Title
Non-uniqueness of weak solutions to hyperviscous Navier–Stokes equations: on sharpness of J.-L. Lions exponent

Journal
Calculus of Variations and Partial Differential Equations, 59(3)

ISSN
0944-2669

Authors
Luo, Tianwen
Titi, Edriss S

Publication Date
2020-06-01

DOI
10.1007/s00526-020-01742-4

NON-UNIQUENESS OF WEAK SOLUTIONS TO HYPERVISCOUS NAVIER-STOKES EQUATIONS - ON SHARPNESS OF J.-L. LIONS EXPONENT

TIANWEN LUO* AND EDRISS S. TITI†

Abstract. Using the convex integration technique for the three-dimensional Navier-Stokes equations introduced by T. Buckmaster and V. Vicol, it is shown the existence of non-unique weak solutions for the 3D Navier-Stokes equations with fractional hyperviscosity $(-\Delta)^\theta$, whenever the exponent θ is less than J.-L. Lions' exponent 5/4, i.e., when $\theta < 5/4$.

1. Introduction

In this paper we consider the question of non-uniquness of weak solutions to the 3D Navier-Stokes equations with fractional viscosity (FVNSE) on $\mathbb{T}^3$

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^\theta v = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{1}$$

where $\theta \in \mathbb{R}$ is a fixed constant.

Definition (weak solutions). *A vector field $v \in C^0_{weak}(\mathbb{R}; L^2(\mathbb{T}^3))$ is called a weak solution to the FVNSE if it solves* (1) *in the sense of distribution.*

When $\theta = 1$, FVNSE (1) is the standard Navier-Stokes equations. J.-L. Lions first considered FVNSE (1) in [16], and showed the existence and uniqueness of weak solutions for $\theta \in [5/4, \infty)$ in [17]. Moreover, an analogue of the Caffarelli-Kohn-Nirenberg [6] result was established in [15] for the FVNSE system (1), showing that the Hausdorff dimension of the singular set, in space and time, is bounded by $5 - 4\theta$ for $\theta \in (1, 5/4)$. The existence, uniqueness, regularity and stability of solutions to the FVNSE have been studied in [14, 19, 21, 22] and references therein. Very recently, using the method of convex integration introduced in [11], Colombo, De Lellis and De Rosa in [7] showed the non-uniquenss of Leray weak solutions to FVNSE (1) for $\theta \in (0, 1/5)$.

In the recent breakthrough work [5], Buckmaster and Vicol obtained non-uniqueness of weak solutions to the three-dimensional Navier-Stokes equations. They developed a new convex integration scheme in Sobolev spaces using intermittent Beltrami

Date: September 25, 2018.

2010 *Mathematics Subject Classification.* 35Q30.

Key words and phrases. Non-uniqueness, Weak solutions, Wild solutions, Navier-Stokes equations, Hyperviscosity, Convex integration.

*Yau Mathematical Sciences Center, Tsinghua University, China. twluo@mail.tsinghua.edu.cn.

†Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843-3368, USA. Also, Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel. titi@math.tamu.edu; edriss.titi@weizmann.ac.il.

flows which combined concentrations and oscillations. Later, the idea of using intermittent flows was used to study non-uniqueness for transport equations in [18], employing scaled Mikado waves.

The schemes in [5, 18] are based on the convex integration framework in Hölder spaces for the Euler equations, introduced by De Lellis and Székelyhidi in [11], subsequently refined in [2, 3, 9, 12], and culminated in the proof of the second half of the Onsager conjecture by Isett in [13]; also see [4] for a shorter proof. For the first half of the Onsager conjecture, see, e.g., [1, 8], and the references therein.

The main contribution of this note is to that the results in Buckmaster-Vicol's paper hold for FVNSE (1) for $\theta < 5/4$:

Theorem 1. *Suppose $\theta \in (-\infty, 5/4)$ is a given constant. Then Theorem 1.2 and Theorem 1.3 in [5] hold for the 3D FVNSE, i.e., there exist non-unique weak solutions $v \in C^0_t W^{\beta,2}_x$, with a different $\beta > 0$, depending on θ.*

Proof. As it will be shown below, one can use the same constructions as in [5] with a slightly different choice of parameters. □

In the rest of the this paper, using the technique developed in [5] and some ideas in [18], we present a slightly simplified construction to show the following simple result:

Proposition 1. *Assume that $\theta \in (-\infty, 5/4)$. Suppose u is a smooth divergence-free vector field, define on $\mathbb{R}_+ \times \mathbb{T}^3$, with compact support in time and satisfies the condition*

$$\int_{\mathbb{T}^3} u(t,x)dx \equiv 0.$$

Then for any given $\varepsilon_0 > 0$, there exists a weak solution v to the FVNSE (1), with compact support in time, satisfying

$$\|v - u\|_{L^\infty_t W^{2\theta-1,1}_x} < \varepsilon_0.$$

2. Outline

2.1. **Iteration lemma.** Following [5], we consider the approximate system

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) + \nabla p + \nu(-\Delta)^\theta v = \nabla \cdot R, \\ \nabla \cdot v = 0, \end{cases} \tag{2}$$

where R is a symmetric 3×3 matrix.

Lemma 1 (Iteration Lemma for L^2 weak solutions)**.** *Let $\theta \in (-\infty, 5/4)$. Assume (v_q, R_q) is a smooth solution to (2) with*

$$\|R_q\|_{L^\infty_t L^1_x} \le \delta_{q+1}, \tag{3}$$

for some $\delta_{q+1} > 0$. Then for any given $\delta_{q+2} > 0$, there exists a smooth solution (v_{q+1}, R_{q+1}) of (2) with

$$\|R_{q+1}\|_{L^\infty_t L^1_x} \le \delta_{q+2}, \tag{4}$$

$$\text{and} \quad \operatorname{supp}_t v_{q+1} \cup \operatorname{supp}_t R_{q+1} \subset N_{\delta_{q+1}}(\operatorname{supp}_t v_q \cup \operatorname{supp}_t R_q). \tag{5}$$

Here for a given set $A \subset \mathbb{R}$, the δ-neighborhood of A is denoted by

$$N_\delta(A) = \{y \in \mathbb{R} : \exists y' \in A, |y - y'| < \delta\}.$$

Furthermore, the increment $w_{q+1} = v_{q+1} - v_q$ satisfies the estimates

$$\|w_{q+1}\|_{L^\infty_t L^2_x} \le C\delta_{q+1}^{1/2}, \tag{6}$$

$$\|w_{q+1}\|_{L^\infty_t W^{2\theta-1,1}_x} \le \delta_{q+2}, \tag{7}$$

where the positive constant C depends only on θ.

Proof of Proposition 1. Assume Lemma 1 is valid. Let $v_0 = u$. Then

$$\int_{\mathbb{T}^3} \partial_t v_0(t,x)dx = \frac{d}{dt}\int_{\mathbb{T}^3} v_0(t,x)dx \equiv 0.$$

Let

$$R_0 = \mathcal{R}(\partial_t v_0 + \nu(-\Delta)^\theta v_0) + v_0 \otimes v_0 + p_0 I, \quad p_0 = -\frac{1}{3}|v_0|^2,$$

where $\mathcal{R}$ is the symmetric anti-divergence operator established in Lemma 5. Clearly (v_0, R_0) solves (2). Set

$$\delta_1 = \|R_0\|_{L^\infty_t L^1_x},$$

$$\delta_{q+1} = 2^{-q}\varepsilon_0, \quad \text{for } q \ge 1.$$

Apply Lemma 1 iteratively to obtain smooth solution (v_q, R_q) to (2). It follows from (6) that

$$\sum \|v_{q+1} - v_q\|_{L^\infty_t L^2_x} = \sum \|w_{q+1}\|_{L^\infty_t L^2_x} \le C \sum \delta_{q+1}^{1/2} < \infty.$$

Thus v_q converge strongly to some $v \in C^0_t L^2_x$. Since $\|R_{q+1}\|_{L^\infty_t L^1_x} \to 0$, as $q \to \infty$, v is a weak solution to the FVNSE (1). Estimate (7) leads to

$$\|v - v_0\|_{L^\infty_t W^{2\theta-1,1}_x} \le \sum_{q=1}^{\infty} \|w_q\|_{L^\infty_t W^{2\theta-1,1}_x} \le \sum_{q=1}^{\infty} \delta_{q+1} \le \varepsilon_0.$$

Furthermore, it follows from (5) that

$$\operatorname{supp}_t v \subset \cup_{q\ge 0} \operatorname{supp}_t v_q \subset N_{\sum_{q\ge 0}\delta_{q+1}}(\operatorname{supp}_t u) \subset N_{\delta_1+\varepsilon_0}(\operatorname{supp}_t u).$$

This proves the proposition. □

3. Iteration scheme

3.1. **Notations and Parameters.** For a complex number $\zeta \in \mathbb{C}$, we denote by ζ^* its complex conjugate. Let us normalize the volume

$$|\mathbb{T}^3| = 1.$$

Following the notation in [5], we introduce here several parameters σ, r, λ, with

$$0 < \sigma < 1 < r < \lambda < \mu < \lambda^2, \quad \sigma r < 1, \tag{8}$$

where $\lambda = \lambda_{q+1} \in 5\mathbb{N}$ is the 'frequency' parameter; σ is a small parameter such that $\lambda\sigma \in \mathbb{N}$ parameterizes the spacing between frequencies; $r \in \mathbb{N}$ denotes the number of frequencies along edges of a cube; μ measures the amount of temporal oscillation.

Later σ, r, μ will be chosen to be suitable powers of λ_{q+1}. The constants implicitly in the notation '$\lesssim$' may depend on p and s, but are independent of the parameters σ, r, λ.

3.2. **Intermittent Beltrami flows.** We use intermittent Beltrami flows introduced in [5] as the building blocks. Recall some basic facts of Beltrami waves.

Proposition 2. *([5, Proposition 3.1]) Given* $\overline{\xi} \in \mathbb{S}^2 \cap \mathbb{Q}^3$*, let* $A_{\overline{\xi}} \in \mathbb{S}^2 \cap \mathbb{Q}^3$ *be such that*

$$A_{\overline{\xi}} \cdot \overline{\xi} = 0, \quad |A_{\overline{\xi}}| = 1, \quad A_{-\overline{\xi}} = A_{\overline{\xi}}.$$

Let Λ *be a given finite subset of* $\mathbb{S}^2$ *such that* $-\Lambda = \Lambda$*, and* $\lambda \in \mathbb{Z}$ *be such that* $\lambda\Lambda \subset \mathbb{Z}^3$*. Then for any choice of coefficients* $a_{\overline{\xi}} \in \mathbb{C}$ *with* $a_{\overline{\xi}}^* = a_{-\overline{\xi}}$ *the vector field*

$$W(x) = \sum_{\overline{\xi} \in \Lambda} a_{\overline{\xi}} B_{\overline{\xi}} e^{i\lambda \overline{\xi} \cdot x}, \qquad \text{with } B_{\overline{\xi}} = \frac{1}{\sqrt{2}}(A_{\overline{\xi}} + i\overline{\xi} \times A_{\overline{\xi}}),$$

is real-valued, divergence-free and satisfies

$$\nabla \times W = \lambda W, \quad \nabla \cdot (W \otimes W) = \nabla \frac{|W|^2}{2}.$$

Furthermore,

$$\langle W \otimes W \rangle := \fint_{\mathbb{T}^3} W \otimes W d\xi = \frac{1}{2} |a_{(\overline{\xi})}|^2 (\mathrm{Id} - \overline{\xi} \otimes \overline{\xi}).$$

Let $\Lambda, \Lambda^+, \Lambda^- \subset \mathbb{S}^2 \cap \mathbb{Q}^3$ be defined by

$$\Lambda^+ = \{\frac{1}{5}(3e_1 \pm 4e_2), \frac{1}{5}(3e_2 \pm 4e_3), \frac{1}{5}(3e_3 \pm 4e_1)\},$$
$$\Lambda^- = -\Lambda^+, \quad \Lambda = \Lambda^+ \cup \Lambda^-.$$

Clearly we have

$$5\Lambda \in \mathbb{Z}^3, \qquad \text{and} \qquad \min_{\overline{\xi}', \overline{\xi} \in \Lambda, \overline{\xi}' + \overline{\xi} \neq 0} |\overline{\xi}' + \overline{\xi}| \geq \frac{1}{5}. \tag{9}$$

Also it is direct to check that

$$\frac{1}{4} \sum_{\overline{\xi} \in \Lambda} (\mathrm{Id} - \overline{\xi} \otimes \overline{\xi}) = \mathrm{Id}.$$

In fact, representations of this form exist for symmetric matrices close to the identity. We have the following simple variant of [5, Proposition 3.2].

Proposition 3. *Let* $B_\varepsilon(\mathrm{Id})$ *denote the ball of symmetric matrices, centered at the identity, of radius* ε*. Then there exist a constant* $\varepsilon_\gamma > 0$ *and smooth positive functions* $\gamma_{(\overline{\xi})} \in C^\infty(B_{\varepsilon_\gamma}(\mathrm{Id}))$*, such that*

1. $\gamma_{(\overline{\xi})} = \gamma_{(-\overline{\xi})}$*;*
2. *for each* $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ *we have the identity*

$$R = \frac{1}{2} \sum_{\overline{\xi} \in \Lambda} \left(\gamma_{(\overline{\xi})}(R) \right)^2 (\mathrm{Id} - \overline{\xi} \otimes \overline{\xi}).$$

Define the Dirichlet kernel

$$D_r(x) = \frac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}, \quad \Omega_r = \{(j,k,l) : j,k,l \in \{-r, \cdots, r\}\}.$$

It has the property that, for $1 < p \leq \infty$,

$$\|D_r\|_{L^p} \lesssim r^{3/2 - 3/p}, \quad \|D_r\|_{L^2} = (2\pi)^3.$$

Following [5], for $\overline{\xi} \in \Lambda^+$, define a directed and rescaled Dirichlet kernel by

$$\eta_{(\overline{\xi})}(t,x) = \eta_{\overline{\xi},\lambda,\sigma,r,\mu}(t,x) = D_r(\lambda\sigma(\overline{\xi}\cdot x + \mu t, A_{\overline{\xi}}\cdot x, (\overline{\xi}\times A_{\overline{\xi}})\cdot x)), \tag{10}$$

and for $\overline{\xi} \in \Lambda^-$, define

$$\eta_{(\overline{\xi})}(t,x) = \eta_{-(\overline{\xi})}(t,x).$$

Note the important identity

$$\frac{1}{\mu}\partial_t \eta_{(\overline{\xi})}(t,x) = \pm(\overline{\xi}\cdot\nabla)\eta_{(\overline{\xi})}(t,x), \quad \overline{\xi}\in\Lambda^{\pm}. \tag{11}$$

Since the map $x \mapsto \lambda\sigma(\overline{\xi}\cdot x + \mu t, A_{\overline{\xi}}\cdot x, (\overline{\xi}\times A_{\overline{\xi}})\cdot x)$ is the composition of a a rotation by a rational orthogonal matrix mapping $\{e_1, e_2, e_3\}$ to $\{\overline{\xi}, A_{\overline{\xi}}, \overline{\xi}\times A_{\overline{\xi}}\}$, a translation, and a rescaling by integers, for $1 < p \le \infty$, we have

$$\fint_{\mathbb{T}^3} \eta_{(\overline{\xi})}(t,x)^2(t,x)dx = 1, \quad \|\eta_{(\overline{\xi})}\|_{L^\infty_x L^p_x(\mathbb{T}^3)} \lesssim r^{3/2-3/p}.$$

Let $W_{(\overline{\xi})}$ be the Beltrami plane wave at frequency λ,

$$W_{(\overline{\xi})} = W_{\overline{\xi},\lambda}(x) = B_{\overline{\xi}} e^{i\lambda\overline{\xi}\cdot x}.$$

Define the intermittent Beltrami wave $\mathbb{W}_{(\overline{\xi})}$ as

$$\mathbb{W}_{(\overline{\xi})}(t,x) := \mathbb{W}_{\overline{\xi},\lambda,\sigma,r,\mu}(t,x) = \eta_{(\overline{\xi})}(t,x) W_{(\overline{\xi})}(x). \tag{12}$$

It follows from the definitions and (9) that

$$\mathbb{P}_{[\lambda/2,2\lambda]}\mathbb{W}_{(\overline{\xi})} = \mathbb{W}_{(\overline{\xi})}, \tag{13}$$

$$\mathbb{P}_{[\lambda/5,4\lambda]}(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}) = \mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}, \quad \overline{\xi}' \ne -\overline{\xi}. \tag{14}$$

The following properties are immediate from the definitions.

Proposition 4. *([5, Proposition 3.4]) If $a_{\overline{\xi}} \in \mathbb{C}$ are constants with $a^*_{\overline{\xi}} = a_{-\overline{\xi}}$. Let*

$$W(x) = \sum_{\overline{\xi}\in\Lambda} a_{\overline{\xi}} \mathbb{W}_{(\overline{\xi})}(x).$$

Then $W(x)$ is real valued. Moreover, for each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have

$$\sum_{\overline{\xi}\in\Lambda}\Big(\gamma_{(\overline{\xi})}(R)\Big)^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(-\overline{\xi})} = \sum_{\overline{\xi}\in\Lambda}\Big(\gamma_{(\overline{\xi})}(R)\Big)^2 B_{\overline{\xi}}\otimes B_{-\overline{\xi}} = R.$$

Proposition 5. *([5, Proposition 3.5]) For any $1 < p \le \infty, N \ge 0, K \ge 0$:*

$$\|\nabla^N \partial_t^K \mathbb{W}_{(\overline{\xi})}\|_{L^\infty_t L^p_x} \lesssim \lambda^N (\lambda\sigma r\mu)^K r^{3/2-3/p}, \tag{15}$$

$$\|\nabla^N \partial_t^K \eta_{(\overline{\xi})}\|_{L^\infty_t L^p_x} \lesssim (\lambda\sigma r)^N (\lambda\sigma r\mu)^K r^{3/2-3/p}. \tag{16}$$

3.3. **Perturbations.** Let $\psi(t)$ be a smooth cut-off function such that

$$\psi(t) = 1 \text{ on } \operatorname{supp}_t R_q, \quad \operatorname{supp} \psi(t) \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q), \quad |\psi'(t)| \le 2\delta_{q+1}^{-1}. \tag{17}$$

Take a smooth increasing function χ such that

$$\chi(s) = \begin{cases} 1, & 0 \le s < 1 \\ s, & s \ge 2 \end{cases},$$

and set

$$\rho(t,x) = \varepsilon_\gamma^{-1} \delta_{q+1} \chi(\delta_{q+1}^{-1} |R_q(t,x)|) \psi^2(t).$$

where ε_γ is the constant in Proposition 3. Then clearly

$$\operatorname{supp}_t \rho \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q). \tag{18}$$

It follows from the above definition that

$$|R_q|/\rho = \varepsilon_\gamma \frac{|R_q|}{\delta_{q+1}\chi(\delta_{q+1}^{-1}|R_q(t,x)|)\psi^2} \le \varepsilon_\gamma \implies \mathrm{Id} - R_q/\rho \in B_\varepsilon(\mathrm{Id}) \text{ on } \operatorname{supp} R_q.$$

Therefore, the amplitude functions

$$a_{(\overline{\xi})}(t,x) := \rho^{1/2}(t,x)\gamma_{(\overline{\xi})}(\mathrm{Id} - \rho(t,x)^{-1} R_q(t,x))$$

are well-defined and smooth. Define the velocity perturbation to be $w = w_{q+1}$:

$$\begin{aligned} w &= w^{(p)} + w^{(c)} + w^{(t)}, \\ w^{(p)} &= \sum_{\overline{\xi}\in\Lambda} a_{(\overline{\xi})} \mathbb{W}_{(\overline{\xi})} = \sum_{\overline{\xi}\in\Lambda} a_{(\overline{\xi})}(t,x)\eta_{(\overline{\xi})}(t,x) B_{\overline{\xi}} e^{i\lambda\overline{\xi}\cdot x}, \\ w^{(c)} &= \frac{1}{\lambda_{q+1}} \sum_{\overline{\xi}\in\Lambda} \nabla \left(a_{(\overline{\xi})}\eta_{(\overline{\xi})} \right) \times W_{(\overline{\xi})}, \\ w^{(t)} &= \frac{1}{\mu} \sum_{\overline{\xi}\in\Lambda^+} \mathbb{P}_H \mathbb{P}_{\neq 0} \left(a_{(\overline{\xi})}^2 \eta_{(\overline{\xi})}^2 \overline{\xi} \right), \end{aligned}$$

where $\mathbb{P}_H = \mathrm{Id} - \nabla\Delta^{-1} \operatorname{div}$ is the Helmholtz projection into divergence-free vector field, and $\mathbb{P}_{\neq 0} f = f - \fint_{\mathbb{T}^3} f\,dx$. It follows from Proposition 3 that

$$\sum_{\overline{\xi}\in\Lambda} a_{(\overline{\xi})}^2 \fint_{\mathbb{T}^3} \mathbb{W}_{(\overline{\xi})} \otimes \mathbb{W}_{(-\overline{\xi})} dx = \rho \mathrm{Id} - R_q. \tag{19}$$

3.4. **Estimates for perturbations.**

Lemma 2. *The following bounds hold:*

$$\|\rho\|_{L_t^\infty L_x^1} \le C\delta_{q+1}, \tag{20}$$

$$\|\rho^{-1}\|_{C^0(\operatorname{supp} R_q)} \lesssim \delta_{q+1}^{-1}, \tag{21}$$

$$\|\rho\|_{C_{t,x}^N} \le C(\delta_{q+1}, \|R_q\|_{C^N}), \tag{22}$$

$$\|a_{(\overline{\xi})}\|_{L_t^\infty L_x^2} \lesssim \|\rho\|_{L_t^\infty L_x^1}^{1/2} \lesssim \delta_{q+1}^{1/2}, \tag{23}$$

$$\|a_{(\overline{\xi})}\|_{C_{t,x}^N} \le C(\delta_{q+1}, \|R_q\|_{C^N}). \tag{24}$$

Proof. It follows from (3) that

$$\|\rho(t,\cdot)\|_{L^1_x} = \int_{|R_q|\le\delta_{q+1}} \rho + \int_{|R_q|>\delta_{q+1}} \rho \lesssim \delta_{q+1} + \int_{|R_q|>\delta_{q+1}} |R_q| \\ \le C\delta_{q+1}.$$

It is direct to verify (21) and (23), while (22) and (24) follow from (17) and (21). □

Now we can estimate the time support of w_{q+1}:

$$\operatorname{supp}_t w_{q+1} \subset \operatorname{supp}_t \rho \subset \operatorname{supp} \psi \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q). \tag{25}$$

We need the following Lemma, which is a variant of [5, Lemma 3.6].

Lemma 3. *([18, Lemma 2.1]) Let $f, g \in C^\infty(\mathbb{T}^d)$, and g is $(\mathbb{T}/N)^3$ periodic, $N \in \mathbb{N}$. Then for $1 \le p \le \infty$,*

$$\|fg\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^p} + C_p N^{-1/p}\|f\|_{C^1}\|g\|_{L^p}.$$

Let us denote

$$\mathcal{C}_N = C(\sup_{\overline{\xi}\in\Lambda} \|a_{(\overline{\xi})}\|_{C^N_{t,x}}) \tag{26}$$

to be some polynomials depending on $\sup_{\overline{\xi}\in\Lambda} \|a_{(\overline{\xi})}\|_{C^N_{t,x}}$.

Lemma 4. *Suppose the parameters satisfy*

$$r^{3/2} \le \mu \le \sigma^{-1/2} r. \tag{27}$$

Then the following estimates for the perturbations hold:

$$\|w^{(p)}_{q+1}\|_{L^\infty_t L^2_x} \lesssim \delta^{1/2}_{q+1} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1, \tag{28}$$

$$\|w_{q+1}\|_{L^\infty_t L^p_x} \lesssim r^{3/2-3/p}\mathcal{C}_1, \tag{29}$$

$$\|w^{(c)}_{q+1}\|_{L^\infty_t L^p_x} + \|w^{(t)}_{q+1}\|_{L^\infty_t L^p_x} \lesssim (\sigma r + \mu^{-1} r^{3/2}) r^{3/2-3/p}\mathcal{C}_1, \tag{30}$$

$$\|\partial_t w^{(p)}_{q+1}\|_{L^\infty_t L^p_x} + \|\partial_t w^{(c)}_{q+1}\|_{L^\infty_t L^p_x} \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2, \tag{31}$$

$$\||\nabla|^N w_{q+1}\|_{L^\infty_t L^p_x} \lesssim r^{3/2-3/p}\lambda^N_{q+1}\mathcal{C}_{N+1}, \tag{32}$$

for $1 < p < \infty, N \ge 1$.

Proof. Since $\mathbb{W}_{(\overline{\xi})}$ is $(\mathbb{T}/\lambda\sigma)^3$ periodic, it follows from (15), (23), and Lemma 3 that

$$\|w^{(p)}_{q+1}\|_{L^\infty_t L^2_x} \lesssim \sum_{\overline{\xi}\in\Lambda} (\|a_{(\overline{\xi})}\|_{L^\infty_t L^2_x} + (\lambda_{q+1}\sigma)^{-1/2}\|a_{(\overline{\xi})}\|_{C^1})\|\mathbb{W}_{(\overline{\xi})}\|_{L^\infty_t L^2_x} \\ \lesssim \delta^{1/2}_{q+1} + (\lambda_{q+1}\sigma)^{-1/2}\mathcal{C}_1.$$

In view of (8), (15) and (16) yield that

$$\|w^{(p)}_{q+1}\|_{L^\infty_t L^p_x} \lesssim \sum_{\overline{\xi}\in\Lambda} \|a_{(\overline{\xi})}\|_{C^0}\|\mathbb{W}_{(\overline{\xi})}\|_{L^\infty_t L^p_x} \lesssim r^{3/2-3/p}\mathcal{C}_0,$$

$$\|w^{(c)}_{q+1}\|_{L^\infty_t L^p_x} + \|w^{(t)}_{q+1}\|_{L^\infty_t L^p_x} \lesssim \sum_{\overline{\xi}\in\Lambda} \sigma r\|a_{(\overline{\xi})}\|_{C^1}\|\mathbb{W}_{(\overline{\xi})}\|_{L^\infty_t L^p_x} + \mu^{-1}\|a^2_{(\overline{\xi})}\|_{C^0}\|\eta_{(\overline{\xi})}\|^2_{L^\infty_t L^{2p}_x} \\ \lesssim (\sigma r + \mu^{-1} r^{3/2}) r^{3/2-3/p}\mathcal{C}_1,$$

$$
\begin{aligned}
\|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda} \|\partial_t a_{(\overline{\xi})}\|_{C^0}\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} + \|a_{(\overline{\xi})}\|_{C^0}\|\partial_t \mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_1, \\
\|\partial_t w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1}\sum_{\overline{\xi}\in\Lambda} \|a_{(\overline{\xi})}\|_{C_{t,x}^2}\|\eta_{(\overline{\xi})} + \nabla\eta_{(\overline{\xi})} + \partial_t\eta_{(\overline{\xi})} + \partial_t\nabla\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \sigma r\lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2 \lesssim \lambda_{q+1}\sigma\mu r^{5/2-3/p}\mathcal{C}_2.
\end{aligned}
$$

For $N \geq 1$, Using (15) and (16), we obtain that

$$
\begin{aligned}
\|\nabla^N w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} &\lesssim \sum_{\overline{\xi}\in\Lambda}\sum_{k=0}^{N} \|\nabla^k a_{(\overline{\xi})}\|_{C^0}\|\nabla^{N-k}\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_N, \\
\|\nabla^N w_{q+1}^{(c)}\|_{L_t^\infty L_x^p} &\lesssim \lambda_{q+1}^{-1}\sum_{\overline{\xi}\in\Lambda}\sum_{m=0}^{N}\sum_{k=0}^{m} \lambda_{q+1}^{N-m}\|\nabla^{k+1}a_{(\overline{\xi})}\|_{C^0}\|\nabla^{m-k}\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\quad + \lambda_{q+1}^{-1}\sum_{\overline{\xi}\in\Lambda}\sum_{m=0}^{N}\sum_{k=0}^{m} \lambda_{q+1}^{N-m}\|\nabla^{k}a_{(\overline{\xi})}\|_{C^0}\|\nabla^{m-k+1}\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_{N+1}, \\
\|\nabla^N w_{q+1}^{(t)}\|_{L_t^\infty L_x^p} &\lesssim \mu^{-1}\sum_{\overline{\xi}\in\Lambda}\sum_{m=0}^{N} \|\nabla^{N-m}(a_{(\overline{\xi})}^2)\|_{C^0}\sum_{k=0}^{m}\|\nabla^k\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^{2p}}\|\nabla^{m-k}\eta_{(\overline{\xi})}\|_{L_t^\infty L_x^{2p}} \\
&\lesssim \lambda_{q+1}^N r^{3/2-3/p}\frac{(\sigma r)^N r^{3/2}}{\mu}\mathcal{C}_N \lesssim \lambda_{q+1}^N r^{3/2-3/p}\mathcal{C}_N,
\end{aligned}
$$

where we use (8) and (27). □

3.5. **Estimates for the stress.** Let us recall the following operator in [11].

Lemma 5 (symmetric anti-divergence)**.** *There exists a linear operator $\mathcal{R}$, of order -1, mapping vector fields to symmetric matrices such that*

$$
\nabla\cdot\mathcal{R}(u) = u - \fint_{\mathbb{T}^3} u, \tag{33}
$$

with standard Calderon-Zygmund and Schauder estimates, for $1 < p < \infty$,

$$
\|\mathcal{R}\|_{L^p\to W^{1,p}} \lesssim 1, \quad \|\mathcal{R}\|_{C^0\to C^0} \lesssim 1, \quad \|\mathcal{R}\mathbb{P}_{\neq 0}u\|_{L^p} \lesssim \||\nabla|^{-1}\mathbb{P}_{\neq 0}u\|_{L^p}. \tag{34}
$$

We have the following variant of [5, Lemma B.1] in [5].

Lemma 6. *Let $a \in C^2(\mathbb{T}^3)$. For $1 < p < \infty$, and any $f \in L^p(\mathbb{T}^3)$, we have*

$$
\||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k}f)\|_{L^p} \lesssim k^{-1}(\|a\|_{L^\infty} + \|\nabla^2 a\|_{L^\infty})\|f\|_{L^p}. \tag{35}
$$

Proof of Lemma 6. We follow the proof in [5]. Note that

$$
|\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k}f) = |\nabla|^{-1}\mathbb{P}_{\geq k/2}(\mathbb{P}_{\leq k/2}a\mathbb{P}_{\geq k}f) + |\nabla|^{-1}\mathbb{P}_{\neq 0}(\mathbb{P}_{\geq k}a\mathbb{P}_{\geq k}f).
$$

Using the bounds

$$
\||\nabla|^{-1}\mathbb{P}_{\geq k/2}\|_{L^p\to L^p} \lesssim k^{-1}, \quad \||\nabla|^{-1}\mathbb{P}_{\neq 0}\|_{L^p\to L^p} \lesssim 1,
$$

and the embedding $W^{1,4}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, we obtain

$$
\begin{aligned}
&\||\nabla|^{-1}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq k}f)\|_{L^p} \lesssim k^{-1}\|\mathbb{P}_{\leq k/2}a\mathbb{P}_{\geq k}f\|_{L^p} + \|\mathbb{P}_{\geq k/2}a\mathbb{P}_{\geq k}f\|_{L^p} \\
&\lesssim k^{-1}(\|a\|_{L^\infty} + k\|\nabla\mathbb{P}_{\geq k/2}a\|_{L^4})\|f\|_{L^p} \lesssim k^{-1}(\|a\|_{L^\infty} + \|\nabla^2\mathbb{P}_{\geq k/2}a\|_{L^4})\|f\|_{L^p} \\
&\lesssim k^{-1}(\|a\|_{L^\infty} + \|\nabla^2 a\|_{L^\infty})\|f\|_{L^p}.
\end{aligned}
$$

□

It follows from the definition that

$$\int (-\Delta)^\theta w_{q+1} dx = 0 = \int \partial_t w_{q+1} dx.$$

We obtain R_{q+1} by plugging $v_{q+1} = v_q + w_{q+1}$ in (2), using (33) and the assumption that (v_q, R_q) solves (2):

$$
\begin{aligned}
\nabla \cdot R_{q+1} = \nabla \cdot &\Big[\mathcal{R}(\nu(-\Delta)^\theta w_{q+1} + \partial_t w_{q+1}^{(p)} + \partial_t w_{q+1}^{(c)}) + v_q \otimes w_{q+1} + w_{q+1} \otimes v_q\Big] \\
&+ \nabla \cdot \Big[(w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\Big] \\
&\Big[\nabla \cdot (w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - R_q) + \partial_t w_{q+1}^{(t)}\Big] + \nabla(p_{q+1} - p_q) \\
&:= \nabla \cdot (\widetilde{R}_{linear} + \widetilde{R}_{corrector} + \widetilde{R}_{oscillation}) + \nabla(p_{q+1} - p_q).
\end{aligned}
$$

First, we estimate the supports of R_{q+1}. Using (25) we obtain

$$\operatorname{supp}_t R_{q+1} \subset \operatorname{supp}_t w_{q+1} \cup \operatorname{supp}_t R_q \subset N_{\delta_{q+1}}(\operatorname{supp}_t R_q).$$

It follows from Lemma 4 that

$$
\begin{aligned}
\|\widetilde{R}_{corrector}\|_{L_t^\infty L_x^p} &\lesssim \left(\|w_{q+1}^{(c)}\|_{L_t^\infty L_x^{2p}} + \|w_{q+1}^{(c)}\|_{L_t^\infty L_x^{2p}}\right)\|w_{q+1}\|_{L_t^\infty L_x^{2p}} \\
&\lesssim (\sigma r + \mu^{-1}r^{3/2})r^{3-3/p}\mathcal{C}_1.
\end{aligned}
$$

Noting that $\nabla \times \frac{w_{q+1}^{(p)}}{\lambda_{q+1}} = w_{q+1}^{(p)} + w_{q+1}^{(c)}$, Lemma 4 and (34) yield that

$$
\begin{aligned}
&\|\widetilde{R}_{linear}\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^{-1}\|\partial_t\mathcal{R}\nabla \times (w_{q+1}^{(p)})\|_{L_t^\infty L_x^p} + \|\mathcal{R}(\nu(-\Delta)^\theta w_{q+1})\|_{L_t^\infty L_x^p} \\
&\quad + \|v_q \otimes w_{q+1} + w_{q+1} \otimes v_q\|_{L_t^\infty L_x^p} \\
&\lesssim \lambda_{q+1}^{-1}\|\partial_t w_{q+1}^{(p)}\|_{L_t^\infty L_x^p} + \||\nabla|^{2\theta-1}w_{q+1}\|_{L_t^\infty L_x^p} + \|v_q\|_{C^0}\|w_{q+1}\|_{L_t^\infty L_x^p} \\
&\lesssim \sigma\mu r^{5/2-3/p}\mathcal{C}_2 + r^{3/2-3/p}(\lambda_{q+1}^{2\theta-1} + \|v_q\|_{C^0})\mathcal{C}_3.
\end{aligned}
$$

Note that, for fixed $\theta \in (-\infty, 5/4)$, one can choose $p > 1$, close enough to 1, so that

$$3/2 - 3/p + 2\theta - 1 < 0.$$

It remains to estimate $\widetilde{R}_{oscillation}$, which can be handled in the same way as in [5]. It follows from (19) that

$$\begin{aligned}
&\nabla\cdot(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}-R_q)=\nabla\cdot(\sum_{\overline{\xi},\overline{\xi}'\in\Lambda}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{W}_{\overline{\xi}}\otimes\mathbb{W}_{(\overline{\xi}')}-R_q)\\
&=\nabla\cdot(\sum_{\overline{\xi},\overline{\xi}'\in\Lambda}a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')})+\nabla\rho\\
&:=\sum_{\overline{\xi},\overline{\xi}'\in\Lambda}E_{(\overline{\xi},\overline{\xi}')}+\nabla\rho.
\end{aligned}$$

Since $E_{(\overline{\xi},\overline{\xi}')}$ has zero mean, we can split it as

$$\begin{aligned}
E_{(\overline{\xi},\overline{\xi}')}+E_{(\overline{\xi}',\overline{\xi})}&=\mathbb{P}_{\neq0}\left(\nabla(a_{(\overline{\xi})}a_{(\overline{\xi}')})\cdot(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi'}}\otimes\mathbb{W}_{(\overline{\xi})}))\right)\\
&\quad+\mathbb{P}_{\neq0}\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi'}}\otimes\mathbb{W}_{(\overline{\xi})})\right)\\
&:=E_{(\overline{\xi},\overline{\xi}',1)}+E_{(\overline{\xi},\overline{\xi}',2)}.
\end{aligned}$$

Using (15), (34) and (35), we obtain

$$\begin{aligned}
\|\mathcal{R}E_{(\overline{\xi},\overline{\xi}',1)}\|_{L_t^\infty L_x^p}&\lesssim\||\nabla|^{-1}E_{(\overline{\xi},\overline{\xi}',1)}\|_{L_t^\infty L_x^p}\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}(\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\|_{C^1}+\|\nabla^2(a_{(\overline{\xi})}a_{(\overline{\xi}')})\|_{C^1})\|\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}\|_{L_t^\infty L_x^p}\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\|_{C^3}\|\mathbb{W}_{(\overline{\xi})}\|_{L_t^\infty L_x^{2p}}\|\mathbb{W}_{(\overline{\xi}')}\|_{L_t^\infty L_x^{2p}}\\
&\lesssim(\lambda_{q+1}\sigma)^{-1}r^{3-3/p}\mathcal{C}_3.
\end{aligned}$$

Recall the vector identity $A\cdot\nabla B+B\cdot\nabla B=\nabla(A\cdot B)-A\times(\nabla\times B)-B\times(\nabla\times A)$. For $\overline{\xi},\overline{\xi}'\in\Lambda$, using the anti-symmetry of the cross product, we can write

$$\begin{aligned}
&\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{(\overline{\xi}')}\otimes\mathbb{W}_{(\overline{\xi})})\\
&=\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)+\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\left(W_{(\overline{\xi})}\cdot\nabla W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\cdot\nabla W_{(\overline{\xi})}\right)\\
&=\left(W_{(\overline{\xi}')}\cdot\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right)W_{(\overline{\xi})}+\left(W_{(\overline{\xi})}\cdot\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right)W_{\overline{\xi}'}+\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\nabla\left(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\right).
\end{aligned}$$

For the term $E_{(\overline{\xi},\overline{\xi}',2)}$, first consider the case $\overline{\xi}+\overline{\xi'}\neq0$. (14) implies that

$$\begin{aligned}
&a_{(\overline{\xi})}a_{(\overline{\xi}')}\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi}')}+\mathbb{W}_{\overline{\xi'}}\otimes\mathbb{W}_{(\overline{\xi})})\\
&=a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\cdot\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)\\
&\quad+\nabla\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\right)-\nabla\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\right)\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\mathbb{W}_{(\overline{\xi})}\cdot\mathbb{W}_{(\overline{\xi}')}\right)\\
&\quad-a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\left(W_{(\overline{\xi})}\cdot W_{(\overline{\xi}')}\right)\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\right),
\end{aligned}$$

where the second term is a pressure, the third can be estimated analogously to $E_{(\overline{\xi},\overline{\xi}',1)}$. Also note that the first and fourth term can estimated analogously. Using (16), (34) and (35), we obtain

$$\begin{aligned}
&\|\mathcal{R}\left(a_{(\overline{\xi})}a_{(\overline{\xi}')}\mathbb{P}_{\geq\lambda_{q+1}/10}\left(\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\cdot\left(W_{(\overline{\xi})}\otimes W_{(\overline{\xi}')}+W_{(\overline{\xi}')}\otimes W_{(\overline{\xi})}\right)\right)\right)\|_{L_t^\infty L_x^p}\\
&\lesssim\lambda_{q+1}^{-1}(\|a_{(\overline{\xi})}a_{(\overline{\xi}')}\|_{C^1}+\|\nabla^2(a_{(\overline{\xi})}a_{(\overline{\xi}')})\|_{C^1})\|\nabla\left(\eta_{(\overline{\xi})}\eta_{(\overline{\xi}')}\right)\|_{L_t^\infty L_x^p}\\
&\lesssim\sigma r^{4-3/p}\mathcal{C}_3.
\end{aligned}$$

Now consider $E_{(\overline{\xi},-\overline{\xi},2)}$. We can write

$$\nabla\cdot(\mathbb{W}_{(\overline{\xi})}\otimes\mathbb{W}_{(-\overline{\xi})}+\mathbb{W}_{(-\overline{\xi})}\otimes\mathbb{W}_{(\overline{\xi})})=\left(W_{(-\overline{\xi})}\cdot\nabla\eta^2_{(\overline{\xi})}\right)W_{(\overline{\xi})}+\left(W_{(\overline{\xi})}\cdot\nabla\eta^2_{(\overline{\xi})}\right)W_{(-\overline{\xi})}$$

$$=(A_{\overline{\xi}}\cdot\nabla\eta^2_{(\overline{\xi})})A_{\overline{\xi}}+((\overline{\xi}\times A_{\overline{\xi}})\cdot\nabla\eta^2_{(\overline{\xi})})(\overline{\xi}\times A_{\overline{\xi}})=\nabla\xi^2_{(\overline{\xi})}-(\overline{\xi}\cdot\nabla\eta^2_{(\overline{\xi})})\overline{\xi}=\nabla\eta^2_{(\overline{\xi})}-\frac{\overline{\xi}}{\mu}\partial_t\eta^2_{(\overline{\xi})},$$

where we use (11) and the fact that $\{\overline{\xi},A_{\overline{\xi}},\overline{\xi}\times A_{\overline{\xi}}\}$ forms an orthonormal basis of $\mathbb{R}^3$. Therefore, we can write

$$\begin{aligned}E_{(\overline{\xi},-\overline{\xi},2)}&=\mathbb{P}_{\neq 0}\left(a^2_{(\overline{\xi})}\nabla\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\overline{\xi})}-a^2_{(\overline{\xi})}\frac{\overline{\xi}}{\mu}\partial_t\eta^2_{(\overline{\xi})}\right)\\&=\nabla\left(a^2_{(\overline{\xi})}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\overline{\xi})}\right)-\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)\\&\quad-\mu^{-1}\partial_t\mathbb{P}_{\neq 0}\left(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\right)+\mu^{-1}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right).\end{aligned}$$

Using the identity $\mathrm{Id}-\mathbb{P}_H=\nabla\Delta^{-1}\operatorname{div}$, we obtain

$$\sum_{\overline{\xi}}E_{(\overline{\xi},-\overline{\xi},2)}+\partial_t w^{(t)}_{q+1}=\nabla\sum_{\overline{\xi}}\left(a^2_{(\overline{\xi})}\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}\eta^2_{(\overline{\xi})}\right)-\nabla\sum_{\overline{\xi}}\mu^{-1}\Delta^{-1}\cdot\partial_t\left(a^2_{(\overline{\xi})}\eta^2_{(\overline{\xi})}\overline{\xi}\right)$$
$$-\sum_{\overline{\xi}}\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)+\mu^{-1}\sum_{\overline{\xi}}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right),$$

where the first and second terms are pressure terms. Using (16), (34) and (35), we obtain

$$\begin{aligned}\|\mathcal{R}\mathbb{P}_{\neq 0}\left(\mathbb{P}_{\geq\lambda_{q+1}\sigma/2}(\eta^2_{(\overline{\xi})})\nabla a^2_{(\overline{\xi})}\right)\|_{L^\infty_tL^p_x}&\lesssim(\lambda_{q+1}\sigma)^{-1}\|\eta_{(\overline{\xi})}\|^2_{L^\infty_tL^{2p}_x}\mathcal{C}_3\\&\lesssim(\lambda_{q+1}\sigma)^{-1}r^{3-3/p}\mathcal{C}_3.\end{aligned}$$

It follows from (16) and (34) that

$$\begin{aligned}\mu^{-1}\|\mathcal{R}\mathbb{P}_{\neq 0}\left(\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\right)\|_{L^\infty_tL^p_x}&\lesssim\mu^{-1}\|\partial_t\left(a^2_{(\overline{\xi})}\right)\eta^2_{(\overline{\xi})}\overline{\xi}\|_{L^\infty_tL^p_x}\\&\lesssim\mu^{-1}r^{3-3/p}\mathcal{C}_1.\end{aligned}$$

Now we choose the parameters r,σ,μ. Fix α so that

$$\max\{0,\frac{2}{3}(2\theta-1)\}<\alpha<1,$$

which is possible since $\theta\in(-\infty,5/4)$. Fix

$$r=\lambda^{\alpha}_{q+1},\quad\sigma=\lambda^{-(\alpha+1)/2}_{q+1},\quad\mu=\lambda^{(5\alpha+1)/4}_{q+1}.\tag{36}$$

Clearly (27) is satisfied. Choose $p>1$ sufficiently close to 1 so that

$$-\frac{\alpha+1}{2}+\frac{5\alpha+1}{4}+\left(\frac{5}{2}-\frac{3}{p}\right)\alpha<0,\quad\left(\frac{3}{2}-\frac{3}{p}\right)\alpha+\max(0,2\theta-1)<0,$$
$$-\frac{5\alpha+1}{4}+\left(\frac{9}{2}-\frac{3}{p}\right)\alpha<0,\quad-\frac{1-\alpha}{2}+\left(3-\frac{3}{p}\right)\alpha<0.$$

Note that $\mathcal{C}_N$ is independent of λ_{q+1}, due to (24). Combining the above estimates with Lemma 4, it is easy to check that, by taking λ_{q+1} sufficiently large, we arrive at (4), (6) and (7). This completes the proof of Lemma 1.

Acknowledgement The authors would like to thank the "The Institute of Mathematical Sciences", Chinese University of Hong Kong, for the warm and kind hospitality during which part of this work was completed. The work of T.L. is supported in part by NSFC Grants 11601258. The work of E.S.T. is supported in part by the ONR grant N00014-15-1-2333, the Einstein Stiftung/Foundation - Berlin, through the Einstein Visiting Fellow Program, and by the John Simon Guggenheim Memorial Foundation.

References

[1] C. Bardos and E.S. Titi, *Onsager's conjecture for the incompressible Euler equations in bounded domains*, Arch. Ration. Mech. Anal. **228** (2018), 197-207.

[2] T. Buckmaster, C. De Lellis, P. Isett, L. Székelyhidi, Jr., *Anomalous dissipation for* 1/5-*Hölder Euler flows*, Ann. of Math. **182:1** (2015), 127–172.

[3] T. Buckmaster, C. De Lellis, L. Székelyhidi, *Dissipative Euler flows with Onsager-critical spatial regularity*, Comm. Pure Appl. Math. **69:9** (2016), 16131670.

[4] T. Buckmaster, C. De Lellis, L. Székelyhidi, V. Vicol, *Onsager's conjecture for admissible weak solutions*, Comm. Pure Appl. Math. (2018). https://doi.org/10.1002/cpa.21781. arXiv:1701.08678v1.

[5] T. Buckmaster, V. Vicol, *Nonuniqueness of weak solutions to the Navier-Stokes equation*, arXiv:1709.10033v3.

[6] L. Caffarelli, R. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions to the NavierStokes equations*, Comm. Pure Appl. Math. **35** (1982), 771831.

[7] M. Colombo, C. De Lellis, L. De Rosa,*Ill-Posedness of Leray Solutions for the Hypodissipative NavierStokes Equations*, Comm. Math. Phys. 362 (2018), no. 2, 659688.

[8] P. Constantin, W. E, E. S. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. **165:1** (1994), 207–209.

[9] S. Daneri, L. Székelyhidi, Jr., *Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations*. Arch. Ration. Mech. Anal. **224:2** (2017), 471–514.

[10] C. De Lellis, L. Székelyhidi, Jr., *The Euler equations as a differential inclusion*. Ann. of Math. **170:3** (2009), 1417–1436.

[11] C. De Lellis, L. Székelyhidi, Jr., *Dissipative continuous Euler flows*, Invent. Math. **193:2** (2013), 377–407.

[12] P. Isett, *Holder continuous Euler flows with compact support in time.* Thesis (Ph.D.)Princeton University. 2013.

[13] P. Isett, *A Proof of Onsager's Conjecture*, Ann. of Math. **188:3** (2018), 1–93. arXiv:1608.08301v1.

[14] Q. Jiu, Y. Wang, *On possible time singular points and eventual regularity of weak solutions to the fractional Navier-Stokes equations*. Dyn. Partial Differ. Equ. **11:4** (2014), 321–343.

[15] N. H. Katz, N. A. Pavlović, *A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation*. Geom. Funct. Anal. **12:2** (2002), 355–379.

[16] J. L. Lions, *Quelques rsultats d'existence dans des quations aux drives partielles non linaires*. Bull. Soc. Math. France **87** (1959), 245–273.

[17] J. L. Lions, *Quelques Méthodes de Resolution des Problémes aux Limites Non linéaires*, Vol 1. Dunod, Paris, 1969.

[18] S. Modena, L. Székelyhidi, *Non-uniqueness for the transport equation with Sobolev vector fields*, arXiv:1712.03867v3.

[19] E. Olson, E. S. Titi, *Viscosity versus vorticity stretching: Global well-posedness for a family of NavierStokes-alpha-like models*, Nonlinear Analysis: Theory, Methods & Applications **66:11** (2007), 2427–2458.

[20] V. Scheffer, *An inviscid flow with compact support in space-time*, J. Geom. Anal., **3:4** (1993), 343–401.

[21] J. Wu, *Generalized MHD equations*, J. Differential Equations **195** (2003), 284312.

[22] T. Tao, *Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation*, Analysis & PDE **3** (2009), 361-366.