

NON-UNIQUENESS FOR PLATEAU'S PROBLEM. A BIFURCATION PROCESS

JOHANNES C. C. NITSCHÉ

The discussion to follow will deal with aspects of Plateau's problem about which no exhaustive information is available today, and particular attention will be paid to the phenomenon of non-uniqueness. It is well known that the area of a minimal surface, its suggestive name notwithstanding, need not furnish a minimum (absolute or relative) among the areas of all surfaces having the same boundary. Let us consider a minimal surface $S = \{ \mathbf{r} = \mathbf{r}(u, v); u^2 + v^2 < R^2 \}$ which lies imbedded in Euclidean 3-space. We shall assume that u and v are isothermal parameters on S so that $r_u^2(u, v) = r_v^2(u, v) \equiv E(u, v) > 0$ and $r_u(u, v) r_v(u, v) \equiv F(u, v) = 0$. Let us further denote by $\mathfrak{X}(u, v)$ the unit normal vector of our surface and by $K(u, v)$ its Gaussian curvature. On S we consider the one-parameter family of Jordan curves

$$\Gamma_r = \{ \mathbf{r} = \mathbf{r}(r \cos \vartheta, r \sin \vartheta); 0 \leq \vartheta \leq 2\pi \}, \quad 0 < r < R,$$

which bound expanding portions

$$S(r) = \{ \mathbf{r} = \mathbf{r}(u, v); (u, v) \in \bar{P}_r \}$$

of S . Here \bar{P}_r is used as an abbreviation for the closure of the disc $P_r = \{ u, v; u^2 + v^2 < r^2 \}$. For sufficiently small values of r two assertions are valid:

- i) $S(r)$ is the unique solution of Plateau's problem for Γ_r .
- ii) The area $A[S(r)]$ of $S(r)$ furnishes the absolute minimum among the areas of all disc-type surfaces bounded by Γ_r .

As r increases, either one of these statements may become false. As for i), $S(r)$ will remain unique as long as Γ_r retains a simply covered convex curve as its parallel, or central, projection on a plane (see [10]); or as long as the total curvature of Γ_r does not exceed the value 4π (see [7]). On the other hand, the minimizing property of the surface $S(r)$ – more precisely, the question whether $A[S(r)]$ remains at least a relative

minimum — depends on the second variation of its area which, in turn, is closely related to the eigenvalue problem

$$(1) \quad \Delta \xi - 2 \lambda E K \xi = 0 \quad \text{in } P_r, \quad \xi = 0 \quad \text{on } \partial P_r.$$

Because $K(u, v)$ is non-positive and vanishes at most in isolated points, this problem will have a sequence of eigenvalues $\{\lambda_n(r)\}$ satisfying the inequalities $0 < \lambda_1(r) < \lambda_2(r) \leq \lambda_3(r) \leq \dots$ and corresponding eigenfunctions $\xi_1(u, v; r)$, $\xi_2(u, v; r)$, ... subject to the ortho-normality relations

$$2 \int_{P_r} \int E |K| \xi_m \xi_n \, du \, dv = \delta_{mn}.$$

The smallest eigenvalue $\lambda_1(r)$ is a continuous and strictly decreasing function of r .

If $\eta(u, v) = \sum_{n=1}^{\infty} c_n \xi_n(u, v; r)$ denotes an arbitrary regular function, vanishing on ∂P_r , then the area of the comparison surface $S^{(\varepsilon)}(r) = \{ \mathbf{r} = \mathbf{r}(u, v) + \varepsilon \eta(u, v) \mathbf{X}(u, v); (u, v) \in \bar{P}_r \}$ is found to be

$$\begin{aligned} A[S^{(\varepsilon)}(r)] &= A[S(r)] + \frac{1}{2} \varepsilon^2 \int_{P_r} \int (\eta_u^2 + \eta_v^2 + 2 E K \eta^2) \, du \, dv + O(\varepsilon^3) \\ &= A[S(r)] + \frac{1}{2} \varepsilon^2 \sum_{n=1}^{\infty} (\lambda_n(r) - 1) c_n^2 + O(\varepsilon^3) \\ &\geq A[S(r)] + \varepsilon^2 (\lambda_1(r) - 1) \int_{P_r} \int E |K| \eta^2 \, du \, dv + O(\varepsilon^3). \end{aligned}$$

If $\lambda_1(r) > 1$, then $S(r)$ can be imbedded in a field of minimal surfaces (a detailed construction can be found in [7]), and by a classical argument going back to H. A. Schwarz ([11], pp. 224–240, 332–334) it is seen that the area of $S(r)$ furnishes a strong relative minimum. Moreover, since the boundary value problem $\Delta \eta - 2 E K \eta = 0$ in P_r , $\eta = 0$ on ∂P_r under the assumption $\lambda_1(r) > 1$ has the unique solution $\eta(u, v) \equiv 0$, it follows (see e.g. [2], Lemma 3.4) that $S(r)$ is isolated or, as we shall also say, locally unique; see [8]. By [1] the inequality $\lambda_1(r)$ is guaranteed at least as long as the total curvature $\iint_{P_r} E |K| \, du \, dv$ of $S(r)$ remains smaller than 2π .

If $\lambda_1(r) < 1$, then (by choosing $\eta(u, v) = \xi_1(u, v; r)$) disc-type surfaces bounded by Γ_r and lying arbitrarily close to $S(r)$ can be constructed whose area is smaller than the area of $S(r)$. $S(r)$ is thus seen to be unstable. It is a matter of record, however, that Γ_r also bounds a solution surface of least area for Plateau's problem. Obviously, this surface

must be distinct from $S(r)$. Whether it is close to $S(r)$ or not is, of course, another question. In any case, Γ_r bounds at least two solutions of Plateau's problem whenever $\lambda_1(r) < 1$.

The preceding remarks suggest the possible existence of two positive numbers r_1 and r_2 defined as follows: r_1 is the supremum of all those values r for which the surface $S(r)$ represents the unique solution of Plateau's problem for Γ_r . The number r_2 is the supremum of all those values r for which the area of $S(r)$ is a strong relative minimum among the areas of all disc-type surfaces bounded by Γ_r , i.e., $\lambda_1(r_2) = 1$. Obviously, the inequalities $0 < r_1 \leq r_2$ hold. Since it is not clear whether the local uniqueness of $S(r)$ for $\lambda_1(r) > 1$ implies its global uniqueness, the strict inequality $r_1 < r_2$ cannot be excluded apriori. Our interest here is directed not so much to this question but rather to the study of the *bifurcation process* in which a second or more surfaces branch off from $S(r)$ as r passes through the critical value $r = r_2$. Considering that the treasure of general insights is still hidden today, we shall pursue this bifurcation process in detail for the case of an explicit example — Enneper's minimal surface whose position vector is given with the help of isothermal parameters u and v by

$$\mathbf{r}(u, v) = \begin{pmatrix} u + u v^2 - \frac{1}{3} u^3 \\ -v - u^2 v + \frac{1}{3} v^3 \\ u^2 - v^2 \end{pmatrix}.$$

We have $E(u, v) = (1 + u^2 + v^2)^2$ and $K(u, v) = -4(1 + u^2 + v^2)^{-4}$. It is known (see [9], § 90) that the part of Enneper's surface which corresponds to the disc $u^2 + v^2 < 3$ is free of self-intersections. In particular, the image of every concentric circle $u^2 + v^2 = r^2$, $0 < r < \sqrt{3}$, is a Jordan curve $\Gamma_r = \{ \mathbf{r} = \mathfrak{z}(\vartheta; r); 0 \leq \vartheta \leq 2\pi \}$ with the position vector

$$\mathfrak{z}(\vartheta; r) = \mathbf{r}(r \cos \vartheta, r \sin \vartheta) = \begin{pmatrix} r \cos \vartheta - \frac{1}{3} r^3 \cos 3\vartheta \\ -r \sin \vartheta - \frac{1}{3} r^3 \sin 3\vartheta \\ r^2 \cos 2\vartheta \end{pmatrix}.$$

Pictures of Γ_r can be found in [9], pp. 76, 79. The points $(u = \pm \sqrt{3}, v = 0)$ on the circle $u^2 + v^2 = 3$ are mapped by $\mathbf{r}(u, v)$ into the same point $\{0, 0, 3\}$ in space. Accordingly, we shall choose $R = \sqrt{3}$.

The curvature of the projection γ_r of Γ_r onto the (x, y) -plane is given by

$$\frac{(1 + r^2)(1 - 3r^2) + 4r^2 \sin^2 2\vartheta}{r [1 + 2r^2 \cos 4\vartheta + r^4]^{3/2}}.$$

γ_r is a convex curve for $0 < r < 1/\sqrt{3}$ so that $S(r)$ is the unique solution of Plateau's problem for Γ_r if $0 < r \leq 1/\sqrt{3}$. The second criterion for uniqueness mentioned above leads to a better result. A computation shows that the total curvature $\kappa(\Gamma_r)$ of Γ_r is equal to

$$\kappa(\Gamma_r) = \frac{8r}{1+r^2} \frac{1}{k(r)} E(k(r)), \quad k(r) = \frac{2r}{\sqrt{1+10r^2+9r^4}}$$

and that

$$\frac{d}{dr} \kappa(\Gamma_r) \geq 6\pi \frac{k(r)}{1+r^2}.$$

Here $E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \vartheta} d\vartheta$ denotes the complete elliptic integral of the second kind. We have $\kappa(\Gamma_r) = 4\sqrt{3} E(1/2) = 10.167... < 4\pi$ for $r = 1/\sqrt{3}$ and $\kappa(\Gamma_r) = 4\sqrt{5} E(1/\sqrt{5}) = 13.369... > 4\pi$ for $r = 1$. The equality $\kappa(\Gamma_r) = 4\pi$ is achieved for $r = 0.882$ (see [9], § 828). It is therefore certain that $r_1 \geq 0.882$, although we conjecture that $r_1 = 1$.

The function $\zeta(u, v) = (1-u^2-v^2)/(1+u^2+v^2)$, which is positive for $u^2+v^2 < 1$ and zero for $u^2+v^2 = 1$, is a solution of the differential equation

$$\Delta \zeta - 2EK\zeta \equiv \Delta \zeta + \frac{8}{(1+u^2+v^2)^2} \zeta = 0.$$

From this fact it can be concluded that $\lambda_1(1) = 1$ so that $r_2 = 1$. Further properties of the curves Γ_r are discussed in [9], §§ 91, 390–396. It has been proved in [4] that Γ_r bounds three distinct solutions of Plateau's problem for r in a certain interval $r_0 < r < \sqrt{3}$. (A crude estimate for r_0 is $r_0 \leq 1.682$.) From the following developments it will be seen that *as r passes increasingly through the critical value $r = 1$, the curves Γ_r acquire the capability of bounding, in addition to Enneper's surface, two further minimal surfaces which appear in a continuous bifurcation process.*

It is now necessary to interject some basic definitions. Let $\Gamma = \{r = \mathfrak{z}(\tau); 0 \leq \tau \leq 2\pi\}$ be a Jordan curve in Euclidean 3-space. We are concerned with vectors $r(u, v)$ defined in the closure \bar{P} of the unit disc $P = \{u, v; u^2 + v^2 < 1\}$ which map the boundary ∂P onto Γ . Setting $u + iv = w = \varrho e^{i\vartheta}$ we shall henceforth interchangeably use the notations $r(u, v)$, or $r(w)$, or $r(\varrho, \vartheta)$ (and later also $r(w; r)$ instead of $r(u, v; r)$, etc.) – whichever is most convenient. Denote by $\mathfrak{F} = \mathfrak{F}(\Gamma)$ the set of vectors $r(w) \in C^2(P) \cap C^0(\bar{P})$ which are harmonic in P and which map ∂P onto Γ monotonically so that three fixed distinct points w_1, w_2, w_3 on ∂P are transformed into three fixed distinct points $\mathfrak{z}_j = \mathfrak{z}(\tau_j)$, ($j = 1, 2, 3$), on Γ , respectively. Once and for all we shall choose $w_1 = 1$,

$w_2 = i, w_3 = -1$. Then $r(e^{i\vartheta}) = \mathfrak{z}(\tau(\vartheta))$ where $\tau(\vartheta)$ is a continuous monotone function satisfying the conditions $\tau(0) = \tau_1, \tau(\pi/2) = \tau_2, \tau(\pi) = \tau_3$ and $\tau(\vartheta + 2\pi) = \tau(\vartheta) + 2\pi$. Endowed with the distance between two vectors,

$$|r_2 - r_1| = \max_{(u,v) \in \bar{P}} |r_2(u, v) - r_1(u, v)|,$$

the set \mathfrak{S} becomes a metric space. Each element $r(u, v)$ of \mathfrak{S} defines a harmonic surface $S = \{r = r(u, v); (u, v) \in \bar{P}\}$. By definition, a solution of Plateau's problem for the curve Γ is a harmonic surface whose position vector satisfies in P the additional relations $r_u^2 = r_v^2, r_u r_v = 0$. We denote by \mathfrak{M} the set of all such vectors. It is a matter of record that an element $r(u, v)$ of \mathfrak{M} provides a topological mapping between ∂P and Γ .

Since we shall benefit from working with the fixed parameter domain P , a slight change of the introductory notation is advisable: The portion of Enneper's surface bounded by Γ_r will now be denoted by $S^{(0)}(r) = \{r = r^{(0)}(w; r); w \in \bar{P}\}$ where

$$r^{(0)}(\varrho, \vartheta; r) = \left\{ \begin{array}{l} r \varrho \cos \vartheta - \frac{1}{3} r^3 \varrho^3 \cos 3\vartheta \\ -r \varrho \sin \vartheta - \frac{1}{3} r^3 \varrho^3 \sin 3\vartheta \\ r^2 \varrho^2 \cos 2\vartheta \end{array} \right\} = \operatorname{Re} \left\{ \begin{array}{l} r w - \frac{1}{3} r^3 w^3 \\ i r w + i \frac{1}{3} r^3 w^3 \\ r^2 w^2 \end{array} \right\}$$

and $r^{(0)}(1, \vartheta; r) = \mathfrak{z}(\vartheta; r)$.

Let $S(r) = \{r = r(w; r); w \in \bar{P}\}$ be another solution of Plateau's problem for the curve Γ_r . By a theorem of H. Lewy [3] the position vector of $S(r)$ is analytic in \bar{P} . Moreover, owing to the results of [6], the derivatives of $r(w; r)$ can be estimated uniformly for all possible solutions. We shall refer here to the validity of the inequality

$$\max_{0 \leq \vartheta \leq 2\pi} \{ |r(1, \vartheta; r)|, |r_\vartheta(1, \vartheta; r)|, |r_{\vartheta\vartheta}(1, \vartheta; r)| \} \leq K(r)$$

with a universal constant $K = K(r)$ depending on r alone, $0 < r < \sqrt{3}$. From the proofs in [6] it would not be difficult to extract an explicit expression for $K(r)$ in terms of the geometric properties of Γ_r . For the purposes at hand knowledge of the precise form of $K(r)$ would only be of minor advantage, however.

For the time being a value $r, 1 \leq r < \sqrt{3}$ is chosen and then kept fixed. A computation shows that $|r_\vartheta^{(0)}(1, \vartheta; r)| = r(1+r^2)$ and

$$|r_{\vartheta\vartheta}^{(0)}(1, \vartheta; r)| = r[1 + r^2(6 + 4 \cos^2 2\vartheta) + 9 r^4]^{1/2} \leq r \sqrt{1+r^2} \sqrt{1+9 r^2}.$$

From [8], p. 406, we know that $|\mathbf{r}_\vartheta(1, \vartheta)| \geq r(1+r^2)/2$ as long as $|\mathbf{r} - \mathbf{r}^{(0)}| < \varepsilon_1 \equiv \min(\pi/50, 1/175K)$. Since Γ_r is the topological image of ∂P under the mapping of the position vectors of both surfaces $S^{(0)}$ and S , the boundary values of \mathbf{r} are related to those of $\mathbf{r}^{(0)}$ through an expression of the form

$$(2) \quad \mathbf{r}(1, \vartheta; r) = \mathbf{r}^{(0)}(1, \vartheta + \lambda(\vartheta); r).$$

Here $\lambda(\vartheta)$ is a periodic analytic function satisfying the inequality $\lambda'(\vartheta) \geq -1$. We write $\lambda(\vartheta)$ as a Fourier series,

$$\lambda(\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\vartheta + b_n \sin n\vartheta)$$

with the norm $\|\lambda\|$ defined by

$$\|\lambda\|^2 = \frac{1}{\pi} \int_0^{2\pi} \lambda^2(\vartheta) d\vartheta = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

In view of the "three point condition" above, $\lambda(\vartheta)$ is subject to the conditions $\lambda(0) = \lambda(\pi/2) = \lambda(\pi) = 0$. The relation $\mathbf{r}_\vartheta(1, \vartheta; r) = (1 + \lambda'(\vartheta)) \mathbf{r}_\vartheta^{(0)}(1, \vartheta + \lambda(\vartheta); r)$ which follows from (2) implies the inequalities

$$\frac{1}{2} \leq 1 + \lambda'(\vartheta) \leq \frac{1}{2} K \quad \text{whenever} \quad |\mathbf{r} - \mathbf{r}^{(0)}| < \varepsilon_1.$$

From [8], p. 406, we also conclude that

$$\frac{1}{8} |\mathbf{r} - \mathbf{r}^{(0)}| \leq |\lambda(\vartheta)| \leq |\mathbf{r} - \mathbf{r}^{(0)}| \quad \text{for} \quad |\lambda(\vartheta)| \leq \frac{\pi}{3}.$$

It follows that the size of the neighborhood of $\mathbf{r}^{(0)}$, as measured according to the metric of \mathfrak{H} , is also governed by the magnitude of $L = \max_{0 \leq \vartheta \leq 2\pi} |\lambda(\vartheta)|$. A series expansion of (2) leads to

$$(3) \quad \mathbf{r}(1, \vartheta; r) = \mathbf{r}^{(0)}(1, \vartheta; r) + \lambda(\vartheta) \mathbf{r}_\vartheta^{(0)}(1, \vartheta; r) + \frac{1}{2} \lambda^2(\vartheta) \mathbf{r}_{\vartheta\vartheta}^{(0)}(1, \vartheta; r) + \dots$$

We denote by $\mathbf{r}^{(k)}(\varrho, \vartheta; r)$ the harmonic vector in \bar{P} with boundary values

$$(4) \quad \mathbf{r}^{(k)}(1, \vartheta; r) = \frac{1}{k!} \lambda^k(\vartheta) \frac{\partial^k}{\partial \vartheta^k} \mathbf{r}^{(0)}(1, \vartheta; r).$$

The complex-valued expression

$$(5) \quad \begin{aligned} \Phi(\varrho, \vartheta) + i \Psi(\varrho, \vartheta) &= \frac{1}{2} (\mathbf{r}_\vartheta^2 - \varrho^2 \mathbf{r}_\vartheta^2) + i \varrho \mathbf{r}_\vartheta \mathbf{r}_\vartheta \\ &= \frac{1}{2} \varrho^2 e^{2i\vartheta} [(\mathbf{r}_v^2 - \mathbf{r}_u^2) + i 2 \mathbf{r}_u \mathbf{r}_v] \end{aligned}$$

is an analytic function in P for every element of \mathfrak{S} , i.e., for every surface whose position vector is harmonic in P . It is therefore easily seen that the surface S is a solution of Plateau's problem if, and only if, $\Psi(1, \vartheta) = 0$. From [8] we know that this equation is equivalent with the condition

$$(6) \quad \sum_{m=1}^{\infty} A^{(m)}(1, \vartheta; r, \lambda) = 0.$$

Here the $A^{(m)}$ are defined by

$$(7) \quad A^{(m)}(1, \vartheta; r, \lambda) = \sum_{k=0}^m \frac{1}{k!} \lambda^k(\vartheta) r_e^{(m-k)}(1, \vartheta; r) \frac{\partial^{k+1}}{\partial \vartheta^{k+1}} r^{(0)}(1, \vartheta; r).$$

Applying the estimating procedures used in [5], [8] we find that there are positive constants \mathcal{C} and M such that for $\max_{0 \leq \vartheta \leq 2\pi} |\lambda(\vartheta)| \leq L$, $\max_{0 \leq \vartheta \leq 2\pi} |\bar{\lambda}(\vartheta)| \leq L$

$$(8) \quad \|A^{(m)}(1, \vartheta; r, \lambda)\| \leq \mathcal{C} \frac{(ML)^{m-1}}{(m-1)!} \|\lambda'\|,$$

$$\|A^{(m)}(1, \vartheta; r, \bar{\lambda}) - A^{(m)}(1, \vartheta; r, \lambda)\| \leq \mathcal{C} \frac{(ML)^{m-2}}{(m-2)!} (\|\lambda'\| + \|\bar{\lambda}'\|) \|\bar{\lambda}' - \lambda'\|.$$

From [8], p. 408 we take over the computation of $\sigma(\vartheta; r, \lambda) \equiv A^{(1)}(1, \vartheta; r, \lambda)$:

$$(9) \quad \sigma(\vartheta; r, \lambda) = r^2(r^2 + 1) \left\{ a_2 c_2 - \frac{r^2 - 1}{r^2 + 1} b_2 s_2 + \sum_{n=3}^{\infty} [(r^2 + 1)n - (3r^2 + 1)] (a_n c_n + b_n s_n) \right\}.$$

In this formula the abbreviations $c_n = \cos n\vartheta$ and $s_n = \sin n\vartheta$ have been used. Condition (6) can be written as

$$(10) \quad \sigma(\vartheta; r, \lambda) = - \sum_{m=2}^{\infty} A^{(m)}(1, \vartheta; r, \lambda) \equiv A(1, \vartheta; r, \lambda).$$

We turn now to the attempt to find a solution $\lambda = \lambda(\vartheta; r)$ of (10) for values $r > 1$ close to $r = 1$. For this purpose we set $\varepsilon = \sqrt{r^2 - 1}$ and assume λ to be of the form

$$\lambda(\vartheta; r) = \varepsilon \lambda^{(1)}(\vartheta) + \varepsilon^2 \lambda^{(2)}(\vartheta) + \dots$$

where

$$\lambda^{(k)}(\vartheta) = \frac{1}{2} a_0^{(k)} + \sum_{n=1}^{\infty} (a_n^{(k)} \cos n\vartheta + b_n^{(k)} \sin n\vartheta).$$

Some preparatory heuristic remarks are in order. A lengthy computation¹ shows that

$$(11) \quad \begin{aligned} A^{(2)}(1, \vartheta; r, p \sin 2\vartheta) &= -\frac{r^2}{4} (1 + 14r^2 + 3r^4) p^2 \sin 4\vartheta \\ &= -\frac{9}{2} r^2 p^2 \sin 4\vartheta + p^2 O(\varepsilon^2). \end{aligned}$$

Accordingly, we shall choose the first two terms in the expansion of $\lambda(\vartheta; r)$ as follows:

$$\begin{aligned} \lambda^{(1)}(\vartheta) &= b_2^{(1)} \sin 2\vartheta, \\ \lambda^{(2)}(\vartheta) &= b_2^{(2)} \sin 2\vartheta + b_4^{(2)} \sin 4\vartheta. \end{aligned}$$

Then

$$\sigma(\vartheta; r, \lambda) = 8 \varepsilon^2 b_4^{(2)} \sin 4\vartheta + O(\varepsilon^3)$$

and

$$A(1, \vartheta; r, \lambda) = \frac{9}{2} \varepsilon^2 [b_2^{(1)}]^2 \sin 4\vartheta + O(\varepsilon^3)$$

so that

$$(12) \quad b_4^{(2)} = \frac{9}{16} [b_2^{(1)}]^2.$$

Now the necessity arises to compare certain terms in ε^3 . For this purpose we note the following relations which can be obtained only by extensive computations (remember the abbreviations $s_n = \sin n\vartheta$):

$$\begin{aligned} &A^{(2)}(1, \vartheta; r, p s_2 + q s_4) \\ &= -r^2 \left\{ \frac{9}{2} p^2 s_4 + 9 p q (s_2 + s_6) + \frac{1}{16} q^2 (22 s_4 + 117 s_8 + 2 s_{12}) \right\} \\ &\quad + p^2 O(\varepsilon^2) + p q O(\varepsilon^2) + q^2 O(\varepsilon^2), \end{aligned}$$

$$A^{(3)}(1, \vartheta; r, p s_2) = r^2 \frac{11}{6} p^3 (3 s_2 - s_6) + p^3 O(\varepsilon^2).$$

It is now seen that the coefficients of $\sin 2\vartheta$ are

$$\begin{aligned} \text{in } \frac{1}{r^2} \sigma(1, \vartheta; r, \lambda) &: \quad -\varepsilon^3 b_2^{(1)} + O(\varepsilon^4), \\ \text{in } \frac{1}{r^2} A(1, \vartheta; r, \lambda) &: \quad 9 \varepsilon^3 b_2^{(1)} b_4^{(2)} - \frac{11}{2} \varepsilon^3 [b_2^{(1)}]^3 + O(\varepsilon^4). \end{aligned}$$

¹Unfortunately, neither this computation nor the even more extensive explicit computations, which will become necessary later on, can be reproduced here.

Consequently, we shall have

$$-b_2^{(1)} = 9 b_2^{(1)} b_4^{(2)} - \frac{11}{2} [b_2^{(1)}]^3$$

or, combined with (12),

$$b_2^{(1)} + \left(\frac{81}{16} - \frac{11}{2} \right) [b_2^{(1)}]^3 = \left(1 - \frac{7}{16} [b_2^{(1)}]^2 \right) b_2^{(1)} = 0.$$

This condition has three distinct solutions

$$b_2^{(1)} = 0, \quad b_2^{(1)} = \frac{4}{\sqrt{7}}, \quad b_2^{(1)} = -\frac{4}{\sqrt{7}}$$

and, corresponding to these solutions, $b_4^{(2)} = 0$ in the first case and $b_4^{(2)} = 9/7$ in the other two cases.

The preceding heuristic remarks indicate that a bifurcation appears as ε increases from zero to small positive values. i.e., as r increases from one to values larger than one. The choice $b_2^{(1)} = 0$ leads, of course, to $\lambda(\vartheta; r) \equiv 0$, that is to say, back to our original surface $S^{(0)}(r)$. Of the other two cases we shall now discuss rigorously the second for which $b_2^{(1)} = 4/\sqrt{7}$. The third case can be treated similarly.

We shall try to find a solution $\lambda(\vartheta; r)$ of (10) for small positive values of ε in the form

$$(13) \quad \lambda(\vartheta; r) = \left[\frac{4}{\sqrt{7}} \varepsilon + \varepsilon^2 b_2(\varepsilon) \right] s_2 + \left[\frac{9}{7} \varepsilon^2 + \varepsilon^3 b_4(\varepsilon) \right] s_4 + \varepsilon^3 \left\{ \frac{1}{2} a_0(\varepsilon) + a_1(\varepsilon) c_1 + b_1(\varepsilon) s_1 + \mu(\vartheta; \varepsilon) \right\}$$

where

$$\mu(\vartheta; \varepsilon) = a_2(\varepsilon) c_2 + a_3(\varepsilon) c_3 + b_3(\varepsilon) s_3 + a_4(\varepsilon) c_4 + \sum_{n=5}^{\infty} (a_n(\varepsilon) c_n + b_n(\varepsilon) s_n)$$

and the $a_n(\varepsilon)$, $b_n(\varepsilon)$ are themselves power series in ε . In view of the three point condition we must have

$$\begin{aligned} \frac{1}{2} a_0(\varepsilon) + a_1(\varepsilon) &+ \mu(0; \varepsilon) = 0, \\ \frac{1}{2} a_0(\varepsilon) &+ b_1(\varepsilon) + \mu(\pi/2; \varepsilon) = 0, \\ \frac{1}{2} a_0(\varepsilon) - a_1(\varepsilon) &+ \mu(\pi; \varepsilon) = 0. \end{aligned}$$

From these equations the three coefficients a_0 , a_1 , b_1 can be determined:

$$\begin{aligned}
 (14) \quad a_0(\varepsilon) &= -2 [a_2(\varepsilon) + a_4(\varepsilon) + a_6(\varepsilon) + \dots] \\
 a_1(\varepsilon) &= - [a_3(\varepsilon) + a_5(\varepsilon) + a_7(\varepsilon) + \dots] \\
 b_1(\varepsilon) &= 2 [a_2(\varepsilon) + a_6(\varepsilon) + a_{10}(\varepsilon) + \dots] \\
 &\quad + [b_3(\varepsilon) - b_5(\varepsilon) + b_7(\varepsilon) - \dots]
 \end{aligned}$$

so that

$$(15) \quad \frac{1}{2} a_0^2(\varepsilon) + a_1^2(\varepsilon) + b_1^2(\varepsilon) \leq \frac{5}{2} \|\mu'(\vartheta; \varepsilon)\|^2.$$

After substitution of $\lambda(\vartheta; r)$ from (13), $(1/r^2) A(1, \vartheta; r, \lambda)$ attains an expansion

$$\frac{1}{r^2} A(1, \vartheta; r, \lambda) = \sum_{n=2}^{\infty} [A_n(\varepsilon, \lambda) c_n + B_n(\varepsilon, \lambda) s_n].$$

From (5) it is clear, that neither a constant term nor terms with $\cos \vartheta$ or $\sin \vartheta$ can appear on the right hand side. In determining the coefficients A_n and B_n it is necessary to carry the computations to the point that explicit expressions become available for all terms up to those of order ε^4 in B_2 and all terms up to those of order ε^3 in the other coefficients. The result of these computations is as follows:

$$B_2(\varepsilon, \lambda) = -\frac{4}{\sqrt{7}} \varepsilon^3 + \left[\frac{180}{49} - \frac{183}{7} b_2(\varepsilon) + \frac{36}{\sqrt{7}} b_4(\varepsilon) \right] \varepsilon^4 + \varepsilon^5 \tilde{B}_2(\varepsilon, \lambda),$$

$$B_4(\varepsilon, \lambda) = \frac{72}{7} \varepsilon^2 + \frac{36}{\sqrt{7}} \varepsilon^3 b_2(\varepsilon) + \varepsilon^4 \tilde{B}_4(\varepsilon, \lambda),$$

$$B_6(\varepsilon, \lambda) = \frac{1324}{21 \sqrt{7}} \varepsilon^3 + \varepsilon^4 \tilde{B}_6(\varepsilon, \lambda).$$

All the other coefficients have the form

$$\begin{aligned}
 A_n(\varepsilon, \lambda) &= \varepsilon^4 \tilde{A}_n(\varepsilon, \lambda), \quad n \geq 2, \\
 B_n(\varepsilon, \lambda) &= \varepsilon^4 \tilde{B}_n(\varepsilon, \lambda), \quad n \geq 2, \quad n \neq 2, 4, 6.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \frac{1}{r^2} \sigma(\vartheta; r, \lambda) &= (2 + \varepsilon^2) \varepsilon^3 a_2(\varepsilon) c_2 - \left(\frac{4}{7} \varepsilon^3 + \varepsilon^4 b_2(\varepsilon) \right) s_2 \\
 &\quad + 2 \varepsilon^3 (2 + \varepsilon^2) [a_3(\varepsilon) c_3 + b_3(\varepsilon) s_3] \\
 &\quad + \varepsilon^3 (2 + \varepsilon^2) (4 + \varepsilon^2) a_4(\varepsilon) c_4
 \end{aligned}$$

$$\begin{aligned}
& + (2 + \varepsilon^2) (4 + \varepsilon^2) \left(\frac{9}{7} \varepsilon^2 + \varepsilon^3 b_4(\varepsilon) \right) s_4 \\
& + \varepsilon^3 (2 + \varepsilon^2) \sum_{n=5}^{\infty} [(2 + \varepsilon^2) n - (4 + 3 \varepsilon^2)] (a_n(\varepsilon) c_n + b_n(\varepsilon) s_n).
\end{aligned}$$

The conditions for the coefficients $a_0(\varepsilon)$, $a_1(\varepsilon)$, $b_1(\varepsilon)$ are contained in (14). A comparison leads to a system of equations for the Fourier coefficients of $\lambda(\theta; r)$. The three "special" equations are

$$\begin{aligned}
-b_2(\varepsilon) &= \frac{180}{49} - \frac{183}{7} b_2(\varepsilon) + \frac{36}{\sqrt{7}} b_4(\varepsilon) + \varepsilon \tilde{B}_2(\varepsilon, \lambda), \\
(8 + 6 \varepsilon^2 + \varepsilon^4) b_4(\varepsilon) &= \frac{36}{\sqrt{7}} b_2(\varepsilon) - \varepsilon (6 + \varepsilon^2) + \varepsilon \tilde{B}_4(\varepsilon, \lambda), \\
(16 + 14 \varepsilon^2 + 3 \varepsilon^4) b_6(\varepsilon) &= \frac{1324}{21 \sqrt{7}} + \varepsilon \tilde{B}_6(\varepsilon, \lambda).
\end{aligned}$$

The remaining equations are

$$\begin{aligned}
(2 + \varepsilon^2) a_2(\varepsilon) &= \varepsilon \tilde{A}_2(\varepsilon, \lambda), \\
(2 + \varepsilon^2) [(2 + \varepsilon^2) n - (4 + 3 \varepsilon^2)] a_n(\varepsilon) &= \varepsilon \tilde{A}_n(\varepsilon, \lambda), \quad n \geq 3, \\
(2 + \varepsilon^2) [(2 + \varepsilon^2) n - (4 + 3 \varepsilon^2)] b_n(\varepsilon) &= \varepsilon \tilde{B}_n(\varepsilon, \lambda), \quad n \geq 3, \quad n \neq 4, 6.
\end{aligned}$$

For $\varepsilon = 0$ we obtain

$$\begin{aligned}
a_2(\varepsilon) &= a_3(\varepsilon) = \dots = 0, \\
b_3(\varepsilon) &= b_5(\varepsilon) = b_7(\varepsilon) = b_8(\varepsilon) = \dots = 0,
\end{aligned}$$

and

$$\begin{aligned}
-b_2(0) &= \frac{180}{49} - \frac{183}{7} b_2(0) + \frac{36}{\sqrt{7}} b_4(0), \\
8 b_4(0) &= \frac{36}{\sqrt{7}} b_2(0), \\
16 b_6(0) &= \frac{1324}{21 \sqrt{7}},
\end{aligned}$$

or

$$\begin{aligned} \frac{176}{7} b_2(0) - \frac{36}{\sqrt{7}} b_4(0) &= \frac{180}{49}, \\ -\frac{36}{\sqrt{7}} b_2(0) + 8 b_4(0) &= 0 \end{aligned}$$

and $b_6(0) = 331 / (84 \sqrt{7})$, so that

$$b_2(0) = \frac{90}{49}, \quad b_4(0) = \frac{405}{49 \sqrt{7}}, \quad b_6(0) = \frac{331}{84 \sqrt{7}}.$$

In view of the estimates (8), (15) and the solvability for $\varepsilon = 0$ it can be shown that our system of equations has, for all small non-negative values of ε , in fact a solution $\lambda^{(+)}(\vartheta; r)$ of the form (13) whose derivative has a uniformly bounded norm:

$$\lambda^{(+)}(\vartheta; r) = \varepsilon \frac{4}{\sqrt{7}} s_2 + \varepsilon^2 \left[\frac{90}{49} s_2 + \frac{9}{7} s_4 \right] + O(\varepsilon^3).$$

Let us summarize: We have seen that equations (10) have three distinct solutions for values of $r \geq 1$ close to $r = 1$:

$$\begin{aligned} \lambda^{(0)}(\vartheta; r) &= 0, \\ \lambda^{(+)}(\vartheta; r) &= \frac{4}{\sqrt{7}} \sqrt{r^2-1} \sin 2\vartheta + O(r^2-1), \\ \lambda^{(-)}(\vartheta; r) &= -\frac{4}{\sqrt{7}} \sqrt{r^2-1} \sin 2\vartheta + O(r^2-1). \end{aligned}$$

Each generates a solution of Plateau's problem for the Jordan curve Γ_r . $\lambda^{(0)} = 0$ leads us back to Enneper's surface $S^{(0)}(r)$ with the position vector $\mathbf{r}^{(0)}(\varrho, \vartheta; r)$. But in addition to Enneper's surface two new surfaces $S^{(+)}(r)$ and $S^{(-)}(r)$ appear. In view of (3) the position vectors of these surfaces are, respectively,

$$\mathbf{r}^{(\pm)}(\varrho, \vartheta; r) = \mathbf{r}^{(0)}(\varrho, \vartheta; r) + \frac{2}{\sqrt{7}} \sqrt{r^2-1} \begin{pmatrix} \varrho^3 c_3 - \varrho^5 c_5 \\ -\varrho^3 s_3 - \varrho^5 s_5 \\ -2 + 2 \varrho^4 c_4 \end{pmatrix} + O(r^2-1).$$

We know that Enneper's surface is not stable for $r > 1$. The surfaces $S^{(+)}(r)$ and $S^{(-)}(r)$ will be the two surfaces of least area bounded by Γ_r . (It is the special symmetry of Γ_r which gives rise to *two* distinct, but congruent surfaces of least area.) While the point $u = v = 0$ is mapped by $\mathbf{r}^{(0)}$ into the origin of 3-space for all values of r , its image under the mapping by the vectors $\mathbf{r}^{(+)}$ and $\mathbf{r}^{(-)}$, roughly the point

$$\left(0, 0, -\frac{4}{\sqrt{7}}\sqrt{r^2-1}\right) \text{ or } \left(0, 0, \frac{4}{\sqrt{7}}\sqrt{r^2-1}\right),$$

moves along the z -axis, down or up, as r increases. In this way the surfaces $S^{(+)}$ and $S^{(-)}$ arrange to decrease the surface area of $S^{(0)}$.

Acknowledgement. The preceding research was supported in part by the National Science Foundation through Grant No. MPS 75-05418.

References

- [1] BARBOSA, J. L., and M. DO CARMO: On the size of a stable minimal surface in R^3 . - Amer. J. Math. 98, 1976, 515-528.
- [2] BÖHME, R., and F. TOMI: Zur Struktur der Lösungsmenge des Plateauproblems. - Math. Z. 133, 1973, 1-29.
- [3] LEWY, H.: On the boundary behavior of minimal surfaces. - Proc. Nat. Acad. Sci. U.S.A. 37, 1951, 103-110.
- [4] NITSCHKE, J. C. C.: Contours bounding at least three solutions of Plateau's problem. - Arch. Rational Mech. Anal. 30, 1968, 1-11.
- [5] —»— Concerning the isolated character of solutions of Plateau's problem. - Math. Z. 109, 1969, 393-411.
- [6] —»— The boundary behavior of minimal surfaces. Kellogg's theorem and branch points on the boundary. - Invent. Math. 8, 1969, 313-333.
- [7] —»— A new uniqueness theorem for minimal surfaces. - Arch. Rational Mech. Anal. 52, 1973, 319-329.
- [8] —»— Local uniqueness for Plateau's problem. - Istituto Nazionale di Alta Matematica, Symposia Mathematica 14, Academic Press, London-New York, 1974, 403-411.
- [9] —»— Vorlesungen über Minimalflächen. - Die Grundlehren der mathematischen Wissenschaften 199, Springer-Verlag, Berlin-Heidelberg-New York, 1975.
- [10] RADÓ, T.: Contributions to the theory of minimal surfaces. - Acta Litt. Sci. Univ. Szeged. Sect. Sci. Math. 6, 1932, 1-20.
- [11] SCHWARZ, H. A.: Gesammelte mathematische Abhandlungen. I. - Verlag von Julius Springer, Berlin, 1890.

University of Minnesota
 School of Mathematics
 Minneapolis, Minnesota 55455
 USA

Received 15 August 1975