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Non-triviality of the vacancy phase transition for the Boolean model

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Abstract

In the spherical Poisson Boolean model, one takes the union of random balls centred on the points of a Poisson process in Euclidean d-space with $d \geq 2$. We prove that whenever the radius distribution has a finite d-th moment, there exists a strictly positive value for the intensity such that the vacant region percolates.

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1 Introduction

The Boolean model [6, 8] is a classic model of continuum percolation [11, 3] and more general stochastic geometry [9, 4, 14, 10]. In the spherical version of this model, an *occupied region* in Euclidean d-space is defined as a union of balls (sometimes called *grains*) of fixed or random radius centred on the points of a Poisson process of intensity λ. One may define a critical value λ_c of λ, depending on the radius distribution, above which the occupied region percolates, and a further critical value λ_c^*, below which the complementary *vacant region* percolates. It is a fundamental question whether these critical values are *non-trivial*, i.e. strictly positive and finite.

For fixed or bounded radii, the non-triviality of λ_c and λ_c^* for $d \geq 2$ is well known and may be proved using discretization and counting arguments from lattice percolation theory. For unbounded radii, it took some years to fully characterize those radius distributions for which λ_c is non-trivial [8, 7]. In the present work we carry out a similar task for λ_c^*.

We now describe the model in more detail (for yet more details we refer the reader to [11] or [10]). Let $d \in \mathbb{N}$ with $d \geq 2$. Let μ be a probability measure on $[0, \infty)$ with $\mu(\{0\}) < 1$. Let $\lambda \in (0, \infty)$. On a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (with associated expectation operator $\mathbb{E}$), let $\mathcal{P}_\lambda = \{y_k : k \in \mathbb{N}\}$ be a homogeneous Poisson point process in $\mathbb{R}^d$ of intensity λ (here viewed as a random subset of $\mathbb{R}^d$ enumerated in order of increasing distance from the origin), and let $\rho, \rho_1, \rho_2, \ldots$ be independent nonnegative random variables with common distribution μ, independent of $\mathcal{P}_\lambda$. For $x \in \mathbb{R}^d$ and $r \geq 0$ we let $B(x, r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$, where $\| \cdot \|$ is the Euclidean norm. The occupied and vacant regions of the (Poisson, spherical) Boolean model are random sets $Z_\lambda \subset \mathbb{R}^d$ and $Z_\lambda^* \subset \mathbb{R}^d$, given respectively by

$$Z_\lambda = \cup_{y_k \in \mathcal{P}_\lambda} B(y_k, \rho_k); \qquad Z_\lambda^* = \mathbb{R}^d \setminus Z_\lambda.$$

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Let U_λ be the event that Z_λ *percolates*, i.e. has an unbounded connected component, and let U^*_λ be the event that Z^*_λ percolates. By an ergodicity argument (see [11], or [10], Exercise 10.1), $\mathbb{P}[U_\lambda] \in \{0,1\}$ and $\mathbb{P}[U^*_\lambda] \in \{0,1\}$. Also $\mathbb{P}[U_\lambda]$ is increasing in λ, while $\mathbb{P}[U^*_\lambda]$ is decreasing in λ. Define the critical values

$$\lambda_c := \inf\{\lambda : \mathbb{P}[U_\lambda] = 1\}; \qquad \lambda^*_c := \inf\{\lambda : \mathbb{P}[U^*_\lambda] = 0\}.$$

It is well known that λ_c and λ^*_c are finite, and that if $\mathbb{E}\left[\rho^d\right] = \infty$ then $Z_\lambda = \mathbb{R}^d$ almost surely, for any $\lambda > 0$ (see [8], [11] or [10]), so that $\lambda_c = \lambda^*_c = 0$. Hence $\mathbb{E}\left[\rho^d\right] < \infty$ is a necessary condition for λ_c or λ^*_c to be strictly positive. In the case of λ_c, Gouéré [7] has shown that this condition is also sufficient:

Theorem 1. [7] If $\mathbb{E}\left[\rho^d\right] < \infty$ then $\lambda_c > 0$.

We here present a similar result for λ^*_c:

Theorem 2. If $\mathbb{E}\left[\rho^d\right] < \infty$ then $\lambda^*_c > 0$.

Theorem 2 says that for the spherical Poisson Boolean model with $\mathbb{E}\left[\rho^d\right] < \infty$, there exists a non-zero value of the intensity λ for which the vacant region percolates. In fact we can say more:

Theorem 3. For any μ, if $d = 2$ then $\lambda^*_c = \lambda_c$. If $d \geq 3$ then $\lambda^*_c \geq \lambda_c$.

Sarkar [13] has proved the strict inequality $\lambda^*_c > \lambda_c$ for $d \geq 3$ when ρ is deterministic, i.e. when μ is a Dirac measure.

Theorem 2 could be seen as a trivial corollary of Theorems 1 and 3. However, we would like to prove Theorems 2 and 3 separately, to emphasise that our proof of Theorem 2 is self-contained (and quite short), whereas our proof of Theorem 3 is not, as we now discuss.

In parallel and independent work, Ahlberg, Tassion and Teixeira [2] prove a similar set of results to our Theorems 2 and 3; their proof seems to be completely different from ours. Earlier, in [1] they proved for $d = 2$ that (among other things) $\lambda^*_c = \lambda_c$ whenever $\mathbb{E}\left[\rho^2 \log \rho\right] < \infty$.

We prove Theorem 2 in the next two sections. The proof of Theorem 3 is given by adapting our proof of Theorem 2 using results in [1], and is therefore heavily reliant on [1]; we give this argument in Section 4.

Finally, we consider the relation between λ^*_c and a different percolation threshold, defined in terms of expected diameter. For non-empty $B \subset \mathbb{R}^d$, let $D(B) := \sup_{x,y \in B}(\|x - y\|)$, the Euclidean diameter of B, and set $D(\emptyset) = 0$. Let W_λ be the connected component of Z_λ containing the origin, and set

$$\lambda_D := \inf\{\lambda : \mathbb{E}\left[D(W_\lambda)\right] = \infty\}.$$

It is easy to see that that $\lambda_D \leq \lambda_c$. Therefore by Theorem 3, for any μ we have

$$\lambda^*_c \geq \lambda_D. \tag{1.1}$$

In Section 5 we present an alternative, rather simple, direct proof of (1.1) (not reliant on any other results, either here or in [1]).

A further result in [7] says that $\lambda_D > 0$, if and only if $\mathbb{E}\left[\rho^{d+1}\right] < \infty$. Therefore (1.1) provides an alternative proof that $\lambda^*_c > 0$ under this stronger moment condition. Moreover, it is known in many cases that $\lambda_D = \lambda_c$ (see e.g. [11, 15, 5]), and in all such cases our proof of (1.1) provides another way to show that $\lambda^*_c \geq \lambda_c$.

Our proof of Theorems 2 and 3 for $d = 2$ uses a form of multiscale methodology, inspired by [7], which may be of use in other settings. We conclude this section with an outline of the method. At length-scale r, we define functions $f(r)$ and $g(r)$. Up to a constant multiple, $f(r)$ is the probability of a 'local' event (defined in terms of a box-crossing, using only grains centred near the box) while $g(r)$ is the probability of an 'outside influence' event that is still determined at length-scale r.

We show that $g(10^n)$ is summable in n (see Lemma 2 below), and also that $f(10^{n+1}) \leq f(10^n)^2 + g(10^{n+1})$ (see (2.2) and (2.6) below). From this we can deduce that there exists n_0 such that $\sum_{n \geq n_0}(f(10^n) + g(10^n)) < 1$, if only we can get started by showing $f(n_0)$ is sufficiently small. This can be done either by taking λ small (in the proof of Theorem 2) or for general $\lambda < \lambda_c$, by taking n_0 large and using a result from [1] (in the proof of Theorem 3). Finally, we can take a sequence of boxes of length 10^{n+n_0}, such that if none of these is crossed then Z_λ^* percolates.

We let o denote the origin in $\mathbb{R}^d$, and for $r > 0$ put $B(r) := B(o, r)$.

2 Preparation for the proof

Throughout this section we assume that $d = 2$. We give some definitions and lemmas required for our proof of Theorem 2.

Given $\lambda > 0$, for each Borel set $A \subset \mathbb{R}^2$ we define the random set

$$Z_\lambda^A := \cup_{\{k: y_k \in \mathcal{P}_\lambda \cap A\}} B(y_k, \rho_k).$$

Also, for $r > 0$ set $A_r := \cup_{x \in A} B(x, r)$, the (deterministic) r-neighbourhood of A.

Given $r > 0$, let $S(r) := [-5r, 5r] \times [-r/2, r/2]$, the closed $10r \times r$ horizontal rectangle (or 'strip') centred at o. Note that $S(r)_r$ is a $12r \times 3r$ rectangle with its corners smoothed (this smoothing is not important to us).

Let $F_\lambda(r)$ be the event that there is a short-way crossing of $S(r)$ by $Z_\lambda^{S(r)_r}$ (that is, by grains centred within the r-neighbourhood of $S(r)$). Also define the event

$$G_\lambda(r) = \{Z_\lambda^{B(10^6 r) \setminus S(r)_r} \cap S(r) \neq \emptyset\}. \tag{2.1}$$

Lemma 1. There is a constant $C_1 \geq 1$ such that for all $\lambda > 0$ and $r > 0$,

$$\mathbb{P}[F_\lambda(10r)] \leq C_1(\mathbb{P}[F_\lambda(r)]^2 + \mathbb{P}[G_\lambda(r)]). \tag{2.2}$$

Proof. Fix (λ, r). Set $S := S(10r) = [-50r, 50r] \times [-5r, 5r]$. Let $T := [-50r, 50r] \times [-4.5r, -3.5r]$ and $\tilde{T} := [-50r, 50r] \times [3.5r, 4.5r]$, so that T and $\tilde{T}$ are horizontal $100r \times r$ thin strips along S near the bottom and top of S, respectively.

We shall now define a collection $R_1, \ldots, R_{37}$ of horizontal $10r \times r$ and vertical $r \times 10r$ rectangles that knit together in such a way that if there is a long-way vacant crossing of each of $R_1, \ldots, R_{37}$ then there is a long-way vacant crossing of T (this is a well known technique in these kinds of proof). We shall arrange that they are all contained within the band $\mathbb{R} \times [-12r, -2r]$ and their r-neighbourhoods $(R_1)_r, \ldots, (R_{37})_r)$ all lie within the lower half of the region $S_{10r} := (S(10r))_{10r}$.

Here are the details. Let $R_1, R_2, \ldots, R_{19}$ be horizontal $10r \times r$ rectangles centred on $(-45r, -4r), (-40r, -4r), \ldots, (45r, -4r)$ respectively. Let $R_{20}, \ldots, R_{37}$ be vertical $r \times 10r$ rectangles centred at $(-42.5r, -7r), (-37.5r, -7r), \ldots, (42.5, -7r)$ respectively.

Similarly, we define a collection $R_{38}, \ldots, R_{74}$ of $10r \times r$ and $r \times 10r$ rectangles, such that if each of these has a long-way vacant crossing then there is a long-way vacant crossing of $\tilde{T}$. Each rectangle R_{37+i}, $1 \leq i \leq 37$, is defined simply as the reflection of R_i in the x-axis.

For $1 \leq i \leq 74$, let D_i be the disk of radius $10^6 r$ with the same centre as R_i. Let A_i denote the event that there exists a grain of the Boolean model that intersects R_i and has its centre in the region $D_i \setminus (R_i)_r$. Let B_i denote the event that the rectangle R_i can be crossed the short way in the union of grains that are centred inside $(R_i)_r$. If R_i is crossed the short way in the union of grains centred in D_i, then $A_i \cup B_i$ must occur.

Suppose $F_\lambda(10r)$ occurs, i.e. there is a short-way occupied crossing of S, using grains centred in S_{10r}. Then there is no long-way vacant crossing of S, and hence no long-way vacant crossing either of T or of $\tilde{T}$. Hence

$$\begin{aligned} F_\lambda(10r) &\subset \cup_{(i,j)\in\{1,\dots,37\}^2} ((A_i \cup B_i) \cap (A_{37+j} \cup B_{37+j})) \\ &\subset \left(\cup_{i=1}^{74} A_i\right) \cup \left(\cup_{(i,j)\in\{1,\dots,37\}^2} (B_i \cap B_{37+j})\right). \end{aligned} \tag{2.3}$$

For $i, j \in \{1, \dots, 37\}$, since $(R_i)_r \cap (R_j)_r = \emptyset$ the events B_i and B_{37+j} are independent. Hence by (2.3) and the union bound we have (2.2), taking $C_1 = 37^2$. □

Lemma 2. Suppose $\mathbb{E}[\rho^2] < \infty$. Let $\lambda_0 \in (0, \infty)$. Then

$$\sum_{n\ge1} \sup_{\lambda\in(0,\lambda_0]} \mathbb{P}[G_\lambda(10^n)] < \infty. \tag{2.4}$$

Proof. Given $\lambda, r > 0$, if $G_\lambda(r)$ occurs then there exists a point $y_k \in \mathcal{P}_\lambda \cap B(10^6 r) \setminus S(r)_r$ with associated radius $\rho_k > r$. Therefore by Markov's inequality $\mathbb{P}[G_\lambda(r)]$ is bounded above by the expected number of such points y_k. Therefore

$$\mathbb{P}[G_\lambda(r)] \le \lambda\pi(10^6 r)^2 \mathbb{P}[\rho > r] = 10^{12}\lambda\pi r^2 \mathbb{P}[\rho^2 > r^2].$$

Hence,

$$\sum_{n\ge1} \sup_{\lambda\in(0,\lambda_0]} \mathbb{P}[G_\lambda(10^n)] \le 10^{12}\lambda_0\pi \sum_{n=1}^{\infty} 100^n \mathbb{P}[\rho^2 > 100^n]$$

which is finite because we assume $\mathbb{E}[\rho^2] < \infty$. □

Lemma 3. Suppose $\mathbb{E}[\rho^2] < \infty$. Then there exist $b > 0$ and $\lambda > 0$ such that

$$\sum_{n=1}^{\infty} (\mathbb{P}[F_\lambda(10^n b)] + \mathbb{P}[G_\lambda(10^n b)]) \le 1/2. \tag{2.5}$$

Proof. Let $C_1 \ge 1$ be as in Lemma 1. Given $\lambda, r > 0$ we define

$$f_\lambda(r) := C_1 \mathbb{P}[F_\lambda(r)]; \qquad g_\lambda(r) := C_1^2 \mathbb{P}[G_\lambda(r/10)].$$

Then by (2.2) we have

$$\begin{aligned} f_\lambda(r) &\le C_1^2 (\mathbb{P}[F_\lambda(r/10)]^2 + \mathbb{P}[G_\lambda(r/10)]) \\ &= f_\lambda(r/10)^2 + g_\lambda(r). \end{aligned} \tag{2.6}$$

Let $C_2 = 9$. Using (2.4), we can choose b to be a big enough power of 10 so that for all $\lambda \in (0, 1]$, we have

$$\sum_{n=1}^{\infty} g_\lambda(10^n b) \le C_2^{-2}. \tag{2.7}$$

Now fix this b. Choose $\lambda \le 1$ to be small enough so that $f_\lambda(b) \le C_2^{-1}$. Using (2.6) repeatedly, we have $f_\lambda(10^n b) \le C_2^{-1}$ for all n. Then using (2.6) repeatedly again, we have for $n \in \mathbb{N}$ that

$$\begin{aligned} f_\lambda(10^n b) &\le \frac{f_\lambda(10^{n-1} b)}{C_2} + g_\lambda(10^n b) \le \cdots \\ &\le C_2^{-n-1} + \frac{g_\lambda(10b)}{C_2^{n-1}} + \frac{g_\lambda(100b)}{C_2^{n-2}} + \cdots + \frac{g_\lambda(10^n b)}{C_2^0}, \end{aligned}$$

and therefore

$$\sum_{n\geq 1} f_\lambda(10^n b) \leq (C_2^{-2} + g_\lambda(10b) + g_\lambda(100b) + g_\lambda(1000b) + \cdots) \times (1 + C_2^{-1} + C_2^{-2} + \cdots)$$

so by (2.7) and the fact that $\sum_{k=0}^{\infty} C_2^{-k} \leq 2$, we have

$$\begin{aligned}\sum_{n\geq 1}(\mathbb{P}[F_\lambda(10^n b)] + \mathbb{P}[G_\lambda(10^n b)]) &\leq \sum_{n\geq 1}(f_\lambda(10^n b) + g_\lambda(10^n b)) \\ &\leq 2C_2^{-2} + 3\sum_{n\geq 1} g_\lambda(10^n b) \leq 5C_2^{-2},\end{aligned}$$

and hence (2.5). □

3 Proof of Theorem 2

We can now complete the proof of Theorem 2. We assume from now on that

$$\mathbb{E}\,[\rho^d] < \infty. \tag{3.1}$$

Consider first the case with $d = 2$. Let b and λ be as given in Lemma 3.

Let $S_1, S_2, S_3, \ldots$ be a sequence of 'strips', i.e. closed rectangles of aspect ratio 10, with successive lengths (the short way) $10b, 100b, 1000b, \ldots$ alternating between horizontal and vertical strips with each strip S_n centred at the origin. Then each strip S_n crosses the next one S_{n+1} the short way.

For each $n \in \mathbb{N}$, define the events

$$H_n := \{S_n \text{ is crossed by } Z_\lambda^{S_{n+2}} \text{ the short way}\};$$

$$J_n := \{Z_\lambda^{S_{n+4}\setminus S_{n+2}} \cap S_n \neq \emptyset\}.$$

Lemma 4. If none of the events $H_1, J_1, H_2, J_2, \ldots$ occurs then Z_λ^* percolates.

Proof. Suppose none of the events $H_1, J_1, H_2, J_2, \ldots$ occurs.

We claim for each $n \in \mathbb{N}$ that $Z_\lambda^{\mathbb{R}^2\setminus S_{n+2}} \cap S_n = \emptyset$. Indeed, if $Z_\lambda^{\mathbb{R}^2\setminus S_{n+2}} \cap S_n \neq \emptyset$ then for some integer $m \geq n+2$ with $m-n$ even we have $Z_\lambda^{S_{m+2}\setminus S_m} \cap S_n \neq \emptyset$, and then since $n \leq m$ we also have $S_n \subset S_m$ so that $Z_\lambda^{S_{m+2}\setminus S_m} \cap S_m \neq \emptyset$, contradicting the assumed non-occurrence of J_{m-2}.

For each n, by the assumed non-occurrence of H_n along with the preceding claim there is no short-way crossing of S_n by Z_λ so there is a long-way crossing of S_n by Z_λ^*, i.e. a path $\gamma_n \subset S_n \cap Z_\lambda^*$ that crosses S_n the long way.

Then for each n we have $\gamma_n \cap \gamma_{n+1} \neq \emptyset$, so $\cup_n \gamma_n$ is an unbounded connected set contained in Z_λ^*. Therefore Z_λ^* percolates. □

Proof of Theorem 2. Suppose $d = 2$. Let λ and b be as given in Lemma 3. Recall the definition of events $F_\lambda(r)$ and $G_\lambda(r)$ at (2.1). We claim now for each n that

$$\mathbb{P}[H_n \cup J_n] \leq \mathbb{P}[F_\lambda(10^n b)] + \mathbb{P}[G_\lambda(10^n b)]. \tag{3.2}$$

Indeed, suppose the parity of n is such that S_n is horizontal. Then, in terms of earlier notation, $S_n = S(10^n b)$. Since $S_{n+4} \subset B(10^{6+n} b)$ we have $J_n \subset G_\lambda(10^n b)$ and $H_n \subset F_\lambda(10^n b) \cup G_\lambda(10^n b)$. Then (3.2) follows from the union bound.

Using first Lemma 4, then (3.2), and finally (2.5), we have

$$\begin{aligned} 1-\mathbb{P}[U^*_\lambda] &\leq \mathbb{P}[\cup_{n=1}^\infty (H_n \cup J_n)] \\ &\leq \sum_{n=1}^\infty (\mathbb{P}[F_\lambda(10^n b)] + \mathbb{P}[G_\lambda(10^n b)]) \leq 1/2. \end{aligned}$$

Therefore by ergodicity $\mathbb{P}[U^*_\lambda] = 1$ so $\lambda \leq \lambda^*_c$. Hence we have $\lambda^*_c > 0$ as required.

Now suppose $d \geq 3$. Let $\tilde{Z}_\lambda$, be the intersection of Z_λ with the two-dimensional subspace $\mathbb{R}^2 \times \{o''\}$ of $\mathbb{R}^d$, where o'' denotes the origin in $\mathbb{R}^{d-2}$.

Let ω_{d-2} denote the volume of the unit ball in $\mathbb{R}^{d-2}$. It can be seen that $\tilde{Z}_\lambda$ is a two-dimensional Boolean model with intensity

$$\lambda \omega_{d-2}(d-2) \int_0^\infty \mathbb{P}[\rho \geq r] r^{d-3} dr = \lambda \omega_{d-2} \mathbb{E}\left[\rho^{d-2}\right] =: \lambda',$$

which is finite by our assumption (3.1). Moreover if σ denotes a random variable with the radius distribution in this planar Boolean model we claim that $\mathbb{E}\left[\sigma^2\right] < \infty$. This can be demonstrated by a computation, but it is more quickly seen using the fact that, since $\mathbb{P}[o \in Z_\lambda] < 1$ for the original Boolean model by (3.1), also $\mathbb{P}[o \in \tilde{Z}_\lambda] < 1$, which would not be the case if $\mathbb{E}\left[\sigma^2\right]$ were infinite.

Therefore by the two-dimensional case already considered, for small enough $\lambda > 0$ we have λ' small enough so that the complement (in the space $\mathbb{R}^2 \times \{o''\}$) of $\tilde{Z}_\lambda$ percolates. Hence Z^*_λ percolates for small enough $\lambda > 0$, so $\lambda^*_c > 0$. □

4 Proof of Theorem 3

As mentioned in Section 1, if $\mathbb{E}\left[\rho^d\right] = \infty$ then $\lambda_c = \lambda^*_c = 0$, so without loss of generality we assume (3.1).

First suppose $d = 2$. We need to prove that $\lambda^*_c = \lambda_c$.

Suppose $\lambda > \lambda_c$. Let V^*_λ be the event that there is an unbounded component of Z^*_λ intersecting with $B(1)$. For $n \in \mathbb{N}$, set $Q(n) := [-n, n]^2$. Let $E(n)$ be the event that there exists a path in Z^*_λ from $Q(n)$ to $\mathbb{R}^2 \setminus Q(3n)$.

The annulus $Q(3n) \setminus Q(n)$ can be written as the union of two $3n \times n$ and two $n \times 3n$ rectangles, and if Z_λ crosses each of these four rectangles the long way then $Q(n)$ is surrounded by an occupied circuit contained in $Q(3n)$ so $E(n)$ does not occur. Hence by Theorem 1.1 (i) of [1] and the union bound, $\mathbb{P}[E(n)] \to 0$ as $n \to \infty$. Since $V^*_\lambda \subset \cap_{n=1}^\infty E(n)$, we therefore have $\mathbb{P}[V^*_\lambda] = 0$ and hence $\mathbb{P}[U^*_\lambda] = 0$. Hence $\lambda \geq \lambda^*_c$ so $\lambda^*_c \leq \lambda_c$.

Now suppose $\lambda < \lambda_c$. Then by Theorem 1.1(iii) of [1], in the proof of our Lemma 3 we can choose b large enough so that we have both (2.7), and the inequality $f_\lambda(b) < C_2^{-1}$. Then the rest of the proof of Lemma 3 carries through for this (b, λ), so the conclusion of Lemma 3 holds for this (b, λ). Then the proof (for $d = 2$) in Section 3 works for this (b, λ), showing that $\lambda \leq \lambda^*_c$ for any $\lambda < \lambda_c$ and hence that $\lambda^*_c \geq \lambda_c$. Thus $\lambda^*_c = \lambda_c$ for $d = 2$.

Now suppose that $d \geq 3$ and $\lambda < \lambda_c$. Then as discussed in Section 3, $Z_\lambda \cap (\mathbb{R}^{d-2} \times \{o''\})$ is a two-dimensional Boolean model possessing no infinite component, so the radius distribution for this two-dimensional Boolean model has finite second moment and the intensity λ' of this two-dimensional Boolean model is subcritical (in fact, strictly subcritical since we can repeat the argument for any $\lambda_1 \in (\lambda, \lambda_c)$). Therefore by the argument just given for $d = 2$, the complement (in $\mathbb{R}^{d-2} \times \{o''\}$) of this Boolean model percolates, and therefore the original Z^*_λ also percolates so $\lambda \leq \lambda^*_c$. Hence $\lambda^*_c \geq \lambda_c$, and the proof is complete.

5 Alternative proof of (1.1)

We divide the nonnegative x-axis into unit intervals $I_0, I_1, I_2, \dots$ where $I_k = [k, k+1) \times \{o'\}$ (here o' is the origin in $\mathbb{R}^{d-1}$). For each $k \in \mathbb{N}$ let $W_{k,\lambda}$ be the union of I_k and all components of Z_λ which intersect I_k.

Lemma 5. If $0 < \lambda < \lambda_D$, then $\mathbb{E}\left[D(W_{0,\lambda})\right] < \infty$.

Proof. Fix $\lambda \in (0, \lambda_D)$. Then $\mathbb{E}\left[D(W_\lambda)\right] < \infty$. Let F be the event that $I_0 \subset Z_\lambda$, and set $F^c := \Omega \setminus F$. Then $0 < \mathbb{P}[F] < 1$. If F occurs then $W_{0,\lambda} = W_\lambda$. Hence by the Harris-FKG inequality (see [11] or [10]),

$$\mathbb{E}\left[D(W_{0,\lambda})\right] \leq \mathbb{E}\left[D(W_{0,\lambda})|F\right] = \mathbb{E}\left[D(W_\lambda)|F\right] < \infty,$$

as required. □

Given $\lambda > 0$, define the event

$$E_\lambda := (\cap_{k=2}^{\infty}\{D(W_{k,\lambda}) \leq k/2\}) \cap \{Z_\lambda \cap (I_0 \cup I_1) = \emptyset\}.$$

Lemma 6. If $0 < \lambda < \lambda_D$, then $\mathbb{P}[E_\lambda] > 0$.

Proof. Fix $\lambda \in (0, \lambda_D)$. Then by Lemma 5.

$$\sum_{k\geq 1}\mathbb{P}[D(W_{k,\lambda}) > k/2] = \sum_{k\geq 1}\mathbb{P}[D(W_{0,\lambda}) > k/2] \leq \mathbb{E}\left[2D(W_{0,\lambda})\right] < \infty.$$

Choose $k_0 \in \mathbb{N}$ with $k_0 > 2$, such that $\sum_{k\geq k_0}\mathbb{P}[D(W_{k,\lambda}) > k/2] < 1/2$. Then by the union bound and complementation, $\mathbb{P}[\cap_{k=k_0}^{\infty}\{D(W_{k,\lambda}) \leq k/2\}] \geq 1/2$. Moreover $\mathbb{P}[\cap_{k=0}^{k_0}\{Z_\lambda \cap I_k = \emptyset\}] > 0$. Hence by the Harris-FKG inequality,

$$\mathbb{P}[E_\lambda] \geq \mathbb{P}\left[\left(\cap_{k=k_0}^{\infty}\{D(W_{k,\lambda}) \leq k/2\}\right) \cap \left(\cap_{k=0}^{k_0}\{Z_\lambda \cap I_k = \emptyset\}\right)\right] > 0.$$ □

Lemma 7. Suppose that $A \subset \mathbb{R}^d$ is closed, connected and unbounded, and that $\mathbb{R}^d \setminus A$ has an unbounded connected component. Then ∂A, the boundary of A, has an unbounded connected component.

Proof. Let B be an unbounded component of $\mathbb{R}^d \setminus A$. Denote the closure of B by $\overline{B}$. Then both $\overline{B}$ and $\mathbb{R}^d \setminus B$ are closed and connected. By the unicoherence of $\mathbb{R}^d$ [12], the set $\overline{B} \cap A = \overline{B} \cap (\mathbb{R}^d \setminus B)$ is connected. Moreover it is unbounded, and contained in ∂A. □

Given $\varepsilon > 0$, let $\tilde{Z}_{\lambda,\varepsilon} := \cup_{k:\rho_k>0}B(y_k, \varepsilon\rho_k)$ and let $\tilde{Z}^*_{\lambda,\varepsilon} := \mathbb{R}^d \setminus \tilde{Z}_{\lambda,\varepsilon}$. Let $Z^0_\lambda := \cup_{\{k:r_k=0\}}\{y_k\}$, the union of balls of radius zero contributing to our Boolean model ($\tilde{Z}_{\lambda,\varepsilon}$ is the union of all the other balls, scaled by ε). If $\mathbb{P}[\rho = 0] > 0$ then Z^0_λ is (almost surely) non-empty but locally finite.

Lemma 8. Let $\varepsilon \in (0, 1)$. If E_λ occurs then $\tilde{Z}^*_{\lambda,\varepsilon} \cup Z^0_\lambda$ percolates.

Proof. Suppose E_λ occurs. Let A be the union of the half-line $[1, \infty) \times \{o'\}$, with all components of Z_λ intersecting this half-line. Then A is connected, unbounded, and contained in the half-space $[1, \infty) \times \mathbb{R}^{d-1}$ so o lies in an unbounded component of $\mathbb{R}^d \setminus A$. Therefore by Lemma 7, ∂A has an unbounded connected component.

No point of ∂A lies in the interior of any of the balls $B(y_k, \rho_k)$. Therefore $\partial A \subset \tilde{Z}^*_{\lambda,\varepsilon} \cup Z^0_\lambda$. Thus $\tilde{Z}^*_{\lambda,\varepsilon} \cup Z^0_\lambda$ has an unbounded connected subset. □

Proof of (1.1). Assume $\lambda_D > 0$ (else there is nothing to prove). Suppose $\lambda \in (0, \lambda_D)$ and $\varepsilon \in (0,1)$. By the last two lemmas, with strictly positive probability the set $\tilde{Z}^*_{\lambda,1-\varepsilon} \cup Z^0_\lambda$ percolates. Almost surely, $\tilde{Z}^*_{\lambda,1-\varepsilon}$ is open, Z^0_λ is locally finite and all points of Z^0_λ lie either in $\tilde{Z}^*_{\lambda,1-\varepsilon}$ or in the interior of $\tilde{Z}_{\lambda,1-\varepsilon}$. Therefore if the set $\tilde{Z}^*_{\lambda,1-\varepsilon} \cup Z^0_\lambda$ percolates, so does $\tilde{Z}^*_{\lambda,1-\varepsilon}$, and so does $\tilde{Z}^*_{\lambda,1-\varepsilon} \setminus Z^0_\lambda$. Thus the set $\tilde{Z}^*_{\lambda,1-\varepsilon} \setminus Z^0_\lambda$, which is equal to $\mathbb{R}^d \setminus \cup_k B(y_k, (1-\varepsilon)\rho_k)$, percolates with strictly positive probability, and hence by ergodicity, with probability 1. Hence by scaling (see [11]) the set $Z^*_{(1-\varepsilon)^d\lambda}$ also percolates almost surely, so that $\lambda^*_c \geq (1-\varepsilon)^d \lambda$, and therefore $\lambda^*_c \geq \lambda_D$. □

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