_\epsilon(1)^{\frac{1}{3}}$ around 1 for ϵ small. In polar coordinates let us remark that $\rho e^{i\theta} + 1 \in B_\epsilon(1)^{\frac{1}{3}}$ is equivalent to:

$$|(\rho e^{i\theta} + 1)^3 - 1|^2 = |3\rho e^{i\theta} + 3\rho^2 e^{2i\theta} + \rho^3 e^{3i\theta}|^2 = g(\rho, \theta) \leq \epsilon^2,$$

where

$$g(\rho, \theta) = 9\rho^2 + 18\rho^3 \cos\theta + 3\rho^4 \left(1 + 4\cos^2\theta\right) + 6\rho^5 \cos\theta + \rho^6.$$

Observe that for δ_0 small

$$\frac{\partial g}{\partial \rho} = 18\rho + O(\rho^2) > 0 \quad \forall\, 0 \leq \rho \leq \delta_0, \ \theta \in [0, 2\pi].$$

Since $g(0, \theta) = 0$ and $g_\epsilon(\delta_0, \theta) \geq \delta_0^2$ for any $\theta \in [-\pi, \pi]$ and δ_0 small, we get that for any $0 < \epsilon < \delta_0$ and $\theta \in [0, 2\pi]$ there exists an unique $\rho_\epsilon = \rho_\epsilon(\theta)$ so that

$$\{\rho \in [0, \delta_0] : g_\epsilon(\rho, \theta) \leq \epsilon^2\} = [0, \rho_\epsilon(\theta)].$$

We need to identify the asymptotic of ρ_ϵ as $\epsilon \to 0$. To this aim, introduce

$$\rho_\pm = \rho_\pm(\theta) = \frac{\epsilon}{3} \left(1 - \frac{\epsilon}{3} \cos\theta + \frac{11\cos^2\theta - 1}{54} \epsilon^2 \pm \epsilon^3\right)$$

and compute

$$g_\epsilon(\rho_\pm, \theta) = \epsilon^2 + \epsilon^5 \left(\pm 2 + \frac{8}{27} \cos^3\theta - \frac{4}{81} \cos\theta\right) + O(\epsilon^6)$$

uniformly for $\theta \in [0, 2\pi]$. Since $|\frac{8}{27} \cos^3\theta - \frac{4}{81} \cos\theta| \leq \frac{28}{81}$, we get that for ϵ small $\pm[g_\epsilon(\rho_\pm, \theta) - \epsilon^2] > 0$ for any $\theta \in [0, \pi]$. Therefore, for ϵ small $\rho_- < \rho_\epsilon < \rho_+$ or equivalently

$$\rho_\epsilon(\theta) = \frac{\epsilon}{3} \left(1 - \frac{\epsilon}{3} \cos\theta + \frac{11\cos^2\theta - 1}{54} \epsilon^2 + O(\epsilon^3)\right) \tag{5.11}$$

does hold uniformly on $\theta \in [0, 2\pi]$ as $\epsilon \to 0$.

We are now in position to determine the value of I_0:

$$\begin{aligned} I_0 &= \frac{2\pi}{3}\int_\epsilon^\infty r^{-3}dr - \frac{1}{27}\int_0^{2\pi} d\theta \int_{\rho_\epsilon(\theta)}^{1/2} r^{-3}dr + o(1) \\ &= \frac{\pi}{3}\epsilon^{-2} + \frac{1}{54}\int_0^{2\pi} \left(4 - \rho_\epsilon^{-2}(\theta)\right) d\theta + o(1) \\ &= \frac{4\pi}{27} + \frac{\pi}{3}\epsilon^{-2} - \frac{1}{6}\epsilon^{-2}\int_0^{2\pi} \left(1 - \frac{\epsilon}{3}\cos\theta + \frac{11\cos^2\theta - 1}{54}\epsilon^2 + O(\epsilon^3)\right)^{-2} d\theta + o(1) \\ &= \frac{4\pi}{27} + \frac{\pi}{3}\epsilon^{-2} - \frac{1}{6}\epsilon^{-2}\int_0^{2\pi} \left(1 + \frac{2\cos\theta}{3}\epsilon - \frac{2\cos^2\theta - 1}{27}\epsilon^2 + O(\epsilon^3)\right) d\theta + o(1) \\ &= \frac{4\pi}{27} + \frac{1}{162}\int_0^{2\pi} \cos(2\theta)d\theta + o(1) = \frac{4\pi}{27} + o(1) \to \frac{4\pi}{27} \end{aligned}$$

as $\epsilon \to 0$, by means of (5.11). The validity of Lemma 4.2 is completely established.

6 Appendix B

Let us recall the definition of the operator L:

$$L(\phi) = \Delta\phi + \rho^2(e^{PU} + e^{-PU})\phi - \frac{\rho^2}{\pi}\int_B (e^{PU} + e^{-PU})\phi,$$

which acts on $\phi \in \mathcal{S}_0$. Our final aim is to show the validity of Proposition 3.1 and we will follow the approach in [4,6,7]. It makes a crucial use of comparison arguments for the linearized operator and the first main difficulty is that L in general does not satisfy the Maximum Principle. Indeed, L is the sum of a differential operator $\tilde{L} = \Delta + \rho^2(e^{PU} + e^{-PU})$ and an integral term $c(\phi) = -\frac{\rho^2}{\pi}\int_B (e^{PU} + e^{-PU})\phi$. According to [4,6,7], the operator $\tilde{L}$ will satisfy the Maximum Principle and by comparison some a-priori estimates will hold. The main goal will be to get rid of the presence of the term $c(\phi)$.

Letting $\Sigma_R = B_{R\delta\rho}(0) \cup \bigcup_{j=0}^2 B_{R\epsilon\rho}(la_j)$, we have the following:

Proposition 6.1 *Assume* (2.3). *There exist* $C > 0$ *and* $R > 0$ *large such that every solution* ϕ *of* $\tilde{L}\phi = h$ *in* $B_{1/2}\setminus\Sigma_R$ *satisfies*

$$\|\phi\|_{\infty, B_{1/2}\setminus\Sigma_R} \le C\left(\|h\|_* + \|\phi\|_{\infty,\partial B_{1/2}\cup\partial\Sigma_R}\right) \tag{6.12}$$

for $l > 0$ *small.*

Proof The proof is adapted from [6] and only minor changes take place. For reader's convenience, we recall the basic steps and refer to [6] for all the details.

First step. The operator $\tilde{L}$ satisfies the Maximum Principle in $B_{1/2}\setminus\Sigma_R$, for R large independent on l small:

$$\tilde{L}(\psi) \le 0 \ \text{ in } B_{1/2}\setminus\Sigma_R \ \text{ and } \ \psi \ge 0 \ \text{ on } \partial B_{1/2}\cup\partial\Sigma_R \Rightarrow \psi \ge 0 \ \text{ in } B_{1/2}\setminus\Sigma_R.$$

It is sufficient to construct a strictly positive super-solution M as a comparison function. The function

$$M(x) = 2\frac{a^2|x|^2 - \delta^2\rho^2}{\delta^2\rho^2 + a^2|x|^2} + 2\sum_{j=0}^{2}\frac{a^2|x - la_j|^2 - \epsilon^2\rho^2}{\epsilon^2\rho^2 + a^2|x - la_j|^2}$$

satisfies

$$\begin{cases} \tilde{L}(M) < 0 & \text{in } B_{1/2}\setminus\Sigma_R \\ \frac{8}{3} \le M \le 8 & \text{in } B_{1/2}\setminus\Sigma_R \end{cases}$$

for $0 < a < \frac{1}{\sqrt{27D_0}}$ and $R > \frac{\sqrt{2}}{a}$, where D_0 is the constant in (2.20).

Second step. Let $R > 0$ be given and $0 < \eta < \frac{3}{4R}$. Letting

$$A_\eta = 32(\frac{4\eta}{3})^{\frac{1}{4}}, \quad B_\eta = \left(\frac{32}{R^{\frac{1}{4}}} - A_\eta\right)\frac{1}{\ln\frac{4R\eta}{3}} < 0,$$

define

$$\psi_\eta(x) = -32\frac{\eta^{\frac{1}{4}}}{|x|^{\frac{1}{4}}} + A_\eta + B_\eta \ln\frac{4|x|}{3},$$

a solution of

$$\begin{cases} -\Delta\psi_\eta = 2\frac{\eta^{\frac{1}{4}}}{|x|^{\frac{9}{4}}} & \text{for } R\eta < |x| < \frac{3}{4} \\ \psi_\eta = 0 & \text{for } |x| = R\eta \quad \text{and} \quad |x| = \frac{3}{4} \end{cases}$$

so that $0 < \psi_\eta < \frac{64}{R^{\frac{1}{4}}}$. The function

$$T(x) = \psi_{\delta\rho}(x) + \sum_{j=0}^{2}\psi_{\epsilon\rho}(x - la_j)$$

then satisfies

$$\tilde{L}(T) \le -W^{-1} \quad \text{in } B_{1/2}\setminus\Sigma_R, \quad 0 < T \le \frac{256}{R^{\frac{1}{4}}}$$

for any $R \ge D_0^4 2^{44}$.

Third step. Estimate (6.12) does hold for $R > 0$ large. Indeed, introduce the comparison function $\frac{3}{8}\|\phi\|_{\infty,\partial B_{1/2}\cup\partial\Sigma_R}M + \|h\|_* T$. We have that

$$\tilde{L}\left(\frac{3}{8}\|\phi\|_{\infty,\partial B_{1/2}\cup\partial\Sigma_R}M + \|h\|_* T\right) \le -\|h\|_* W^{-1} \le -|h| \quad \text{in } B_{1/2}\setminus\Sigma_R$$
$$\frac{3}{8}\|\phi\|_{\infty,\partial B_{1/2}\cup\partial\Sigma_R}M + \|h\|_* T \ge |\phi| \quad \text{on } \partial B_{1/2}\cup\partial\Sigma_R$$

and, by the Maximum Principle,

$$|\phi|(x) \le C(\|\phi\|_{\infty,\partial B_{1/2}\cup\partial\Sigma_R} + \|h\|_*) \quad \text{in } B_{1/2}\setminus\Sigma_R$$

for R large, where C depends on R. □

We want to extend now (6.12) to solutions of $L(\phi) = h$. Letting as before $c(\phi) = -\frac{\rho^2}{\pi}\int_B (e^{PU}+e^{-PU})\phi$, the operator L rewrites as $L = \tilde{L} + c(\cdot)$. We can introduce the function $\tilde{\phi} = \phi + \frac{c(\phi)}{4}|x|^2$ in order to get that

$$\tilde{L}(\tilde{\phi}) = h + \frac{c(\phi)}{4}|x|^2\rho^2(e^{PU}+e^{-PU}).$$

We can apply (6.12) to $\tilde{\phi}$ and, taking into account $\rho^2 W(e^{PU}+e^{-PU}) \le 8D_0$ in view of (2.20), it follows:

Corollary 6.2 *Assume* (2.3). *There exist $C>0$ and $R>0$ large such that every solution ϕ of $L\phi = h$ in $B_{1/2}\setminus\Sigma_R$ satisfies*

$$\|\phi\|_{\infty, B_{1/2}\setminus\Sigma_R} \le C\left(\|h\|_* + \|\phi\|_{\infty,\partial B_{1/2}\cup\partial\Sigma_R} + |c(\phi)|\right) \tag{6.13}$$

for $l>0$ small.

We consider now the problem (3.1) when $c=0$:

$$\begin{cases} L(\phi) = h & \text{in } B\\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial B\\ \int_B \Delta PZ\phi = 0, & \end{cases} \tag{6.14}$$

with $h \in \mathcal{S}_0$. We are now in position to show:

Proposition 6.3 *Assume* (2.3). *There exists $C>0$ such that for every solution $\phi\in\mathcal{S}_0$ of* (6.14) *there holds*

$$\|\phi\|_\infty \le C|\ln l|\|h\|_* \tag{6.15}$$

for $l>0$ small.

Proof By contradiction, assume the existence of sequences $\rho_n, l_n, \phi_n \in \mathcal{S}_0$ and $h_n\in\mathcal{S}_0$ so that ϕ_n is a solution of (6.14) associated to ρ_n and h_n, $\|\phi_n\|_\infty = 1$, $l_n\to 0$ and $|\ln l_n|\|h_n\|_*\to 0$ as $n\to+\infty$. We will denote by $\epsilon_n\rho_n$, $\delta_n\rho_n$ the concentration parameters associated to l_n and by $U_n = (U_n)_+ - \sum_{j=0}^2 (U_n)_-^j$ the corresponding approximating solution.
First claim. There hold

$$\phi_n \rightharpoonup 0 \text{ weakly in } H^1(B) \text{ and strongly in } C^1_{\rm loc}(\bar{B}\setminus\{0\}) \tag{6.16}$$

$$c(\phi_n) = -\frac{1}{\pi}\rho_n^2\int_B (e^{PU_n}+e^{-PU_n})\phi_n \to 0 \tag{6.17}$$

as $n\to+\infty$.

Multiply (6.14) by ϕ_n and integrate on B:

$$\int_B |\nabla\phi_n|^2 = \rho_n^2\int_B (e^{PU_n}+e^{-PU_n})\phi_n^2 - \int_B h_n\phi_n$$

in view of $\int_B \phi_n = 0$. By (2.20) we get that

$$\rho_n^2\int_B (e^{PU_n}+e^{-PU_n})\phi_n^2 \le D_0\rho_n^2\int_B (e^{(U_n)^+} + \sum_{j=0}^2 e^{(U_n)_j^-}) \le C.$$

Since

$$\left|\int_B h_n\phi_n\right| \le \int_B |h_n| \le \|h_n\|_* \int_B W_n^{-1} \le C\|h_n\|_*,$$

we get that $\sup_{n\in\mathbb{N}} \int_B |\nabla\phi_n|^2 < +\infty$. Since $\int_B \phi_n = 0$, the sequence ϕ_n is bounded in $H^1(B)$. Moreover, by elliptic regularity theory $\|\phi_n\|_\infty = 1$ implies that ϕ_n is bounded in $C^{1,\alpha}_{\text{loc}}(\bar{B}\setminus\{0\})$, $\alpha \in (0,1)$.

By Ascoli–Arzelá Theorem, let us consider a subsequence of ϕ_n (still denoted by ϕ_n) so that $\phi_n \rightharpoonup \phi_0$ weakly in $H^1(B)$, strongly in $C^1_{\text{loc}}(\bar{B}\setminus\{0\})$ and $c(\phi_n) = -\frac{1}{\pi}\rho_n^2 \int_B (e^{PU_n} + e^{-PU_n})\phi_n \to c_0$ as $n \to +\infty$. Since $h_n - \rho_n^2(e^{PU_n} + e^{-PU_n})\phi_n \to 0$ in $C_{\text{loc}}(\bar{B}\setminus\{0\})$, we get that $\phi_0 \in H^1(B)$ is a weak solution of

$$\Delta\phi_0 = -c_0 \quad \text{in } B\setminus\{0\}, \qquad \frac{\partial\phi_0}{\partial\nu} = 0 \quad \text{on } \partial B$$

so that $\|\phi_0\| \le 1$. Hence, the origin is a removable singularity and the equation holds in the whole B. By $-\pi c_0 = \int_B \Delta\phi_0 = 0$ we get that $c_0 = 0$ and then, $\phi_0 = 0$. Since it holds along any convergent subsequence of ϕ_n, it is true for all the sequence ϕ_n and the claim is established.

Second claim. There exist $R > 0$ large and $\eta > 0$ so that

$$\|\phi_n\|_{\infty,\Sigma_R} \ge \eta \tag{6.18}$$

for n large.

Let us note that (6.16) implies

$$\|\phi_n\|_{\infty,B\setminus B_{1/2}} \to 0 \quad \text{as } n \to +\infty. \tag{6.19}$$

Fix now $R > 0$ large. If $\|\phi_n\|_{\infty,\Sigma_R} \to 0$ as $n \to +\infty$ (up to a subsequence), we can use (6.16), (6.17) and $\|h_n\|_* \to 0$ in (6.13) to get

$$\|\phi_n\|_{\infty,B_{1/2}\setminus\Sigma_R} \to 0$$

as $n \to +\infty$. Hence, we get that $\|\phi_n\|_\infty \to 0$, in contradiction with $\|\phi_n\|_\infty = 1$. Hence, (6.18) holds and the claim is proved.

Introduce now $\Phi_n(y) = \phi_n(\delta_n\rho_n y)$ in $B_n = B_{1/\delta_n\rho_n}$ and $\Phi_{j,n}(y) = \phi_n(\epsilon_n\rho_n y + l_n a_j)$ in $B_{j,n} = B_{1/\epsilon_n\rho_n}\left(-\frac{l_n}{\epsilon_n\rho_n}a_j\right)$, $j = 0,1,2$. The function Φ_n satisfies

$$\Delta\Phi_n + \delta_n^2\rho_n^4(e^{PU_n} + e^{-PU_n})(\delta_n\rho_n y)\Phi_n - \delta_n^2\rho_n^2 c(\phi_n) = \delta_n^2\rho_n^2 h_n(\delta_n\rho_n y) \quad \text{in } B_n.$$

Note that for every $M > 0$

$$\begin{aligned}\|\delta_n^2\rho_n^2 h_n(\delta_n\rho_n y)\|_{\infty,B_M} &\le \delta_n^2\rho_n^2\|h_n\|_*\|W_n^{-1}(\delta_n\rho_n y)\|_{\infty,B_M}\\ &\le \left(1 + O(\delta_n^2\rho_n^2\frac{(\epsilon_n\rho_n)^{\frac{1}{4}}}{l_n^{\frac{9}{4}}}\right)\|h_n\|_* \le 2\|h_n\|_* \to 0\end{aligned}$$

and $B_n \to \mathbb{R}^2$ as $n \to +\infty$ (to estimate $\|W_n^{-1}(\delta_n\rho_n y)\|_{\infty,B_M}$ we are using that the distance among $0, l_n a_0, l_n a_1, l_n a_2$ is of order l_n and is much bigger than $\epsilon_n\rho_n$ and $\delta_n\rho_n$). Since

$\|\Phi_n\|_\infty \leq 1$, up to a subsequence and a diagonal process, by elliptic regularity theory $\Phi_n \to \Phi$ in $C_{\rm loc}(\mathbb{R}^2)$, where Φ is a bounded solution of

$$\Delta\Phi + \frac{8}{(1+|y|^2)^2}\Phi = 0, \tag{6.20}$$

by means of (2.21). According to [1], the function Φ is a linear combination of

$$\frac{1-|y|^2}{1+|y|^2}, \quad \frac{y_1}{1+|y|^2}, \quad \frac{y_2}{1+|y|^2}.$$

Since $\phi_n \in \mathcal{S}$, the function Φ is $\frac{2\pi}{3}-$periodic and then

$$\Phi(y) = E\frac{1-|y|^2}{1+|y|^2},$$

for some coefficient $E \in \mathbb{R}$. Similarly, the function $\Phi_{j,n} \to \Phi_j$ in $C_{\rm loc}(\mathbb{R}^2)$, where Φ_j is a bounded solution of (6.20).
We use now the assumption $\int_B \Delta P Z_n \phi_n = 0$, which rewrites by symmetries as $(x \to \bar{a_j}x)$

$$\begin{aligned} 0 &= \rho_n^2 \sum_{j=0}^{2} \int_B e^{(U_n)_j^-}\phi_n Z_{j,n}\cdot a_j = 3\rho_n^2 \int_B e^{(U_n)_0^-}\phi_n Z_{0,n}\cdot a_0 \\ &= 3\int_{B_{0,n}} \frac{8}{(1+|y|^2)^2}\frac{4y\cdot a_0}{1+|y|^2}\Phi_{0,n}. \end{aligned}$$

By Lebesgue's Theorem, letting $n \to +\infty$ we get that

$$\int_{\mathbb{R}^2} \frac{y_1}{(1+|y|^2)^3}\Phi_0 = 0. \tag{6.21}$$

Since $\phi_n(x) = \phi_n(\bar{x})$, the function $\Phi_{0,n}$ is also invariant by conjugation in $B_{0,n}$. In the limit, $\Phi_0(y) = \Phi_0(\bar{y})$ in $\mathbb{R}^2$ and the following relation follows

$$\int_{\mathbb{R}^2} \frac{y_2}{(1+|y|^2)^3}\Phi_0 = 0. \tag{6.22}$$

Since ϕ_n is $\frac{2\pi}{3}-$periodic, observe that

$$\Phi_{0,n}(y) = \phi_n(\epsilon_n\rho_n y + l_n a_0) = \phi_n(\epsilon_n\rho_n a_j y + l_n a_j) = \Phi_{j,n}(a_j y),$$

which gives in the limit $\Phi_0(y) = \Phi_j(a_j y)$ in $\mathbb{R}^2$. Using this relation in (6.21)–(6.22), by the change of variable $y \to a_j y$ we get that

$$\int_{\mathbb{R}^2} \frac{y\cdot a_j}{(1+|y|^2)^3}\Phi_j = \int_{\mathbb{R}^2} \frac{y\cdot(ia_j)}{(1+|y|^2)^3}\Phi_j = 0.$$

These two relations are linearly independent and lead to

$$\int_{\mathbb{R}^2} \frac{y_1}{(1+|y|^2)^3}\Phi_j = \int_{\mathbb{R}^2} \frac{y_2}{(1+|y|^2)^3}\Phi_j = 0. \tag{6.23}$$

By (6.21)–(6.23) we get that $\Phi_j = F_j \frac{1-|y|^2}{1+|y|^2}$. Since $\Phi_0(y) = \Phi_j(a_j y)$, we have that $F_0 = F_1 = F_2$ and hence

$$\Phi_j(y) = F \frac{1-|y|^2}{1+|y|^2},$$

for some coefficient $F \in \mathbb{R}$. By the second claim as stated in (6.18) we get that Φ, Φ_j can't be both trivial and a contradiction would arise if $E = F = 0$. Based on the assumption $|\ln l_n| \|h_n\|_* \to 0$, this will be the content of next claim.

Third claim. $E = F = 0$

We will us an idea developed first in [5] and then exploited in [6,7]. We construct suitable test functions to recover the additional orthogonality relation:

$$\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} \Phi = \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} \Phi_j = 0,$$

which clearly would imply $E = F = 0$ as claimed.

Let us perform the following construction with respect to the origin. Define

$$w_n(x) = \frac{4}{3} \ln(\delta_n^2 \rho_n^2 + |x|^2) \frac{\delta_n^2 \rho_n^2 - |x|^2}{\delta_n^2 \rho_n^2 + |x|^2} + \frac{8}{3} \frac{\delta_n^2 \rho_n^2}{\delta_n^2 \rho_n^2 + |x|^2}$$

and $t_n(x) = -2 \frac{\delta_n^2 \rho_n^2}{\delta_n^2 \rho_n^2 + |x|^2}$. They solve $-\Delta w_n - \rho_n^2 e^{(U_n)^+} w_n = \rho_n^2 e^{(U_n)^+} (Y_{0,n})$ and $-\Delta t_n - \rho_n^2 e^{(U_n)^+} t_n = \rho_n^2 e^{(U_n)^+}$ respectively.

The good test function in the origin will be $P z_n$, where $z_n = w_n - 2t_n$. Observe that

$$\frac{\partial}{\partial \nu} \left(P z_n - z_n - \frac{16\pi}{3} H(\cdot, 0) \right) = O(\delta_n^2 \rho_n^2) \quad \text{on } \partial B$$
$$\int_B \left(P z_n - z_n - \frac{16\pi}{3} H(\cdot, 0) \right) = O\left(\delta_n^2 \rho_n^2 \ln^2(\delta_n \rho_n)\right).$$

Since it holds

$$\begin{aligned}
-\Delta \left(P z_n - z_n - \frac{16\pi}{3} H(\cdot, 0) \right) &= \frac{1}{\pi} \int_B \Delta z_n + \frac{16}{3} = \frac{1}{\pi} \int_{\partial B} \frac{\partial z_n}{\partial \nu} + \frac{16}{3} \\
&= -\frac{16}{3} \int_{\partial B} \frac{\partial H(\cdot, 0)}{\partial \nu} + \frac{16}{3} + O(\delta_n^2 \rho_n^2) = -\frac{16}{3} \int_B \Delta H(\cdot, 0) + \frac{16}{3} + O(\delta_n^2 \rho_n^2) \\
&= O(\delta_n^2 \rho_n^2),
\end{aligned}$$

by the representation's formula we get that

$$P z_n - z_n - \frac{16\pi}{3} H(\cdot, 0) = O\left(\delta_n^2 \rho_n^2 \ln^2(\delta_n \rho_n)\right) \tag{6.24}$$

uniformly in Ω. Hence, we have that

$$\begin{aligned}
\Delta P z_n + \rho_n^2 e^{(U_n)^+} P z_n &= \Delta z_n + \frac{16}{3} + \rho_n^2 e^{(U_n)^+} P z_n + O\left(\delta_n^2 \rho_n^2 \ln^2(\delta_n \rho_n)\right) \\
&= -\rho_n^2 e^{(U_n)^+} (Y_{0,n} + \rho_n^2 e^{(U_n)^+} (P z_n - z_n + 2) \\
&\quad + \frac{16}{3} + O\left(\delta_n^2 \rho_n^2 \ln^2(\delta_n \rho_n)\right)
\end{aligned}$$

Since $\int_B Pz_n = \int_B \phi_n = 0$, multiply (6.14) by Pz_n and integrate on B to get:

$$\begin{aligned}\int_B h_n Pz_n &= \int_B \phi_n \left(\Delta Pz_n + \rho_n^2 e^{(U_n)^+} Pz_n\right) + \rho_n^2 \int_B \left(e^{PU_n} + e^{-PU_n} - e^{(U_n)^+}\right) \phi_n Pz_n \\ &= -\rho_n^2 \int_B e^{(U_n)^+} \phi_n (Y_{0,n} + \rho_n^2 \int_B e^{(U_n)^+} (Pz_n - z_n + 2)\, \phi_n \\ &\quad + \rho_n^2 \int_B \left(e^{PU_n} + e^{-PU_n} - e^{(U_n)^+}\right) \phi_n Pz_n + O\left(\delta_n^2 \rho_n^2 \ln^2(\delta_n \rho_n)\right) \qquad (6.25)\end{aligned}$$

As for the L.H.S., by (6.24) we get that $Pz_n = z_n + O(1) = O(|\ln \delta_n^2 \rho_n^2|) = O(|\ln l_n|)$ and then

$$\left|\int_B h_n Pz_n\right| = O\left(|\ln l_n| \int_B |h_n|\right) = O(|\ln l_n| \|h_n\|_*) \to 0$$

as $n \to +\infty$, by our assumption by contradiction on h_n. As for the first term in the R.H.S., we can write now

$$-\rho_n^2 \int_B e^{(U_n)^+} \phi_n Y_{0,n} = 2 \int_{B_n} \frac{8}{(1+|y|^2)^2} \frac{1-|y|^2}{1+|y|^2} \Phi_n \to 2E \int_{\mathbb{R}^2} \frac{8(1-|y|^2)^2}{(1+|y|^2)^4}$$

as $n \to +\infty$, by means of Lebesgue Theorem and $\Phi_n \to E\frac{1-|y|^2}{1+|y|^2}$ in $C_{\text{loc}}(\mathbb{R}^2)$. By (6.24) the second term in the R.H.S. gives a contribution

$$\begin{aligned}\rho_n^2 \int_B e^{(U_n)^+} (Pz_n - z_n + 2)\, \phi_n &= \frac{16\pi}{3} \rho_n^2 \int_B e^{(U_n)^+} (H(x,0) - H(0,0))\, \phi_n \\ &\quad + O\left(\delta_n^2 \rho_n^2 \ln^2(\delta_n \rho_n)\right) = O(\delta_n \rho_n) \to 0 \qquad \text{as } n \to +\infty.\end{aligned}$$

Since $Pz_n = O(|\ln l_n|)$, for the third term in the R.H.S. by (2.11), (2.18) and (6.24) we get that

$$\begin{aligned}&\rho_n^2 \int_B \left(e^{PU_n} + e^{-PU_n} - e^{(U_n)^+}\right) \phi_n Pz_n = \rho_n^2 \sum_{j=0}^{2} \int_B e^{(U_n)_j^-} \phi_n Pz_n + O(l_n^2 \ln^2 l_n) \\ &= \sum_{j=0}^{2} \int_{B_{j,n}} \frac{8}{(1+|y|^2)^2} \Phi_{j,n} z_n(\epsilon_n \rho_n y + l_n a_j) \\ &\quad + \frac{16\pi}{3} \sum_{j=0}^{2} \int_{B_{j,n}} \frac{8}{(1+|y|^2)^2} \Phi_{j,n} H(\epsilon_n \rho_n y + l_n a_j, 0) + O(l_n^2 \ln^2 l_n) \\ &= 8F \ln l_n \int_{\mathbb{R}^2} \frac{8(|y|^2 - 1)}{(1+|y|^2)^3} - 6F H(0,0) \int_{\mathbb{R}^2} \frac{8(1-|y|^2)}{(1+|y|^2)^3} + o(1) \to 0\end{aligned}$$

as $n \to +\infty$, by means of Lebesgue Theorem and $\Phi_{j,n} \to F\frac{1-|y|^2}{1+|y|^2}$ in $C_{\text{loc}}(\mathbb{R}^2)$. We have used that $\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} = 0$. In conclusion, (6.25) leads to $E = 0$.

A similar argument can be carried out by using the test function $Pz_{j,n}$, where $z_{j,n} = w_{j,n} + \frac{16\pi}{3} H(l_n a_j, l_n a_j) t_{j,n}$. Here, the functions $w_{j,n}$ and $t_{j,n}$ are defined as follows:

$$w_{j,n}(x) = \frac{4}{3}\ln(\epsilon_n^2\rho_n^2 + |x - l_n a_j|^2)\frac{\epsilon_n^2\rho_n^2 - |x - l_n a_j|^2}{\epsilon_n^2\rho_n^2 + |x - l_n a_j|^2} + \frac{8}{3}\frac{\epsilon_n^2\rho_n^2}{\epsilon_n^2\rho_n^2 + |x - l_n a_j|^2}$$

and $t_{j,n}(x) = -2\frac{\epsilon_n^2\rho_n^2}{\epsilon_n^2\rho_n^2+|x-l_n a_j|^2}$. □

It is now easy to include a term $c\Delta PZ$ in the R.H.S. of $L(\phi) = h$ and obtain:

Corollary 6.4 *Assume* (2.3). *There exists* $C > 0$ *such that for every solution* $\phi \in \mathcal{S}_0$ *of* (3.1) *there holds*

$$\|\phi\|_\infty \le C|\ln l|\|h\|_* \tag{6.26}$$

for $l > 0$ *small.*

Proof We need an estimate on the value of c. To this aim, multiply (3.1) by PZ and integrate on B:

$$\int_B hPZ + c\int_B \Delta PZPZ = \rho^2\int_B (e^{PU} + e^{-PU})\phi PZ, \tag{6.27}$$

because $\int_B \Delta\phi PZ = \int_B \Delta PZ\phi = 0$ and $\int_B PZ = 0$. By (3.8) we get that $PZ = O(1)$ and $\left|\int_B hPZ\right| = O\left(\int_B |h|\right) = O(\|h\|_*)$. Moreover, by (2.11), (2.18) and (2.20) we deduce that

$$\rho^2\int_B (e^{PU} + e^{-PU})\phi PZ = \rho^2\int_B (e^{PU} + e^{-PU})\phi Z + O(\epsilon\rho l\|\phi\|_\infty)$$
$$= \rho^2\int_B \left(e^{U^+} + \sum_{j=0}^{2} e^{U_j^-}\right)\phi Z + O(l^2|\ln l|\|\phi\|_\infty)$$

in view of (3.8). We have that for any $j = 0, 1, 2$

$$\rho^2\int_B e^{U^+}\phi Z_j\cdot a_j = O\left(\|\phi\|_\infty\int_{|y|\le 1/\delta\rho}\frac{8}{(1+|y|^2)^2}\frac{\epsilon\delta^{-1}|y - l\delta^{-1}\rho^{-1}a_j|}{\epsilon^2\delta^{-2} + |y - l\delta^{-1}\rho^{-1}a_j|^2}\right)$$
$$= O\left(\epsilon\rho l^{-1}\|\phi\|_\infty\int_{|y|\le l^{\frac{3}{2}}/\delta\rho}\frac{8}{(1+|y|^2)^2}\right) + O\left(\|\phi\|_\infty\int_{|y|\ge l^{\frac{3}{2}}/\delta\rho}\frac{8}{(1+|y|^2)^2}\right)$$
$$= O\left(l\|\phi\|_\infty\right)$$

and for any $k \neq j$

$$\rho^2 \int_B e^{U_j^-} \phi Z_k \cdot a_k = O\left(\|\phi\|_\infty \int_{|y| \le 1/\epsilon\rho} \frac{8}{(1+|y|^2)^2} \frac{|y + l\epsilon^{-1}\rho^{-1}(a_j - a_k)|}{1 + |y + l\epsilon^{-1}\rho^{-1}(a_j - a_k)|^2} \right)$$

$$+ O(\epsilon^2\rho^2 \|\phi\|_\infty) = O\left(\epsilon\rho l^{-1} \|\phi\|_\infty \int_{|y| \le l^{\frac{3}{2}}/\epsilon\rho} \frac{8}{(1+|y|^2)^2} \right)$$

$$+ O(\|\phi\|_\infty \int_{|y| \ge l^{\frac{3}{2}}/\epsilon\rho} \frac{8}{(1+|y|^2)^2}) + O(\epsilon^2\rho^2\|\phi\|_\infty) = O(l\|\phi\|_\infty)$$

as $l \to 0$, uniformly on ϕ. In conclusion, we have that

$$\rho^2 \int_B (e^{PU} + e^{-PU})\phi PZ = \rho^2 \sum_{j=0}^{2} \int_B e^{U_j^-} \phi Z_j \cdot a_j + O(l\|\phi\|_\infty)$$

$$= -\int_B \Delta PZ\phi + O(l\|\phi\|_\infty) = O(l\|\phi\|_\infty)$$

as $l \to 0$. By (3.9) and (6.27) we deduce that

$$c \int_{\mathbb{R}^2} \frac{128 y_1^2}{(1+|y|^2)^4} + o(1)|c| = O(\|h\|_* + l\|\phi\|_\infty)$$

as $l \to 0$. Then the following estimate on c does hold

$$|c| = O(\|h\|_* + l\|\phi\|_\infty),$$

as $l \to 0$. By Proposition 6.3 and the estimate on c we get that

$$\|\phi\|_\infty \le C|\ln l|\|h + c\Delta PZ\|_* \le C^{'}|\ln l|\|h\|_* + O(l|\ln l|\|\phi\|_\infty)$$

and then, the validity of (6.26) easily follows because $O(l|\ln l|)$ is small independently on ϕ. □

Corollary 6.4 now yields easily to the validity of Proposition 3.1. Indeed, let us introduce the operator $(\Delta)^{-1}$ with Neumann boundary condition: given $f \in L^p(B)$ for some $p > 1$, the function $u = (\Delta)^{-1}(f) \in H^1(B)$ is the unique solution of

$$\begin{cases} \Delta u = f - \frac{1}{\pi}\int_B f & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \\ \int_B u = 0. \end{cases}$$

By uniqueness, observe that $u \in \mathcal{S}$ whenever $f \in \mathcal{S}$. Thanks to $(\Delta)^{-1}$ we can rewrite problem (3.1) as $\phi + K(\phi) = (\Delta)^{-1}(h) + cPZ$, where by elliptic regularity

$$K(\phi) = (\Delta)^{-1}\left(\rho^2(e^{PU} + e^{-PU})\phi\right)$$

is a compact operator from $H^1(B) \cap \mathcal{S}_0$ into itself. In the space $H^1(B) \cap \mathcal{S}_0$, define Π and $\Pi^{\perp} = \mathrm{Id} - \Pi$ as the projection operators onto PZ and $\{PZ\}^{\perp}$ respectively. Problem (3.1) can be further rewritten in an equivalent way as

$$\phi + \Pi^{\perp} K(\phi) = \Pi^{\perp}(\Delta)^{-1}(h).$$

Observe that, by Corollary 6.4, $\mathrm{Id} + \Pi^{\perp} \circ K$ is injective in $H^1(B) \cap \mathcal{S}_0$, where $\Pi^{\perp} \circ K$ is a compact operator. For any $h \in L^{\infty}(B) \cap \mathcal{S}_0$, Fredholm alternative then provides the existence of a unique solution $\phi \in H^1(B) \cap \mathcal{S}_0$ of (3.1) satisfying the bound $\|\phi\|_{\infty} \leq C|\ln l| \|h\|_*$ for l small. Moreover, by elliptic regularity theory $\phi \in W^{2,2}(B)$ and there holds:

$$\int_B |\nabla \phi|^2 = -\int_B h\phi + \rho^2 \int_B (e^{PU} + e^{-PU})\phi^2 \leq C(\|\phi\|_{\infty} + \|h\|_*)^2,$$

by Young inequality and (2.20). Proposition 3.1 is completely established.

Acknowledgments The first named author would like to thank Prof. Wei for the kind invitation in the Department of Mathematics, Chinese University of Hong Kong. The present paper is the outcome of the useful discussions we had. The authors would also like to thank the referee for the careful reading of the manuscript and for the many critical corrections he suggested to us.

References

1. Baraket, S., Pacard, F.: Construction of singular limits for a semilinear elliptic equation in dimension 2. Calc. Var. Partial Diff. Equ. **6**(1), 1–38 (1998)
2. Bartolucci, D., Pistoia, A.: Existence and qualitative properties of concentrating solutions for the sinh-Poisson equation. IMA J. Appl. Math. **72**(6), 706–729 (2007)
3. Chen, X.: Remarks on the existence of branch bubbles on the blowup analysis of equation $-\Delta u = e^{2u}$ in dimension two. Comm. Anal. Geom. **7**, 295–302 (1999)
4. del Pino, M., Kowalczyk, M., Musso, M.: Singular limits in Liouville-type equations. Calc. Var. Partial Diff. Equ. **24**(1), 47–81 (2005)
5. Esposito, P., Grossi, M., Pistoia, A.: On the existence of blowing-up solutions for a mean field equation. Ann. IHP Anal. Non Lineaire **22**(2), 127–157 (2005)
6. Esposito, P., Musso, M., Pistoia, A.: Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent. J. Diff. Equ. **227**(1), 29–68 (2006)
7. Esposito, P., Pistoia, A., Wei, J.: Concentrating solutions for the Hénon equation in $\mathbb{R}^2$. J. Anal. Math. **100**, 249–280 (2006)
8. Jost, J., Wang, G., Ye, D., Zhou, C.: The blow up analysis of solutions to the elliptic sinh-Gordon equation. Calc. Var. Partial Diff. Equ. **31**(2), 263–276 (2008)
9. Li, Y.Y., Shafrir, I.: Blow up analysis for solutions of $-\Delta u = V(x)e^u$ in dimension two. Indiana Univ. Math. J. **43**, 1255–1270 (1994)
10. Ohtsuka, H., Suzuki, T.: Mean field equation for the equilibrium turbulence and a related functional inequality. Adv. Diff. Equ. **11**, 281–304 (2006)
11. Wei, J., Ye, D., Zhou, F.: Bubbling solutions for anisotropic Emden–Fowler equation. Calc. Var. Partial Diff. Equ. **28**(2), 217–247 (2007)