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Non-very ample configurations arising from contingency tables

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Abstract In this paper, it is proved that, if a toric ideal possesses a fundamental binomial none of whose monomials is squarefree, then the corresponding semigroup ring is not very ample. Moreover, very ample semigroup rings of Lawrence type are discussed. As an application, we study very ampleness of configurations arising from contingency tables.

Keywords Fundamental binomial · Toric ring · Very ample configuration · Lawrence lifting · Combinatorial pure subring

1 Introduction

A configuration in $\mathbb{R}^d$ is a finite set $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d_{\geq 0}$ such that there exists a vector $\mathbf{w} \in \mathbb{R}^d$ satisfying $\mathbf{w} \cdot \mathbf{a}_i = 1$ for all i. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ denote the polynomial ring in d variables over a field K. We associate a configuration A with the semigroup ring $K[A] = K[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$, where $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$ if $\mathbf{a} = (a_1, \ldots, a_d)$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K. The toric ideal I_A of A is the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \longrightarrow K[A]$ defined by setting $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i}$ for $1 \leq i \leq n$.

We are interested in the following conditions:

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(i) A is *unimodular*, i.e., the initial ideal of I_A is generated by squarefree monomials with respect to any monomial order;
(ii) A is *compressed*, i.e., the initial ideal of I_A is generated by squarefree monomials with respect to any reverse lexicographic order;
(iii) there exists a monomial order $<$ such that the initial ideal of I_A with respect to $<$ is generated by squarefree monomials;
(iv) $K[A]$ is *normal*, i.e., $\mathbb{Z}_{\geq 0}A = \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A$;
(v) $K[A]$ is *very ample*, i.e., $(\mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A) \setminus \mathbb{Z}_{\geq 0}A$ is a finite (or empty) set.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) holds and each of the converse of them is false in general. If $K[A]$ is not normal, then an element of $(\mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A) \setminus \mathbb{Z}_{\geq 0}A$ is called *hole*.

Let P_A denote the convex hull of A. For a subset $B \subset A$, $K[B]$ is called *combinatorial pure subring* (Ohsugi et al. 2000; Ohsugi 2007) of $K[A]$ if there exists a face F of P_A such that $B = A \cap F$. For example, if $K[B] = K[A] \cap K[t_{i_1}, \ldots, t_{i_s}]$, then $K[B]$ is a combinatorial pure subring of $K[A]$. (This is the original definition of a combinatorial pure subring in Ohsugi et al. (2000).) A binomial $f \in I_A$ is called *fundamental* if there exists a combinatorial pure subring $K[B]$ of $K[A]$ such that I_B is generated by f. In Sect. 2, it will be proved that, if I_A possesses a fundamental binomial none of whose monomials is squarefree, then $K[A]$ is not very ample. The *Lawrence lifting* $\Lambda(A)$ of the configuration A is the configuration arising from the matrix

$$\Lambda(A) = \begin{pmatrix} A & \mathbf{0} \\ I_n & I_n \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{0}$ is the $d \times n$ zero matrix. A configuration A is called *Lawrence type* if there exists a configuration B such that $\Lambda(B) = A$. In Sect. 2, it will be proved that a configuration of Lawrence type is very ample if and only if it is unimodular.

In Sect. 3, by using the results in Sect. 2, we study very ample configurations arising from no n-way interaction models for $r_1 \times r_2 \times \cdots \times r_n$ contingency tables, where $r_1 \geq r_2 \geq \cdots \geq r_n \geq 2$. Let $A_{r_1 r_2 \cdots r_n}$ be the set of vectors $\mathbf{e}^{(1)}_{i_2 i_3 \cdots i_n} \oplus \mathbf{e}^{(2)}_{i_1 i_3 \cdots i_n} \oplus \cdots \oplus \mathbf{e}^{(n)}_{i_1 i_2 \cdots i_{n-1}}$, where each i_k belongs to $[r_k] = \{1, 2, \ldots, r_k\}$ and $\mathbf{e}^{(k)}_{j_1 j_2 \cdots j_{n-1}}$ is a unit coordinate vector of $\mathbb{Z}^{d_k}$ with $d_k = \frac{\prod_{\ell=1}^{n} r_\ell}{r_k}$. The toric ideal $I_{A_{r_1 r_2 \cdots r_n}}$ is the kernel of the homomorphism

$$\pi : K[\{x_{i_1 i_2 \cdots i_n} \mid i_k \in [r_k]\}] \longrightarrow K\left[\left\{t^{(k)}_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_n} \;\middle|\; k \in [n], i_k \in [r_k]\right\}\right]$$

defined by $\pi(x_{i_1 i_2 \cdots i_n}) = t^{(1)}_{i_2 i_3 \cdots i_n} t^{(2)}_{i_1 i_3 \cdots i_n} \cdots t^{(n)}_{i_1 i_2 \cdots i_{n-1}}$. Table 1 is known.

By virtue of the results in Sect. 2, we will prove that configurations in "otherwise" part are not very ample.

2 Fundamental binomials

The following lemma plays an important role in the present paper.

Table 1 Algebraic properties of configurations of contingency tables

$r_1 \times r_2$ or $r_1 \times r_2 \times 2 \times \cdots \times 2$	Unimodular
$r_1 \times 3 \times 3$	Compressed, not unimodular
$4 \times 4 \times 3$	Normal, not compressed
$5 \times 5 \times 3$ or $5 \times 4 \times 3$	Not compressed (normality is unknown)
Otherwise, i.e., $n \geq 4$ and $r_3 \geq 3$; $n = 3$ and $r_3 \geq 4$; $n = 3, r_3 = 3, r_1 \geq 6$ and $r_2 \geq 4$	Not normal

Lemma 1 *Let $K[B]$ be a combinatorial pure subring of $K[A]$. If $K[A]$ is normal (resp. very ample), then $K[B]$ is normal (resp. very ample).*

Proof Let $K[B]$ be a combinatorial pure subring of $K[A]$. It is enough to show that $(\mathbb{Z}B \cap \mathbb{Q}_{\geq 0}B) \setminus \mathbb{Z}_{\geq 0}B \subset (\mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A) \setminus \mathbb{Z}_{\geq 0}A$.

Let $\alpha \in (\mathbb{Z}B \cap \mathbb{Q}_{\geq 0}B) \setminus \mathbb{Z}_{\geq 0}B$. Since B is a subset of A, we have $\alpha \in \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A$. Suppose that $\alpha \in \mathbb{Z}_{\geq 0}A$. Then $\alpha = \sum_{\mathbf{a} \in A} z_{\mathbf{a}} \mathbf{a}$ with $0 \leq z_{\mathbf{a}} \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Z}_{\geq 0}B$, $0 < z_{\mathbf{a}}$ for some $\mathbf{a} \in A \setminus B$. Moreover, since $\alpha \in \mathbb{Q}_{\geq 0}B$, $\alpha = \sum_{\mathbf{a} \in B} q_{\mathbf{a}} \mathbf{a}$ with $0 \leq q_{\mathbf{a}} \in \mathbb{Q}$. Thus $\alpha = \sum_{\mathbf{a} \in A} z_{\mathbf{a}} \mathbf{a} = \sum_{\mathbf{a} \in B} q_{\mathbf{a}} \mathbf{a}$. Since $K[B]$ is a combinatorial pure subring of $K[A]$, there exists a face F of P_A such that $B = A \cap F$. Then there exist $\mathbf{v} \in \mathbb{R}^d$ and $c \in \mathbb{R}$ satisfying

$$F = P_A \cap \{\, \mathbf{b} \in \mathbb{R}^d \mid \mathbf{v} \cdot \mathbf{b} = c \,\},$$

$$P_A \subset \{\, \mathbf{b} \in \mathbb{R}^d \mid \mathbf{v} \cdot \mathbf{b} \leq c \,\}.$$

Then $\mathbf{v} \cdot \mathbf{a} = c$ for all $\mathbf{a} \in B$ and $\mathbf{v} \cdot \mathbf{a} < c$ for all $\mathbf{a} \in A \setminus B$. Hence $\mathbf{v} \cdot \alpha = c \sum_{\mathbf{a} \in B} q_{\mathbf{a}} < c \sum_{\mathbf{a} \in A} z_{\mathbf{a}}$. Thus we have $c \neq 0$ and $\sum_{\mathbf{a} \in B} q_{\mathbf{a}} \neq \sum_{\mathbf{a} \in A} z_{\mathbf{a}}$. On the other hand, since A is a configuration, there exists a vector $\mathbf{w} \in \mathbb{R}^d$ satisfying $\mathbf{w} \cdot \mathbf{a} = 1$ for all $\mathbf{a} \in A$. Hence $\mathbf{w} \cdot \alpha = \sum_{\mathbf{a} \in B} q_{\mathbf{a}} = \sum_{\mathbf{a} \in A} z_{\mathbf{a}}$. This is a contradiction. Thus $\alpha \in (\mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A) \setminus \mathbb{Z}_{\geq 0}A$ as desired. □

It is known (Ohsugi et al. 2000, Lemma 3.1) that

Proposition 2 *If $g = u - v \in K[\mathbf{x}]$ is a binomial such that neither u nor v is squarefree and if $I_A = (g)$, then $K[A]$ is not normal.*

We extend Proposition 2 as follows:

Lemma 3 *If $g = u - v \in K[\mathbf{x}]$ is a binomial such that neither u nor v is squarefree and if $I_A = (g)$, then $K[A]$ is not very ample.*

Proof Let $g = x_1^2 u' - x_2^2 v'$. Since g is irreducible, u' $(\neq 1)$ is not divided by x_2 and v' $(\neq 1)$ is not divided by x_1. Since $\pi(x_1^2 u') = \pi(x_2^2 v')$, we have $\sqrt{\pi(u'v')} = \frac{\pi(x_1 u')}{\pi(x_2)}$. Let x_k be a variable with $k \neq 1, 2$. Then the monomial $\pi(x_k^m)\sqrt{\pi(u'v')}$ belongs to the quotient field of $K[A]$ and is integral over $K[A]$ for all positive integer m. Suppose that there exists a monomial w such that $\pi(w) = \pi(x_k^m)\sqrt{\pi(u'v')}$. It then follows that the binomial $g' = x_1u'x_k^m - x_2w$ belongs to I_A. Since $I_A = (g)$ and $x_1u'x_k^m$ is divided by neither x_1^2u' nor x_2^2v', we have $g' = 0$. Hence x_2 must divide u', a contradiction. Thus $\pi(x_k^m)\sqrt{\pi(u'v')}$ is a hole for all m and $K[A]$ is not very ample. □

Theorem 4 *If I_A possesses a fundamental binomial $g = u - v$ such that neither u nor v is squarefree, then $K[A]$ is not very ample.*

Proof Since g is fundamental, there exists a combinatorial pure subring $K[B]$ of $K[A]$ such that I_B is generated by g. Thanks to Lemma 3, $K[B]$ is not very ample. Since $K[B]$ is a combinatorial pure subring of $K[A]$, $K[A]$ is not very ample by Lemma 1. □

Thanks to Theorem 4 together with the results in Ohsugi et al. (2000), we extend (Ohsugi et al. 2000, Theorem 3.4) as follows:

Corollary 5 *Let $K[A]$ be a semigroup ring and let $K[\Lambda(A)]$ its Lawrence lifting. Then, the following conditions are equivalent:*

(i) *$K[A]$ is unimodular;*
(ii) *$K[\Lambda(A)]$ is unimodular;*
(iii) *$K[\Lambda(A)]$ is very ample.*

Proof First, (ii) $\Rightarrow$ (iii) is well-known. On the other hand, (i) $\Leftrightarrow$ (ii) is proved in (Ohsugi et al. 2000, Theorem 3.4).

In order to show (iii) $\Rightarrow$ (i), suppose that $K[A]$ is not unimodular. Then, by the same argument in Proof of (Ohsugi et al. 2000, Theorem 3.4), $I_{\Lambda(A)}$ has a fundamental binomial $\overline{g}$ none of whose monomials is squarefree. Thanks to Theorem 4, $K[\Lambda(A)]$ is not very ample as desired. □

Remark 6 A binomial f belonging to I_A is called *indispensable* if, for an arbitrary system $\mathcal{F}$ of binomial generators of I_A, either f or $-f$ appears in $\mathcal{F}$. In particular, every fundamental binomial is indispensable. However, Theorem 4 is not true if we replace "fundamental" with "indispensable." Let $K[A] = K[t_2, t_1t_2, t_1^3t_2, t_1^4t_2] \subset K[t_1, t_2]$. Then $K[A]$ is very ample and I_A is generated by the set of indispensable binomials $\{x_1x_4 - x_2x_3,\ x_2^3 - x_1^2x_3,\ x_3^3 - x_2x_4^2,\ x_1x_3^2 - x_2^2x_4\}$. (The toric ideal I_A has no fundamental binomials.)

3 Configurations arising from contingency tables

Configurations in "otherwise" part of Table 2 are studied in Ohsugi and Hibi (2007) by using the notion of combinatorial pure subring and indispensable binomials. For $6 \times 4 \times 3$ case, non-normality is shown in Vlach (1986) and it was proved Hemmecke et al. (2009) that it is not very ample. On the other hand, compressed configurations are classified in Sullivant (2006). For $4 \times 4 \times 3$ case, it was announced in (Hemmecke et al., 2009, p. 87) that Ruriko Yoshida verified that it is normal by using the software NORMALIZ (Bruns and Ichim 2008).

The basic facts on $A_{r_1 \cdots r_n}$ are (Ohsugi and Hibi 2007, Proposition 3.1 and Proposition 3.2):

Table 2 Algebraic properties of configurations of n way contingency tables

$r_1 \times r_2$ or $r_1 \times r_2 \times 2 \times \cdots \times 2$	Unimodular
$r_1 \times 3 \times 3$	Compressed, not unimodular
$4 \times 4 \times 3$	Normal, not compressed
$5 \times 5 \times 3$ or $5 \times 4 \times 3$	Not compressed (normality is unknown)
Otherwise, i.e., $n \geq 4$ and $r_3 \geq 3$ $n = 3$ and $r_3 \geq 4$ $n = 3, r_3 = 3, r_1 \geq 6$ and $r_2 \geq 4$	Not normal

Proposition 7 *The configuration $A_{r_1 \cdots r_n 2}$ is the Lawrence lifting of $A_{r_1 \cdots r_n}$.*

Proposition 8 *Suppose that $A_{r_1 \cdots r_n}$ and $A_{s_1 \cdots s_n}$ satisfy $s_i \leq r_i$ for all $1 \leq i \leq n$. Then $K[A_{s_1 \cdots s_n}]$ is a combinatorial pure subring of $K[A_{r_1 \cdots r_n}]$.*

Theorem 9 *Work with the same notation as above. Then, each configuration in "otherwise" part is not very ample.*

Proof Let A be a configuration in "otherwise" part. Thanks to Proposition 8, $K[A]$ has at least one of $K[A_{444}]$, $K[A_{643}]$ and $K[A_{3332 \cdots 2}]$ as a combinatorial pure subring. It is easy to check that $I_{A_{444}}$ has a fundamental binomial

$$\begin{aligned} &x_{111}^2 x_{133} x_{144} x_{223} x_{224} x_{232} x_{242} x_{313} x_{322} x_{341} x_{414} x_{422} x_{431} \\ &\quad - x_{113} x_{114} x_{131} x_{141} x_{222}^2 x_{233} x_{244} x_{311} x_{323} x_{342} x_{411} x_{424} x_{432}, \end{aligned}$$

and $I_{A_{643}}$ has a fundamental binomial

$$\begin{aligned} &x_{111} x_{221} x_{331} x_{641} x_{212} x_{522} x_{432} x_{642} x_{413} x_{323} x_{633}^2 x_{143} x_{543} \\ &\quad - x_{211} x_{321} x_{631} x_{141} x_{412} x_{222} x_{632} x_{542} x_{113} x_{523} x_{333} x_{433} x_{643}^2. \end{aligned}$$

Since none of the monomials appearing above is squarefree, both $K[A_{444}]$ and $K[A_{643}]$ are not very ample by Theorem 4. Moreover, since A_{333} is not unimodular, $K[A_{3332 \cdots 2}]$ is not very ample by Corollary 5 together with Proposition 7. Thus, $K[A]$ is not very ample by Lemma 1. □

We close the present paper with an interesting problem.

Problem 10. Find natural classes of configurations appearing in statistics which is not normal but very ample.

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