Nonassociative Ricci flows, star product and R-flux deformed black holes, and swampland conjectures

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April 30, 2023

Abstract

We extend to a theory of nonassociative geometric flows a string-inspired model of nonassociative gravity determined by star product and R-flux deformations. The nonassociative Ricci tensor and curvature scalar defined by (non) symmetric metric structures and generalized (non) linear connections are used for defining nonassociative versions of Grigori Perelman F- and W-functionals for Ricci flows and computing associated thermodynamic variables. We develop and apply the anholonomic frame and connection deformation method, AFCDM, which allows us to construct exact and parametric solutions describing nonassociative geometric flow evolution scenarios and modified Ricci soliton configurations with quasi-stationary generic off-diagonal metrics. There are provided explicit examples of solutions modelling geometric and statistical thermodynamic evolution on a temperature-like parameter of modified black hole configurations encoding nonassociative star-product and R-flux deformation data. Further perspectives of the paper are motivated by nonassociative off-diagonal geometric flow extensions of the swampland program, related conjectures and claims on geometric and physical properties of new classes of quasi-stationary Ricci flow and black hole solutions.

Keywords: nonassociative geometric flows; nonassociative gravity; nonholonomic star product; R-flux deformations; Perelman F- and W-functionals; geometric flow thermodynamics; exact and parametric solutions; nonassociative black holes; modified swampland program; swampland conjectures.

Contents

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1 Introduction, preliminaries, and motivations

1.1 On nonassociative geometric flows, gravity, new methods of constructing exact/ parametric solutions, and the swampland program

One of the most important results in modern mathematics consists from the proof of the Poincaré-Thurston conjecture due to Grigori Perelman [1, 2, 3]. The approach involved a study of geometric flow evolution equations of Riemannian metrics introduced independently by R. Hamilton [4], in mathematics, and D. Friedan [5]), in physics. We note that a new concept of W-entropy and a respective statistical thermodynamic model for geometric flows [1] were elaborated, when the thermodynamic variables are defined in terms of the Riemannian metric volume forms with a normalizing function and using the Ricci tensor and scalar curvature. Such ideas and geometric methods are very important for elaborating new directions and applications in modern physics, cosmology and astrophysics, and quantum information theory; see new results, research programs and references from [6, 7, 8]. Comprehensive reviews of advanced topological and geometric analysis methods involved in the Ricci flow theory can be found in monographs [9, 10, 11].[1]

A challenging problem in modern particle physics and gravity is to develop the fundamental mathematical results on geometric flows and G. Perelman's thermodynamics in a relativistic form, for metrics with Lorentz signature and/or for non-Riemannian geometric objects derived, for instance, for string modifications of Einstein gravity. There is a substantial difference between the original mathematical methods with Riemannian metrics and the directions for elaborating new methods and research on relativistic geometric flows, supersymmetric and (non) commutative generalizations, and recent applications [14, 15, 7], see also references therein.[2] Even, at present, certain variants of the Poincaré-Thurston conjecture for pseudo-Riemannian metrics and/or generalized connections were not formulated and proven, we can elaborate on self-consistent causal geometric flow evolution models in physics and information theory if we apply the anholonomic frame and connection deformation methods, AFCDM, and construct/select more general classes of solutions with well-defined and verifiable classical and quantum properties [16, 17].

[1]Readers may familiarize themselves with the outstanding scientific and social impact of such results presented, respectively, in a magazine article and a YouTube clip: S. Nasar and D. Gruber, Manifold Destiny – A legendary problem and the battle over who solved it. The New Yorker, Annals of Mathematics, August 28, 2006,
https://www.newyorker.com/magazine/2006/08/28/manifold-destiny ; and "Grigory Perelman documentary" [Russian with English subtitles] https://www.youtube.com/watch?v=Ng1W2KUHI2s

[2]Some hundred of works devoted to possible implications in modern physics of the Ricci flow theory were published during the last 15 years and it is not possible to analyse all new and original results in this research article. We cite and discuss only a series of most relevant papers which motivate our research program on "nonassociative geometric flows and applications in physics and information theory" stated in [12, 13] and allow to develop the geometric and analytic methods which are used for constructing new classes of exact and parametric solutions of geometric evolution and modified gravity field equations.

Geometric flow theories are closely related to the RG flow models and underlying nonlinear sigma-models with beta-functions computed in a framework of string gravity theory, or a modified/quantum gravity model with ultraviolet, UV, completion and UV/ IR correspondence (IR, from infrared) [18]. In connection to this, we note the swampland program [19, 20, 21] which main goal is to elaborate rigorous criteria how to distinguish the low-energy effective field theories that can be completed in the UV from those that cannot. The swampland hypothesis in quantum gravity, QG, the infinite distance conjecture and various other someway related conjectures, were revisited recently in a series of works [22, 23, 24, 25, 26, 27] using (non) commutative geometric flow, exact solutions in gravity theories, and quantum field methods. Here we emphasize that to elaborate on rigorous algebraic and geometric approaches to mathematical particle physics and QG, we have to consider models of nonassociative quantum mechanics, QM, [30, 31] and further developments with nonassicative and noncommutative algebras [32, 33, 15, 34, 35, 36]. We cite [37, 38, 39, 12, 13, 40, 41] for reviews and recent results on nonassociative and noncommutative geometry and physics. An important task in string and M-theory, and modern quantum field theory, QFT, and QG, is to extend the swampland program in a form incorporating geometric and physical models with nonassociative/ noncommutative structures. *A general scope of this work is to investigate how exact and parametric solutions in nonassociative gravity can be correlated to the infinite distance conjecture and corresponding criteria/conjectures/ claims involving nontrivial running cosmological constants, non-Riemannian and pseudo-Riemannian geometric flows, and modified gravity theories, MGTs.* From a plethora of above mentioned nonassociative and noncommutative geometric and field theories, we study an explicit class of models with nonassociative twist deformations defined by R-flux deformations in string theory. In such an approach, we are able to elaborate on physically well-defined geometric flow evolution and gravity theories encoding nonassociative data, when the results are verifiable in linear order on a small deformation parameter. Various classes of new parametric solutions and applications in modern cosmology, astrophysics, information thermodynamics etc. can be also considered. At the end of subsection 7.2, we discuss the validity of our methods and claims and further perspectives for general nonassociative theories. We argue that fundamental geometric flow equations and important statistical thermodynamic functionals can be always postulated in abstract geometric form and then all order decompositions on deformation parameter can be performed. In the linear approximation, certain additional variational principles can be formulated and then recurrently extended to higher order decompositions on a small deformation parameter.

In [38, 39], two quite similar and self-consistent approaches to nonassociative gravity (defined by star product R-flux deformations in string theory) were elaborated up to levels of definition and parametric computation of the nonassociative Ricci tensor $\mathcal{R}ic^{\star}[\nabla^{\star}]$ and corresponding curvature scalar $\mathcal{R}s^{\star}[\nabla^{\star}]$.[3] Such nonassociative geometric objects are determined by a nonassociative Levi-Civita, LC, connection $\nabla^{\star}[\mathfrak{g}^{\star}]$, constructed as a nonlinear functional using nonassociative symmetric, $g^{\star}$, and nonsymmetric, $\breve{g}^{\star}$, components of a star-metric, $\star$-metric, $\mathfrak{g}^{\star} = (g^{\star}, \breve{g}^{\star})$. As in string gravity and M-theory [34, 35, 36], the nonassociative geometric constructions and (vacuum) gravity theories involve nonassociative star, $\star$, product deformations computed for a prescribed Moyal–Weyl tensor product and determined by non-geometric fluxes (R-fluxes). Such a $\star$–product allows us to define and compute nonassociative deformations of the LC-connection in (pseudo) Riemannian geometry, $\nabla \to \nabla^{\star}$, and (for more general constructions) to elaborate on various models of nonassociative non-Riemannian geometry with nontrivial torsion and non-metricity involving nonsymmetric metric tensors.

We follow in nonassociative geometry and gravity a symbolic abstract geometric formalism [39, 12, 13] which is similar to that for GR [43] but (in our approach) is more formalized and adapted to nonassociative

[3]Formulas of type ...$[\nabla^{\star}]$ state a functional dependence involving possible (star) products, partial derivations etc. We follow the notations and conventions from [12, 13] generalizing for arbitrary nonholonomic frames the formulas from [38, 39]. This allows us to use in the Introduction section certain abstract nonassociative formulas if they are analogs of associative and commutative ones. In result, the motivations and purposes of the work can be formulated in a more compact form. Of course, in index and coordinate/frame forms, which are necessary for finding solutions of physically important systems of PDEs, the formulas for nonassociative geometry/gravity are more sophisticate because they involve terms from star product and R-flux deformations. Such computations will be considered in next sections. In Appendix A, there are provided necessary details and explanations. We recommend readers to study the mentioned works and summaries of previous results before reading the main part of the article.

and noncommutative geometric structures. It simplifies the geometric constructions which in many cases are formal re-definitions and more sophisticate transforms/ deformations of certain fundamental geometric objects and formulas into similar ones with star labels. The nonassociative vacuum gravitations equations can be postulated/ derived, and computed in abstract geometric form, or using cumbersome nonholonomic frame / coordinate formulas, as respective star product R-flux deformations of some standard commutative ones. In abstract (non) associative geometric form, we can postulate the $\star$–deformed vacuum Einstein equations:

$$ {}^{\shortmid}\mathcal{R}ic^{\star}[\ {}^{\shortmid}\nabla^{\star}] = \ {}^{\shortmid}\Lambda_0\ {}^{\shortmid}\mathfrak{g}^{\star}, \quad (1) $$

where ${}^{\shortmid}\Lambda_0$ is a conventional cosmological constant. The nonassociative geometric objects in these equations (i.e. ${}^{\shortmid}\mathfrak{g}^{\star}$, ${}^{\shortmid}\nabla^{\star}$, and ${}^{\shortmid}\mathcal{R}ic^{\star}$) are defined on a $\star$-product deformed phase space ${}^{\shortmid}\mathcal{M} = T^{*}V \rightarrow\ {}^{\shortmid}\mathcal{M}^{\star}$, where $T^{*}V$ is the cotangent bundle of a Lorentz manifold V.[4]

Similar assumptions and a respective abstract geometric formalism for star product R-flux deformations can be used in order to postulate nonassociative generalizations of the R. Hamilton equations [4, 5, 1],

$$ \frac{\partial\ {}^{\shortmid}\mathfrak{g}^{\star}}{\partial\tau} = -2\ {}^{\shortmid}\mathcal{R}ic^{\star}[\ {}^{\shortmid}\nabla^{\star}] + ..., \quad (2) $$

describing the flow evolution on ${}^{\shortmid}\mathcal{M}$ of a family of nonassociative metrics ${}^{\shortmid}\mathfrak{g}^{\star}(\tau)$ parameterized by a positive parameter $0 \leq \tau \leq \tau_0$. In these formulas, dots are used for additional terms which can be defined/computed for respective star product R-flux deformations of a corresponding variational calculus [1, 9, 10, 11, 7], see references therein and details in section 2.3.[5]

The coefficients of geometric objects, effective and matter field sources, and respective physically important geometric evolution/ dynamical equations in such theories can be decomposed into real and complex terms and parametric forms using decompositions on $\hbar$, the Planck constant, and $\kappa := \ell_s^3/6\hbar$, the string constant, were ℓ_s is a length parameter [39].[6] In [12, 13], the AFCDM was generalized for finding exact and parametric solutions of systems of nonlinear partial differential equations, PDEs, of type (1). We proved a general splitting and integration property for quasi-stationary configurations (with Killing symmetry on a time like coordinate) of such nonassociative nonlinear dynamical systems and shown how to construct nonassociative four dimensional, 4-d, and 8-d, black hole, BH, and black ellipsoid, BE, solutions in [13, 40, 41]. Generic off-diagonal quasi-stationary solutions can be generated for κ-linear parametric decompositions transforming the real part of (1) into associative and commutative modified Einstein equations (see appendix 2.1.1),

$$ {}^{\shortmid}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s} = \delta^{\beta_s}{}_{\gamma_s}\ {}^{\shortmid}_{s}\mathcal{K}. \quad (3) $$

[4]To elaborate on nonassociative star product and R-flux deformations of the Einstein equations in general relativity, GR, we consider a basic associative and commutative spacetime manifold V, of dimension $\dim V = 4$, for instance, enabled with metrics of signature $(+++-)$. We suppose that readers are familiar with the basic concepts from the mathematical relativity, and methods of constructing exact solutions, as well with the formalism of (non) linear connections in (co) vector/ tangent bundle geometry [43, 44, 45, 46, 12, 13]. Here we note that the label " ${}^{\shortmid}$" is used in order to emphasize that the geometric objects are defined on cotangent bundles and not on usual tangent bundles $\mathcal{M} = TV$, when respective geometric/physical objects are written $\mathcal{R}ic^{\star}[\nabla^{\star}] = \Lambda_0\ \mathfrak{g}^{\star}$.

[5]Any point ${}^{\shortmid}u = (x,p) = \{\ {}^{\shortmid}u^{\alpha} = (x^i, p_a)\}$ in a phase space ${}^{\shortmid}\mathcal{M}$ is parameterized by spacetime coordinates $x = (x^i)$ and cofiber (momentum like) coordinates $p = (p_a)$, which are dual to conventional velocities $v = (v^a)$, for indices $i, j, k, ... = 1, 2, 3, 4$ and $a, b, c, ... = 5, 6, 7, 8$. In a similar form, we label a point $u = (x, v) = \{u^{\alpha} = (x^i, v^a)\}$ in $\mathcal{M}$. In this work, small Greek indices run values $\alpha, \beta, ... = 1, 2, ...8$, but they may take values up to 10, 11, ... for a corresponding modified gravity theory, MGT, (super) string/ gravity models etc.

[6]Nonassociative geometric and physical theories can be formulated in certain forms encoding quasi-Hopf [42, 39, 12] and/or exceptional algebraic structures, for instance, with octonionic and Clifford configurations [32, 33, 36, 37]. For different nonassociative algebraic and geometric configurations and respective generalized nonassocitative and noncommutative differential and integral calculi, we can elaborate on different types of classical and quantum physical theories. In our works on nonassociative geometry and physics, we follow an explicit approach with quasi-Hopf nonholonomic geometric structures determined by nonassociative star products and R-fluxes [38, 39, 12, 13, 40, 41].

In this formula, the canonical Ricci s-tensor ${}^{\shortmid}\widehat{\mathcal{R}}ic = \{ {}^{\shortmid}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}\}$ is a tensor adapted to a so-called nonholonomic shell structure, s-structure, with a conventional (2+2)+(2+2)-splitting containing 2-d shells labeled in abstract form by $s = 1, 2, 3, 4$. It is defined by a respective canonical s-connection

$$ {}^{\shortmid}\widehat{\mathbf{D}}_{\gamma_s} = {}^{\shortmid}\nabla_{\gamma_s} + {}^{\shortmid}\widehat{\mathbf{Z}}_{\gamma_s}, \qquad (4) $$

where the distortion s-tensor ${}^{\shortmid}\widehat{\mathbf{Z}}_{\gamma_s}$ is such way chosen that the system of nonlinear PDEs (3) can be decoupled and integrated in general form with respect to a correspondingly defined nonholonomic s-frames ${}^{\shortmid}\mathbf{e}_{\gamma_s}$, see details in [13].[7] The effective sources in (3) encodes contributions from nonassociative $\star$-product with nontrivial R-flux terms, $\mathcal{R}^{\cdot\cdot}{}_{\cdot} \sim \mathcal{R}^{\tau_s \xi_s}{}_{\alpha_2}$, being parameterized in s-adapted form as ${}^{\shortmid}\mathbf{K}^{\beta_s}{}_{\gamma_s} = \delta^{\beta_s}{}_{\gamma_s}\ {}^{\shortmid}_{s}\mathcal{K}(\kappa, \hbar, \mathcal{R}^{\cdot\cdot}{}_{\cdot}, {}^{\shortmid}_{s}u)$. General classes of solutions (they are generic off-diagonal because such solutions can't be diagonalized in a general form in a finite phase space region via coordinate transforms) are determined by s-metrics

$$ \begin{aligned} g = {}^{\shortmid}_{s}\mathbf{g} &= (h_1\ {}^{\shortmid}\mathbf{g},\ v_2\ {}^{\shortmid}\mathbf{g},\ c_3\ {}^{\shortmid}\mathbf{g}, c_4\ {}^{\shortmid}\mathbf{g}) \in T\mathbf{T}^*\mathbf{V} \otimes T\mathbf{T}^*\mathbf{V} \\ &= {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}({}^{\shortmid}_{s}u)\ {}^{\shortmid}\mathbf{e}^{\alpha_s} \otimes_s {}^{\shortmid}\mathbf{e}^{\beta_s} = \{ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} = ({}^{\shortmid}\mathbf{g}_{i_1 j_1},\ {}^{\shortmid}\mathbf{g}_{a_2 b_2},\ {}^{\shortmid}\mathbf{g}^{a_3 b_3},\ {}^{\shortmid}\mathbf{g}^{a_4 b_4})\}. \end{aligned} \qquad (5) $$

adapted to a corresponding nonlinear connection, N-connection, structure and for respective s-adapted tensor products $\otimes_s$, see geometric preliminaries in next section.

The AFCDM allows us to construct/find various classes of quasi-stationary solutions of (3), and of (1), when the coefficients of (5) are computed as functionals

$$ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}({}^{\shortmid}_{s}u) = {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}[\,{}^{\shortmid}_{s}\mathcal{K},\ {}^{\shortmid}_{s}\Psi\,] = {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}[\,{}^{\shortmid}_{s}\mathcal{K},\ {}^{\shortmid}_{s}\Phi,\ {}^{\shortmid}_{s}\Lambda_0\,] \qquad (6) $$

determined by effective sources ${}^{\shortmid}_{s}\mathcal{K}$ and generating functions ${}^{\shortmid}_{s}\Psi(\kappa, \hbar, \mathcal{R}^{\cdot\cdot}{}_{\cdot}, {}^{\shortmid}_{s}u)$, or ${}^{\shortmid}_{s}\Phi(\kappa, \hbar, \mathcal{R}^{\cdot\cdot}{}_{\cdot}, {}^{\shortmid}_{s}u)$.[8] In general, such solutions depend also on integration functions and constants, effective sources, broken (non) linear symmetries etc., which should be prescribed/defined from certain experimental/ observational data, boundary conditions etc. A very important property of the functionals for s-metrics (6) is that they posses certain general nonlinear symmetries for generating functions and effective sources and give possibilities to introduce into consideration some effective cosmological constants ${}^{\shortmid}_{s}\Lambda_0$ on any shell $s = 1, 2, 3, 4$. For nonassociative vacuum gravitational equations, such a proof is provided in section 5.4 of [13]. Here, we present, for simplicity, only the formulas for the spacetime shell $s = 2$, for changing the generating data, $({}_{2}\Psi(\hbar, \kappa, x^{i_1}, y^3),\ {}^{\shortmid}_{2}\mathcal{K}(\hbar, \kappa, x^{i_1}, y^3)) \leftrightarrow ({}_{2}\Phi(\hbar, \kappa, x^{i_1}, y^3),\ {}_{2}\Lambda_0)$, (see appendix A.2 with formulas for all shells $s = 1, 2, 3, 4$),

$$ \begin{aligned} \frac{\partial_3[({}_{2}\Psi)^2]}{{}^{\shortmid}_{2}\mathcal{K}} &= \frac{\partial_3[({}_{2}\Phi)^2]}{{}_{2}\Lambda_0}, \text{ which can be integrated as} \\ ({}_{2}\Phi)^2 &= {}_{2}\Lambda_0 \int dy^3 ({}^{\shortmid}_{2}\mathcal{K})^{-1} \partial_3[({}_{2}\Psi)^2] \text{ and/or } ({}_{2}\Psi)^2 = ({}_{2}\Lambda_0)^{-1} \int dy^3 ({}^{\shortmid}_{2}\mathcal{K}) \partial_3[({}_{2}\Phi)^2]. \end{aligned} \qquad (7) $$

We can chose in such formulas ${}^{\shortmid}_{s}\Lambda_0 = \Lambda_0 = const$, or to study running cosmological constants ${}^{\shortmid}_{s}\Lambda(\tau)$, for instance, on a geometric flow parameter τ; and phase space polarizations via ${}^{\shortmid}_{s}\Lambda_0 \rightarrow {}^{\shortmid}_{s}\Lambda(\tau, {}^{\shortmid}_{s}u)$ for more general considerations. Such nonlinear transforms allow us to rewrite equivalently the nonassociative modified

[7] In s-adapted form, we follow such a convention of indices and local coordinates: For instance, $\beta_2 = (j_1, b_2)$, where $j_1 = 1, 2; b_2 = 3, 4$, for the shell $s = 2$ and the coordinate $u^4 = y^4 = t$ considered as a time like one, t, but $u^{\beta_2} = (x^{i_1}, y^3, y^4)$, for (x^{i_1}, y^3) being space like coordinates. In a similar form, we split the indices and coordinates, for instance, on the shall $s = 4$, when $\beta_4 = (j_3, b_4)$, for $j_3 = 1, 2, ...6$ and $b_4 7, 8$; and coordinates ${}^{\shortmid}_{4}u = \{ {}^{\shortmid}u^{\beta_4} = (x^{i_1}, y^3, y^4 = t, p_5, p_6, p_7, p_8 = E)\}$, with E being a conventional energy type coordinate for a relativistic phase space ${}^{\shortmid}\mathcal{M}$. Here we note also that nonholonomic frames ${}^{\shortmid}\mathbf{e}_{\gamma_s} = {}^{\shortmid}\mathbf{e}^{\gamma'_s}{}_{\gamma_s}\ {}^{\shortmid}\partial_{\gamma'_s}$ can be related via frame transforms, using matrices ${}^{\shortmid}\mathbf{e}^{\gamma'_s}{}_{\gamma_s}$, to local coordinate bases ${}^{\shortmid}\partial_{\gamma'_s}$, when the Einstein convention on repeating "up-low" indices is applied. In a similar form, we can consider s-splitting of dual frames ${}^{\shortmid}\mathbf{e}^{\beta_s}$, when ${}^{\shortmid}\mathbf{e}^{\beta_s}\ {}^{\shortmid}\mathbf{e}_{\gamma_s} = \delta^{\beta_s}{}_{\gamma_s}$, with $\delta^{\beta_s}{}_{\gamma_s}$ being the Kronecker symbol.

[8] we note that in this work there are used for explicit computations certain real co-fiber coordinates with labels "${}^{\shortmid}_{s}$" even in nonassociative gravity [39, 12, 13] it is necessary to consider for various purposes complex coordinates with labels "${}^{\shortparallel}_{s}$"

vacuum gravitational equations (3) in a form with effective constants in the right part and redefined functional dependence of the s-metrics in the Ricci s-tensor

$$ {}^{\shortmid}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}\ [\ {}^{\shortmid}_{s}\mathcal{K}(\kappa,\hbar,\mathcal{R}^{\cdot\cdot}\ .,\ {}^{\shortmid}_{s}u),\ {}^{\shortmid}_{s}\Phi(\kappa,\hbar,\mathcal{R}^{\cdot\cdot}\ .,\ {}^{\shortmid}_{s}u)] = \ \delta^{\beta_s}{}_{\gamma_s}\ \ {}^{\shortmid}_{s}\Lambda_0. \qquad (8)$$

These equations define a particular class of nonholonomic Ricci solitons encoding nonassociative data, see section 2.3.

We emphasize that using a class of quasi-stationary solutions ${}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}[\ {}^{\shortmid}_{s}\mathcal{K},\ {}^{\shortmid}_{s}\Phi,\ {}^{\shortmid}_{s}\Lambda_0\ ]$ (6) we can compute in general functional form respective components of a nonassociative $\star$-metric, $\mathfrak{g}^{\star}[...] = (g^{\star}[...], \breve{g}^{\star}[...])$ as it is explained in [13] and, for respective 4-d and 8-d BH/BE solutions, in [40, 41]. In result, we generate parametric solutions for nonassociative vacuum Einstein equations (1) if we restrict the geometric constructions for a subclass of generating functions when the distortion s-tensor ${}^{\shortmid}\widehat{\mathbf{Z}}_{\gamma_s}$ (4) is constrained to zero and ${}^{\shortmid}\widehat{\mathbf{D}}_{\gamma_s} \to {}^{\shortmid}\nabla_{\gamma_s}$.[9]

1.2 The structure, aims, and the main hypothesis of the paper

This work is a natural and logical development of a series of articles on nonassociative geometry and physics [37, 38, 39] and a new research program on nonassociative geometric and quantum information flows and gravity [12, 13, 40, 41]. It is related to the swampland program [19, 20, 21] and revised conjectures [23, 24, 25, 26, 27] following such five aims:

The first aim stated for section 2 is to prove that using nonassociative star products determined by R-flux deformations we can formulate and provide a physical motivation for the generalized nonassociative R. Hamilton equations (2). In general, such nonassociative geometric flow equations can be derived for certain canonical data $\left[\ {}^{\shortmid}_{s}\mathfrak{g}^{\star}(\tau),\ {}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}(\tau)\right]$ stated for generalized nonassociative G. Perelman F- and W-functionals, ${}^{\shortmid}_{s}\widehat{\mathcal{F}}^{\star}(\tau)$ and ${}^{\shortmid}_{s}\widehat{\mathcal{W}}^{\star}(\tau)$. Formulas (54) in [41] present κ-linear parametric versions of such functionals encoding string star product R-flux data which, in this work, are generalized to describe nonassociative geometric flow evolution and/or nonholonomic Ricci solitons (8) (in particular, they include nonassociative modified vacuum equations).

The second aim, for section 3, is to elaborate on statistical thermodynamic models for nonassociative geometric flows with thermodynamic variables derived from the W-entropy ${}^{\shortmid}\widehat{\mathcal{W}}^{\star}(\tau)$. We extend the constructions from [41] and provide general formulas and respective κ-linear parametric versions of effective canonical energy, ${}^{\shortmid}_{s}\widehat{\mathcal{E}}^{\star}(\tau) \to {}^{\shortmid}_{s}\widehat{\mathcal{E}}(\tau)$; entropy, ${}^{\shortmid}_{s}\widehat{\mathcal{S}}^{\star}(\tau) \to {}^{\shortmid}_{s}\widehat{\mathcal{S}}(\tau)$; and quadratic fluctuations, ${}^{\shortmid}_{s}\widehat{\sigma}^{\star}(\tau) \to {}^{\shortmid}_{s}\widehat{\sigma}(\tau)$, all encoding nonassociative star and R-flux deformed data. We emphasize that the off-diagonal quasi-stationary and BH/ BE solutions in nonassociative gravity are not described, in general, in the framework of the Bekenstein-Hawking thermodynamics paradigm for BHs [47, 48, 49, 50, 43, 44, 45] if there are not defined certain hypersurface horizons, duality and holography conditions. Contrary to this, modified versions of G. Perelman W-entropy ${}^{\shortmid}\widehat{\mathcal{W}}(\tau)$ and geometric-statistic thermodynamic entropy ${}^{\shortmid}_{s}\widehat{\mathcal{S}}(\tau)$ can be defined and computed for any class of exact/ parametric solutions in GR and MGTs.

Then, **the third** aim, in section 4, is to extend the AFCDM [16, 17, 13, 40, 41] in such forms which allows us to decouple and integrate the nonassociative R. Hamilton equations in certain generalized forms describing the geometric evolution of quasi-stationary Ricci soliton and vacuum gravitational structures (with effective sources encoding star product and R-flux deformations). We analyze a class of nonlinear symmetries relating different types of generating functions, generating (effective) sources and cosmological constants running on

[9] Here we note that in a similar (in certain sense) dual form one can be constructed nonassociative locally anisotropic and inhomogeneous cosmological solutions with Killing symmetry, for instance, on ∂_3, when the s-metrics in certain s-adapted frame do not depend on coordinate y^3 but depend on $y^4 = t$ and other space and (co) fiber coordinates. In associative and commutative forms, such solutions are discussed in [17, 8, 6], see references therein. Cosmological models encoding nonassociative star product R-flux deformed data will be studied in our future works as it was stated in [12], see also query Q5 at the end of conclusion section.

temperature like geometric flow parameter. There are provided explicit formulas for quadratic linear elements for nonassociative geometric evolution in κ-linear parametric forms of quasi-stationary generic off-diagonal metrics and gravitational polarization functions.

The forth aim, in section 5, is to elaborate on parametric geometric flows and related thermodynamics models of quasi-stationary solutions describing nonassociative evolution of star R-flux deformed BHs. Additionally to the classes of nonassociative BH solutions of Tangherlini and double BH phase space solutions studied in [41], we construct two other types nonassociative generic off-diagonal κ-linear parametric BHs subjected to geometric evolution. There are analyzed nonassociative phase space double Schwarzschild–AdS black ellipsoid, BE, configurations and nonassociative flows of phase space Reisner-Nordström BHs. We show also how to compute G. Perelman thermodynamic variables for generic off-diagonal solutions related via nonlinear symmetries (7) to effective running on temperature τ cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$. Then, there are stated the conditions when the concept Bekenstein-Hawking entropy can be applied for certain particular examples of such nonassociative BHs and their deformations which may have different physical interpretations.

The fifth aim, in section 6, is to study how the swampland program should be generalized/modified in order to include nonassociative geometric flows and related exact/ parametric solutions. We note that the main goal of the swampland program is to elaborate certain criteria which allow to distinguish low-energy effective field theories which can be completed into QG in the UV from those theories that cannot. In fact, for the AdS spaces, the Ricci flow swampland conjecture is equivalent to the anti- de Sitter distance conjecture (ADS) [20, 23, 24]. Swampland conjectures were studied recently in connection to BH physics, extra dimensions and geometric flow conjectures, when the concept of Bekenstein-Hawking entropy was applied [25, 26, 27]. A modern approach to QG and string and M-theory involves models with nonassocitative structures for QM, QFT and MGTs. This modifies substantially the mathematical formalism and methods for constructing exact/parametric solutions and quantization and providing a physical interpretation of nonassociative geometric flows and gravity. The Bekenstein–Hawking thermodynamic paradigm should be completed with a more general one which is based on the concept of G. Perelman W-entropy and derived (non) associative/ commutative geometric thermodynamics.

In section 7, we discuss and conclude the main results based on:
The Main Hypothesis, MH, *of this work is that the Swampland Program and related conjectures have to be generalized and modified following above Aims 1-5 (objectives of this work) with the purpose to formulate well-defined criteria how to include nonassociative and noncommutative geometric flows, QM, QFTs, and MGTs in elaborating QG theories related to M-theory and string gravity. Self-consistent geometric and physical models and solutions should encode at least in parametric form certain nonassociative star product and R-flux data in low-energy limits of corresponding effective geometric flow evolution and field theories which can be completed into QG in the UV forms and distinguished from another classes of theories which do not have such properties.*

In section 2.1, we outline the main concepts and most important formulas on nonassociative differential geometry with symmetric and nonsymmetric metrics and (non) linear connections [38, 39, 12, 13, 40, 41]. Finally, in Appendix A, we summarize the main ideas and steps for constructing generic off-diagonal quasi-stationary and BH solutions using the AFCDM [16, 17, 13, 40, 41]. Such formulas describe as particular cases exact/parametric solutions for nonassociative Ricci solitons and quasi-stationary gravitational polarizations for κ-linear geometric flows studied in the main text of the paper.

2 A model of nonassociative Ricci flows with star product R-flux deformations

G. Perelman elaborated his approach to the Ricci flow theory [1] by introducing (postulating) the concepts of F- and W-functionals for a familiy of geometric flows of of Riemannian metrics ${}^{n}g(\tau) = \{g_{ij}(\tau) \simeq g_{ij}(\tau, x^k)\}$

on a closed manifold ${}^nV$, dim ${}^nV = n$ (in this work, we can consider $n = 3$),

$$F(\tau) \simeq F[\tau, \ {}^nRsc(\tau), \ {}^n\nabla(\tau), \ {}^ng(\tau), f(\tau)] \text{ and } W(\tau) = W[\tau, \ {}^nRsc(\tau), \ {}^n\nabla(\tau), \ {}^ng(\tau), f(\tau)]. \quad (9)$$

In such formulas, it is used a flow parameter $\tau, 0 \leq \tau \leq \tau_0$, which can be treated as **temperature**; the scalar curvatures ${}^nRsc(\tau) = {}^nRsc(\tau, x^k)$ are determined by a corresponding family of LC–connections ${}^n\nabla(\tau) = {}^n\nabla(\tau, x^k)$. A normalizing function $f(\tau) \simeq f(\tau, x^k)$ is used for defining integration measures $(4\pi\tau)^{-n/2} e^{-f} d\mathcal{V}ol(\tau)$ with volume elements $d\mathcal{V}ol(\tau) = \sqrt{| \ {}^ng(\tau)|} \ d^n \ x^i$. It should be emphasized that a type of $f(\tau)$ may have different implications and interpretations in topology and/or geometric analysis, and differential geometry theories.[10]

Using a (3+1) splitting on 4-d, Lorentz manifolds, the functionals (9) can be generalized in relativistic form, which allows us to prove (using standard variational procedures, or abstract geometric methods) respective generalizations of the R. Hamilton equations [4] and elaborate on relativistic geometric thermodynamic models. This does not results in a formulation and proof of a relativistic version of the Poincaré–Thorston conjecture. Nevertheless, we can elaborate on important physical models for certain classes of solutions of relativistic flow equations with well-defined causal evolution and describing important physical processed. Such solutions can be found in nonassociative geometric (flow) and gravity theories using the AFCDM [13, 40, 41].

In [38, 39], the nonassociative geometry and gravity with star product and R-flux deformations were formulated up to defining and computing nonassociative versions and for κ-parametric decompositions of respective Ricci tensors and scalar curvatures defined by nonassociative Levi Civita, LC, connections. In principle, those constructions allow to elaborated on nonassociative versions of G. Perelmans functionals (9). Applying rigorous geometric methods and respective variational procedures generalizing the geometric analysis formalism from [1, 9, 10, 11, 7], we can derive respective nonassociative geometric flow equaitons. Such star deformed R. Hamilton equations and, in particular nonassociative Ricci solitons and/or nonassociative vacuum Einstein equations consists very sophisticate systems of nonlinear partial differential equations, PDEs. It is a very difficult technical problem to solve and analyse possible physical implications of such nonassociative geometric evolution and/or star deformed dynamical gravitational field equations. In [13], we proved that nonassociative vacuum Einstein equations can be decoupled and intergrated in very general forms for the so-called canonical s-connection structure. The same geometric methods can be generalized for generating solutions of nonassociative geometric flow equations. We shall study such applications of the AFCDM in section 4. The goal of this section is to outline necessary geometric methods and formulate a nonassociative generalization of F- and W-functionals (9) is some forms which allow to derive nonassociative versions of R. Hamilton equations which can be solved in certain general off-diagonal forms of nonassociative metrics and (non) linear connections which can be nonholonomically constrained to LC-configurations.

2.1 Nonassociative differential geometry with (non) linear connections

The nonassociative differential geometry on phase spaces enabled with a nonholonomic dyadic shell adapted (s-adapted) star product and R-flux deformations and respective symmetric and nonsymmetric metric and (non) linear connection structures [12, 13] is reviewed. Some necessary definitions and constructions from [38, 39] are also considered but in a form which will allow extensions to so-called exactly and κ-parametric solvable nonassociative geometric flow models. We follow both an abstract (index and coordinate free) description of nonassociative geometric objects and formulas and present certain s-adapted frame (co) bases and index formulas which are used for providing exact/ parametric solutions and swampland conjectures in the sections 4-6. It is supposed that readers are familiar with the main concepts on mathematical relativity and (non)

[10] We can fix it, for instance, in order to elaborate certain geometric/ thermodynamic models for some prescribed topological configurations, or to simplify the procedure of finding exact/parametric solutions of respective geometric flow/ Ricci soliton equations and associated thermodynamic values [17, 7, 8, 6, 41]. In a series of recent works [23, 24, 25, 26, 27], respective physical applications are elaborated for $f(\tau, x^k)$ considered as scalar/Higgs/moduli fields.

linear connection formalism described in [43, 44, 45, 17]. The notations and definitions were stated in partner works [12, 13, 40, 41].[11]

2.1.1 Associative and commutative dyadic and nonlinear connection formalism

The associative and commutative geometric arena consists from a phase space modeled as a cotangent Lorentz bundle $\mathcal{M} = T^{*}V$ on a spacetime manifold V of signature $(+++-)$. Such a phase space can be enabled with conventional (2+2)+(2+2) splitting determined by a nonholonomic (equivalently, anholonomic/non-integrable) dyadic, 2-d, decomposition into four oriented shells $s = 1,2,3,4$ (in brief, s-decomposition). A s-splitting is defined by a nonlinear connection, N-connection (equivalently, s-connection), structure:

$$\begin{array}{rcl} {}_{s}^{\shortmid}\mathbf{N}: \ {}_{s}T\mathbf{T}^{*}\mathbf{V} &=& {}^{1}hT^{*}V \oplus {}^{2}vT^{*}V \oplus {}^{3}cT^{*}V \oplus {}^{4}cT^{*}V, \text{ which is dual to} \\ {}_{s}\mathbf{N}: \ {}_{s}T\mathbf{TV} &=& {}^{1}hTV \oplus {}^{2}vTV \oplus {}^{3}vTV \oplus {}^{4}vTV, \text{ for } s = 1,2,3,4. \end{array} \qquad (10)$$

We use ${}^{1}h$ for a conventional 2-d shell (dyadic) splitting on cotangent bundle, with x^{i_1} local coordinates; then ${}^{2}v$ for a 2-d vertical like splitting with y^{a_2} coordinates on the shell $s = 2$; at the next shell $s = 3$, the splitting is convetional co-vertical, we write ${}^{3}c$ and use local coordinates p_{a_3}; for the 4th shell $s = 4$, the respective symbols are ${}^{4}c$ and p_{a_4}. Such s-splitting will allow to decouple and integrate in general off-diagonal form nonassociative geometric and physically important systems of nonlinear PDEs. In a local coordinate basis (see conventions from footnotes 7, 8, and 9), a nonlinear s-connection defined by a Whitney sum $\oplus$ as in (10), for instance, is characterized by coefficients ${}_{s}^{\shortmid}\mathbf{N} = \{ {}^{\shortmid}N_{i_s a_s}({}^{\shortmid}u)\}$, for $u = (x,p) = {}^{\shortmid}u = ({}_{1}x, {}_{2}y, {}_{3}p, {}_{4}p)$. Such coefficients allow us to construct N-elongated bases (N-/ s-adapted bases) as linear N-operators:

$${}^{\shortmid}\mathbf{e}_{\alpha_s}[{}^{\shortmid}N_{i_s a_s}] = ({}^{\shortmid}\mathbf{e}_{i_s} = \frac{\partial}{\partial x^{i_s}} - {}^{\shortmid}N_{i_s a_s}\frac{\partial}{\partial p_{a_s}}, \ {}^{\shortmid}e^{b_s} = \frac{\partial}{\partial p_{b_s}}) \text{ on } {}_{s}T\mathbf{T}^{*}_{\shortmid}\mathbf{V}, \qquad (11)$$

and, dual s-adapted bases, s-cobases,

$${}^{\shortmid}\mathbf{e}^{\alpha_s}[{}^{\shortmid}N_{i_s a_s}] = ({}^{\shortmid}\mathbf{e}^{i_s} = dx^{i_s}, \ {}^{\shortmid}\mathbf{e}_{a_s} = d\,p_{a_s} + {}^{\shortmid}N_{i_s a_s}dx^{i_s}) \text{ on } {}_{s}T^{*}\mathbf{T}^{*}_{\shortmid}\mathbf{V}. \qquad (12)$$

Such s-frames are not integrable, i.e. nonholonomic (equivalently, anholonomic) because, in general, they satisfy certain anholonomy conditions, ${}^{\shortmid}\mathbf{e}_{\beta_s}{}^{\shortmid}\mathbf{e}_{\gamma_s} - {}^{\shortmid}\mathbf{e}_{\gamma_s}{}^{\shortmid}\mathbf{e}_{\beta_s} = {}^{\shortmid}w^{\tau_s}_{\beta_s\gamma_s}{}^{\shortmid}\mathbf{e}_{\tau_s}$, see details in [12, 13]. For a 4+4 splitting, we write, for instance, ${}^{\shortmid}\mathbf{N} = \{ {}^{\shortmid}N_{ia}(x^{j}, p_{b})\}$, and use the term N-connection. We shall put a left label s for corresponding spaces and geometric objects (labeled with bold letters if they are written in a N-adapted form) if, for instance, a phase space is enabled with a s-adapted dyadic structure, ${}_{s}\mathcal{M}$, and use terms like s-tensor, s-metric, s-connection etc.

The geometric s-objects and respective formulas (10)-(12) can be generalized for additional running on a geometric flow evolution parameter τ, for instance, writting ${}^{\shortmid}\mathbf{N}(\tau) \simeq {}^{\shortmid}\mathbf{N}(\tau, {}^{\shortmid}u) = \{ {}^{\shortmid}N_{ia}(\tau) \simeq {}^{\shortmid}N_{ia}(\tau, x^{j}, p_{b})\}$ and, respectively, ${}^{\shortmid}\mathbf{e}_{\alpha_s}(\tau)$, ${}^{\shortmid}\mathbf{e}^{\alpha_s}(\tau)$, etc. For running of geometric/ physical objects, we shall write only the τ-dependence if that will not result in ambiguities. Here, we note that in a similar form we can introduce and write formulas for geometric objects on ${}_{s}T\mathbf{TV}$, i.e. when the total space coordinates are of spacetime-velocity type. In such case, we omit the labels " ${}^{\shortmid}$" and write, for instance, $\mathbf{e}_{\alpha_s}(\tau)$ and $\mathbf{e}^{\alpha_s}(\tau)$, for local coordinates

[11] Unfortunately, it is not possible to simplify such notations because we have to distinguish various abstract and frame constructions in associative/ commutative nonholonomic geometry, their nonassociative and nonassociative generalizations, κ-parametric decompostions and generating solutions with dependences on spacetime and momentum type coordinates (which in nonassociative geometry can be complex, or almost complex), star product and R-flux deformations of certain prime metrics into noncommutative ones with symmetric and nonsymmetric components, consider nonlinear symmetries etc. Here we also note that there are different styles/ traditions for notations in geometric and statistical thermodynamics, MGTs, nonassociative geometry and coventions for constructing solutions in the theory of nonlinear PDEs. We have to use various abstract left/write, up/low labels, boldface symbols and shell coordinates in orde to show how the AFCDM can be applied and physically important thermodynamic variables can be computed.

$u=(x,v)= {}^{\shortmid}u=(\ _1x,\ _2y,\ _3v,\ _4v)$. In general, such coordinates are not just dual like fibe and co-fiber ones but may include certain Legendre transforms [70, 8]. In this work, we shall work on nonassociative phase spaces as in [38, 39] and [12, 13, 40, 41] using labels " ${}^{\shortmid}$" in order to follow an unified system of notations which will allow in furthe partner works to elaborate on nonassociative models of Finsler-Lagrange spaces, which are important in quantum information theory.

A metric field in a phase space $\mathcal{M}$ is a second rank symmetric tensor ${}^{\shortmid}g=\{{}^{\shortmid}g_{\alpha\beta}\}\in TT^{*}V\otimes TT^{*}V$ of local signature $(+,+,+,-;+,+,+,-)$. It can be written in equivalent form as a s-metric ${}_{s}^{\shortmid}\mathbf{g}=\{{}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\}$ (5). For τ-families of pahse space metric and s-metrics , we shall use notations of type ${}^{\shortmid}g(\tau)=\{{}^{\shortmid}g_{\alpha\beta}(\tau)\}$ and, respectively, ${}_{s}^{\shortmid}\mathbf{g}(\tau)=\{{}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)\}$

Another important geometric concept is that of s-connection with a (2+2)+(2+2) splitting (we use the term distinguished connection, d-connection, for a (4+4)-splitting), which is a linear connection preserving under parallel transports a respective N-connection splitting (10):

$$ {}_{s}^{\shortmid}\mathbf{D}=(h_1\ {}^{\shortmid}\mathbf{D},\ v_2\ {}^{\shortmid}\mathbf{D},\ c_3\ {}^{\shortmid}\mathbf{D},\ c_4\ {}^{\shortmid}\mathbf{D})=\{{}^{\shortmid}\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}\}, \qquad (13)$$

where indices split into respective dyadic components of a respective h_1,v_2,c_3,c_4 decomposition. Using standard definitions from differential geometry, we can introduce and compute in standard form[12] as for any linear connection but for s-adapted ${}_{s}^{\shortmid}\mathbf{D}$ such fundamental geometric s-objects:

$$\begin{aligned}
{}_{s}^{\shortmid}\mathcal{T} &= \{{}^{\shortmid}\mathbf{T}^{\alpha_s}_{\ \beta_s\gamma_s}\}, \text{ the s-torsion };\\
{}_{s}^{\shortmid}\mathcal{R} &= \{{}^{\shortmid}\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}, \text{ the Riemannian s-curvature };\\
{}_{s}^{\shortmid}\mathcal{R}ic &= \{{}^{\shortmid}\mathbf{R}_{\ \beta_s\gamma_s}:={}^{\shortmid}\mathbf{R}^{\alpha_s}_{\ \beta_s\gamma_s\alpha_s}\neq {}^{\shortmid}\mathbf{R}_{\ \gamma_s\beta_s}\}, \text{ the Ricci s-tensor;}\\
{}_{s}^{\shortmid}\mathcal{R}sc &= \{{}^{\shortmid}\mathbf{g}^{\beta_s\gamma_s}\ {}^{\shortmid}\mathbf{R}_{\ \beta_s\gamma_s}\}, \text{the Riemannian scalar }.
\end{aligned} \qquad (14)$$

Geometric data $({}_{s}^{\shortmid}\mathbf{g},{}_{s}^{\shortmid}\mathbf{D})$ of type (5) and (13) enable a ${}_{s}\mathcal{M}$ with a dyadic metric-affine s-structure which is a N-adapted phase space version of metric-affine geometry [43, 46]. Additionally to geometric s-objects (14), such spaces are characterized by a nonmetricity s-tensor, ${}_{s}^{\shortmid}\mathcal{Q}=\{{}^{\shortmid}\mathbf{Q}_{\gamma_s\alpha_s\beta_s}={}^{\shortmid}\mathbf{D}_{\gamma_s}\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\}$. Above formulas for d-/ s-connections and respective geometric s-objects, can be defined and computed for geometric flows, for insance, in the forms ${}_{s}^{\shortmid}\mathbf{D}(\tau)=\{{}^{\shortmid}\Gamma^{\alpha_s}_{\ \beta_s\gamma_s}(\tau)\}$, ${}_{s}^{\shortmid}\mathcal{R}ic(\tau)$ etc. We have to keep a s-label for indices or abstract geometric s-objects in order to emphasize that the goemetric constructions are performed for a nonholonomic dyadic formalism. For splitting of type (4+4), the nonholonomic geometry is different and such decompositions do not allow general decoupling and integration of fundamenta geometric and physical systems of PDEs.

Using a s-metric ${}^{\shortmid}g={}_{s}^{\shortmid}\mathbf{g}$ (5), we can define and compute in abstract and component forms two important linear connection structures (the Levi-Civita, LC, connection and the canonical s-connection):

$$({}_{s}^{\shortmid}\mathbf{g},\ {}_{\mathbf{s}}^{\shortmid}\mathbf{N})\rightarrow\left\{\begin{array}{lll}
{}^{\shortmid}\nabla: & {}^{\shortmid}\nabla\ {}_{s}^{\shortmid}\mathbf{g}=0;\ {}_{\nabla}^{\shortmid}\mathcal{T}=0, & \text{LC–connection };\\
{}_{s}^{\shortmid}\widehat{\mathbf{D}}: & {}_{s}^{\shortmid}\widehat{\mathbf{Q}}=0;\ h_1\ {}^{\shortmid}\widehat{\mathcal{T}}=0, v_2\ {}^{\shortmid}\widehat{\mathcal{T}}=0, c_3\ {}^{\shortmid}\widehat{\mathcal{T}}=0, c_4\ {}^{\shortmid}\widehat{\mathcal{T}}=0, & \text{canonical}\\
 & h_1v_2\ {}^{\shortmid}\widehat{\mathcal{T}}\neq 0, h_1c_s\ {}^{\shortmid}\widehat{\mathcal{T}}\neq 0, v_2c_s\ {}^{\shortmid}\widehat{\mathcal{T}}\neq 0, c_3c_4\ {}^{\shortmid}\widehat{\mathcal{T}}\neq 0, & \text{s-connection }.
\end{array}\right. \qquad (15)$$

In this work, "hat" labels are used for geometric s-objects written in canonical form, for instance, ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$, ${}_{s}^{\shortmid}\widehat{\mathcal{R}}=\{{}^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$ etc. There are canonical distortion relations for linear connections (of type (4)) which allow to compute canonical distortions of fundamental geometric objects (14) and relate, for instance, two different curvature tensors, for instance, ${}_{\nabla}^{\shortmid}\mathcal{R}=\{{}_{\nabla}^{\shortmid}R^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$ and ${}_{s}^{\shortmid}\widehat{\mathcal{R}}=\{{}^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}\}$; ${}_{\nabla}^{\shortmid}\mathcal{R}ic$ and ${}_{s}^{\shortmid}\widehat{\mathcal{R}}ic$ etc. For τ-families such formulas can written, for instance, ${}^{\shortmid}\nabla(\tau)$, ${}_{s}^{\shortmid}\widehat{\mathbf{D}}(\tau)$, ${}_{s}^{\shortmid}\widehat{\mathcal{R}}(\tau)=\{{}^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s}_{\ \beta_s\gamma_s\delta_s}(\tau)\}$, ${}_{\nabla}^{\shortmid}\mathcal{R}ic(\tau)$, etc.

The modified Einstein equations for ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$ (15) can be derived in abstract geometric form as in GR [43] but on phase space ${}_{s}\mathcal{M}$,

$$ {}^{\shortmid}\widehat{\mathbf{R}}ic_{\alpha_s\beta_s}={}^{\shortmid}\Upsilon_{\alpha_s\beta_s}, \qquad (16)$$

[12] see details, proofs, and references in [17, 7, 8, 6], when the coefficient formulas are provided in s-adapted forms with respect to s-frames (11) and (12); a number of important abstract and coefficient formulas with nonassociative generalizations are contained in [12, 13, 40, 41]

where the s-tensor for effective and/or matter field sources can be postulated in the forms

$$
{}^{\shortmid}\Upsilon_{\beta_s\gamma_s} = \left\{ \begin{array}{c} {}_{s}^{\shortmid}\Lambda_0 \ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} = \frac{1}{2}\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\ {}_{s}^{\shortmid}\widehat{\mathbf{R}}sc + \ {}_{s}^{\shortmid}\lambda \ \ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}, \text{vacuum with shell cosmological constants } {}_{s}^{\shortmid}\Lambda_0 \text{ or } {}_{s}^{\shortmid}\lambda; \\ {}_{s}^{\shortmid}\Lambda(\tau,\ {}^{\shortmid}u)\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}, \text{ for polarized constants from geometric flow/ string / quantum theories;} \\ {}^{\shortmid}\mathbf{Y}_{\beta_s\gamma_s}, \text{ from variational/ geometric principles of interactions on } {}_{s}\mathcal{M}; \\ {}^{\shortmid}\mathbf{K}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil, \text{ for effective parametric star R-flux corrections, in this work and [13, 40, 41] }. \end{array} \right. \tag{17}
$$

The gravitational field equations (16) can be written in terms of the LC-connection ∇_α if we consider distortion relations (4). Imposing additional zero s-torsion conditions,

$$
{}_{s}^{\shortmid}\widehat{\mathbf{Z}} = 0, \text{ which is equivalent to } {}_{s}^{\shortmid}\widehat{\mathbf{D}}_{|\ {}_{s}^{\shortmid}\widehat{\mathbf{T}}=0} = \ {}^{\shortmid}\nabla, \tag{18}
$$

we can extract LC-configurations from various classes of solutions of nonholonomic phase space generalized Einstein equations. Conservation laws can be formulated as in GR using ${}^{\shortmid}\nabla$ on $\mathcal{M}$, for instance,

$$
{}^{\shortmid}\nabla(\ {}_{\nabla}^{\shortmid}\mathcal{R}ic_{\alpha_s\beta_s} - \frac{1}{2}\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\ \ {}_{\nabla}^{\shortmid}Rsc) = 0,
$$

but such laws are written in a more cumbersome forms if we distort the geometrical objects and this equations in terms of ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$ using formulas (4). This is a typical property of nonholonomic systems in geometric mechanics and gravity theories. Here we note that notations for nonholonomic constraints of type ${}_{s}^{\shortmid}\widehat{\mathbf{D}}_{|\ {}_{s}^{\shortmid}\widehat{\mathbf{T}}=0}$ (18) are used in our parner works [12, 13, 40] even the editors of some journals request a symplified version for notations like ${}_{s}^{\shortmid}\widehat{\mathbf{D}} = \ {}^{\shortmid}\nabla$ for ${}_{s}^{\shortmid}\widehat{\mathbf{T}} = 0$, when the zero s-torsion conditions are considered as certain nonholonomic constraints on a class of some generic off-diagonal soluions.

The main motivation to use the canonical s-connection ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$ and respective phase space equations (16) with nonholonomic 2+2+2+2 splitting is that in such geometric variables we can decouple and integrate in very general forms various classes of (modified) geometric flow and gravitational field equations. Using the AFCDM, this is proven in [8, 13] and references therein, see a summary of results in Appendix A. Here we note that it is not possible to decouple such systems of nonlinear PDEs written in terms of ∇_α. The main idea is to use ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$ in order to find explicit exact/parametric solutions and then to impose additional constraints of type (18) in order to extract LC-configurations if it is important for elaborating certain physical models. We can model τ-evolution of families of equations of type (16)- (18) for so-called geometric evolution of nonholonomic Einstein systems, NES, studied in [7].

2.1.2 Nonassociative vacuum Einstein equations for the canonical s-connection

The geometric constructions performed in this work are based on the concept of star product $\star_s$ defined in s-adapted form in our works [12, 13] and using the previous constructions from [38, 39]:

$$
\begin{aligned}
f \star_s q :=& \quad \cdot[\mathcal{F}_s^{-1}(f,q)] \\
=& \quad \cdot[\exp(-\frac{1}{2}i\hbar(\ {}^{\shortmid}\mathbf{e}_{i_s}\otimes\ {}^{\shortmid}e^{i_s} - \ {}^{\shortmid}e^{i_s}\otimes\ {}^{\shortmid}\mathbf{e}_{i_s}) + \frac{i\ell_s^4}{12\hbar}R^{i_s j_s a_s}(p_{a_s}\ {}^{\shortmid}\mathbf{e}_{i_s}\otimes\ {}^{\shortmid}\mathbf{e}_{j_a} - \ {}^{\shortmid}\mathbf{e}_{j_s}\otimes p_{a_s}\ {}^{\shortmid}\mathbf{e}_{i_s}))]f\otimes q \\
=& \quad f\cdot q - \frac{i}{2}\hbar[(\ {}^{\shortmid}\mathbf{e}_{i_s}f)(\ {}^{\shortmid}e^{i_s}q) - (\ {}^{\shortmid}e^{i_s}f)(\ {}^{\shortmid}\mathbf{e}_{i_s}q)] + \frac{i\ell_s^4}{6\hbar}R^{i_s j_s a_s}p_{a_s}(\ {}^{\shortmid}\mathbf{e}_{i_s}f)(\ {}^{\shortmid}\mathbf{e}_{j_s}q) + \ldots.
\end{aligned} \tag{19}
$$

In this formula, there are considered actions of ${}^{\shortmid}\mathbf{e}_{i_s}$ on some functions $f(x,p)$ and $q(x,p)$, see formulas for N-elongated derivatives and differentials (11) and (12); a constant ℓ characterizes the R-flux contributions determined by an antisymmetric $R^{i_s j_s a_s}$ background in string theory, when the tensor product $\otimes$ can be written also in a s-adapted form $\otimes_s$. For explicit computations and small parametric decompositions on $\hbar$ and $\kappa = \ell_s^3/6\hbar$, the tensor products turn into usual multiplications as in the third line of above formula. A phase space ${}_{s}\mathcal{M}$ enabled with a star product (19) transforms into a nonassociative one, ${}_{s}^{\star}\mathcal{M}$, when the s-adapted geometric objects and (physical) equations are star-deformed.

Considering geometric flows on a parameter τ of s-frames ${}^{\shortmid}\mathbf{e}_{i_s}(\tau)$ (11), we obtain define respective flow families s-adapted star product $\star_s(\tau)$ even the functions f and q may not depend on evolution parameter. Similar τ-dependences of geometric/ physical s-objects and structures have to be defined for evolution on nonassociative and associative geometric models.

For ${}_s\mathcal{M} \to {}_s^{\star}\mathcal{M}$, a $\star_s$-structure transforms any symmetric metric ${}_s^{\shortmid}\mathbf{g}$ into a general nonsymmetric one with respective symmetric, ${}_{\star s}^{\shortparallel}\mathbf{g}$, and nonsymmetric, ${}_{\star s}^{\shortparallel}\mathfrak{g}$, components. We use labels ${}_{\star s}^{\shortparallel}$ instead of ${}_s^{\shortmid}$ because such metrics may contain complex terms. On (co) tangent bundles, it is always possible to elaborate almost complex models (with real basic manifolds and real (co) fibers), or certain decompositions into pure real and imaginary components (the last ones are not considered for geometric constructions with real variables). So, labels ${}_s^{\shortmid}$ involve a procedure of transforming geometric constructions into certain s-objects on real manifolds and bundle spaces. We studied in [12, 13] the nonassociative (non) symmetric and generalized connection s-structures on ${}_s^{\star}\mathcal{M}$ endowed also with quasi-Hopf s-structure determined by a nonassociative algebra $\mathcal{A}_s^{\star}$ (generalizing the constructions from [39, 42]). A nonassociative symmetric, ${}_{\star s}^{\shortmid}\mathbf{g}$, and nonsymmetric metric, ${}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}$, s-tensor on a phase space ${}_s\mathcal{M}$ with star and R-flux induced terms on a Lorentz base spacetime manifold can be represented in the forms

$$\begin{aligned} {}_{\star s}^{\shortmid}\mathbf{g} &= {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} \star_s ({}^{\shortmid}\mathbf{e}^{\alpha_s} \otimes_{\star s} {}^{\shortmid}\mathbf{e}^{\beta_s}), \text{ where } {}_{\star}^{\shortmid}\mathbf{g}({}^{\shortmid}\mathbf{e}_{\alpha_s}, {}^{\shortmid}\mathbf{e}_{\beta_s}) = {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} = {}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\alpha_s} \in \mathcal{A}_s^{\star} \\ {}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s} &= {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} - \kappa \mathcal{R}^{\tau_s\xi_s}{}_{\alpha_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\tau_s} = {}_{\star}^{\shortmid}\mathfrak{g}^{[0]}_{\alpha_s\beta_s} + {}_{\star}^{\shortmid}\mathfrak{g}^{[1]}_{\alpha_s\beta_s}(\kappa) = {}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s} + {}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}, \end{aligned} \quad (20)$$

where $\mathcal{R}^{\tau_s\xi_s}{}_{\alpha_s}$ are related to $R^{i_s j_s a_s}$ from (19) via certain frame transforms and multiplications on some real/complex coefficients. In (20), we consider that ${}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s}$ is the symmetric part and ${}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}$ is the anti-symmetric part computed,

$$\begin{aligned} {}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s} := & \frac{1}{2}({}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s} + {}_{\star}^{\shortmid}\mathfrak{g}_{\beta_s\alpha_s}) = {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} - \frac{\kappa}{2}\left(\mathcal{R}^{\tau_s\xi_s}{}_{\beta_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\tau_s\alpha_s} + \mathcal{R}^{\tau_s\xi_s}{}_{\alpha_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\tau_s}\right) \quad (21) \\ = & \ {}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s} + {}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[1]}_{\alpha_s\beta_s}(\kappa), \\ & \text{for } {}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s} = {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} \text{ and } {}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[1]}_{\alpha_s\beta_s}(\kappa) = -\frac{\kappa}{2}\left(\mathcal{R}^{\tau_s\xi_s}{}_{\beta_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\tau_s\alpha_s} + \mathcal{R}^{\tau_s\xi_s}{}_{\alpha_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\tau_s}\right); \\ {}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s} := & \frac{1}{2}({}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s} - {}_{\star}^{\shortmid}\mathfrak{g}_{\beta_s\alpha_s}) = \frac{\kappa}{2}\left(\mathcal{R}^{\tau_s\xi_s}{}_{\beta_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\tau_s\alpha_s} - \mathcal{R}^{\tau_s\xi_s}{}_{\alpha_s} \ {}^{\shortmid}\mathbf{e}_{\xi_s} \ {}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\tau_s}\right) \\ = & \ {}_{\star}^{\shortmid}\mathfrak{a}^{[1]}_{\alpha_s\beta_s}(\kappa) = \frac{1}{2}({}_{\star}^{\shortmid}\mathfrak{g}^{[1]}_{\alpha_s\beta_s}(\kappa) - {}_{\star}^{\shortmid}\mathfrak{g}^{[1]}_{\beta_s\alpha_s}(\kappa)). \quad (22) \end{aligned}$$

Respective nonsymmetric inverse s-metrics can be parameterized in the form ${}_{\star}^{\shortmid}\mathfrak{g}^{\alpha_s\beta_s} = {}_{\star}^{\shortmid}\check{\mathfrak{g}}^{\alpha_s\beta_s} + {}_{\star}^{\shortmid}\mathfrak{a}^{\alpha_s\beta_s}$, when ${}_{\star}^{\shortmid}\check{\mathfrak{g}}^{\alpha_s\beta_s}$ is not the inverse to ${}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s}$ and ${}_{\star}^{\shortmid}\mathfrak{a}^{\alpha_s\beta_s}$ is not inverse to ${}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}$. We emphasize that to compute inverse metrics and s-metrics, define s-adapted geometric objects using commutators and anti-commutator, and contractions with s-tensors and s-metrics on ${}_s^{\star}\mathcal{M}$ for such nonassociative geometric models, we have to apply more sophisticate procedures, see details in [39, 12, 13]. For modelling geometric flow evoluton of symmetric and nonsymmetric components of star product deformed metrics, we have to consider respective families of s-objects and their s-adapted components, for instance, ${}_{\star s}^{\shortmid}\mathbf{g}(\tau)$, ${}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\alpha_s}(\tau)$, ${}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau) = {}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s}(\tau) + {}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}(\tau)$ etc.

Nonassociative star deformations $\star_s$ of respective LC- and canonical s-connections from (15), adapted to a nonlinear s–connection structures ${}_s^{\shortmid}\mathbf{N}$, also involve a canonical s-splitting for nonassociative LC-connection and canonical s-connection

$$ {}_s^{\shortmid}\widehat{\mathbf{D}} \to {}_s^{\shortmid}\widehat{\mathbf{D}}^{\star} = (h_1 \ {}^{\shortmid}\widehat{\mathbf{D}}^{\star}, \ v_2 \ {}^{\shortmid}\widehat{\mathbf{D}}^{\star}, \ c_3 \ {}^{\shortmid}\widehat{\mathbf{D}}^{\star}, \ c_4 \ {}^{\shortmid}\widehat{\mathbf{D}}^{\star}) = {}^{\shortmid}\nabla^{\star} + {}_{\star s}^{\shortmid}\widehat{\mathbf{Z}}, \quad (23)$$

where

$$
({}^{\shortmid}_{\star s}\mathbf{g}, {}^{\shortmid}_{s}\mathbf{N}) \rightarrow \left\{ \begin{array}{ll} {}^{\shortmid}_{\star}\nabla : & \boxed{ {}^{\shortmid}_{\star}\nabla \ {}^{\shortmid}_{\star s}\mathbf{g} = 0;\ {}^{\shortmid}_{\nabla}\mathcal{T}^{\star} = 0 } \text{ star LC-connection;} \\ {}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star} : & \boxed{ \begin{array}{c} {}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star} \ {}^{\shortmid}_{\star s}\mathbf{g} = 0;\ h_1 \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} = 0, v_2 \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} = 0, c_3 \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} = 0, c_4 \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} = 0, \\ h_1 v_2 \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} \neq 0, h_1 c_s \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} \neq 0, v_2 c_s \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} \neq 0, c_3 c_4 \ {}^{\shortmid}\widehat{\mathcal{T}}^{\star} \neq 0, \end{array} } \text{ canonical s-connection .} \end{array} \right. \quad (24)
$$

We note that in the definition of ${}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}$ we use the s-tensor ${}^{\shortmid}_{\star s}\mathbf{g}$. There are alternative possibilities, for instance, to involve directly a nonsymmetric metric ${}^{\shortmid}_{\star}\mathfrak{g}_{\alpha_s\beta_s}$, which makes the procedure of constructing parametric solutions more sophisticate than the variant with ${}^{\shortmid}_{\star s}\mathbf{g}$. Working only up to κ-linear terms, such canonical s-connections are equivalent for those configurations when ${}^{\shortmid}_{\star}\mathfrak{g}_{\alpha_s\beta_s} = {}^{\shortmid}_{\star}\check{\mathfrak{g}}_{\alpha_s\beta_s} + {}^{\shortmid}_{\star}\mathfrak{a}_{\alpha_s\beta_s}$, with ${}^{\shortmid}_{\star}\check{\mathfrak{g}}_{\alpha_s\beta_s} = {}^{\shortmid}_{\star}\mathbf{g}_{\ \alpha_s\beta_s}$ and ${}^{\shortmid}_{\star}\mathfrak{a}_{\alpha_s\beta_s|\kappa\rightarrow 0} \rightarrow 0$, but there are non-vanishing terms for ${}^{\shortmid}_{\star}\mathfrak{a}_{\alpha_s\beta_s}(\kappa \neq 0)$. Such conditions can be always stated for certain commutative nonholonomic configurations on which the star product deformations are applied to keep such conditions. After certain classes of physically important κ-parametric solutions are constructed in explicitly form, we can consider general nonassociative frame and coordinate transforms. Families of nonassociative canonical s-connections ${}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}(\tau) = {}^{\shortmid}\nabla^{\star}(\tau) + {}^{\shortmid}_{\star s}\widehat{\mathbf{Z}}(\tau)$ have to be considered for elaborating nonassociative geometric flow models, when all formulas from (23) and (24) are re-defined with τ-parametric dependence.

Nonassociative LC-configurations can be extracted similarly to (18) if we impose additional zero s-torsion conditions,

$$
{}^{\shortmid}_{\star s}\widehat{\mathbf{Z}} = 0, \text{ which is equivalent to } {}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}_{\ |\ {}^{\shortmid}_{s}\widehat{\mathbf{T}}=0} = {}^{\shortmid}_{\star}\nabla. \quad (25)
$$

In general, all type of metrics on ${}^{\star}_{s}\mathcal{M}$, and related s-metrics ${}^{\shortmid}_{\star s}\mathbf{g}$, subjected/ or not to some conditions of type (25) contain certain nonzero anholonomy coefficients of frame structures. In such cases, respective symmetric and nonsymmetric s-metrics can be written in local coordinate forms as generic off-diagonal matrices. For τ-families, such conditions for extracting and flow evolution of LC-connections can be written ${}^{\shortmid}_{\star s}\widehat{\mathbf{Z}}(\tau) = 0$ and ${}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}(\tau) = {}^{\shortmid}_{\star}\nabla(\tau)$ for ${}^{\shortmid}_{s}\widehat{\mathbf{T}}(\tau) = 0$. Here we note that we do not obtain equalities of some linear connections (by definition, two different linear connections have different transformation laws under frame/coordinate transforms) but certain equalities of coefficients in certain s-adapted fromes.

To define and compute geometric and physical objects on a familiy nonassociative phase spaces ${}^{\star}_{s}\mathcal{M}(\tau)$ defined by star product R-flux deformations, we follow:

Convention 2 (see details in [12, 13, 40, 41]; we can consider that in those works all definitions and formulas were stated for a fixed value τ_0 and in this work all results are extended for arbitrary τ): The commutative and nonassociative geometric data derived for a star product $\star_s$ (19), are related by such s-adapted transforms:

$$
\begin{array}{ccc} \boxed{(\star_N(\tau),\ \mathcal{A}^{\star}_N(\tau),\ {}^{\shortmid}_{\star}\mathbf{g}(\tau),\ {}^{\shortmid}_{\star}\mathfrak{g}(\tau),\ {}^{\shortmid}\mathbf{N}(\tau),\ {}^{\shortmid}\mathbf{e}_{\alpha}(\tau),\ {}^{\shortmid}\mathbf{D}^{\star}(\tau))} & \Leftrightarrow & \boxed{(\star_s(\tau),\ \mathcal{A}^{\star}_s(\tau),\ {}^{\shortmid}_{\star s}\mathbf{g}(\tau),\ {}^{\shortmid}_{\star s}\mathfrak{g}(\tau),\ {}^{\shortmid}_{s}\mathbf{N}(\tau),\ {}^{\shortmid}\mathbf{e}_{\alpha_s}(\tau),\ {}^{\shortmid}_{s}\mathbf{D}^{\star}(\tau))} \\ & \Uparrow & \\ \boxed{(\ {}^{\shortmid}\mathbf{g}(\tau),\ {}^{\shortmid}\mathbf{N}(\tau),\ {}^{\shortmid}\mathbf{e}_{\alpha}(\tau),\ {}^{\shortmid}\widehat{\mathbf{D}}(\tau))} & \Leftrightarrow & \boxed{(\ {}^{\shortmid}_{s}\mathbf{g}(\tau),\ {}^{\shortmid}_{s}\mathbf{N}(\tau),\ {}^{\shortmid}\mathbf{e}_{\alpha_s}(\tau),\ {}^{\shortmid}_{s}\widehat{\mathbf{D}}(\tau)),} \end{array} \quad (26)
$$

for certain canonical distortions ${}^{\shortmid}\mathbf{D}^{\star}(\tau) = {}^{\shortmid}_{\star}\nabla(\tau) + {}^{\shortmid}\widehat{\mathbf{Z}}^{\star}(\tau)$, for respective nonholonomic splitting 4+4, and ${}^{\shortmid}_{s}\mathbf{D}^{\star}(\tau) = {}^{\shortmid}_{\star}\nabla(\tau) + {}^{\shortmid}_{s}\widehat{\mathbf{Z}}^{\star}(\tau)$, for corresponding nonholonomic s-splitting. For simplicity, hereafter we shall not write τ-dependencies of geometric objects and structures if that will not result in ambiguities.

Applying the rule of Convention 2, we can define and compute star-product deformations of fundamental geometric s-objects (14),

$$
\begin{array}{lll} {}^{\shortmid}_{s}\mathcal{T} & \rightarrow & {}^{\shortmid}_{s}\widehat{\mathcal{T}}^{\star} = \{ {}^{\shortmid}\widehat{\mathbf{T}}^{\alpha_s}_{\ \star\beta_s\gamma_s} \},\ \text{nonassociative canonical s-torsion ;} \\ {}^{\shortmid}_{s}\mathcal{R} & \rightarrow & {}^{\shortmid}_{s}\widehat{\mathcal{R}}^{\star} = \{ {}^{\shortmid}\widehat{\mathbf{R}}^{\star\alpha_s}_{\ \ \beta_s\gamma_s\delta_s} \}, \text{nonassociative canonical Riemannian s-curvature ;} \\ {}^{\shortmid}_{s}\mathcal{R}ic & \rightarrow & {}^{\shortmid}_{s}\widehat{\mathcal{R}}ic^{\star} = \{ {}^{\shortmid}\widehat{\mathbf{R}}^{\star}_{\ \beta_s\gamma_s} := {}^{\shortmid}\widehat{\mathbf{R}}^{\star\alpha_s}_{\ \ \beta_s\gamma_s\alpha_s} \neq {}^{\shortmid}\widehat{\mathbf{R}}^{\star}_{\ \gamma_s\beta_s} \},\ \text{nonassociative canonical Ricci s-tensor;} \\ {}^{\shortmid}_{s}\mathcal{R}sc & \rightarrow & {}^{\shortmid}_{s}\widehat{\mathcal{R}}sc^{\star} = \{ {}^{\shortmid}\mathbf{g}^{\beta_s\gamma_s}\ {}^{\shortmid}\widehat{\mathbf{R}}^{\star}_{\ \beta_s\gamma_s} \},\ \text{nonassociative canonical Riemannian scalar ;} \\ {}^{\shortmid}_{s}\mathcal{Q} & \rightarrow & {}^{\shortmid}_{s}\mathcal{Q}^{\star} = \{ {}^{\shortmid}\widehat{\mathbf{Q}}^{\star}_{\gamma_s\alpha_s\beta_s} = {}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\gamma_s}\ {}^{\shortmid}_{\star}\mathbf{g}_{\alpha_s\beta_s} \},\ \text{zero nonassociative canonical nonmetricity s-tensor .} \end{array} \quad (27)
$$

The nonassociative canonical Riemann s-tensor $\,^{\shortmid}\widehat{\Re}^{\star} = \{ \,^{\shortmid}\widehat{\Re}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}\}$ from (27) can be defined and computed for the data $(\,^{\shortmid\shortmid}_{\star s}\mathfrak{g} = \{ \,^{\shortmid\shortmid}_{\star}\breve{\mathfrak{g}}_{\alpha_s\beta_s} = \,^{\shortmid}_{\star}\mathbf{g}_{\alpha_s\beta_s}\}, \,^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star} = \{ \,^{\shortmid}\widehat{\mathbf{\Gamma}}^{\gamma_s}_{\star\alpha_s\beta_s}\})$, see details in [13, 41],

$$
\begin{aligned}
\,^{\shortmid}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s} &= \,^{\shortmid}_{\mathbf{1}}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s} + \,^{\shortmid}_{\mathbf{2}}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}, \text{ where} \\
\,^{\shortmid}_{\mathbf{1}}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s} &= \,^{\shortmid}\mathbf{e}_{\gamma_s}\,^{\shortmid}\widehat{\Gamma}^{\mu_s}_{\star\alpha_s\beta_s} - \,^{\shortmid}\mathbf{e}_{\beta_s}\,^{\shortmid}\widehat{\Gamma}^{\mu}_{\star\alpha_s\gamma_s} + \,^{\shortmid}\widehat{\Gamma}^{\mu_s}_{\star\nu_s\tau_s}\star_s(\delta^{\tau_s}_{\gamma_s}\,^{\shortmid}\widehat{\Gamma}^{\nu_s}_{\star\alpha_s\beta_s} - \delta^{\tau_s}_{\beta_s}\,^{\shortmid}\widehat{\Gamma}^{\nu_s}_{\star\alpha_s\gamma_s}) + \,^{\shortmid}w^{\tau_s}_{\beta_s\gamma_s}\star_s\,^{\shortmid}\widehat{\Gamma}^{\mu_s}_{\star\alpha_s\tau_s}, \\
\,^{\shortmid}_{2}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s} &= i\kappa\,^{\shortmid}\widehat{\Gamma}^{\mu_s}_{\star\nu_s\tau_s}\star_s(\mathcal{R}^{\tau_s\xi_s}{}_{\gamma_s}\,^{\shortmid}\mathbf{e}_{\xi_s}\,^{\shortmid}\widehat{\Gamma}^{\nu_s}_{\star\alpha_s\beta_s} - \mathcal{R}^{\tau_s\xi_s}{}_{\beta_s}\,^{\shortmid}\mathbf{e}_{\xi_s}\,^{\shortmid}\widehat{\Gamma}^{\nu_s}_{\star\alpha_s\gamma_s}).
\end{aligned} \tag{28}
$$

Using parametric decompositions of the star canonical s-connection in (28),

$$
\,^{\shortmid}\widehat{\mathbf{\Gamma}}^{\gamma_s}_{\star\alpha_s\beta_s} = \,^{\shortmid}_{[0]}\widehat{\mathbf{\Gamma}}^{\nu_s}_{\star\alpha_s\beta_s} + i\kappa\,^{\shortmid}_{[1]}\widehat{\mathbf{\Gamma}}^{\nu_s}_{\star\alpha_s\beta_s} = \,^{\shortmid}_{[00]}\widehat{\Gamma}^{\nu_s}_{*\alpha_s\beta_s} + \,^{\shortmid}_{[01]}\widehat{\Gamma}^{\nu_s}_{*\alpha_s\beta_s}(\hbar) + \,^{\shortmid}_{[10]}\widehat{\Gamma}^{\nu_s}_{*\alpha_s\beta_s}(\kappa) + \,^{\shortmid}_{[11]}\widehat{\Gamma}^{\nu_s}_{*\alpha_s\beta_s}(\hbar\kappa) + O(\hbar^2,\kappa^2...), \tag{29}
$$

we can compute such parametric decompositions of the nonassociative canonical curvature tensor,

$$
\,^{\shortmid}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s} = \,^{\shortmid}_{[00]}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s} + \,^{\shortmid}_{[01]}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}(\hbar) + \,^{\shortmid}_{[10]}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}(\kappa) + \,^{\shortmid}_{[11]}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}(\hbar\kappa) + O(\hbar^2,\kappa^2,...).
$$

Contracting the first and forth indices of (28), we define the nonassociative canonical Ricci s-tensor,

$$
\begin{aligned}
\,^{\shortmid}_{\mathbf{s}}\widehat{\Re}ic^{\star} &= \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}\star_s(\,^{\shortmid}\mathbf{e}^{\alpha_s}\otimes_{\star s}\,^{\shortmid}\mathbf{e}^{\beta_s}), \text{ where} \\
& \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} := \,^{\shortmid}_{s}\widehat{\Re}ic^{\star}(\,^{\shortmid}\mathbf{e}_{\alpha_s}, \,^{\shortmid}\mathbf{e}_{\beta_s}) = \langle \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\mu_s\nu_s}\star_s(\,^{\shortmid}\mathbf{e}^{\mu_s}\otimes_{\star_s}\,^{\shortmid}\mathbf{e}^{\nu_s}), \,^{\shortmid}\mathbf{e}_{\alpha_s}\otimes_{\star\mathbf{s}}\,^{\shortmid}\mathbf{e}_{\beta_s}\rangle_{\star_s},
\end{aligned}
$$

$$
\begin{aligned}
\text{and } \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} := \quad & \,^{\shortmid}\widehat{\Re}^{\star\mu_s}{}_{\alpha_s\beta_s\mu_s} = \,^{\shortmid}_{[00]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} + \,^{\shortmid}_{[01]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}(\hbar) + \,^{\shortmid}_{[10]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}(\kappa) \\
& + \,^{\shortmid}_{[11]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}(\hbar\kappa) + O(\hbar^2,\kappa^2,...), \text{where} \\
& \,^{\shortmid}_{[00]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} = \,^{\shortmid}_{[00]}\widehat{\Re}^{\star\mu_s}{}_{\alpha_s\beta_s\mu_s}, \,^{\shortmid}_{[01]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} = \,^{\shortmid}_{[01]}\widehat{\Re}^{\star\mu_s}{}_{\alpha_s\beta_s\mu_s}, \\
& \,^{\shortmid}_{[10]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} = \,^{\shortmid}_{[10]}\widehat{\Re}^{\star\mu_s}{}_{\alpha_s\beta_s\mu_s}, \,^{\shortmid}_{[11]}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} = \,^{\shortmid}_{[11]}\widehat{\Re}^{\star\mu_s}{}_{\alpha_s\beta_s\mu_s}.
\end{aligned} \tag{30}
$$

Because of nonholonomic structure, canonical Ricci s-tensors are not symmetric for general (non) commutative and nonassociative cases.

Further h1-v2-c3-c4 decompositions in abstract and coefficient s-adapted forms are also possible for formulas (28) and (30) (we omit such details in this paper).

Contracting the indices of (30) with the inverse nonassociative and nonsymmetric s-metric $\,^{\shortmid}_{\star}\mathfrak{g}^{\mu_s\nu_s}$, we define and compute the nonassociative nonholonomic canonical Ricci scalar curvature:

$$
\begin{aligned}
\,^{\shortmid}_{s}\widehat{\mathbf{R}}sc^{\star} := \quad & \,^{\shortmid}_{\star}\mathfrak{g}^{\mu_s\nu_s}\,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\mu_s\nu_s} = (\,^{\shortmid}_{\star}\breve{\mathfrak{g}}^{\mu_s\nu_s} + \,^{\shortmid}_{\star}\mathfrak{a}^{\mu_s\nu_s})\left(\,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{(\mu_s\nu_s)} + \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{[\mu_s\nu_s]}\right) = \,^{\shortmid}_{s}\widehat{\mathbf{R}}ss^{\star} + \,^{\shortmid}_{s}\widehat{\mathbf{R}}sa^{\star}, \\
& \text{where } \,^{\shortmid}_{s}\widehat{\mathbf{R}}ss^{\star} =: \,^{\shortmid}_{\star}\breve{\mathfrak{g}}^{\mu_s\nu_s}\,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{(\mu_s\nu_s)} \text{ and } \,^{\shortmid}_{s}\widehat{\mathbf{R}}sa^{\star} := \,^{\shortmid}_{\star}\mathfrak{a}^{\mu_s\nu_s}\,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{[\mu_s\nu_s]}.
\end{aligned} \tag{31}
$$

In (31), the respective symmetric (...) and anti-symmetric [...] operators are defined using the multiple 1/2 when, for instance, $\,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\mu_s\nu_s} = \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{(\mu_s\nu_s)} + \,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{[\mu_s\nu_s]}$.

The nonassociative phase space vacuum Einstein equations with a nontrivial at least one shell cosmological constant ($\,^{\shortmid}_{s}\lambda \neq 0$ for any s, or some shells) can be defined and computed for the canonical s-connection $\,^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}$,

$$
\,^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s} - \frac{1}{2}\,^{\shortmid}_{\star}\mathfrak{g}_{\alpha_s\beta_s}\,^{\shortmid}_{s}\widehat{\mathbf{R}}sc^{\star} = \,^{\shortmid}_{s}\lambda\,^{\shortmid}_{\star}\mathfrak{g}_{\alpha_s\beta_s}, \tag{32}
$$

where the nonassociative Ricci s-tensor and scalar curvature are defined respectively by formulas (30) and (31).

2.1.3 Parametric decomposition of nonassociative and vacuum gravitational equations

The procedure of parametric decompositions of geometric s-objects ${}^{\shortmid}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}$ (28), ${}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}$ (30) and ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}$ (31) with $[01,10,11]:=\lceil\hbar,\kappa\rceil$ components, in parametric form of the canonical s-connections (29) is elaborated in [12, 13]. Such constructions extend the formalism for the LC-connections provided originally in [39]. In both cases, the nonassociative and noncommutative of the Riemann and Ricci tensors contains contributions from star product deformations which can be real or complex ones. In our approach, we can consider nonholonomic distributions on phase space when the (almost) complex structures are separated and for the real parts ${}_{[00]}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}= {}^{\shortmid}\widehat{\mathbf{R}}\ {}_{\alpha_s\beta_s}$ and such coefficients are determined by an associative and commutative s-adapted canonical s-connection ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$ (15). As a result, the star s-deformed Ricci s-tensor (30) can be expressed in parametric form,

$$
{}_{s}^{\shortmid}\widehat{\mathcal{R}}ic^{\star}=\{{}^{\shortmid}\widehat{\mathbf{R}}^{\star}_{\beta_s\gamma_s}\}={}_{s}^{\shortmid}\widehat{\mathcal{R}}ic+{}_{s}^{\shortmid}\widehat{\mathcal{K}}ic\lceil\hbar,\kappa\rceil=\{{}^{\shortmid}\widehat{\mathbf{R}}\ {}_{\beta_s\gamma_s}+{}^{\shortmid}\widehat{\mathbf{K}}\ {}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil\}, \qquad (33)
$$

there the distortion tensor

$$
{}_{s}^{\shortmid}\widehat{\mathcal{K}}ic=\{{}^{\shortmid}\widehat{\mathbf{K}}\ {}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil={}_{[01]}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\beta_s\gamma_s}+{}_{[10]}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\beta_s\gamma_s}+{}_{[11]}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\beta_s\gamma_s}\}
$$

encodes nonassociative parametric deformations of the canonical Ricci s-tenor.

We can adapt the nonholonomic s-structure that the nonassociative canonical Ricci scalar is conventionally with a sum of some effective shell polarized cosmological constants ${}^{s}\Lambda({}_{s}^{\shortmid}u)$ depending respectively on shell coordinates,

$$
{}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}={}^{1}\Lambda({}_{1}^{\shortmid}u)+{}^{2}\Lambda({}_{2}^{\shortmid}u)+{}^{3}\Lambda({}_{3}^{\shortmid}u)+{}^{4}\Lambda({}_{4}^{\shortmid}u).
$$

Choosing effective ${}^{s}\Lambda({}_{s}^{\shortmid}u)$ and fixing, for simplicity, ${}_{s}^{\shortmid}\lambda={}^{\shortmid}\lambda$ in a form that ${}^{\shortmid}\lambda+\frac{1}{2}\,{}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}\,{}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}=0$, when the nonsymmetric metric (20) decouple into two independent symmetric and antisymmetric computed respectively by formulas ${}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s}$ (21) and ${}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}$ (22),

$$
{}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}={}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s}+{}_{\star}^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}={}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s}+{}_{\star}^{\shortmid}\mathfrak{a}^{[1]}_{\alpha_s\beta_s},
$$

and determined in explicit form respectively by ${}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s}={}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}$ and ${}_{\star}^{\shortmid}\mathfrak{a}^{[1]}_{\alpha_s\beta_s}=i\kappa\mathcal{R}^{\tau_s\xi_s}{}_{[\alpha_s}\,{}^{\shortmid}\mathbf{e}_{|\xi_s}\,{}^{\shortmid}\mathbf{g}_{\tau_s|\beta_s]}$, where $|\xi_s\tau_s|$ means that such indices are not involved in anti-symmetrization.

Using formulas (33), we express the nonassociative vacuum gravitational field equations (32) in the form ${}^{\shortmid}\widehat{\mathbf{R}}ic_{\alpha_s\beta_s}={}^{\shortmid}\Upsilon_{\alpha_s\beta_s}$ (16), where ${}^{\shortmid}\Upsilon_{\alpha_s\beta_s}=-{}_{s}^{\shortmid}\widehat{\mathcal{K}}ic_{\alpha_s\beta_s}$. Such an effective source of type (17) encodes nonassociative star R-flux deformations,

$$
\begin{aligned}
{}^{\shortmid}\mathbf{K}_{\beta_s\gamma_s} &= {}_{[0]}^{\shortmid}\Upsilon_{\beta_s\gamma_s}+{}_{[1]}^{\shortmid}\mathbf{K}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil,\ \text{where} \qquad (34)\\
& {}_{[0]}^{\shortmid}\Upsilon_{\beta_s\gamma_s}={}^{s}\Lambda({}^{\shortmid}u^{\gamma_s})\,{}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\gamma_s}\ \text{and}\ {}_{[1]}^{\shortmid}\mathbf{K}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil={}^{s}\Lambda({}^{\shortmid}u^{\gamma_s})\,{}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[1]}_{\beta_s\gamma_s}(\kappa)-{}^{\shortmid}\widehat{\mathbf{K}}\ {}_{\beta_s\gamma_s}\lceil\hbar,\kappa\rceil,
\end{aligned}
$$

is an effective parametric source with coefficients proportional to $\hbar,\kappa$ and $\hbar\kappa$. For computing R-flux κ-linear effects, it is enough to consider nonholonomic distributions and effective sources generated by data

$$
\begin{aligned}
{}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s} &= {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}={}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s};\ {}_{\star}^{\shortmid}\check{\mathfrak{g}}^{[1]}_{\beta_s\gamma_s}(\kappa)=0,\ \ {}_{\star}^{\shortmid}\mathfrak{a}^{[0]}_{\alpha_s\beta_s}=0,\ {}_{\star}^{\shortmid}\mathfrak{a}^{[1]}_{\alpha_s\beta_s}=i\kappa\mathcal{R}^{\tau_s\xi_s}{}_{[\alpha_s}\,{}^{\shortmid}\mathbf{e}_{|\xi_s}\,{}^{\shortmid}\mathbf{g}_{\tau_s|\beta_s]},\\
\text{and}\ {}^{\shortmid}\Upsilon_{\alpha_s\beta_s} &= -{}_{s}^{\shortmid}\widehat{\mathcal{K}}ic_{\alpha_s\beta_s}=-{}^{\shortmid}\widehat{\mathbf{K}}_{\alpha_s\beta_s}.
\end{aligned}
$$

The effective sources can be parameterized for nontrivial real quasi-stationary 8-d configurations[13] of s-metrics using coordinates $(x^{k_3},\ {}^{\shortmid}p_8=E)$, with ${}_{\star}^{\shortmid}\mathbf{g}_{\beta_s\gamma_s|\hbar,\kappa=0}={}^{\shortmid}\mathbf{g}_{\beta_s\gamma_s}$, in various forms depending on prescribed

[13] A s-metric is quasi-stationary if the corresponding (non) associative phase spacetime geometric s-objects possess a Killing symmetry on $\partial_4=\partial_t$ on shell $s=2$ and on ${}^{\shortmid\shortmid}\partial_7$, or ${}^{\shortmid\shortmid}\partial_8$, for all shells up to $s=4$.

shell Killing symmetries. Nonassociative effects are determined additionally as some induced nonsymmetric components $\ _{\star}^{\shortmid}\mathfrak{a}^{[1]}_{\alpha_s\beta_s}$.

In this work, we consider such quasi-stationary shell by shell adapted distributions on $\ _{s}^{\star}\mathcal{M}$ when

$$ ^{\shortmid}\mathbf{K}^{\alpha_s}_{\ \beta_s} = [\ _{1}^{\shortmid}\mathcal{K}(\kappa, x^{k_1})\delta^{j_1}_{i_1},\ _{2}^{\shortmid}\mathcal{K}(\kappa, x^{k_1}, x^3)\delta^{a_2}_{b_2},\ _{3}^{\shortmid}\mathcal{K}(\kappa, x^{k_2},\ ^{\shortmid}p_6)\delta^{b_3}_{a_3},\ _{4}^{\shortmid}\mathcal{K}(\kappa, x^{k_3},\ ^{\shortmid}p_8)\delta^{b_4}_{a_4}] \quad (35) $$

contain as functionals certain κ-linear terms with $\mathcal{R}^{\tau_s\xi_s}_{\quad\ \alpha_s}$. Prescribing certain values for effective sources $\ _{s}^{\shortmid}\mathcal{K}$ (35) as **generating sources**, we constrain nonholonomically the gravitational dynamics and effective and possible matter sources. Such generating sources can be related to conventional cosmological constants via nonlinear symmetries, when the nonsymmetric parts of the s-metrics and the canonical Ricci s-tensors can be computed as R-flux deformations of some off-diagonal symmetric metric configurations. Finally, we note that using necessary types of frame s-adapted transform, $^{\shortmid}\widehat{\Upsilon}_{\alpha'_s\beta'_s} = e^{\alpha_s}_{\ \alpha'_s} e^{\beta_s}_{\ \beta'_s}\ ^{\shortmid}\mathcal{K}_{\alpha_s\beta_s}$ we can transform certain general sources into a subset of four generating soucres $^{\shortmid}\mathcal{K}_{\beta_s\gamma_s} = \{\ _{s}^{\shortmid}\mathcal{K}\}$. We use the label "$^{\shortmid}$" for such sources in order to emphasize that they are determined by generating sources encoding in certain general nonholonomic forms certain noncommutative data for star-product and R-flux deformations.

2.2 Nonassociative generalizations of Perelman's F- and W-functionals

We consider families of nonassociative R-flux deformed phase spaces, $_{s}\mathcal{M} \rightarrow\ _{s}^{\star}\mathcal{M}(\tau)$ determined by star product $\star_s(\tau)$ structure (19) adapted to a nonholonomic (2+2)+(2+2) decomposition (i.e. s-structure) of a cotangent Lorentz bundle $\mathcal{M} = T^*V, \dim V = 4$, as in [12, 13, 41]. Star product R-flux deformations of fundamental geometry s-objects (27), determined by nonassociative geometric data $\left[\ _{s}^{\shortmid}\mathfrak{g}^{\star}(\tau),\ _{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau)\right]$, are performed following Convention 2 (26) with κ-linear parametric decompositions when $\ _{\star}^{\shortmid}\check{\mathfrak{g}}^{[0]}_{\alpha_s\beta_s}(\tau) =\ _{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau) =\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)$. Geometric flows on a parameter τ are described in [0]-approximation (zero power on κ) by flows of some canonical data $(\ _{s}^{\shortmid}\mathbf{g}(\tau),\ _{s}^{\shortmid}\widehat{\mathbf{D}}(\tau))$, star product flows $\star_s(\tau)$, determined by s-adapted frames $^{\shortmid}\mathbf{e}_{i_s}(\tau)$ in (19), and flows of volume elements

$$ d\ ^{\shortmid}\mathcal{V}ol(\tau) = \sqrt{|\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\ (\tau)|}\ \delta^8\ ^{\shortmid}u^{\gamma_s}(\tau) \quad (36) $$

are computed for N-elongated s-differentials $\delta^8\ ^{\shortmid}u^{\gamma_s}(\tau)$ using $^{\shortmid}N_{\ i_s a_s}\ (\tau)$ as in (12).

We can elaborate on relativistic thermodynamics models and their nonassociative generalizations if $\ _{s}^{\shortmid}\mathbf{g} = \{\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\}$ is adapted to a causal (3+1)+(3+1) splitting (in GR, such a formalism for Einstein manifolds is considered, for instance, in [43]). Geometric flows of a s-metric can be parameterized in the form

$$ \begin{aligned} \ _{s}^{\shortmid}\mathbf{g}(\tau) &= \ ^{\shortmid}\mathbf{g}_{\alpha'\beta'}(\tau,\ _{s}^{\shortmid}u) d\ ^{\shortmid}\mathbf{e}^{\alpha'}(\tau) \otimes d\ ^{\shortmid}\mathbf{e}^{\beta'}(\tau) \\ &= q_i(\tau, x^k) dx^i \otimes dx^i + q_3(\tau, x^k, y^3)\mathbf{e}^3(\tau) \otimes \mathbf{e}^3(\tau) - \breve{N}^2(\tau, x^k, y^3)\mathbf{e}^4(\tau) \otimes \mathbf{e}^4(\tau) \\ &\quad + \ ^{\shortmid}q^{a_2}(\tau, x^k, y^3, p_{b_2})\ ^{\shortmid}\mathbf{e}_{a_2}(\tau) \otimes\ ^{\shortmid}\mathbf{e}_{a_2}(\tau) \\ &\quad + \ ^{\shortmid}q^7(\tau, x^k, y^3, p_{b_2}, p_{b_3})\ ^{\shortmid}\mathbf{e}_7(\tau) \otimes\ ^{\shortmid}\mathbf{e}_7(\tau) -\ ^{\shortmid}\breve{N}^2(\tau, x^k, y^3, p_{b_2}, p_{b_3})\ ^{\shortmid}\mathbf{e}_8(\tau) \otimes\ ^{\shortmid}\mathbf{e}_8(\tau). \end{aligned} $$

For computations in a phase space point, such an ansatz is written as an extension of a couple of 3–d metrics, $q_{ij} = diag(q_{\grave{i}}) = (q_i, q_3)$ on a hyper-surface $\widehat{\Xi}_t$ and $^{\shortmid}q^{\grave{a}\grave{b}} = diag(\ ^{\shortmid}q^{\grave{a}}) = (\ ^{\shortmid}q^{a_2},\ ^{\shortmid}q^7)$ on a hyper-surface $^{\shortmid}\widehat{\Xi}_E$, i.e. on $\ _{s}^{\shortmid}\widehat{\Xi} = \left(\widehat{\Xi}_t,\ ^{\shortmid}\widehat{\Xi}_E\right)$, when

$$ q_1 = g_1, q_2 = g_2, q_3 = g_3, \breve{N}^2 = -g_4 \text{ and }\ ^{\shortmid}q^5 =\ ^{\shortmid}g^5,\ ^{\shortmid}q^6 =\ ^{\shortmid}g^6,\ ^{\shortmid}q^7 =\ ^{\shortmid}g^7,\ ^{\shortmid}\breve{N}^2 = -\ ^{\shortmid}g^8, \quad (37) $$

where $\breve{N}$ is a lapse function on the base manifold and $^{\shortmid}\breve{N}^2$ is a lapse function in the cofiber.

The Perelman type functionals (9) can be generalized for nonassociative canonical data $\left[{}_{s}^{\shortmid}\mathbf{g}^{\star}(\tau),\ {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau)\right]$ following the Convention 2 (26 and using formulas (14),

$$
\begin{aligned}
{}_{s}^{\shortmid}\widehat{\mathcal{F}}^{\star}(\tau) &= \int_{{}_{s}^{\shortmid}\widehat{\Xi}} ({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star} + | {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}\ {}_{s}^{\shortmid}\widehat{f}|^{2}) \star e^{-\ {}_{s}^{\shortmid}\widehat{f}}\ d\ {}^{\shortmid}\mathcal{V}ol(\tau), \text{ and} \quad (38)\\
{}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}(\tau) &= \int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}\ [\tau({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star} + \sum\nolimits_{s} | {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star} \star\ {}_{s}^{\shortmid}\widehat{f}|)^{2} + {}_{s}^{\shortmid}\widehat{f} - 8] \star e^{-\ {}_{s}^{\shortmid}\widehat{f}}\ d\ {}^{\shortmid}\mathcal{V}ol(\tau), \quad (39)
\end{aligned}
$$

where the integrals and normalizing functions ${}_{s}^{\shortmid}\widehat{f}(\tau,\ {}_{s}^{\shortmid}u)$ are stated to satisfy the condition

$$
\int_{{}_{s}^{\shortmid}\widehat{\Xi}} {}_{s}^{\shortmid}\widehat{\nu}\ \ d\ {}^{\shortmid}\mathcal{V}ol(\tau) := \int_{t_1}^{t_2} \int_{\widehat{\Xi}_t} \int_{{}^{\shortmid}\widehat{\Xi}_E} {}_{s}^{\shortmid}\widehat{\nu}\ \ d\ {}^{\shortmid}\mathcal{V}ol(\tau) = 1, \quad (40)
$$

for integration measures ${}_{s}^{\shortmid}\widehat{\nu} = (4\pi\tau)^{-4}\, e^{-\ {}_{s}^{\shortmid}\widehat{f}}$ parameterized for shells on 8-d phase spaces. The nonassociative canonical s-connection ${}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}$ (24) and respective canonical Ricci scalar ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}$ (31) are computed for a parametrization (37).[14] We can consider star-deformations of the volume form (36),

$$
e^{-\ {}_{s}^{\shortmid}\widehat{f}} d\ {}^{\shortmid}\mathcal{V}ol(\tau) \to e^{-\ {}_{s}^{\shortmid}\widehat{f}} d\ {}^{\shortmid}\mathcal{V}ol^{\star}(\tau) = e^{-\ {}_{s}^{\shortmid}\widehat{f}} \sqrt{|\ {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}\ (\tau)|}\delta\ {}^{\shortmid}u^{\gamma_s}(\tau),
$$

for other types of adapted integration measures and nonholonomic s-shells with ${}_{s}^{\shortmid}\mathfrak{g}^{\star}$. Such transforms can be encoded into a normalizing function ${}_{s}^{\shortmid}\widehat{f}$ and respective separation of nonsymmetric components of s-metrics for κ–linear parameterizations. We simplify further computations if the star products and integration measures, and the orders for performing covariant derivations and integration, are stated as in functionals (38) and (39).

2.3 Nonassociative geometric flow and Ricci soliton equations

In this section, we consider two methods for deriving nonassociative geometric flow equations (using abstract geometric methods and/or elaborating a nonassociative onholonomic s-adapted variational procedure).

2.3.1 Abstract nonassociative geometric star-deformations of R. Hamilton equations

Following abstract geometric principles as in [43], we can derive necessary type geometric/physical important questions considering in symbolic coordinate/frame forms corresponding fundamental associative geometric objects. We can consider any variant of Ricci tensor and scalar curvature defined by respective metric and covariant derivative structures (and, if it is important for certain constructions, nonlinear Laplace, d'Alambert operators). This way, for instance, we can derive the Einstein equations in pure geometric form as in pseudo-Riemannian geometry. Such gravitational filed equations can be written in terms of the Ricci tensor (the left side) and postulating (for the right side) certain types of (effective) sources determined by corresponding physically important energy-momentum tensors. This geometric approach, can be generalized for the canonical s-connection structure ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$ (15) which results in modified phase space Einstein equations (16). Written in "hat" variables, such nonholonomically distorted gravitational field equations can be decoupled and integrated

[14] It should be noted that the F- and W-functionals were postulated by G. Perelman [1] in such forms that they allowed him to perform a variational calculus and prove certain forms of the R. Hamilton equations [4], to define an associated statistical/geometric thermodynamics models for Ricci flows of Riemannian metrics, and to prove the Poincaré-Thurston conjecture. Those constructions can be generalized on nonholonomic Lorentz manifolds and (co) tangent bundles which allow to prove relativistic variants of geometric flows and elaborated on respective thermodynamics models even (as we emphasized in the Introduction and [6, 7, 8]) some relativistic variants of the Poincaré-Thurston conjecture were not formulated/proven in modern mathematics. The Perelman functionals and respective thermodynamic models can be generalized for various non-Riemannian geometries including nonassociative models if the τ-parametric star product (19) and Convention 2 are considered. This allows us to elaborate on theories of nonassociative geometric flows defined by R-flux deformations in string theory.

in certain general forms applying the AFCDM. In terms of the LC-connection $\,^{\shortmid}\nabla$ such systems of nonlinear PDEs do not possess any general decoupling properties for generic off-diagonal metrics depending, in principle, on all phase space coordinates.

The abstract geometric approach allows us to derive in symbolic form certain (associative and commutative) nonholonomic geometric flow equations [14, 17, 7, 8, 6] for τ-families of geometric s-objects $\left(\,_{s}^{\shortmid}\mathbf{g}(\tau), \,_{s}^{\shortmid}\widehat{\mathbf{D}}(\tau)\right)$. For s-metrics $\,_{s}^{\shortmid}\mathbf{g}(\tau) = \{ \,^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)\}$ (5), we can construct nonholonomic canonical s-deformations of the the R. Hamilton equations [4] postulated for various research in modern geometric analysis. Such equations were originally considered in connection to string theory and condensed matter physics in [5]. Then, applying the Convention 2 (26) we can analyze and solve the issue on deriving nonassociative geometric flow equations for the star deformed data $\left(\,_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau), \,_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau)\right)$. Such equations can be postulated (using appropriate diffeomorphysms and s-adapted frame structures) in the form

$$\begin{aligned} \partial_\tau \,_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau) &= -2 \,^{\shortmid}\widehat{\mathbf{R}}^{\star}{}_{\alpha_s\beta_s}(\tau), \\ \partial_\tau \,_{s}^{\shortmid}\widehat{f}(\tau) &= \,_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}(\tau) - \,^{\star}\widehat{\triangle}(\tau) \star \,_{s}^{\shortmid}\widehat{f}(\tau) + (\,_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau) \star \,_{s}^{\shortmid}\widehat{f}(\tau))^{2}(\tau), \end{aligned} \tag{41}$$

where $\,^{\star}\widehat{\triangle}$ is the Laplace operator constructed for $\,_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}$ and the nonsymmetric components of $\,_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}$ are computed using κ-linear parameterizations (20)–(22).

The nonassociative geometric flow equations (41) include as associative and commutative parts (for LC-configurations and Riemannian signatures) certain phase space variants of the evolution equations (1.3) studied in [1]. The noncommutative part of such equations is different from that considered in [15] because that work was devoted to a different type of noncommutative Ricci flow theory based on spectral triples following the A. Connes approach. We postulated above nonassociative system of nonlinear PDEs in such a form that it can be decoupled and solved in certain general forms at least in κ-linear form (see details below: in subsection 2.4, section 4 and appendix A).

2.3.2 A s-adapted variational procedure for deriving nonassociative geometric flow equations

In paragraphs 1.1 and 1.2 of Section 1 in [1], it is considered in brief a variational procedure to prove the associative and commutative variants of geometric flow equations (41) using Riemannian data $(g_{\alpha\beta}(\tau), \nabla_\gamma(\tau))$ for a normalizing function $f(\tau)$ on a closed manifold M of dimension, $\dim M = n$ and $\tau \in [0, \tau_1]$. In this work, we use our system of notations for nonholonomic manifolds and/or phase spaces, $\,_{s}^{\shortmid}\mathcal{M} \to \,_{s}^{\star}\mathcal{M}$, and perform canonical s-adapted geometric constructions on certain closed spacelike regions with respective double $(3+1)+(3+1)$ and $(2+2)+(2+2)$ fibrations extended under spacetime/ phase space paths covering such regions. We note that formal definitions of geometric s-objects and respective covariant and integral calculus do not depend on the signature of s-metrics. Such signatures are important for providing proofs of the Poincaré-Thurston conjecture and generalizations (which is not a goal for this work), see details in [1, 2, 3].

In nonholonomic form, certain N-adapted variational geometric flow methods were considered in [14] for various developments and applications in [14, 17, 7, 8, 6] when $\nabla \to \widehat{\mathbf{D}}$ and d-metrics of arbitrary signatures subjected to modified/ generalized R. Hamilton equations can be found by applying the AFCDM. To develop such a N-adapted variational calculus for nonholonomic Ricci flow theories in a rigorous mathematical form is possible (such proofs are on hundreds of pages, with respective distortions of d-connections and geometric d-objects, like in monographs [9, 10, 11]). We omit such technical details in this work and sketch the proof in a form similar to [1] but with respective geometric symbolic re-definitions of (non) associative nonholonomic s-objects following the Convention 2 (26), stating formal nonholonomic measures and performing integration on a closed region of $\,_{s}^{\star}\mathcal{M}$.

It should be noted that arbitrary deformations induced by a twist operator are not general compatible with the variational principle to be used directly, for instance, in nonassociative field theory. Nevertheless, this is not an unsolved conceptual/ technical problem if we work with star product R-flux deformations defined in s-adapted form as in (19). We can elaborate a well defined nonholonomic geometric flow theory (with a self-consistent s-adapted variational calculus) on $_s^{\shortmid}\mathcal{M}$. Then, we can $\star$-deform the constructions on $_s^{\star}\mathcal{M}$; and, for at least for κ-linear parametric solutions, compute respective deformations of F- and W-functionals and their nonassociative geometric flow equations (41). In such a case, a star-functional $_s^{\shortmid}\widehat{\mathcal{F}}^{\star}$ (38) is constructed for an explicit class of κ-linear solutions of nonassociative R. Hamilton equations in canonical s-variables, when the measure and volume forms can be assumed to satisfy the condition (40). A similar assumption is used for the proof of 1.2 Proposition and formulas (1.1) - (1.4) in [1].

So, considering $_s^{\shortmid}\widehat{\mathcal{F}}^{\star}$, we can define and compute a s-adapted variation $\delta\ _s^{\shortmid}\widehat{\mathcal{F}}^{\star}$ for some variations $\delta(\ _{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}) = \delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s})$ and $\delta\ _s^{\shortmid}\widehat{f}$ as follows:

$$\begin{aligned}
\delta\ _s^{\shortmid}\widehat{\mathcal{F}}^{\star}(\tau) &= \int_{_s^{\shortmid}\widehat{\Xi}}\{(-\ ^{\star}\widehat{\triangle}[\ ^{\shortmid}\mathbf{g}^{\alpha_s\beta_s}\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s})] + \ ^{\shortmid}\widehat{\mathbf{D}}^{\alpha_s}_{\star}\ ^{\shortmid}\widehat{\mathbf{D}}^{\beta_s}_{\star}[\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s})] - \ ^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s\beta_s}_{\star}\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}) \\
&\quad -\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s})\ ^{\shortmid}\widehat{\mathbf{D}}^{\alpha_s}_{\star}(\ _s^{\shortmid}\widehat{f})\ ^{\shortmid}\widehat{\mathbf{D}}^{\beta_s}_{\star}(\ _s^{\shortmid}\widehat{f}) + 2\ ^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\alpha_s}(\ _s^{\shortmid}\widehat{f})\ ^{\shortmid}\widehat{\mathbf{D}}^{\alpha_s}_{\star}(\delta\ \ _s^{\shortmid}\widehat{f}) \\
&\quad +(\ _s^{\shortmid}\widehat{\mathbf{R}}sc^{\star} + |\ _s^{\shortmid}\widehat{\mathbf{D}}^{\star}\ _s^{\shortmid}\widehat{f}|^2)(\frac{1}{2}\ ^{\shortmid}\mathbf{g}^{\alpha_s\beta_s}\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}) - \delta\ \ _s^{\shortmid}\widehat{f})\} \star e^{-\ _s^{\shortmid}\widehat{f}} \\
&= \int_{_s^{\shortmid}\widehat{\Xi}}\{-\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s})[\ ^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s\beta_s}_{\star} + \ ^{\shortmid}\widehat{\mathbf{D}}^{\alpha_s}_{\star}\ ^{\shortmid}\widehat{\mathbf{D}}^{\beta_s}_{\star}(\ _s^{\shortmid}\widehat{f})] \\
&\quad +(\frac{1}{2}\ ^{\shortmid}\mathbf{g}^{\alpha_s\beta_s}\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}) - \delta\ \ _s^{\shortmid}\widehat{f})[2\ ^{\star}\widehat{\triangle}(\ _s^{\shortmid}\widehat{f}) + \ _s^{\shortmid}\widehat{\mathbf{R}}sc^{\star} - |\ _s^{\shortmid}\widehat{\mathbf{D}}^{\star}\ _s^{\shortmid}\widehat{f}|^2]\} \star e^{-\ _s^{\shortmid}\widehat{f}}
\end{aligned} \tag{42}$$

Here we note that $\frac{1}{2}\ ^{\shortmid}\mathbf{g}^{\alpha_s\beta_s}\delta(\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}) - \delta\ \ _s^{\shortmid}\widehat{f}) \equiv 0$ if the measure $d\ ^{\shortmid}\widehat{\mathcal{V}} := \ _s^{\shortmid}\widehat{\nu}\ d\ ^{\shortmid}\mathcal{V}ol\ \ e^{-\ _s^{\shortmid}\widehat{f}} = const$. For such s-adapted configurations, we can prescribe the nonholonomic structure and compute (using symmetric coordinate configurations and then re-defining for N-connections) when the symmetric s-tensor

$$-[\ ^{\shortmid}\widehat{\mathbf{R}}^{\alpha_s\beta_s}_{\star} + \ ^{\shortmid}\widehat{\mathbf{D}}^{\alpha_s}_{\star}\ ^{\shortmid}\widehat{\mathbf{D}}^{\beta_s}_{\star}(\ _s^{\shortmid}\widehat{f})] \text{ is the } L^2 \text{ gradient of the functional } \ _s^{\shortmid}\widehat{\mathcal{F}}^{\star}[d\ ^{\shortmid}\widehat{\mathcal{V}}] = \int_{_s^{\shortmid}\widehat{\Xi}}(\ _s^{\shortmid}\widehat{\mathbf{R}}sc^{\star} + |\ _s^{\shortmid}\widehat{\mathbf{D}}^{\star}\ _s^{\shortmid}\widehat{f}|^2 d\ ^{\shortmid}\widehat{\mathcal{V}}.$$

In these formulas, $_s^{\shortmid}\widehat{f}$ can be considered as $\log(d\ ^{\shortmid}\mathcal{V}ol/d\ ^{\shortmid}\widehat{\mathcal{V}})$. So, prescribing a measure $d\ ^{\shortmid}\widehat{\mathcal{V}}$, we can model a nonassociative gradient flow subjected to equations

$$\partial_{\tau}\ \ _{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} = -2[\ ^{\shortmid}\widehat{\mathbf{R}}^{\star}{}_{\alpha_s\beta_s} + \ ^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\alpha_s}\ ^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\beta_s}(\ _s^{\shortmid}\widehat{f})],$$

derived from a $_s^{\shortmid}\widehat{\mathcal{F}}^{\star}[d\ ^{\shortmid}\widehat{\mathcal{V}}]$. Because the right part can be constructed as a nonholonomic nonsymmetric star deformation, we can chose a corresponding $_s^{\shortmid}\widehat{f}$ when $_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s} \rightarrow \ _{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}$. Modifying by an appropriate diffeomorphism and nonholonomic s-adapted structure, we obtain this type of nonassociative geometric evolution equation:

$$\begin{aligned}
\partial_{\tau}\ _{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s} &= -2\ ^{\shortmid}\widehat{\mathbf{R}}^{\star}{}_{\alpha_s\beta_s}, \\
\partial_{\tau}\ _s^{\shortmid}\widehat{f} &= \ _s^{\shortmid}\widehat{\mathbf{R}}sc^{\star} - \ ^{\star}\widehat{\triangle}\ \ _s^{\shortmid}\widehat{f} + |\ _s^{\shortmid}\widehat{\mathbf{D}}^{\star}\ \ _s^{\shortmid}\widehat{f}|^2.
\end{aligned}$$

Such formulas are equivalent (up to certain nonholonomic transforms and re-definitions of the normalizing functions) to the nonassociative geometric flow equations (41), which we postulated/ constructed following abstract/ symbolic principles.

It is important to note that above formulas obtained from a s-adapted variational principle with (42) still have a geometric symbolic character if we consider general star product R-flux deformations. Even in the associative and commutative Riemannian case, gradient flows may not exist for general measures (see related explanations before paragram 1.2 Proposition and formulas (1.1) - (1.4) in [1]). So, we can not prove that (non) associative geometric flow evolution equations of type (41) can be proven in a general, or s-adapted form from a functional ${}_{s}^{\shortmid}\widehat{\mathcal{F}}^{\star}$ (38). This results in various un-determined variants of nonassociative functionals and symbolic geometric flow equations which to not allow even to formulate certain nonassociative variants of the Poincaré-Thurston conjecture. Nevertheless, well-defined star-deformed functionals ${}_{s}^{\shortmid}\widehat{\mathcal{F}}^{\star}$ and ${}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}$ and related variants of nonassociative geometric evolution equations can be introduced in a self-consistent geometric form (and with various important implications in modern gravity and quantum information theories) if we consider κ-linear parametric deformations. In such cases, all geometric and physical objects (and related N-adapted variational procedures) allow to define and compute such values and finding of exact/parametric solutions in explicit forms. This is possible when we apply the AFCDM and construct generic off-diagonal solutions for certain (associative and commutative) nonholonomic s-adapted configurations and then subject such geometric/ physical data to star product deformations (19). If such R-flux deformations are computed following the procedure described in section 4 and appendix A), the s-adapted variational procedure (42) becomes well-defined mathematically even the constructions involve a twist operator. All such tedious and technical computations are performed using the third line of (19) and κ-parametric decompositions of geometric s-objects as in [39, 12, 13, 41]. We may have certain undetermined values for general classes of nonholonomic bases and s-connections. But if we chose certain canonical variables and deformations of physical important and well-defined solutions (for instance, for BH in any nonassociative or associative variant), then the corresponding nonassociative geometric flows can be modelled in a nonassociative gradient form with a well-defined s-adapted variational procedure.

In our series of works [12, 13, 41], we consider that for the nonassociative geometric flow theories the abstract geometric symbolic principles are more fundamental and efficient that standard variational procedures (similar to classical and quantum field theories) which became un-determined, for instance, for general star product R-flux deformations. Such an assumption is based on ideas from [43] that having certain data for a necessary set of fundamental geometric objects (metric-affine structures, nonlinear and linear connections, respective Riemannian, Ricci tensors etc.) we can elaborate always on respective gravity/ geometric flow models following standard geometric principles. The constructions are symbolic, but for explicit variants of nonassociative/ noncommutative/ supersymmetric etc. generalizations we can state an equivalent variational procedure if certain well-defined κ-parametric decompositions are stated for such models following a typical "deformation philosophy".

Nonassociative geometric flow equations can be derived in similar forms if, for instance, a s-adapted variational procedure is performed for ${}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}(\tau)$ (39). Such details for the LC-connection are provided in [1], see analogous constructions for Riemannian geometric flows; all described by respective formulas 3.1 - 3.4 in section 3 of that work. In s-adapted form, the approach was generalized in [14, 17, 7, 8, 6]. We omit such technical details in this work because they can be derived in abstract form following geometric principles and the Convention 2 (26). For recent applications in high energy physics, we cite [23, 24, 25, 26, 27] where the normalizing function is postulated as a dilaton field and associative and commutative versions of metric-dilaton Ricci flows are investigated. Certain geometric flow equations can be also motivated as star product R-flux deformations of a two-dimensional sigma model with beta functions and dilaton field (see equations (79) and (80) in [23]). Here we note that in section 1.4* of [1] it is mentioned that the F-functional and "its variation formula can be found in the literature on the string theory ...", when f can be treated as a dilaton filed. In our works on nonassociative geometric flows, we show that G. Perelman constructions can be generalized for star products with R-flux deformations from string theory. The AFCDM allows to elaborate on such nonassociative geometric flow models in explicit forms considering various classes of physically important solutions.

2.3.3 Nonassociative Ricci soliton equations in canonical s-variables

Ricci solitons are defined as self-similar configurations of gradient geometric flows for a fixed parameter τ_0. For Riemannian and Kaehler Ricci flows, such geometries are studied in details in [9, 10, 11] (where different types of Ricci soliton equations are formulated). In [6, 7, 8] and references therein, the approach was extended to nonholonomic s-adapted constructions. In canonical s-variables on ${}_{s}\mathcal{M}$, the Ricci soliton s-equations derived from a W-functional are of type

$$ {}^{\shortmid}\widehat{\mathbf{R}}_{\ \alpha_s\beta_s} + {}^{\shortmid}\widehat{\mathbf{D}}_{\alpha_s}\ {}^{\shortmid}\widehat{\mathbf{D}}_{\beta_s}\ {}^{\shortmid}\varpi({}^{\shortmid}_{s}u) = \lambda\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}, $$

where ${}^{\shortmid}\varpi$ is a smooth potential function on every shell $s = 1,2,3,4$ and $\lambda = const$. Following the Convention 2 (26), such systems of nonlinear PDEs can be deformed by star products and R-fluxes to nonassociative Ricci solitons defined by equations

$$ {}^{\shortmid}\widehat{\mathbf{R}}^{\star}_{\ \alpha_s\beta_s} + {}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\alpha_s}\ {}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\beta_s}\ {}^{\shortmid}\varpi({}^{\shortmid}_{s}u) = \lambda\ \ {}^{\shortmid}_{\star}\mathfrak{g}_{\alpha_s\beta_s}. \qquad (43) $$

Similar equations (certain differences can be related to different nonholonomic structures and/or different normalizing functions) can derived from a respective s-adapted variational calculus with $\tau = \tau_0$ for ${}^{\shortmid}_{s}\widehat{\mathcal{W}}^{\star}(\tau)$ (39) and/or from (41). We omit such technical details. Here we note that the nonassociative phase space vacuum gravitational equations (32) defined for the canonical s-connection ${}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\alpha_s}$ consist an example of nonassociative Ricci soliton ones (43).

The nonassociative geometric flow constructions provided in this section can be re-defined in terms of respective LC-connections, ${}^{\shortmid}_{\star}\nabla$ and ${}^{\shortmid}\nabla$, if we impose additional nonholonomic constraints of type (25), when ${}^{\shortmid}_{s}\widehat{\mathbf{D}}^{\star}_{\mid {}^{\shortmid}_{s}\widehat{\mathbf{T}}=0} = {}^{\shortmid}_{\star}\nabla$. As a result, the nonassociative equations (41) and (43) transform respectively into (2) and (1) postulated in the Introduction section. For ${}^{\shortmid}_{\star}\nabla$, such nonassociative geometric flow and Ricci soliton equations could be postulated just having the results of papers [38, 39], where the nonassociative Ricci tensors were defined and computed for ${}^{\shortmid}_{\star}\nabla$ (in our notations). The main motivation for elaborating such theories in terms of ${}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\alpha_s}$ (and, with nonholonomic constraints, of ${}^{\shortmid}_{\star}\nabla$) is that using nonholonomic s-adapted variables we can decouple and solve in very general forms such systems of nonlinear PDEs [13, 40, 41]. This is possible if we apply the AFCDM (see main ideas and important formulas in Appendix A). Constructing exact and parametric solutions of nonassociative Ricci flow/ soliton equations, we analyze how the results and methods of nonassociative geometry can be applied in modern particle physics, gravity and information theory.

2.4 Parametric decomposition of nonassociative functionals and geometric flow equations

To elaborate on possible applications in modern gravity and cosmology, the nonassociative F- and W-functionals and related geometric flow equations are considered for a κ-linear parametric decomposition. Using formulas (29), (30), (31) and (33), for respective parametric formulas for the canonical s-connection, nonsymmetric Ricci s-tensor and scalar curvature, we write (38) and (39) in the forms:

$$ \begin{aligned} {}^{\shortmid}_{s}\widehat{\mathcal{F}}^{\star}_{\kappa}(\tau) &= \int_{{}^{\shortmid}_{s}\widehat{\Xi}} ({}^{\shortmid}_{s}\widehat{\mathbf{R}}sc + {}^{\shortmid}_{s}\widehat{\mathbf{K}}sc + |{}^{\shortmid}_{s}\widehat{\mathbf{D}}\ {}^{\shortmid}_{s}\widehat{f}|^2) e^{-\ {}^{\shortmid}_{s}\widehat{f}}\ d\ {}^{\shortmid}\mathcal{V}ol(\tau), \text{ and} & (44)\\ {}^{\shortmid}_{s}\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau) &= \int_{{}^{\shortmid}_{s}\widehat{\Xi}} (4\pi\tau)^{-4}\ [\tau({}^{\shortmid}_{s}\widehat{\mathbf{R}}sc + {}^{\shortmid}_{s}\widehat{\mathbf{K}}sc + \textstyle\sum_s |{}^{\shortmid}_{s}\widehat{\mathbf{D}}\ {}^{\shortmid}_{s}\widehat{f}|)^2 + {}^{\shortmid}_{s}\widehat{f} - 8] e^{-\ {}^{\shortmid}_{s}\widehat{f}}\ d\ {}^{\shortmid}\mathcal{V}ol(\tau), & (45) \end{aligned} $$

where ${}^{\shortmid}_{s}\widehat{\mathbf{R}}sc^{\star} = {}^{\shortmid}_{s}\widehat{\mathbf{R}}sc + {}^{\shortmid}_{s}\widehat{\mathbf{K}}sc$, for ${}^{\shortmid}_{s}\widehat{\mathbf{K}}sc := {}^{\shortmid}_{\star}\mathfrak{g}^{\mu_s\nu_s}\ {}^{\shortmid}\widehat{\mathbf{K}}_{\ \beta_s\gamma_s}\lceil\hbar,\kappa\rceil$ and the normalizing function ${}^{\shortmid}_{s}\widehat{f}$ is re-defined to include $\lceil\hbar,\kappa\rceil$-terms from ${}^{\shortmid}\widehat{\mathbf{D}}^{\star} \to {}^{\shortmid}_{s}\widehat{\mathbf{D}}$ and remaining terms from κ-parametric decompositions.

There are two ways for deriving nonassociative κ-linear generalizations of the R. Hamilton equations. In the first case, we consider κ-parametric decompositions of (41) and, in the second case, we apply a s-adapted nonholonomic variational procedure to ${}^{\shortmid}_{s}\widehat{\mathcal{F}}^{\star}_{\kappa}(\tau)$ (44), or ${}^{\shortmid}_{s}\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau)$ (45). In all cases, adapting corresponding the nonholonomic structure, we obtain such phase geometric flow equations encoding κ-terms,

$$ \begin{aligned} \partial_\tau\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau) &= -2({}^{\shortmid}\widehat{\mathbf{R}}_{\ \alpha_s\beta_s}(\tau) + {}^{\shortmid}\widehat{\mathbf{K}}_{\ \alpha_s\beta_s}(\tau,\lceil\hbar,\kappa\rceil)), & (46)\\ \partial_\tau\ {}^{\shortmid}_{s}\widehat{f}(\tau) &= {}^{\shortmid}_{s}\widehat{\mathbf{R}}sc(\tau) + {}^{\shortmid}_{s}\widehat{\mathbf{K}}sc(\tau) - \widehat{\triangle}(\tau)\ {}^{\shortmid}_{s}\widehat{f}(\tau) + ({}^{\shortmid}_{s}\widehat{\mathbf{D}}(\tau)\ {}^{\shortmid}_{s}\widehat{f}(\tau))^2(\tau), \end{aligned} $$

where $\widehat{\triangle}$ is the Laplace operator constructed for ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$. Positively, the s-adapted variational procedure with κ-linear decompositions in (41) can be performed in a well-defined mathematical form by involving the AFCDM for constructing respective classes of exact/parametric solutions.

For self-similar configurations with $\tau = \tau_0$, the equations (46) transform into a system of nonlinear PDEs for κ–parametric canonical shell Ricci solitons,

$$^{\shortmid}\widehat{\mathbf{R}}_{\ \alpha_s\beta_s} + \ ^{\shortmid}\widehat{\mathbf{K}}_{\ \alpha_s\beta_s}(\tau, \lceil \hbar, \kappa \rceil) + \ ^{\shortmid}\widehat{\mathbf{D}}_{\alpha_s}\ ^{\shortmid}\widehat{\mathbf{D}}_{\beta_s}\ ^{\shortmid}\varpi(\ _{s}^{\shortmid}u) = \lambda\ \ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}. \quad (47)$$

Similar equations can be also obtained from a corresponding κ-linear decompositions of the nonassociative Ricci soliton equations (43). Re-defining $^{\shortmid}\varpi(\ _{s}^{\shortmid}u)$ for some particular choices and for corresponding nonholonomic structures, we obtain from (47) phase space modified gravitational equations $^{\shortmid}\widehat{\mathbf{R}}ic_{\alpha_s\beta_s} = \ ^{\shortmid}\Upsilon_{\alpha_s\beta_s}$ (16), where $^{\shortmid}\Upsilon_{\alpha_s\beta_s} = -\ _{s}^{\shortmid}\widehat{\mathcal{K}}ic_{\alpha_s\beta_s}$. Such systems of nonlinear PDEs can be decoupled and integrated in general off-diagonal forms using the AFCDM if the effective source $^{\shortmid}\Upsilon_{\alpha_s\beta_s}$ is parameterized following the conventions (34) and ${}_{s}^{\shortmid}\mathcal{K}$ (35).

3 Nonassociative geometric thermodynamics

For the Ricci flows of Riemannian metrics, the W-functional (9) can be treated as a "minus entropy" which allows to formulate a statistical thermodynamic model with thermodynamic variables determined by τ-running fundamental geometric objects on Riemann manifolds. In [6, 7, 8] (see also references therein), we investigated possibilities to extend such constructions to relativistic geometric thermodynamic models and (modified) gravity and quantum information theories. A very important result was that modified G. Perelman thermodynamic models can be associated to various classes generic off-diagonal solutions (in general, with non-Riemannian connections and without conventional horizons) when the concept of Bekenstein–Hawking thermodynamics is not applicable.

The goal of this section is to show how nonholonomic geometric thermodynamic models can be elaborated for nonassociative geometric flows and Ricci solitons determined by certain data $[\ _{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau),\ _{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau),\ _{s}^{\shortmid}\widehat{f}(\tau)]$ and κ-linear parametric decompositions.

3.1 Star product and R-flux deformed statistical thermodynamic variables

On ${}_{s}\mathcal{M}$ with an additional nonholonomic (3+1)+(3+1) splitting, we introduce the statistical partition function

$${}_{s}^{\shortmid}\widehat{Z}(\tau) = \exp[\int_{{}_{s}^{\shortmid}\widehat{\Xi}} [-\ _{s}^{\shortmid}\widehat{f} + 4]\ (4\pi\tau)^{-4}\, e^{-\ _{s}^{\shortmid}\widehat{f}}\ ^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau), \quad (48)$$

where the volume element (36) are computed for a s-metric (37) with "shift and lapse" functions,

$$^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau) = \sqrt{|q_1(\tau)q_2(\tau)q_3(\tau)\breve{N}^2(\tau)\ ^{\shortmid}q^5(\tau)\ ^{\shortmid}q^6(\tau)\ ^{\shortmid}q^7(\tau)\ ^{\shortmid}\breve{N}^2(\tau)|}dx^1dx^2\delta y^3\delta y^4\ ^{\shortmid}\delta\ ^{\shortmid}u_5(\tau)\ ^{\shortmid}\delta\ ^{\shortmid}u_6(\tau)\ ^{\shortmid}\delta\ ^{\shortmid}u_7(\tau)\ ^{\shortmid}\delta\ ^{\shortmid}u_8(\tau). \quad (49)$$

The Convention 2 (26) on star product R-flux deformations of s-adapted geometric objects into respective nonassociative ones has to be adapted for parameterizations of type ${}_{s}^{\shortmid t}\widehat{\mathbf{D}} = \ _{s}^{\shortmid}\widehat{\mathbf{D}}_{|\widehat{\Xi}_t} \rightarrow \ _{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{|\widehat{\Xi}_t}$ and ${}_{s}^{\shortmid E}\widehat{\mathbf{D}} = \ _{s}^{\shortmid}\widehat{\mathbf{D}}_{|\widehat{\Xi}_E} \rightarrow \ _{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{|\widehat{\Xi}_E}$. Such transforms have to be considered for hyper-surface restrictions of the canonical s-connections ${}_{s}^{\shortmid}\widehat{\mathbf{D}} \rightarrow \ _{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}$ and computing integrals with volume forms (49). We can define and compute Ricci s-tensors and scalar curvatures determined by ${}_{s}^{\shortmid t}\widehat{\mathbf{D}}^{\star}$ and ${}_{s}^{\shortmid E}\widehat{\mathbf{D}}^{\star}$ are denoted ${}_{s}^{\shortmid t}\widehat{\mathbf{R}}^{\star}_{\grave{i}\grave{j}}$, ${}_{s}^{\shortmid E}\widehat{\mathbf{R}}^{\grave{a}\grave{b}}_{\star}$ and $^{\shortmid t}\widehat{R}^{\star}$, $^{\shortmid E}\widehat{R}^{\star}$ which are useful for computing hyper-surface geometric s-objects for quasi-stationary configurations with redefined normalizing functions (see examples with κ-parametric nonassociative Ricci solitons in [41] and, for nonholonomic associative and commutative configurations in [6, 7, 8]) and next sections).

Using ${}_{s}^{\shortmid}\widehat{Z}$ (48) and ${}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}(\tau)$ (39) as the respective partition function and W-entropy functional on ${}_{s}^{\star}\mathcal{M}$,[15] we can define and compute respective thermodynamic variables (average energy, ${}_{s}^{\shortmid}\widehat{\mathcal{E}}^{\star}$, entropy, ${}_{s}^{\shortmid}\widehat{S}^{\star}$, and fluctuation, ${}_{s}^{\shortmid}\widehat{\sigma}^{\star}$):

$$
\begin{aligned}
{}_{s}^{\shortmid}\widehat{\mathcal{E}}^{\star} &= -\tau^{2}\int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}\left({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star} + |\, {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}\, {}_{s}^{\shortmid}\widehat{f}|^{2} - \frac{4}{\tau}\right)\star e^{-\,{}_{s}^{\shortmid}\widehat{f}}\, {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau), \quad (50)\\
{}_{s}^{\shortmid}\widehat{S}^{\star} &= -\int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}\left(\tau({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star} + |\, {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}\, {}_{s}^{\shortmid}\widehat{f}|^{2}) + {}_{s}^{\shortmid}\tilde{f} - 8\right)\star e^{-\,{}_{s}^{\shortmid}\widehat{f}}\, {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau),\\
{}_{s}^{\shortmid}\widehat{\sigma}^{\star} &= 2\,\tau^{4}\int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}|\, {}^{\shortmid}\widehat{\mathbf{R}}^{\star}_{\alpha_{s}\beta_{s}} + {}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\alpha_{s}}\, {}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\beta_{s}}\, {}_{s}^{\shortmid}\widehat{f} - \frac{1}{2\tau}\mathbf{g}^{\star}_{\alpha_{s}\beta_{s}}|^{2}\star e^{-\,{}_{s}^{\shortmid}\widehat{f}}\, {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau).
\end{aligned}
$$

To prove these formulas we can apply a tedious variational s-adapted calculus on nonassocitative phase space. Following the abstract geometric formalism, such formulas can be derived in a simplified symbolic form when geometric s-objects are generalized into similar ones with star labels.

We can restrict such formulas to 4–d and 6-d shell configurations, for respective redefinitions of ${}_{s}^{\shortmid}\widehat{f}$ into a convenient ${}_{s}^{\shortmid}\tilde{f}$, in order to adapt the geometric thermodynamic constructions to a prescribed both shell and (3+1)+(3+1) splitting. For corresponding fixed value $\tau = \tau_{0}$, the formulas (50) characterize nonassociative Ricci soliton (43) (and, in particular, nonassociative vacuum gravitational (32)) equations. Such thermodynamic values can be computed for any example of exact/parametric solution of nonassociative geometric flow equations (41).

3.2 Parametric decompositions in nonassociative geometric thermodynamics

For κ-linear parametric decompositions as in section 2.4 (following again the Convention 2 (26) and τ-parametric formulas (28)-(31) and (33)-(34)), the formulas for thermodynamic variables (50) encoding data for nonassociative geometric flows transform respectively into

$$
\begin{aligned}
{}_{s}^{\shortmid}\widehat{\mathcal{E}}^{\star}_{\kappa} &= -\tau^{2}\int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}\left({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau) + {}_{s}^{\shortmid}\widehat{\mathbf{K}}sc(\tau) + |\, {}_{s}^{\shortmid}\widehat{\mathbf{D}}\, {}_{s}^{\shortmid}\widehat{f}|^{2}(\tau) - \frac{4}{\tau}\right) e^{-\,{}_{s}^{\shortmid}\widehat{f}}\, {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau), \quad (51)\\
{}_{s}^{\shortmid}\widehat{S}^{\star}_{\kappa} &= -\int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}\left(\tau({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau) + {}_{s}^{\shortmid}\widehat{\mathbf{K}}sc(\tau) + |\, {}_{s}^{\shortmid}\widehat{\mathbf{D}}\, {}_{s}^{\shortmid}\widehat{f}|^{2}(\tau)) + {}_{s}^{\shortmid}\widehat{f}(\tau) - 8\right) e^{-\,{}_{s}^{\shortmid}\widehat{f}}\, {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau),\\
{}_{s}^{\shortmid}\widehat{\sigma}^{\star}_{\kappa} &= 2\,\tau^{4}\int_{{}_{s}^{\shortmid}\widehat{\Xi}} (4\pi\tau)^{-4}|\, {}^{\shortmid}\widehat{\mathbf{R}}_{\alpha_{s}\beta_{s}}(\tau) + {}^{\shortmid}\widehat{\mathbf{K}}_{\alpha_{s}\beta_{s}}(\tau) + {}^{\shortmid}\widehat{\mathbf{D}}_{\alpha_{s}}\, {}^{\shortmid}\widehat{\mathbf{D}}_{\beta_{s}}\, {}_{s}^{\shortmid}\widehat{f}(\tau) - \frac{1}{2\tau}\mathbf{g}_{\alpha_{s}\beta_{s}}(\tau)|^{2} e^{-\,{}_{s}^{\shortmid}\widehat{f}}\, {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau).
\end{aligned}
$$

We emphasize that such variables encode certain nonassociative data in ${}_{s}^{\shortmid}\widehat{\mathbf{K}}sc$ and ${}^{\shortmid}\widehat{\mathbf{K}}_{\alpha_{s}\beta_{s}}$ and certain dependencies in nontrivial κ-linear terms in ${}_{s}^{\shortmid}\widehat{\mathbf{D}}(\tau), \mathbf{g}_{\alpha_{s}\beta_{s}}(\tau), {}^{\shortmid}\delta\, {}^{\shortmid}\mathcal{V}(\tau)$ and ${}_{s}^{\shortmid}\widehat{f}(\tau)$ defined for solutions of canonical nonholonomic Ricci flow (46), or Ricci soliton (47), equations.

The formulas (51) can be derived alternatively using on ${}_{s}^{\star}\mathcal{M}$ a) a s-adapted variational calculus for the statistical generating function ${}_{s}^{\shortmid}\widehat{Z}(\tau)$ (36) and W-entropy ${}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau)$ (45), with a κ-linear parametric decomposition of nonholonomic geometric and thermodynamic variables and/or b) a corresponding abstract nonassociative geometric calculus.

We note that, in general, the nonassociative geometric flow thermodynamic variables may be not well-defined as physical values, for instance, one could be obtained negative entropies etc. This depends on the classes of solutions we consider and compute such values. In κ-linear parametric form we can investigate such issues and select certain self-consistent and relativistic causal nonassociative cosmological scenarios or for some BH like configurations.

[15] To formulate a statistical thermodynamic model, we can consider a partition function $Z = \int \exp(-\beta E) d\omega(E)$ for the canonical ensemble at temperature $\beta^{-1} = \tau$ being defined by the measure taken to be the density of states $\omega(E)$. The thermodynamical variables are computed as the average energy, $\langle E \rangle := -\partial \log Z/\partial\beta$, the entropy $S := \beta \langle E \rangle + \log Z$ and the fluctuation parameter $\sigma := \left\langle (E - \langle E \rangle)^{2} \right\rangle = \partial^{2} \log Z/\partial\beta^{2}$.

Finally, for the last two subsections, it should be noted that similar thermodynamic variables for nonassociative geometric flows or Ricci solitons can be formulated/ in terms of respective LC-connections, ${}_{\star}^{\shortmid}\nabla$ and ${}^{\shortmid}\nabla$, if we impose additional nonholonomic constraints of type (25), when ${}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}_{\mid {}_{s}^{\shortmid}\widehat{\mathbf{T}}=0} = {}_{\star}^{\shortmid}\nabla$. Such constraints result into standard G. Perelman's functionals (9) and related thermodynamic variables but on phase space ${}_{s}^{\star}\mathcal{M}$. In many cases, various distortion of s-connection terms can be encoded into a new type of normalization functions ${}_{s}^{\shortmid}\widehat{f}(\tau)$, or in respective classes of generating functions and generating effective sources.

4 Parametric geometric flows and off-diagonal quasi-stationary solutions

The goal of this section is to prove that κ-linear parametric geometric flow equations (46) (and, in particular, the nonassociative Ricci soliton equations (47)) can be decoupled and integrated in general off-diagonal forms for effective sources encoding nonassociative star product and R-flux data and additional τ-induced coefficients. We follow the AFCDM [6, 7, 8, 13, 40, 41] outlined in Appendix A. There are provided four possible parameterizations of such quasi-stationary solutions and analyzed their nonlinear symmetries.

An effective τ-depending source ${}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau) = -({}^{\shortmid}\widehat{\mathbf{K}}_{\alpha_s\beta_s}(\tau) + \frac{1}{2}\partial_\tau\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau))$, parameterized on ${}_{s}^{\star}\mathcal{M}$ in s-shell adapted form

$$ {}^{\shortmid}\Im^{\alpha_s}_{\ \beta_s}(\tau,\ {}^{\shortmid}u^{\gamma_s}) = [\ {}_{1}^{\shortmid}\Im(\kappa,\tau,x^{k_1})\delta^{j_1}_{i_1},\ {}_{2}^{\shortmid}\Im(\kappa,\tau,x^{k_1},y^3)\delta^{a_2}_{b_2},\ {}_{3}^{\shortmid}\Im(\kappa,\tau,x^{k_2},\ {}^{\shortmid}p_5)\delta^{b_3}_{a_3},\ {}_{4}^{\shortmid}\Im(\kappa,\tau,x^{k_3},\ {}^{\shortmid}p_7)\delta^{b_4}_{a_4}], \quad (52) $$

can be used for generating quasi-stationary solutions with Killing symmetry on $\partial_4 = \partial_t$. For other types of Killing symmetries, we need corresponding type parameterizations. Such families of effective sources contain as functionals certain κ-linear terms with $\mathcal{R}^{\tau_s\xi_s}{}_{\alpha_s}(\tau)$ which for any fixed τ are similar to ${}_{s}^{\shortmid}\mathcal{K}$ (35). We suppose that parameterizations of type (52) can be obtained for certain frame s-adapted transform, ${}^{\shortmid}\widehat{\Im}_{\alpha'_s\beta'_s} = e^{\alpha_s}{}_{\alpha'_s} e^{\beta_s}{}_{\beta'_s}\ {}^{\shortmid}\Im_{\alpha_s\beta_s}$, when some general sources are transformed into a subset of four generating sources ${}^{\shortmid}\Im_{\beta_s\gamma_s} = diag\{\ {}_{s}^{\shortmid}\Im\}$. Any prescribed ${}_{s}^{\shortmid}\Im\ (\tau,\ {}^{\shortmid}u^{\gamma_s})$ imposes a s-shell nonholonomic constraint for τ-derivatives of the metrics s-coefficients $\partial_\tau\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)$. For small parametric deformations, such constraints can be solved in explicit general forms. In other cases, we have to search for some special classes of generating and integration functions which allow to find some examples of exact/ parametric solutions.

Using effective sources (52), we can write the κ-linear parametric geometric flow equations (46) in the form

$$ {}^{\shortmid}\widehat{\mathbf{R}}ic_{\alpha_s\beta_s}(\tau) = {}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau), \quad (53) $$

which are very similar to the modified Einstein equations (16) (see (3) for a running source, ${}^{\shortmid}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}(\tau) = \delta^{\beta_s}{}_{\gamma_s}\ {}_{s}^{\shortmid}\Im(\tau)$). In these equations, there is an additional dependence geometric objects on the geometric flow parameter τ and when the sources of type (17) are corrected with terms of type $\frac{1}{2}\partial_\tau\ {}_{s}^{\shortmid}\mathbf{g}$. Such systems of nonlinear PDEs can be solved in very general off-diagonal forms using the same formulas as in Appendix A but with additional assumptions when all coefficients of s-metrics depend additionally on τ and (considering nonlinear symmetries) on running effective cosmological constants ${}_{s}^{\shortmid}\Lambda\ (\tau)$. We consider that corresponding classes of generic off-diagonal solutions are physically important if they satisfy well-defined causality conditions and self-consistent G. Perelman like thermodynamic variables (51) in certain phase space finite regions. In general, it is not possible to express in explicit functional form all coefficients ${}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau) = {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}[\ {}_{s}^{\shortmid}\Im(\tau,\ {}^{\shortmid}u^{\gamma_s})]$. Nevertheless, using decompositions on small parameters (like on κ and other physical constants) and corresponding s-adapting of geometric constructions, we can construct in explicit forms exact solutions in nonassociative gravity at least to certain levels of approximation including κ-linear terms.

The ansatz for generating quasi-stationary solutions of nonassociative geometric flow equations (53) can be chosen as (A.1) and (A.2) but with additional dependencies on τ,

$$ \begin{aligned} d\widehat{s}^2(\tau) \ = \ & g_{i_1}(\tau,x^{k_1})(dx^{i_1})^2 + g_{a_2}(\tau,x^{i_1},y^3)(\mathbf{e}^{a_2}(\tau))^2 + {}^{\shortmid}g^{a_3}(\tau,x^{i_2},p_6)(\ {}^{\shortmid}\mathbf{e}_{a_3}(\tau))^2 + {}^{\shortmid}g^{a_4}(\tau,\ {}^{\shortmid}x^{i_3},p_7)(\ {}^{\shortmid}\mathbf{e}_{a_4}(\tau))^2, \\ & \text{where } \mathbf{e}^{a_2}(\tau) = dy^{a_2} + N^{a_2}_{k_1}(\tau,x^{i_1},y^3)dx^{k_1},\ {}^{\shortmid}\mathbf{e}_{a_3}(\tau) = dp_{a_3} + {}^{\shortmid}N_{a_3k_2}(\tau,x^{i_2},p_5)dx^{k_2}, \\ & {}^{\shortmid}\mathbf{e}_{a_4}(\tau) = dp_{a_4} + {}^{\shortmid}N_{a_4k_3}(\tau,\ {}^{\shortmid}x^{i_3},p_7)d\ {}^{\shortmid}x^{k_3} \end{aligned} \quad (54) $$

are determined by geometric flows of N-connection coefficients.

4.1 Geometric evolution of quasi-stationary solutions with effective sources

Applying the AFCDM for temperature running of sources in (A.3), $\ _{s}^{\shortmid}\mathcal{K} \to \ _{s}^{\shortmid}\Im(\tau)$ (52), and introducing τ-dependencies for the coefficients of s-metric and N-connection in (A.3), we construct a class of quasi-stationary solutions for nonassociative κ-parametric geometric flows:

$$\begin{aligned} d\widehat{s}^2(\tau) = & \ e^{\psi(\hbar,\kappa;\tau,x^{k_1})}[(dx^1)^2+(dx^2)^2] + \\ & \frac{[\partial_3(\ _2\Psi(\tau))]^2}{4(\ _2^{\shortmid}\Im(\tau))^2\{g_4^{[0]}(\tau)-\int dy^3 \frac{\partial_3[(\ _2\Psi(\tau))^2]}{4(\ _2^{\shortmid}\Im(\tau))}\}}(\mathbf{e}^3(\tau))^2 + (g_4^{[0]}(\tau) - \int dy^3 \frac{\partial_3[(\ _2\Psi(\tau))^2]}{4(\ _2^{\shortmid}\Im(\tau))})(\mathbf{e}^4(\tau))^2 \\ & + \frac{[\ ^{\shortmid}\partial^5(\ _3^{\shortmid}\Psi(\tau))]^2}{4(\ _3^{\shortmid}\Im)^2\{\ ^{\shortmid}g^6_{[0]}(\tau) - \int dp_5 \frac{\ ^{\shortmid}\partial^6[(\ _3^{\shortmid}\Psi(\tau))^2]}{4(\ _3^{\shortmid}\Im(\tau))}\}}(\ ^{\shortmid}\mathbf{e}_5(\tau))^2 + (\ ^{\shortmid}g^6_{[0]}(\tau) - \int dp_5 \frac{\ ^{\shortmid}\partial^5[(\ _3^{\shortmid}\Psi(\tau))^2]}{4(\ _3^{\shortmid}\Im)})(\ ^{\shortmid}\mathbf{e}_6(\tau))^2 \\ & + \frac{[\ ^{\shortmid}\partial^7(\ _4^{\shortmid}\Psi(\tau))]^2}{4(\ _4^{\shortmid}\Im(\tau))^2\{\ ^{\shortmid}g^8_{[0]}(\tau) - \int dp_7 \frac{\ ^{\shortmid}\partial^7[(\ _4^{\shortmid}\Psi(\tau))^2]}{4(\ _4^{\shortmid}\Im(\tau))}\}}(\ ^{\shortmid}\mathbf{e}_7(\tau))^2 + (\ ^{\shortmid}g^8_{[0]}(\tau) - \int dp_7 \frac{\ ^{\shortmid}\partial^7[(\ _4^{\shortmid}\Psi(\tau))^2]}{4(\ _4^{\shortmid}\Im(\tau))})(\ ^{\shortmid}\mathbf{e}_8(\tau))^2. \end{aligned} \quad (55)$$

The nonholonomic s-frames in this formula are computed:

$$\begin{aligned} \mathbf{e}^3(\tau) &= dy^3 + w_{k_1}(\hbar,\kappa,\tau,x^{i_1},y^3)dx^{k_1} = dy^3 + \frac{\partial_{k_1}(\ _2\Psi(\tau))}{\partial_3(\ _2\Psi(\tau))}dx^{k_1}, \\ \mathbf{e}^4(\tau) &= dt + n_{k_1}(\hbar,\kappa,\tau,x^{i_1},y^3)dx^{k_1} \\ &= dy^4 + (\ _1n_{k_1}(\tau) + \ _2n_{k_1}(\tau)\int dy^3 \frac{\partial_3[(\ _2\Psi(\tau))^2]}{4(\ _2^{\shortmid}\Im(\tau))^2|g_4^{[0]}(\tau) - \int dy^3 \frac{\partial_3[(\ _2\Psi(\tau))^2]}{4(\ _2^{\shortmid}\Im(\tau))}|^{5/2}})dx^{k_1}, \end{aligned} \quad (56)$$

$$\begin{aligned} \ ^{\shortmid}\mathbf{e}_5(\tau) &= dp_5 + \ ^{\shortmid}w_{k_2}(\hbar,\kappa,\tau,x^{i_2},p_5)dx^{k_2} = dp_5 + \frac{\partial_{k_2}(\ _3^{\shortmid}\Psi(\tau))}{\ ^{\shortmid}\partial^5(\ _3^{\shortmid}\Psi(\tau))}dx^{k_2}, \\ \ ^{\shortmid}\mathbf{e}_6(\tau) &= dp_6 + \ ^{\shortmid}n_{k_2}(\hbar,\kappa,\tau,x^{i_2},p_5)dx^{k_2} \\ &= dp_6 + (\ _1^{\shortmid}n_{k_2}(\tau) + \ _2^{\shortmid}n_{k_2}(\tau)\int dp_5 \frac{\ ^{\shortmid}\partial^5[(\ _3^{\shortmid}\Psi(\tau))^2]}{4(\ _3^{\shortmid}\Im(\tau))^2|\ ^{\shortmid}g^6_{[0]}(\tau) - \int dp_5 \frac{\ ^{\shortmid}\partial^5[(\ _3^{\shortmid}\Psi(\tau))^2]}{4(\ _3^{\shortmid}\Im(\tau))}|^{5/2}})dx^{k_2}, \end{aligned}$$

$$\begin{aligned} \ ^{\shortmid}\mathbf{e}_7(\tau) &= dp_7 + \ ^{\shortmid}w_{k_3}(\hbar,\kappa,\tau,x^{i_2},p_5,p_7)d\ ^{\shortmid}x^{k_3} = dp_7 + \frac{\ ^{\shortmid}\partial_{k_3}(\ _4^{\shortmid}\Psi(\tau))}{\ ^{\shortmid}\partial^7(\ _4^{\shortmid}\Psi(\tau))}d\ ^{\shortmid}x^{k_3}, \\ \ ^{\shortmid}\mathbf{e}_8(\tau) &= dp_8 + \ ^{\shortmid}n_{k_3}(\hbar,\kappa,\tau,x^{i_2},p_5,p_7)d\ ^{\shortmid}x^{k_3} \\ &= dp_8 + (\ _1^{\shortmid}n_{k_3}(\tau) + \ _2^{\shortmid}n_{k_3}(\tau)\int dp_7 \frac{\ ^{\shortmid}\partial^7[(\ _4^{\shortmid}\Psi(\tau))^2]}{4(\ _4^{\shortmid}\Im(\tau))^2|\ ^{\shortmid}g^8_{[0]}(\tau) - \int dp_7 \frac{\ ^{\shortmid}\partial^7[(\ _4^{\shortmid}\Psi(\tau))^2]}{4(\ _4^{\shortmid}\Im(\tau))}|^{5/2}})\ d\ ^{\shortmid}x^{k_3}. \end{aligned}$$

The generating and integration functions for the class of solutions (55) with N-coefficients (56) are similar to (A.4) but with extended τ-parametric dependence:

$$\begin{array}{lll} \text{generating functions:} & \psi(\tau) \simeq \psi(\hbar,\kappa;\tau,x^{k_1});\ {}_2\Psi(\tau) \simeq {}_2\Psi(\hbar,\kappa;\tau,x^{k_1},y^3); & \\ & {}_3^{\shortmid}\Psi(\tau) \simeq {}_3^{\shortmid}\Psi(\hbar,\kappa;\tau,x^{k_2},p_5);\ {}_4^{\shortmid}\Psi(\tau) \simeq {}_4^{\shortmid}\Psi(\hbar,\kappa;\tau,{}^{\shortmid}x^{k_3},p_7); & \\ \text{generating sources:} & {}_1^{\shortmid}\Im(\tau) \simeq {}_1^{\shortmid}\Im(\hbar,\kappa;\tau,x^{k_1});\ {}_2^{\shortmid}\Im(\tau) \simeq {}_2^{\shortmid}\Im(\hbar,\kappa;\tau,x^{k_1},y^3); & \\ & {}_3^{\shortmid}\Im(\tau) \simeq {}_3^{\shortmid}\Im(\hbar,\kappa;\tau,x^{k_2},p_5);\ {}_4^{\shortmid}\Im(\tau) \simeq {}_4^{\shortmid}\Im(\hbar,\kappa;\tau,{}^{\shortmid}x^{k_3},p_7); & \\ \text{integrating functions:} & & \end{array} \quad (57)$$

$$\begin{array}{lll} g_4^{[0]}(\tau) & \simeq & g_4^{[0]}(\hbar,\kappa;\tau,x^{k_1}),\ {}_1n_{k_1}(\tau) \simeq {}_1n_{k_1}(\hbar,\kappa;\tau,x^{j_1}),\ {}_2n_{k_1}(\tau) \simeq {}_2n_{k_1}(\hbar,\kappa;\tau,x^{j_1}); \\ {}^{\shortmid}g_{[0]}^6\tau) & \simeq & {}^{\shortmid}g_{[0]}^6(\hbar,\kappa;\tau,x^{k_2}),\ {}_1n_{k_2}(\tau) \simeq {}_1n_{k_2}(\hbar,\kappa;\tau,x^{j_2}),\ {}_2n_{k_2}(\tau) \simeq {}_2n_{k_2}(\hbar,\kappa;\tau,x^{j_2}); \\ {}^{\shortmid}g_{[0]}^8(\tau) & \simeq & {}^{\shortmid}g_{[0]}^8(\hbar,\kappa;\tau,{}^{\shortmid}x^{j_3}),\ {}_1n_{k_3}(\tau) \simeq {}_1^{\shortmid}n_{k_3}(\hbar,\kappa;\tau,{}^{\shortmid}x^{j_3}),\ {}_2n_{k_3}(\tau) \simeq {}_2^{\shortmid}n_{k_3}(\hbar,\kappa;\tau,{}^{\shortmid}x^{j_3}). \end{array}$$

The family of generating functions $\psi(\tau)$ are solutions of a respective family of 2-d Poisson equations,

$$\partial_{11}^2\psi(\hbar,\kappa;\tau,x^{k_1}) + \partial_{22}^2\psi(\hbar,\kappa;\tau,x^{k_1}) = 2\ {}_1\Im(\hbar,\kappa;\tau,x^{k_1}), \quad (58)$$

encoding geometric flows of nonassociative data if, in general, ${}_1\Im(\tau)$ contains such nonholonomic dependencies.

Geometric evolution scenarios of quasi-stationary configurations defined above are characterized by four types of additional geometric and thermodynamic flow variables:

1. The geometric evolution of nonsymmetric metrics ${}_\star^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}(\tau) = {}_\star^{\shortmid}\mathfrak{a}_{\alpha_s\beta_s}(\hbar,\kappa;\tau,{}^{\shortmid}u^{\gamma_s})$ is computed in explicit form by introducing in (22) the s-metric and N-connection coefficients, respectively, (55) and (56) (we omit such formulas in this work). For flow evolution of quasi-stationary configurations, it is possible to decouple the symmetric and nonsymmetric components of s-metrics (proven in [13, 40, 41]). This allows us to study independently their nonassociative geometric evolution models.

2. In general, such solutions are with nontrivial geometric flows of nonholonomic torsion. Nevertheless, we can always constrain such geometric flows of s-metrics to subclasses of τ-parametric families of generating data solving the equations (25) and/or (18), which allow to extract configurations with zero torsion. The remarks at the end of appendix A.1 state how to restrict the generating data (57) in order to restrict the nonolonomic flows to families of LC-connections ${}_\star^{\shortmid}\nabla(\tau)$ and ${}^{\shortmid}\nabla(\tau)$.

3. We can compute respective thermodynamic variables (51) associated to such quasi-stationary solutions. In next section, we shall provide such examples for BH configurations and their nonassociative star product R-flux deformatins.

4. The solutions for nonassociative Ricci soliton equations (47) consist self-similar configurations if the geometric flow constant is fixed, $\tau = \tau_0$, after a class of generic off-diagonal solutions is constructed in a form (55) and (56). Such s-metrics are characterized by fixed τ_0 geometric s-adapted values and thermodynamic variables stated in 1-3. Such Ricci soliton configurations can be generated equivalently by solutions constructed using the AFCDM as it is outlined in appendix A.

The class of off-diagonal solutions defined by (55) and (56) involves non-explicit nonholonomic constraints on temperature derivatives of certain s-metric coefficients, $\partial_\tau\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)$, encoded in effective sources ${}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau)$ (52). To decouple completely such formulas is possible only for more special classes of nonholonomic distributions or additional decompositions on small parameters, for instance, for certain types of BH nonassociative deformations to BE ones, ore other type configurations.

Finally, we note that the families of quasi-stationary s-metrics (55) are of type (54) with s-shell Killing symmetries on ∂_t, ${}^{\shortmid}\partial^6$, ${}^{\shortmid}\partial^8$. They are defined by s-adapted coefficients with respective "symbolic" phase co-ordinates and τ–dependencies of coefficients and generating/integration functions and effective sources. In a similar form (we have to change the "symbolic" phase coordinates and parametric dependencies of s-coefficients and respective functions), we can construct solutions with Killing symmetries on $\partial_t,\ {}^{\shortmid}\partial^6,\ {}^{\shortmid}\partial^7$, or $\partial_t,\ {}^{\shortmid}\partial^5,\ {}^{\shortmid}\partial^8$, or $\partial_t,\ {}^{\shortmid}\partial^5,\ {}^{\shortmid}\partial^7$. Such formulas have very similar physical interpretations if they are for metrics of the same signature. If we consider solutions, for instance, with symmetries on $\partial_3,\ {}^{\shortmid}\partial^6,\ {}^{\shortmid}\partial^8$, the s-metric and s-connection coefficients depend generically on a time like variable $y^4=t$. This allows to study the nonassociative geometric evolutions of locally anisotropic cosmological solutions. We plan to elaborate on nonassociative cosmological scenarios in our further partner works (see [17] and references therein for associative and commutative generic off-diagonal cosmological models).

4.2 Nonlinear symmetries and temperature running cosmological constants

Quasi-stationary s-metric (55) with N-connection coefficients (54) posses very important nonlinear symmetries which generalize for nonassociative geometric flow equations the nonlinear symmetries for Ricci solitons and vacuum gravitational equations stated in appendix A.2. Corresponding nonlinear transforms allow:

- to construct nonassociative nonholonomic geometric flow deformations of families of **prime** s-metrics ${}_s^{\shortmid}\mathring{\mathbf{g}}(\tau)$ (they can be arbitrary ones, i.e. not solutions of some (modified) Einstein equations) into a corresponding family of **target** s-metrics ${}_s^{\shortmid}\mathbf{g}(\tau)$ defining a nonassociative geometric flow evolution scenarios of quasi-stationary metrics on ${}_s^{\star}\mathcal{M}$,
$$ {}_s^{\shortmid}\mathring{\mathbf{g}}(\tau)\rightarrow\ {}_s^{\shortmid}\mathbf{g}(\tau)=[\ {}^{\shortmid}g_{\alpha_s}(\tau)=\ {}^{\shortmid}\eta_{\alpha_s}(\tau)\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau),\ {}^{\shortmid}N^{a_s}_{i_{s-1}}(\tau)=\ {}^{\shortmid}\eta^{a_s}_{i_{s-1}}(\tau)\ {}^{\shortmid}\mathring{N}^{a_s}_{i_{s-1}}(\tau)], \quad (59)$$
where the deformations with gravitational running on τ polarization functions are defined be formulas (A.5), (A.6) and (A.7) generalized for τ-dependencies of respective s-coefficients;
- to re-define the geometric flows of generating functions and relate the effective sources to certain effective shell τ-running cosmological constants,
$$\begin{array}{c}(\ {}_s\Psi(\tau),\ {}_s^{\shortmid}\Im(\tau))\leftrightarrow(\ {}_s^{\shortmid}\mathbf{g}(\tau),\ {}_s^{\shortmid}\Im(\tau))\leftrightarrow(\ {}_s^{\shortmid}\eta(\tau)\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau)\sim(\ {}^{\shortmid}\zeta_{\alpha_s}(\tau)(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s}(\tau))\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau),\ {}_s^{\shortmid}\Im(\tau))\leftrightarrow\\ (\ {}_s\Phi(\tau),\ {}_s^{\shortmid}\Lambda(\tau))\leftrightarrow(\ {}_s^{\shortmid}\mathbf{g},\ {}_s^{\shortmid}\Lambda(\tau))\leftrightarrow(\ {}_s^{\shortmid}\eta(\tau)\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau)\sim(\ {}^{\shortmid}\zeta_{\alpha_s}(\tau)(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s}(\tau))\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau),\ {}_s^{\shortmid}\Lambda(\tau)),\end{array}\quad (60)$$
where ${}_s^{\shortmid}\Lambda_0=\ {}_s^{\shortmid}\Lambda(\tau_0)$ for nonassociative Ricci soliton symmetries of type (A.8).

For simplicity, we study in this work only geometric flows with running of effective cosmological constants even the nonlinear symmetries of quasi-stationary solutions can be formulated for generalized phase space polarizations of cosmological constants ${}_s^{\shortmid}\Lambda(\tau,\ {}^{\shortmid}u^{\gamma_s})$. Such models involve re-definitions of effective generating sources ${}_s^{\shortmid}\Im(\tau,\ {}^{\shortmid}u^{\gamma_s})$ into another classes of effective sources/ cosmological constants ${}_s^{\shortmid}\Lambda(\tau,\ {}^{\shortmid}u^{\gamma_s})$.

Nonlinear transforms of flows of quasi-stationary s-metric (55) into equivalent ones with different classes of generating functions are described by introducing additional τ-dependencies in (A.9), when
$$\begin{aligned}\partial_3[(\ {}_2\Psi(\tau))^2] &= -\int dy^3(\ {}_2^{\shortmid}\Im(\tau))\partial_3 g_4(\tau)\simeq-\int dy^3(\ {}_2^{\shortmid}\Im(\tau))\partial_3(\ {}^{\shortmid}\eta_4(\tau)\ \mathring{g}_4(\tau))\\ &\simeq -\int dy^3(\ {}_2^{\shortmid}\Im(\tau))\partial_3[\ {}^{\shortmid}\zeta_4(\tau)(1+\kappa\ {}^{\shortmid}\chi_4(\tau))\ \mathring{g}_4(\tau)], \qquad (61)\\ (\ {}_2\Phi(\tau))^2 &= -4\ {}_2\Lambda(\tau)g_4(\tau)\simeq-4\ {}_2\Lambda(\tau)\ {}^{\shortmid}\eta_4(\tau)\ \mathring{g}_4(\tau)\\ &\simeq -4\ {}_2\Lambda(\tau)\ {}^{\shortmid}\zeta_4(\tau)(1+\kappa\ {}^{\shortmid}\chi_4(\tau))\ \mathring{g}_4(\tau);\end{aligned}$$
$$\begin{aligned}{}^{\shortmid}\partial^5[(\ {}_3^{\shortmid}\Psi(\tau))^2] &= -\int dp_5(\ {}_3^{\shortmid}\Im(\tau))\ {}^{\shortmid}\partial^5\ {}^{\shortmid}g^6(\tau)\simeq-\int dp_5(\ {}_3^{\shortmid}\Im(\tau))\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6(\tau))\\ &\simeq -\int dp_5(\ {}_3^{\shortmid}\Im(\tau))\ {}^{\shortmid}\partial^5[\ {}^{\shortmid}\zeta^6(\tau)(1+\kappa\ {}^{\shortmid}\chi^6(\tau))\ \mathring{g}^6(\tau)],\\ (\ {}_3^{\shortmid}\Phi(\tau))^2 &= -4\ {}_3^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}g^6(\tau)\simeq\ -4\ {}_3^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6(\tau)\\ &\simeq -4\ {}_3^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}\zeta^6(\tau)(1+\kappa\ {}^{\shortmid}\chi^6(\tau))\ {}^{\shortmid}\mathring{g}^6(\tau);\end{aligned}$$

$$
\begin{aligned}
{}^{|}\partial^7[(\,{}^{|}_4\Psi(\tau))^2] &= -\int dp_7(\,{}^{|}_4\Im(\tau))\,{}^{|}\partial^7\,{}^{|}g^8(\tau) \simeq -\int dp_7(\,{}^{|}_4\Im(\tau))\,{}^{|}\partial^7(\,{}^{|}\eta^8(\tau)\,{}^{|}\mathring{g}^8(\tau)) \\
&\simeq -\int dp_7(\,{}^{|}_4\Im(\tau))\,{}^{|}\partial^7[\,{}^{|}\zeta^8(\tau)(1+\kappa\,{}^{|}\chi^8(\tau))\,\mathring{g}^8(\tau)], \\
(\,{}^{|}_4\Phi(\tau))^2 &= -4\,{}^{|}_4\Lambda(\tau)\,{}^{|}g^8(\tau) \simeq -4\,{}^{|}_4\Lambda(\tau)\,{}^{|}\eta^8(\tau)\,{}^{|}\mathring{g}^8(\tau) \\
&\simeq -4\,{}^{|}_4\Lambda(\tau)\,{}^{|}\zeta^8(\tau)(1+\kappa\,{}^{|}\chi^8(\tau))\,{}^{|}\mathring{g}^8(\tau).
\end{aligned}
$$

We present the corresponding quadratic line elements for quasi-stationary geometric flow solutions defined by such transforms in next subsections.

4.3 Parametric solutions for nonassociative geometric flows with running cosmological constants

Nonlinear symmetries (60) allow us to change the generating functions and generating sources into certain new types of generating functions and effective cosmological data, $[\,{}_s\Psi(\tau),\,{}^{|}_s\Im(\tau)] \to [\,{}_s\Phi(\tau),\,{}^{|}_s\Lambda(\tau)]$. In result, the κ-linear parametric nonassociative geometric flow equations (46) (written as ${}^{|}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}(\tau,\,{}_s\Psi(\tau)) = \delta^{\beta_s}{}_{\gamma_s}\,{}^{|}_s\Im(\tau)$ (53) and integrated in quasi-stationary form using ${}_s\Psi(\tau))$ can be re-defined equivalently as a system of functional equations with ${}_s\Phi(\tau)$,

$$
{}^{|}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}(\tau,\,{}_s\Phi(\tau),\,{}^{|}_s\Im(\tau)) = \delta^{\beta_s}{}_{\gamma_s}\,{}^{|}_s\Lambda(\tau). \tag{62}
$$

We suppose that such nonlinear systems of PDEs are derived for certain shell effective τ-running constants ${}^{|}_s\Lambda(\tau)$ introduced for modelling geometric flow evolution processes. The solutions of (62) are τ-parametric generalizations of s-metrics (A.10) and generating data (A.11), ${}^{|}_s\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau),\,{}_s\Psi(\tau),\,{}^{|}_s\Im(\tau)] \to {}^{|}_s\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau),\,{}_s\Phi(\tau),\,{}^{|}_s\Lambda(\tau)]$. They are defined equivalently by the quasi-stationary quadratic element (55) with nonholonomic frames (56) transformed respectively into such values:

Using nonlinear formulas (61), the quasi-stationary solutions of τ-parametric running κ-linear modified Einstein equations (62) are defined by such quadratic linear elements:

$$
\begin{aligned}
d\widehat{s}^2(\tau) &= e^{\psi(\hbar,\kappa;\tau,x^{k_1})}[(dx^1)^2+(dx^2)^2]+ \\
&\quad -\frac{1}{g_4^{[0]}(\tau)-\frac{(\,{}_2\Phi(\tau))^2}{4\,{}_2\Lambda(\tau)}}\frac{(\,{}_2\Phi(\tau))^2[\partial_3(\,{}_2\Phi(\tau))]^2}{|\,{}_2\Lambda(\tau)\int dy^3(\,{}^{|}_2\Im(\tau))[\partial_3(\,{}_2\Phi(\tau))^2]|}(\mathbf{e}^3(\tau))^2+\left(g_4^{[0]}(\tau)-\frac{(\,{}_2\Phi(\tau))^2}{4\,{}_2\Lambda(\tau)}\right)(\mathbf{e}^4(\tau))^2 \\
&\quad -\frac{1}{{}^{|}g^6_{[0]}(\tau)-\frac{(\,{}^{|}_3\Phi(\tau))^2}{4\,{}^{|}_3\Lambda(\tau)}}\frac{(\,{}^{|}_3\Phi(\tau))^2[\,{}^{|}\partial^5(\,{}^{|}_3\Phi(\tau))]^2}{|\,{}^{|}_3\Lambda(\tau)\int dp_5(\,{}^{|}_3\Im(\tau))\,\,{}^{|}\partial^5[(\,{}^{|}_3\Phi(\tau))^2]|}(\,{}^{|}\mathbf{e}_5(\tau))^2+\left(\,{}^{|}g^6_{[0]}(\tau)-\frac{(\,{}^{|}_3\Phi(\tau))^2}{4\,{}^{|}_3\Lambda(\tau)}\right)(\,{}^{|}\mathbf{e}_6(\tau))^2 \\
&\quad -\frac{1}{{}^{|}g^8_{[0]}(\tau)-\frac{(\,{}^{|}_4\Phi(\tau))^2}{4\,{}^{|}_4\Lambda(\tau)}}\frac{(\,{}^{|}_4\Phi(\tau))^2[\,{}^{|}\partial^7(\,{}^{|}_4\Phi(\tau))]^2}{|\,{}^{|}_4\Lambda(\tau)\int dp_7(\,{}^{|}_4\Im)\,\,{}^{|}\partial^7[(\,{}^{|}_4\Phi(\tau))^2]|}(\,{}^{|}\mathbf{e}_7(\tau))^2+\left(\,{}^{|}g^8_{[0]}(\tau)-\frac{(\,{}^{|}_4\Phi(\tau))^2}{4\,{}^{|}_4\Lambda(\tau)}\right)(\,{}^{|}\mathbf{e}_8(\tau))^2,
\end{aligned} \tag{63}
$$

where the nonholonomic s-frames and respective off-diagonal terms are computed:

$$
\begin{aligned}
\mathbf{e}^3(\tau) &= dy^3+\frac{\partial_{k_1}\int dy^3(\,{}^{|}_2\Im(\tau))\,\partial_3[(\,{}_2\Phi(\tau))^2]}{(\,{}^{|}_2\Im(\tau))\,\partial_3[(\,{}_2\Phi(\tau))^2]}dx^{k_1}, \\
\mathbf{e}^4(\tau) &= dt+(\,{}_1n_{k_1}(\tau)+\,{}_2n_{k_1}(\tau)\frac{\int dy^3\frac{(\,{}_2\Phi(\tau))^2[\,\partial_3(\,{}_2\Phi(\tau))]^2}{|\,{}_2\Lambda(\tau)\int dy^3(\,{}^{|}_2\Im(\tau))[\,\partial_3(\,{}_2\Phi(\tau))^2]|}}{\left|g_4^{[0]}(\tau)-\frac{(\,{}_2\Phi(\tau))^2}{4\,{}_2\Lambda(\tau)}\right|^{5/2}})dx^{k_1},
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
{}^{\shortmid}\mathbf{e}_5(\tau) &= dp_5 + \frac{\partial_{k_2} \int dp_5({}_3^{\shortmid}\Im(\tau)) \ {}^{\shortmid}\partial^5[({}_3^{\shortmid}\Phi(\tau))^2]}{({}_3^{\shortmid}\Im(\tau)) \ {}^{\shortmid}\partial^5[({}_3^{\shortmid}\Phi(\tau))^2]} dx^{k_2}, \\
{}^{\shortmid}\mathbf{e}_6(\tau) &= dp_6 + ({}_1^{\shortmid}n_{k_2}(\tau) + {}_2^{\shortmid}n_{k_2}(\tau) \frac{\int dp_5 \frac{({}_3^{\shortmid}\Phi(\tau))^2 [{}^{\shortmid}\partial^5({}_3^{\shortmid}\Phi(\tau))]^2}{| {}_3^{\shortmid}\Lambda(\tau) \int dp_5({}_3^{\shortmid}\Im(\tau))[{}^{\shortmid}\partial^5({}_3^{\shortmid}\Phi(\tau))^2]|}}{\left| {}^{\shortmid}g^6_{[0]}(\tau) - \frac{({}_3^{\shortmid}\Phi(\tau))^2}{4 \ {}_3^{\shortmid}\Lambda(\tau)} \right|^{5/2}}) dx^{k_2},
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}\mathbf{e}_7(\tau) &= dp_7 + \frac{\partial_{k_3} \int dp_7({}_4^{\shortmid}\Im(\tau)) \ {}^{\shortmid}\partial^7[({}_4^{\shortmid}\Phi(\tau))^2]}{({}_4^{\shortmid}\Im(\tau)) \ {}^{\shortmid}\partial^7[({}_4^{\shortmid}\Phi(\tau))^2]} d\ {}^{\shortmid}x^{k_3}, \\
{}^{\shortmid}\mathbf{e}_8(\tau) &= dE + ({}_1^{\shortmid}n_{k_3}(\tau) + {}_2^{\shortmid}n_{k_3}(\tau) \frac{\int dp_7 \frac{({}_4^{\shortmid}\Phi(\tau))^2 [{}^{\shortmid}\partial^7({}_4^{\shortmid}\Phi(\tau))]^2}{| {}_4^{\shortmid}\Lambda(\tau) \int dp_7({}_4^{\shortmid}\Im(\tau))[{}^{\shortmid}\partial^7({}_4^{\shortmid}\Phi(\tau))^2]|}}{\left| {}^{\shortmid}g^8_{[0]}(\tau) - \frac{({}_4^{\shortmid}\Phi(\tau))^2}{4 \ {}_4^{\shortmid}\Lambda(\tau)} \right|^{5/2}}) d\ {}^{\shortmid}x^{k_3}.
\end{aligned}
$$

The conventions for the generating and integration functions and effective sources in s-metric coefficients (63) with nonholonomic frames (64) are those from (A.11) but with ${}_s^{\shortmid}\mathcal{K} \to {}_s^{\shortmid}\Im(\tau)$ and for the data $({}_s^{\shortmid}\Phi(\tau), {}_s^{\shortmid}\Lambda(\tau))$. Respective functionals can be constrained to define LC-configurations and model their nonassociative geometric evolution as we explain in point 2 of subsection 4.1. For all cases of nonlinear transforms (60), the functional representations of off-diagonal solutions allow to encode possible contributions from effective cosmological constants when certain evolution of effective sources is re-distributed into off-diagonal terms of s-metrics and with modifications of the diagonal s-adapted terms. The effective ${}_s^{\shortmid}\Im(\tau)$ are not completely substituted by effective τ-running constants ${}_s^{\shortmid}\Lambda(\tau)$ and both types of values are present in integrals for certain s-connection coefficients $g_3(\tau)$, ${}^{\shortmid}g^5(\tau)$, ${}^{\shortmid}g^8(\tau)$ and all N-connection coefficients in (64). Nevertheless, we can simplify substantially certain classes of solutions using ${}_s^{\shortmid}\Lambda(\tau)$ and then to speculate on their physical properties etc.

4.4 Flows with some s-metric coefficients as generating functions

We can generate quasi-stationary τ-running ansatz of type (A.1) prescribing $g_4(\tau)$, ${}^{\shortmid}g^6(\tau)$ and ${}^{\shortmid}g^8(\tau)$ as generating functions. For constructing κ-parametric Ricci soliton configurations, we can apply directly the procedure described in appendix A.4 and s-metrics (A.12). To study nonassociative geometric flow evolution we can consider such functionals for the generating functions and their nonlinear transforms (61):

$$
\begin{aligned}
g_4(\tau) &= g_4(\tau, x^{k_1}, y^3) = g_4[{}_2\Psi(\tau), {}_2^{\shortmid}\Im(\tau)] = g_4[{}_2\Phi(\tau), {}_2\Lambda(\tau)]; \\
{}^{\shortmid}g^6(\tau) &= {}^{\shortmid}g^6(\tau, x^{i_2}, p_5) = {}^{\shortmid}g^6[{}_3\Psi(\tau), {}_3^{\shortmid}\Im(\tau)] = {}^{\shortmid}g^6[{}_3^{\shortmid}\Phi(\tau), {}_3\Lambda(\tau)]; \\
{}^{\shortmid}g^8(\tau) &= {}^{\shortmid}g^8(\tau, x^{i_2}, p_6, E) = {}^{\shortmid}g^8[{}_4\Psi(\tau), {}_4^{\shortmid}\Im(\tau)] = {}^{\shortmid}g^8[{}_4^{\shortmid}\Phi(\tau), {}_4\Lambda(\tau)].
\end{aligned}
$$

For instance, expressing ${}_s\Psi(\tau) = {}_s\Psi[{}_s^{\shortmid}\Im(\tau), g_4(\tau), {}^{\shortmid}g^6(\tau), {}^{\shortmid}g^8(\tau)]$, we can compute the un-known coefficients of a s-metric (55) and N-coefficients (56). The resulting quadratic linear element is

$$
\begin{aligned}
d\widehat{s}^2(\tau) &= e^{\psi(\tau)}[(dx^1)^2 + (dx^2)^2] - \frac{(\partial_3 g_4(\tau))^2}{| \int dy^3 \partial_3[({}_2^{\shortmid}\Im(\tau)) g_4(\tau)]| \ g_4(\tau)} (\mathbf{e}^3(\tau))^2 + g_4(\tau)(\mathbf{e}^4(\tau))^2 \qquad (65) \\
& - \frac{[{}^{\shortmid}\partial^5({}^{\shortmid}g^6(\tau))]^2}{| \int dp_5 \ {}^{\shortmid}\partial^5[({}_3^{\shortmid}\Im(\tau)) \ {}^{\shortmid}g^6(\tau)] \ | \ {}^{\shortmid}g^6(\tau)} ({}^{\shortmid}\mathbf{e}_5(\tau))^2 + {}^{\shortmid}g^6(\tau)({}^{\shortmid}\mathbf{e}_6(\tau))^2 - \\
& - \frac{[{}^{\shortmid}\partial^7({}^{\shortmid}g^7(\tau))]^2}{| \int dp_7 \ {}^{\shortmid}\partial^7[({}_4^{\shortmid}\Im(\tau)) \ {}^{\shortmid}g^8(\tau)] \ | \ {}^{\shortmid}g^8(\tau)} ({}^{\shortmid}\mathbf{e}_7(\tau))^2 + {}^{\shortmid}g^8(\tau)({}^{\shortmid}\mathbf{e}_8(\tau))^2,
\end{aligned}
$$

where the nonholonomic s-frames and respective off-diagonal terms defined by N-connection coefficients are computed:

$$\mathbf{e}^3(\tau) = dy^3 + \frac{\partial_{k_1}[\int dy^3({}_2^{|}\Im(\tau))\ \partial_3 g_4(\tau)]}{({}_2^{|}\Im(\tau))\ \partial_3 g_4(\tau)} dx^{k_1}, \quad (66)$$

$$\mathbf{e}^4(\tau) = dt + ({}_1n_{k_1}(\tau) + {}_2n_{k_1}(\tau) \int dy^3 \frac{(\partial_3 g_4(\tau))^2}{|\int dy^3 \partial_3[({}_2^{|}\Im(\tau)) g_4(\tau)]|\ [g_4(\tau)]^{5/2}}) dx^{k_1},$$

$$^{|}\mathbf{e}_5(\tau) = dp_5 + \frac{\partial_{k_2}[\int dp_5({}_3^{|}\Im(\tau))\ ^{|}\partial^5(^{|}g^6(\tau))]}{({}_3^{|}\Im(\tau))\ ^{|}\partial^5(^{|}g^6(\tau))} dx^{k_2},$$

$$^{|}\mathbf{e}_6(\tau) = dp_6 + ({}_1^{|}n_{k_2}(\tau) + {}_2^{|}n_{k_2}(\tau) \int dp_5 \frac{[^{|}\partial^5(^{|}g^6(\tau))]^2}{|\int dp_5\ ^{|}\partial^5[({}_3^{|}\Im(\tau))\ ^{|}g^6(\tau)]|\ [^{|}g^6(\tau)]^{5/2}}) dx^{k_2},$$

$$^{|}\mathbf{e}_7(\tau) = dp_7 + \frac{^{|}\partial_{k_3}[\int dp_7({}_4^{|}\Im(\tau))\ ^{|}\partial^7(^{|}g^8(\tau))]}{({}_4^{|}\Im(\tau))\ ^{|}\partial^7(^{|}g^8(\tau))} d\ ^{|}x^{k_3},$$

$$^{|}\mathbf{e}_8(\tau) = dE + ({}_1^{|}n_{k_3}(\tau) + {}_2^{|}n_{k_3}(\tau) \int dp_7 \frac{[^{|}\partial^7(^{|}g^8(\tau))]^2}{|\int dp_7\ ^{|}\partial^7[({}_4^{|}\Im(\tau))\ ^{|}g^8(\tau)]|\ [^{|}g^8(\tau)]^{5/2}}) d\ ^{|}x^{k_3},$$

The s-adapted coefficients (65) and (66) define τ-flows of quasi-stationary solutions of type ${}_s^{|}\mathbf{g}[\hbar, \kappa, \psi, {}_s^{|}\mathcal{K}, g_4, {}^{|}g^6, {}^{|}g^8]$ in (A.9) evolving in s-adapted κ-linear parametric form to s-metrics ${}_s^{|}\mathbf{g}(\tau) \simeq {}_s^{|}\mathbf{g}[\hbar, \kappa, \psi(\tau), {}_s^{|}\Im(\tau), g_4(\tau), {}^{|}g^6(\tau), {}^{|}g^8(\tau)]$.

Above linear quadratic elements with respective s- and N-coefficients can be re-defined to include functional dependencies on running cosmological constants ${}_s^{|}\Lambda(\tau)$ if we begin with (63) and (64) and express, using respective formulas from (61), ${}_s\Phi(\tau) = {}_s\Phi[{}_s^{|}\Lambda(\tau), {}_s^{|}\Im, g_4(\tau), {}^{|}g^6(\tau), {}^{|}g^8(\tau)]$. This allows us to model the τ-evolution of nonassociative Ricci solitons of type ${}_s^{|}\mathbf{g}[\hbar, \kappa, \psi, {}_s^{|}\Lambda_0, {}_s^{|}\mathcal{K}, g_4, {}^{|}g^6, {}^{|}g^8]$, see (A.9), into generic off-diagonal solutions of nonassociative κ-linear parametric geometric flow equations (53) determined by families of quasi-stationary s-metrics ${}_s^{|}\mathbf{g}(\tau) \simeq {}_s^{|}\mathbf{g}[\hbar, \kappa, \psi(\tau), {}_s^{|}\Lambda(\tau), {}_s^{|}\Im(\tau), g_4(\tau), {}^{|}g^6(\tau), {}^{|}g^8(\tau)]$.

4.5 Quasi-stationary nonassociative evolution via gravitational polarizations

How to construct quasi-stationary solutions for nonassociative Ricci soliton and vacuum gravitational equations using the AFCDM with gravitational polarization functions is discussed in appendix A.5. Here, we extend the approach for generating τ-parametric quasi-stationary solutions for nonassociative geometric flow equations (53).

Off-diagonal deformations of a family of prescribed prime metric into other families of target ones, ${}_s^{|}\mathring{\mathbf{g}}(\tau) = [\ ^{|}\mathring{g}_{\alpha_s}(\tau),\ ^{|}\mathring{N}^{a_s}_{i_{s-1}}(\tau)] \to {}_s^{|}\mathbf{g}(\tau)$ (59) described by η-polarizations (A.6), can be defined by such τ-parametric generating functions

$$\psi(\tau) \simeq \psi(\hbar, \kappa; \tau, x^{k_1}), \eta_4(\tau) \simeq \eta_4(\tau, x^{k_1}, y^3),\ ^{|}\eta^6(\tau) \simeq\ ^{|}\eta^6(\tau, x^{i_2}, p_5),\ ^{|}\eta^8(\tau) \simeq\ ^{|}\eta^8(\tau, x^{i_2}, p_5, p_7). \quad (67)$$

As a result, we generalize in real variables and for nonassociative geometric flows the quadratic line element constructed in Appendix B2, formula (B2), to [13]:

$$\begin{aligned} d\ ^{|}\widehat{s}^2(\tau) &= {}^{|}g_{\alpha_s\beta_s}(\hbar, \kappa, \tau, x^k, y^3, p_{a_3}, p_{a_4};\ ^{|}\mathring{g}_{\alpha_s}(\tau); \eta_4(\tau),\ ^{|}\eta^6(\tau),\ ^{|}\eta^8(\tau),\ {}_s^{|}\Lambda(\tau);\ {}_s^{|}\Im(\tau)) d\ ^{|}u^{\alpha_s} d\ ^{|}u^{\beta_s} \quad (68)\\ &= e^{\psi(\tau)}[(dx^1)^2 + (dx^2)^2] - \\ &\frac{[\partial_3(\eta_4(\tau)\ \mathring{g}_4(\tau))]^2}{|\int dy^3\ {}_2\Im(\tau)\partial_3(\eta_4(\tau)\ \mathring{g}_4(\tau))|\ (\eta_4(\tau)\mathring{g}_4(\tau))} \{dy^3 + \frac{\partial_{i_1}[\int dy^3\ {}_2\Im(\tau)\ \partial_3(\eta_4(\tau)\mathring{g}_4(\tau))]}{{}_2\Im(\tau)\partial_3(\eta_4(\tau)\mathring{g}_4(\tau))} dx^{i_1}\}^2 + \\ &\eta_4(\tau)\mathring{g}_4(\tau))\{dt + [\ {}_1n_{k_1}(\tau) + {}_2n_{k_1}(\tau) \int dy^3 \frac{[\partial_3(\eta_4(\tau)\mathring{g}_4(\tau))]^2}{|\int dy^3\ {}_2\Im(\tau)\partial_3(\eta_4(\tau)\mathring{g}_4(\tau))|\ (\eta_4(\tau)\mathring{g}_4(\tau))^{5/2}}] dx^{k_1}\} \end{aligned}$$

$$-\frac{[\ ^{\shortmid}\partial^5(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))]^2}{|\int dp_5\ ^{\shortmid}_3\Im(\tau)\ ^{\shortmid}\partial^5(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))\ |\ (\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))}\{dp_5+\frac{^{\shortmid}\partial_{i_2}[\int dp_5\ ^{\shortmid}_3\Im(\tau)\ ^{\shortmid}\partial^5(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))]}{^{\shortmid}_3\Im(\tau)\ ^{\shortmid}\partial^5(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))}dx^{i_2}\}^2$$
$$+(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))\{dp_6+[\ ^{\shortmid}_1n_{k_2}(\tau)+$$
$$^{\shortmid}_2n_{k_2}(\tau)\int dp_5\frac{[\ ^{\shortmid}\partial^5(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))]^2}{|\int dp_5\ ^{\shortmid}_3\Im(\tau)\ \partial^5(\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))|\ (\ ^{\shortmid}\eta^6(\tau)\ ^{\shortmid}\mathring{g}^6(\tau))^{5/2}}]dx^{k_2}\}$$

$$-\frac{[\ ^{\shortmid}\partial^7(\ ^{\shortmid}\eta^8(\tau)\ ^{\shortmid}\mathring{g}^8(\tau))]^2}{|\int dp_7\ ^{\shortmid}_4\Im(\tau)\ ^{\shortmid}\partial^8(\ ^{\shortmid}\eta^7(\tau)\ ^{\shortmid}\mathring{g}^7(\tau))\ |\ (\ ^{\shortmid}\eta^7(\tau)\ ^{\shortmid}\mathring{g}^7(\tau))}\{dp_7+\frac{^{\shortmid}\partial_{i_3}[\int dp_7\ ^{\shortmid}_4\Im(\tau)\ \ ^{\shortmid}\partial^7(\ ^{\shortmid}\eta^8(\tau)\ \ ^{\shortmid}\mathring{g}^8(\tau))]}{^{\shortmid}_4\Im(\tau)\ ^{\shortmid}\partial^7(\ ^{\shortmid}\eta^8(\tau)\ ^{\shortmid}\mathring{g}^8(\tau))}d\ ^{\shortmid}x^{i_3}\}^2$$
$$+(\ ^{\shortmid}\eta^8(\tau)\ ^{\shortmid}\mathring{g}^8(\tau))\{dE+[\ _1n_{k_3}(\tau)+$$
$$_2n_{k_3}(\tau)\int dp_7\frac{[\ ^{\shortmid}\partial^7(\ ^{\shortmid}\eta^8(\tau)\ ^{\shortmid}\mathring{g}^8(\tau))]^2}{|\int dp_7\ ^{\shortmid}_4\Im(\tau)[\ ^{\shortmid}\partial^7(\ ^{\shortmid}\eta^8(\tau)\ ^{\shortmid}\mathring{g}^8(\tau))]|\ [(\ ^{\shortmid}\eta^8(\tau)\ ^{\shortmid}\mathring{g}^8(\tau))]^{5/2}}]d\ ^{\shortmid}x^{k_3}\}.$$

The gravitational polarization η–functions describe transforms of certain classes of prime s-metrics into other types of target s-metrics. We can prescribe respective geometric/ physical properties and investigate how geometric flows may relate the evolution of such configurations. Respective formulas for small parametric nonassociative geometric flow deformations of type (68) when $^{\shortmid}_s\eta(\tau)\ ^{\shortmid}\mathring{g}_{\alpha_s}(\tau)\sim\ ^{\shortmid}\zeta_{\alpha_s}(\tau)(1+\kappa\ ^{\shortmid}\chi_{\alpha_s}(\tau))\ \ ^{\shortmid}\mathring{g}_{\alpha_s}(\tau)$ (60) are provided in appendix A.5.

5 Modified Bekenstein–Hawking and G. Perelman thermodynamics of BH solutions deformed by nonassociative geometric flows

Applying nonholonomic geometric flow methods [6, 7, 8], we concluded [41] that solutions for nonassociative star product R-flux deformations of the Tangerlini higher dimension BHs and double Schwarzschild BHs can be described in the framework of a (modified) κ-linear parametric Perelman's statistical thermodynamic model encoding nonassociative data. In section 3, we generalized the approach to a theory of nonassociative Ricci flows and formulated a statistical thermodynamic model using decoupling properties of respective systems of nonlinear PDEs involving for the canonical s-connection structure.

The goal of this section is to elaborate on explicit models of nonassociative flow evolution of defined by quasi-stationary parametric solutions. We construct and analyze important geometric and physical properties of two new classes of nonassociative modified BH solutions in the framework of nonassociative geometric evolution theory. For a fixed evolution parameter τ_0, corresponding s-metrics describe κ-linear deformations of the double Reisner-Nordström de Sitter, RN-dS, metrics and their 'dissipation' into off-diagonal terms; or couples of Schwarzschild - AdS BHs deformed to black ellipsoid, BE, configurations; or higher dimension RN anti de Sitter, RN-AdS, configurations. Such quasi-stationary generic off-diagonal solutions present explicit examples of nonassociative Ricci solitons, which can be extended to describe geometric flow evolution scenarios on temperature like τ-parameter.

For very special cases of nonholonomic deformations (for instance, defining BE configurations and/or other variants with geometric evolving hyper-surface horizons), we can apply the concept of Bekenstein-Hawking entropy [47, 48, 49, 50]. In another turn, the generalized G. Perelman W-entropy and thermodynamic variables [1, 6, 7, 8, 41] can be defined and computed for all classes of solutions in nonassociative geometric flow and (modified) gravity theories. We prove this by providing explicit examples how to compute thermodynamic variables for general quasi-stationary τ–deformations of nonassociative modified RN-(A)dS, BE deformations and Schwarzschild-(A)dS metrics.

5.1 Geometric flow thermodynamics of nonassociative quasi-stationary solutions with running cosmological constants

Statistical G. Perelman thermodynamic models can be defined for any nonassociative geometric flow data $[\ _s^{\shortmid}\mathfrak{g}^{\star}(\tau),\ _s^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau),\ _s^{\shortmid}\widehat{f}(\tau)]$ as we explain in section 2. For κ-linear parametric decompositions, the corresponding thermodynamic variables are computed using the formulas for $\ _s^{\shortmid}\widehat{\mathcal{E}}_{\kappa}^{\star}$ and $\ _s^{\shortmid}\widehat{S}_{\kappa}^{\star}$ from (51). In this work, we do not provide cumbersome formulas for computing quadratic fluctuation parameter $\ _s^{\shortmid}\widehat{\sigma}_{\kappa}^{\star})$. The nonassociative thermodynamic variables are derived for a W-entropy $\ _s^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$ (45) using a respective statistical generating function $\ _s^{\shortmid}\widehat{Z}(\tau)$ (48). We can prescribe the nonholonomic structure on phase space $\ _s^{\star}\mathcal{M}$ and the normalizing functions $\ _s^{\shortmid}\widehat{f}(\tau)$ in such forms that all basic formulas are determined by a volume form $^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau)$ (49).

For exact/parametric solutions of nonassociative geometric flow equations, we can compute the thermodynamic variables corresponding to τ–modified Einstein equations (53), with effective sources $\ _s^{\shortmid}\Im(\tau)$, and/or (62) with running effective cosmological constants $\ _s^{\shortmid}\Lambda(\tau)$. In the first approach, we can formulate a G. Perelman thermodynamic model for quasi-stationary solutions taken in the form (55) with N-coefficients (56). This results in cumbersome formulas for thermodynamic variables which are not appropriate for investigating, for instance, physical problems related to the swampland conjecture [20, 23, 24, 25, 26, 27]. Using nonlinear symmetries (60), with $\ _s^{\shortmid}\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau),\ _s\Psi(\tau),\ _s^{\shortmid}\Im(\tau)] \rightarrow\ _s^{\shortmid}\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau),\ _s\Phi(\tau),\ _s^{\shortmid}\Lambda(\tau)]$, we may develop a second approach involving quasi-stationary solutions of type (A.10). This allows us to simplify the formulas for explicit computation of geometric thermodynamic variables and elaborate on physical models with running/fixed effective cosmological constants $\ _s^{\shortmid}\Lambda(\tau)$.

The nonassociative geometric flow thermodynamic variables defined for a temperature parameter $\tau, 0 < \tau' \leq \tau$, with prescribed constant normalizing functions and for a volume form $^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau)$ computed for quasi-stationary data $[_s\Phi(\tau),\ _s^{\shortmid}\Lambda(\tau)]$, are expressed in the form

$$
\begin{aligned}
\ _s^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau) &= \int_{\tau'}^{\tau} \frac{d\tau}{(4\pi\tau)^4} \int_{\ _s^{\shortmid}\widehat{\Xi}} \left(\tau[\textstyle\sum_s\ _s^{\shortmid}\Lambda(\tau)]^2 - 8\right)\ ^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau), \qquad (69)\\
\ _s^{\shortmid}\widehat{\mathcal{Z}}_{\kappa}^{\star}(\tau) &= \exp\left[\int_{\tau'}^{\tau} \frac{d\tau}{(2\pi\tau)^4} \int_{\ _s^{\shortmid}\widehat{\Xi}}\ ^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau)\right],\\
\ _s^{\shortmid}\widehat{\mathcal{E}}_{\kappa}^{\star}(\tau) &= -\int_{\tau'}^{\tau} \frac{d\tau}{(4\pi)^4\tau^2} \int_{\ _s^{\shortmid}\widehat{\Xi}} \left([\textstyle\sum_s\ _s^{\shortmid}\Lambda(\tau)] - \frac{4}{\tau}\right)\ ^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau),\\
\ _s^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau) &= -\int_{\tau'}^{\tau} \frac{d\tau}{(4\pi\tau)^4} \int_{\ _s^{\shortmid}\widehat{\Xi}} \left(\tau[\textstyle\sum_s\ _s^{\shortmid}\Lambda(\tau)] - 8\right)\ ^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau).
\end{aligned}
$$

The integration on τ parameter and respective 8-d hyper-surface integrals in (69) should be defined for nonholonomic s-distributions which result in well-defined relativistic thermodynamic values.[16] Such constructions were considered for a fixed temperature parameter τ_0 and nonassociative Ricci solitons in [41] (see formulas (60) and (61) in that work).

For any prescribed data (τ',τ), $\ _s^{\shortmid}\widehat{\Xi}$ and $\ _s^{\shortmid}\Lambda(\tau)$, the thermodynamic variables (69) are determined by a volume form $^{\shortmid}\delta\ ^{\shortmid}\mathcal{V}(\tau)$ which must be computed for a chosen class of exact/parametric solutions of nonassociative geometric flow equations. We can elaborate an explicit geometric integration formalism adapted to nonassociative and nonholonomic distributions, and parametric deformations, if we choose, for instance some primary data $\ _s^{\shortmid}\mathring{\mathfrak{g}}^{\star}$ and study possible flow evolution scenarios to certain target $\ _s^{\shortmid}\mathfrak{g}^{\star}(\tau)$. For a τ-family of

[16] For instance, we must exclude un-physical configurations with negative entropy; additionally, we should analyze and select on more optimal energetic regimes and stability conditions; and construct thermodynamic models with causal evolution on cotangent Lorentz bundles etc.

s-metrics (63) with respect to nonholonomic frames (64), we can define the volume functional

$$
\begin{aligned}
{}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau) &= {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}[\tau,\ {}_{s}^{\shortmid}\Lambda(\tau),\ {}_{s}^{\shortmid}\Im(\tau);\psi(\tau),\ {}_{s}^{\shortmid}\Phi(\tau)] \qquad (70)\\
&= e^{\psi(\tau)}\frac{|\ {}_{2}\Phi(\tau)\partial_3[\ {}_{2}\Phi(\tau)]^2|}{|\ {}_{2}\Lambda(\tau)\int dy^3\ {}_{2}\Im(\tau)\{\partial_3[\ {}_{2}\Phi(\tau)]^2\}^2|^{1/2}}[dy^3+\frac{\partial_{i_1}\left(\int dy^3\ {}_{2}\Im(\tau)\partial_3[\ {}_{2}\Phi(\tau)]^2\right)}{{}_{2}\Im(\tau)\partial_3[\ {}_{2}\Phi(\tau)]^2}dx^{i_1}]dx^1dx^2dt\\
&\quad \frac{|\ {}_{3}^{\shortmid}\Phi(\tau)\ {}^{\shortmid}\partial^5[\ {}_{3}^{\shortmid}\Phi(\tau)]^2|}{|\ {}_{3}^{\shortmid}\Lambda(\tau)\int dp_5\ {}_{3}^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^5[\ {}_{3}^{\shortmid}\Phi(\tau)]^2|^{1/2}}[dp_5+\frac{{}^{\shortmid}\partial_{i_2}\left(\int dp_5\ {}_{3}^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^5[\ {}_{3}^{\shortmid}\Phi(\tau)]^2\right)}{{}_{3}^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^5[\ {}_{3}^{\shortmid}\Phi(\tau)]^2}dx^{i_2}]dp_6\\
&\quad \frac{|\ {}_{4}^{\shortmid}\Phi(\tau)\ {}^{\shortmid}\partial^7[\ {}_{4}^{\shortmid}\Phi(\tau)]^2|}{|\ {}_{4}^{\shortmid}\Lambda(\tau)\int dp_7\ \ {}_{4}^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7[\ \ {}_{4}^{\shortmid}\Phi(\tau)]^2|^{1/2}}[dp_7+\frac{{}^{\shortmid}\partial_{i_3}\left(\int dp_7\ {}_{4}^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7[\ {}_{4}^{\shortmid}\Phi(\tau)]^2\right)}{{}_{4}^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7[\ {}_{4}^{\shortmid}\Phi(\tau)]^2}dx^{i_3}]dE.
\end{aligned}
$$

To compute in explicit forms and study properties of such volume functionals we can consider, for simplicity, nonholonomic evolution models with trivial integration functions ${}_{1}^{\shortmid}n_{k_s}=0$ and ${}_{2}^{\shortmid}n_{k_s}=0$. The formulas for ${}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau)$ (70) can be computed for other classes of solutions determined by g-generating functions and/or η-/χ-polarization functions using (61), when

$$
\begin{aligned}
{}_{2}\Phi(\tau) &= 2\sqrt{|\ {}_{2}\Lambda(\tau)\ g_4(\tau)|} = 2\sqrt{|\ {}_{2}\Lambda(\tau)\ \eta_4(\tau)\mathring{g}_4(\tau)|} \simeq 2\sqrt{|\ {}_{2}\Lambda(\tau)\ \zeta_4(\tau)\mathring{g}_4(\tau)|}[1-\frac{\kappa}{2}\chi_4(\tau)], \qquad (71)\\
{}_{3}^{\shortmid}\Phi(\tau) &= 2\sqrt{|\ {}_{3}^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}g^6(\tau)|} = 2\sqrt{|\ {}_{3}^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6(\tau)|} \simeq 2\sqrt{|\ {}_{3}^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}\zeta^6(\tau)\ {}^{\shortmid}\mathring{g}^6(\tau)|}[1-\frac{\kappa}{2}\ {}^{\shortmid}\chi^6(\tau)],\\
{}_{4}^{\shortmid}\Phi(\tau) &= 2\sqrt{|\ {}_{4}^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}g^8(\tau)|} = 2\sqrt{|\ {}_{4}^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\mathring{g}^8(\tau)|} \simeq 2\sqrt{|\ {}_{4}^{\shortmid}\Lambda(\tau)\ {}^{\shortmid}\zeta^8(\tau)\ {}^{\shortmid}\mathring{g}^8(\tau)|}[1-\frac{\kappa}{2}\ {}^{\shortmid}\chi^8(\tau)],
\end{aligned}
$$

for a prime s-metric ${}_{\shortmid}\mathring{g}_{\alpha}(\tau)=(\mathring{g}_i(\tau),\ {}_{\shortmid}\mathring{g}^a(\tau))$. Introducing formulas (71) involving η-polarizations in (70), then separating terms with shell τ-running cosmological constants, we express:

$$
\begin{aligned}
{}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V} &= {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}[\tau,\ {}_{s}^{\shortmid}\Lambda(\tau),\ {}_{s}^{\shortmid}\Im(\tau);\psi(\tau),\ g_4(\tau),\ {}^{\shortmid}g^6(\tau),\ {}^{\shortmid}g^8(\tau)] = {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\ {}_{s}\Im(\tau),\ {}_{s}^{\shortmid}\Lambda(\tau),\ {}^{\shortmid}\eta_{\alpha_s}(\tau)\ {}^{\shortmid}\mathring{g}_{\alpha_s})\\
&= \frac{1}{\sqrt{|\ {}_{1}\Lambda(\tau)\ {}_{2}\Lambda(\tau)\ {}_{3}^{\shortmid}\Lambda(\tau)\ {}_{4}^{\shortmid}\Lambda(\tau)|}}\ {}^{\shortmid}\delta\ {}_{\eta}^{\shortmid}\mathcal{V},\ \text{where}\ {}^{\shortmid}\delta\ {}_{\eta}^{\shortmid}\mathcal{V} = {}^{\shortmid}\delta\ {}_{\eta}^{1}\mathcal{V}\times\ {}^{\shortmid}\delta\ {}_{\eta}^{2}\mathcal{V}\times\ {}^{\shortmid}\delta\ {}_{\eta}^{3}\mathcal{V}\times\ {}^{\shortmid}\delta\ {}_{\eta}^{4}\mathcal{V}.
\end{aligned}
$$

In these formulas, there are used the functionals:

$$
\begin{aligned}
{}^{\shortmid}\delta\ {}_{\eta}^{1}\mathcal{V} &= {}^{\shortmid}\delta\ {}_{\eta}^{1}\mathcal{V}[\ {}_{1}\Im(\tau),\eta_2(\tau)\ \mathring{g}_2] \qquad (72)\\
&= \frac{16}{3}e^{\widetilde{\psi}(\tau)}dx^1dx^2 = \frac{3}{16}\sqrt{|\ {}_{1}\Lambda(\tau)|}e^{\psi(\tau)}dx^1dx^2,\ \text{for } \psi(\tau) \text{ being a solution of (58)},\\
{}^{\shortmid}\delta\ {}_{\eta}^{2}\mathcal{V} &= {}^{\shortmid}\delta\ {}_{\eta}^{2}\mathcal{V}[\ {}_{2}\Im(\tau),\eta_4(\tau)\ \mathring{g}_4]\\
&= \frac{16}{3}\frac{\partial_3|\ \eta_4(\tau)\ \mathring{g}_4|^{3/2}}{\sqrt{|\int dy^3\ {}_{2}\Im(\tau)\{\partial_3|\ \eta_4(\tau)\ \mathring{g}_4|\}^2|}}[dy^3+\frac{\partial_{i_1}\left(\int dy^3\ {}_{2}\Im(\tau)\partial_3|\ \eta_4(\tau)\ \mathring{g}_4|\right)dx^{i_1}}{{}_{2}\Im(\tau)\partial_3|\ \eta_4(\tau)\ \mathring{g}_4|}]dt,\\
{}^{\shortmid}\delta\ {}_{\eta}^{3}\mathcal{V} &= {}^{\shortmid}\delta\ {}_{\eta}^{3}\mathcal{V}[\ {}_{3}\Im(\tau),\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6]\\
&= \frac{16}{3}\frac{{}^{\shortmid}\partial^5|\ \ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6|^{3/2}}{\sqrt{|\int dp_5\ {}_{3}\Im(\tau)\{\ {}^{\shortmid}\partial^5|\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6|\}^2|}}[dp_5+\frac{\partial_{i_2}\left(\int dp_5\ {}_{2}\Im(\tau)\ {}^{\shortmid}\partial^5|\ \ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6|\right)dx^{i_2}}{{}_{2}\Im(\tau)\ {}^{\shortmid}\partial^5|\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6|}]dp_6,\\
{}^{\shortmid}\delta\ {}_{\eta}^{4}\mathcal{V} &= {}^{\shortmid}\delta\ {}_{\eta}^{4}\mathcal{V}[\ {}_{4}\Im(\tau),\ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\mathring{g}^8]\\
&= \frac{16}{3}\frac{{}^{\shortmid}\partial^7|\ \ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\mathring{g}^8|^{3/2}}{\sqrt{|\int dp_7\ {}_{4}\Im(\tau)\{\ {}^{\shortmid}\partial^7|\ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\mathring{g}^8|\}^2|}}[dp_7+\frac{\partial_{i_3}\left(\int dp_7\ {}_{3}\Im(\tau)\ {}^{\shortmid}\partial^7|\ \ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\mathring{g}^8|\right)}{{}_{2}\Im(\tau)\ {}^{\shortmid}\partial^7|\ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\mathring{g}^8|}dx^{i_3}]dE.
\end{aligned}
$$

The G. Perelman thermodynamic variables (69) computed for the volume functionals can be expressed as thermodynamic functionals:

$$
\begin{aligned}
{}_s^{\shortmid}\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau) &= \int_{\tau'}^{\tau} \frac{d\tau}{(4\pi\tau)^4} \frac{\tau[\,{}_1\Lambda(\tau) + \,{}_2\Lambda(\tau) + \,{}_3^{\shortmid}\Lambda(\tau) + \,{}_4^{\shortmid}\Lambda(\tau)]^2 - 8}{\sqrt{|\,{}_1\Lambda(\tau)\,{}_2\Lambda(\tau)\,{}_3^{\shortmid}\Lambda(\tau)\,{}_4^{\shortmid}\Lambda(\tau)|}}\ {}_{\eta}^{\shortmid}\mathring{\mathcal{V}}(\tau), \qquad (73)\\
{}_s^{\shortmid}\widehat{\mathcal{Z}}^{\star}_{\kappa}(\tau) &= \exp\left[\int_{\tau'}^{\tau} \frac{d\tau}{(2\pi\tau)^4} \frac{1}{\sqrt{|\,{}_1\Lambda(\tau)\,{}_2\Lambda(\tau)\,{}_3^{\shortmid}\Lambda(\tau)\,{}_4^{\shortmid}\Lambda(\tau)|}}\ {}_{\eta}^{\shortmid}\mathring{\mathcal{V}}(\tau)\right],\\
{}_s^{\shortmid}\widehat{\mathcal{E}}^{\star}_{\kappa}(\tau) &= -\int_{\tau'}^{\tau} \frac{d\tau}{(4\pi)^4\tau^3} \frac{\tau[\,{}_1\Lambda(\tau) + \,{}_2\Lambda(\tau) + \,{}_3^{\shortmid}\Lambda(\tau) + \,{}_4^{\shortmid}\Lambda(\tau)] - 4}{\sqrt{|\,{}_1\Lambda(\tau)\,{}_2\Lambda(\tau)\,{}_3^{\shortmid}\Lambda(\tau)\,{}_4^{\shortmid}\Lambda(\tau)|}}\ {}_{\eta}^{\shortmid}\mathring{\mathcal{V}}(\tau),\\
{}_s^{\shortmid}\widehat{\mathcal{S}}^{\star}_{\kappa}(\tau) &= -\int_{\tau'}^{\tau} \frac{d\tau}{(4\pi\tau)^4} \frac{\tau[\,{}_1\Lambda(\tau) + \,{}_2\Lambda(\tau) + \,{}_3^{\shortmid}\Lambda(\tau) + \,{}_4^{\shortmid}\Lambda(\tau)] - 8}{\sqrt{|\,{}_1\Lambda(\tau)\,{}_2\Lambda(\tau)\,{}_3^{\shortmid}\Lambda(\tau)\,{}_4^{\shortmid}\Lambda(\tau)|}}\ {}_{\eta}^{\shortmid}\mathring{\mathcal{V}}(\tau).
\end{aligned}
$$

In these formulas, we use the running phase space volume functional

$$
{}_{\eta}^{\shortmid}\mathring{\mathcal{V}}(\tau) = \int_{{}_s^{\shortmid}\widehat{\Xi}} {}^{\shortmid}\delta\ {}_{\eta}^{\shortmid}\mathcal{V}(\,{}_s^{\shortmid}\Im(\tau),\ {}^{\shortmid}\mathring{g}_{\alpha_s}) \qquad (74)
$$

determined by prescribed classes of generating η-functions, effective generating sources ${}_s^{\shortmid}\Im(\tau)$, coefficients of a prime s-metric ${}^{\shortmid}\mathring{g}_{\alpha_s}$ and nonholonomic distributions defining the hyper-surface ${}_s^{\shortmid}\widehat{\Xi}$.

We can define the effective volume functionals (72) and geometric thermodynamic variables (73) for further parametric decompositions with κ-linear approximations (71) and χ-polarizations and find parametric formulas for τ-flows and nonassociative R-flux deformations of prime metrics,

$$
\begin{aligned}
{}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V} &= {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}_0[\tau,\ {}_s^{\shortmid}\Lambda(\tau),\ \ {}_s^{\shortmid}\Im(\tau);\psi(\tau),\ \mathring{g}_{i_1},\ \mathring{g}_{a_2},\ {}^{\shortmid}\mathring{g}^{a_3},\ {}^{\shortmid}\mathring{g}^{a_4};\zeta_4(\tau),\ {}^{\shortmid}\zeta^6(\tau),\ {}^{\shortmid}\zeta^8(\tau)] + \qquad (75)\\
& \kappa\ {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}_1[\tau,\ {}_s^{\shortmid}\Lambda(\tau),\ \ {}_s^{\shortmid}\Im(\tau);\psi(\tau),\ \mathring{g}_{i_1},\ \mathring{g}_{a_2},\ {}^{\shortmid}\mathring{g}^{a_3},\ {}^{\shortmid}\mathring{g}^{a_4};\zeta_4(\tau),\ {}^{\shortmid}\zeta^6(\tau),\ {}^{\shortmid}\zeta^8(\tau),\chi_4(\tau),\ {}^{\shortmid}\chi^6(\tau),\ {}^{\shortmid}\chi^8(\tau)].
\end{aligned}
$$

Introducing a quasi-stationary parametric solution in (75) for nonassociative R. Hamilton equations, we can compute corresponding κ-decompositions of the thermodynamic variables (69),

$$
{}_s^{\shortmid}\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau) = {}_s^{\shortmid}\widehat{\mathcal{W}}_0 + \kappa\,{}_s^{\shortmid}\widehat{\mathcal{W}}^{\star}_1(\tau),\ {}_s^{\shortmid}\widehat{\mathcal{Z}}^{\star}_{\kappa}(\tau) = {}_s^{\shortmid}\widehat{\mathcal{Z}}_0\,{}_s^{\shortmid}\widehat{\mathcal{Z}}^{\star}_1(\tau),\ {}_s^{\shortmid}\widehat{\mathcal{E}}^{\star}_{\kappa}(\tau) = {}_s^{\shortmid}\widehat{\mathcal{E}}_0 + \kappa\,{}_s^{\shortmid}\widehat{\mathcal{E}}^{\star}_1(\tau),\ {}_s^{\shortmid}\widehat{\mathcal{S}}^{\star}_{\kappa}(\tau) = {}_s^{\shortmid}\widehat{\mathcal{S}}_0 + \kappa\,{}_s^{\shortmid}\widehat{\mathcal{S}}^{\star}_1(\tau). \qquad (76)
$$

In this work, there are not presented cumbersome computations and incremental formulas with κ–linear decomposition for ${}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V} = {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}_0 + \kappa\ {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}_1$ (75) and (76) (considered for solutions of type (84), with χ-polarization functions). We consider a more general and compact approach when the thermodynamic variables (69) are computed for volume forms (70) using quasi-stationary solutions of type (63), (65), or (68).

5.2 Geometric evolution of nonassociative double Reisner-Nordström-(A)dS BHs in phase spaces

In a series of recent works [23, 24, 25, 26, 27], certain models of geometric flows of the Schwarzschild-AdS, RN and other type metrics were studied in connection to the swampland program [19, 20, 21, 18]. Those papers are devoted to associative and commutative geometric and physical theories with solutions which can be characterized by the Bekenstein-Hawking entropy. In this subsection, we consider nonassociative generalizations and nonholonomic geometric flow deformations of the 4-d RN-dS metrics dubbed both on the base spacetime and typical cofiber and star-deformed by a 8-d phase spaces evolution. We cite [51, 52], for fundamental results on RN BHs (see also monographs [43, 44, 45, 46]), and [53, 54] and references therein, on higher dimension extensions for RN-(A)dS.

The geometric thermodynamic variables (85) and (89) can be used for constructing an effective thermodynamic model for quasi-stationary evolution in a conventional phase space media of double BHs into respective BE configurations. Such constructions for nonassociative geometric flow and/or off-diagonal deformation scenarios can be performed for very special classes of nonholonomic constraints, when the existence of corresponding horizon hyper-surfaces allows us to describe the τ-evolution of such physical objects in the framework of generalized Bekenstein-Hawking thermodynamics. For explicit models non involving prescribed hyper-horizons and/or duality conditions for Ricci solitons and/or geometric flows, even we work with quasi-stationary s-metrics, the concept of Hawking entropy is not applicable and we have to elaborate on other types of statistical and geometric thermodynamic theories. In section 4 of [41], we concluded that in general form we have to change the paradigm and characterize nonassociative geometric flow and gravitational theories (and various classes of related physically important solutions) in the framework of modified G. Perelman thermodynamics [1].

5.2.1 Prime metrics for phase space double RN-dS BHs

Let us consider a 4-d base Lorentz spacetime manifold V on which the Einstein-Maxwell theory is defined by the action for a metric $g_{\alpha\beta}$ and electromagnetic field A_μ,

$$S = \int_V d^4x\sqrt{|g|}[\frac{1}{16\pi G_4}(R - 2\check{\Lambda})] - \frac{1}{4e_0^2}F_{\mu\nu}F^{\mu\nu}], \tag{77}$$

where e_0 is the electromagnetic constant, $F_{\mu\nu}$ is the anti-symmetric strength tensor of A_μ; G_4 is the 4-d gravitational constant, and $\check{\Lambda} > 0$ is the de Sitter, dS, cosmological constant. In this theory, a Reisner-Nordström, RN, BH is constructed as a spherically symmetric and static solution of corresponding gravitational and electromagnetic field equations in GR with zero cosmological constant. The corresponding quadratic line element for such a RN-dS solution with positive cosmological constant, describing an electrically charged BH in an asymptotic dS spacetime, can be parameterized as a prime spacetime metric

$$\begin{aligned} d\,{}^{\flat}s^2_{[4d]} &= {}^{\flat}g_1(r)dr^2 + {}^{\flat}g_2(r)d\theta^2 + {}^{\flat}g_3(r,\theta)d\varphi^2 + {}^{\flat}g_4(r)dt^2 \text{ and } A_\mu = (Q/r,0,0,0), \text{ for} \\ {}^{\flat}g_4(r) &= -(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - \frac{r^2}{r^2_{\check{\Lambda}}}) = -[\,{}^{\flat}g_1(r)]^{-1},\ {}^{\flat}g_2(r) = r^2,\ {}^{\flat}g_3(r,\theta) = r^2\sin^2\theta, \end{aligned} \tag{78}$$

where the constant velocity of light is stated $c = 1$; the Schwarzschild radius $r_s = 2Gm$ is determined by the BH mass; the characteristic electric length $r_Q^2 = Q^2G/4\pi e_0^2$; and $r^2_{\check{\Lambda}} = 3/\check{\Lambda}$. The local spherical coordinates are parameterized $x^1 = r, x^2 = \theta, y^3 = \varphi$ and $y^4 = t$, with $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ being the metric on the unity 2-d sphere, when the conditions for the causal horizon $r = r_h$ are stated for quadric polynomial ${}^{\flat}g_4(r_h) = 0$.

In a 8-d phase space ${}_s\mathcal{M}$, we use double 4-d local spherical coordinates, both on base spacetime manifold V (as in $d\,{}^{\flat}s^2_{[4d]}$ (78)) and 4-d spherical momentum type coordinates $p_1 = p_r, p_2 = p_\theta, p_3 = p_\varphi$ and $p_4 = E$; and consider the quadratic linear element

$$\begin{aligned} d\,{}^{\flat}_{\shortmid}s^2_{[8d]} &= d\,{}^{\flat}s^2_{[4d]} + d\,{}^{\flat}_{\shortmid}s^2, \\ &\text{with } d\,{}^{\flat}_{\shortmid}s^2 = {}^{\flat}_{\shortmid}g^5(p)dp^2 + {}^{\flat}_{\shortmid}g^6(p)dp_\theta{}^2 + {}^{\flat}_{\shortmid}g^7(p,p_\theta)dp^2_\varphi + {}^{\flat}_{\shortmid}g^8(p)dE^2, \text{ for} \\ &{}^{\flat}_{\shortmid}g^8(p) = -(1 - \frac{p_s}{p} + \frac{p_Q^2}{p^2} - \frac{p^2}{p^2_{\shortmid\check{\Lambda}}}) = -[\,{}^{\flat}_{\shortmid}g^5(p)]^{-1},\ {}^{\flat}_{\shortmid}\mathring{g}^6(p) = p^2,\ {}^{\flat}_{\shortmid}\mathring{g}^7(p,p_\theta) = p^2\sin^2 p_\theta. \end{aligned} \tag{79}$$

In such formulas, the dimension for $p = \sqrt{(p_1)^2 + (p_2)^2 + (p_3)}$ is stated (via multiplication on a constant parameter, or working in natural units with $G = c = \hbar = 1$) to be the same as for $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)}$. There are also considered some conventional "horizontal", $\check{\Lambda}$, and "covertical", ${}^{\shortmid}\check{\Lambda}$, cosmological constants. To define a double 4-d BH configuration on ${}_s\mathcal{M}$ we can introduce in the typical co-fiber space a conventional

Schwarzschild radius $p_s = 2\ {}^{\shortmid}G\ {}^{\shortmid}m$ is determined by the co-fiber BH mass ${}^{\shortmid}m$ with a conventional (it can be different from G) gravitational constant ${}^{\shortmid}G$; the characteristic electric length $p_Q^2 = {}^{\shortmid}Q^2\ {}^{\shortmid}G/4\pi\ {}^{\shortmid}e_0^2$ with conventional electric charge in co-fiber, ${}^{\shortmid}e_0$; and $p_{\check{\Lambda}}^2 = 3/\ {}^{\shortmid}\check{\Lambda}$.

For arbitrary frame/coordinate transforms, a diagonal phase space s-metric ${}_{\shortmid}^{\flat}g_\alpha = ({}^{\flat}g_i,\ {}_{\shortmid}^{\flat}g^a)$ (79) can be written in off-diagonal form ${}_{\shortmid}^{\flat}g_{\alpha\beta}({}^{\shortmid}u)$ with a prime shell structure ${}_{\shortmid}^{\flat}\mathbf{g}$ adapted to a prime and trivial N-connection splitting ${}_{\shortmid}^{\flat}\mathbf{N}$. In general, such data $({}_{\shortmid}^{\flat}\mathbf{g},\ {}_{\shortmid}^{\flat}\mathbf{N})$ do not define a solution of vacuum (non) associative/commutative gravitational equations even their horizontal/vacuum components, for instance, (78) can determine certain 4-d electro-vacuum or RNdS BH configurations.

5.2.2 Nonassociative geometric κ-linear evolution of double phase space BH configurations

In this subsection, we construct κ-linear parametric solutions of nonassociative geometric flow equations (62) describing the evolution of a double BH phase space metric ${}_{\shortmid}^{\flat}g_\alpha = ({}^{\flat}g_i,\ {}_{\shortmid}^{\flat}g^a)$ (79). In a trivial s-adapted form (with N-connection coefficients such way defined by some coordinate transforms when the solutions do not contain coordinate singularities), such a primary s-metric can be parameterized by corresponding s-coefficients ${}_{\shortmid s}^{\flat}\mathbf{g} = ({}_{s}^{\flat}g,\ {}_{\shortmid s}^{\flat}g) = \{{}_{\shortmid}^{\flat}g_{\alpha_s} = ({}^{\flat}g_{i_1},\ {}^{\flat}g_{a_2},\ {}_{\shortmid}^{\flat}g^{a_3},)\}$. Using nonlinear symmetries (60) and respective nonlinear transforms (61), we can re-define the generating functions to define nonassociative R-flux deformations of such prime BH s-metrics into τ-families of target quasi-stationary ones, ${}_{\shortmid s}^{\flat}\mathbf{g} \to {}_{s}^{\shortmid}\mathbf{g}(\tau)$ (63) with N-connection coefficients ${}^{\shortmid}N_{i_{s-1}}^{a_s}(\tau)$ in nonholonomic s-frames (64).

We can express the parametric solutions for nonassociative κ-linear geometric flow deformations of type ${}_{\shortmid s}^{\flat}\mathbf{g} \to {}_{s}^{\shortmid}\mathbf{g}(\tau)$ (63) in terms of gravitational η-polarizations and generating functions (67) using phase space local coordinates as for the prime s-metric (79). The corresponding class of quasi-stationary τ-running s-metrics are defined by quadratic linear elements of type (68), with $\mathring{g}_4(\tau) \to {}^{\flat}g_4$, ${}^{\shortmid}\mathring{g}^6(\tau) \to {}_{\shortmid}^{\flat}g^6$, ${}^{\shortmid}\mathring{g}^8(\tau) \to {}_{\shortmid}^{\flat}g^8$, when the primary s-metrics do not depend on τ. To avoid singular coordinate evolution scenarios for a necessary τ-interval we can prescribe a primary s-metric written in respective coordinates and further frame transforms to define certain well-defined data

$$\left({}_{\shortmid s}^{\flat}\mathbf{g};\ {}_{\shortmid s}^{\flat}\mathbf{N}\right) = ({}_{s}^{\flat}g,\ {}_{\shortmid s}^{\flat}g;\ {}_{\shortmid}^{\flat}N_{i_{s-1}}^{a_s}) = \{{}_{\shortmid}^{\flat}g_{\alpha_s} = ({}^{\flat}g_{i_1},\ {}^{\flat}g_{a_2},\ {}_{\shortmid}^{\flat}g^{a_3};\ {}^{\flat}N_{i_1}^{a_2},\ {}_{\shortmid}^{\flat}N_{i_2a_3},\ {}_{\shortmid}^{\flat}N_{i_3a_4})\}.$$

In η–polarized nonsymmetric κ-linear τ-evolving quasi-stationary phase space backgrounds, double RNdS BH configurations are described by quadratic elements:

$$\begin{aligned} d\ {}^{\shortmid}\widehat{s}^2(\tau) &= {}^{\shortmid}g_{\alpha_s\beta_s}(\hbar,\kappa,\tau,x^k,y^3,p_{a_3},p_{a_4};\ {}_{\shortmid}^{\flat}g_{\alpha_s}(\tau);\eta_4(\tau),\ {}^{\shortmid}\eta^6(\tau),\ {}^{\shortmid}\eta^8(\tau),\ {}_{s}^{\shortmid}\Lambda(\tau);\ {}_{s}^{\shortmid}\Im(\tau))d\ {}^{\shortmid}u^{\alpha_s}d\ {}^{\shortmid}u^{\beta_s} \\ &= e^{\psi(\tau)}[(dx^1)^2+(dx^2)^2] - \qquad (80) \\ & \frac{[\partial_3(\eta_4(\tau)\ {}^{\flat}g_4)]^2}{|\int dy^3\ {}_2\Im(\tau)\partial_3(\eta_4(\tau)\ {}^{\flat}g_4)|\ (\eta_4(\tau)\ {}^{\flat}g_4)}\{dy^3 + \frac{\partial_{i_1}[\int dy^3\ {}_2\Im(\tau)\ \partial_3(\eta_4(\tau)\ {}^{\flat}g_4)]}{{}_2\Im(\tau)\partial_3(\eta_4(\tau)\ {}^{\flat}g_4)}dx^{i_1}\}^2 + \\ & \eta_4(\tau)\ {}^{\flat}g_4)\{dt + [\ {}_1n_{k_1}(\tau) + {}_2n_{k_1}(\tau)\int dy^3 \frac{[\partial_3(\eta_4(\tau)\ {}^{\flat}g_4)]^2}{|\int dy^3\ {}_2\Im(\tau)\partial_3(\eta_4(\tau)\ {}^{\flat}g_4)|\ (\eta_4(\tau)\ {}^{\flat}g_4)^{5/2}}]dx^{k_1}\} \\ & - \frac{[\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)]^2}{|\int dp_5\ {}_3^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\mathring{g}^6(\tau))\ |\ (\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)}\{dp_5 + \frac{{}^{\shortmid}\partial_{i_2}[\int dp_5\ {}_3^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)]}{{}_3^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)}dx^{i_2}\}^2 \\ & +(\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)\{dp_6 + [\ {}_1^{\shortmid}n_{k_2}(\tau) + {}_2^{\shortmid}n_{k_2}(\tau)\int dp_5 \frac{[\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)]^2}{|\int dp_5\ {}_3^{\shortmid}\Im(\tau)\partial^5(\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)|\ (\ {}^{\shortmid}\eta^6(\tau)\ {}_{\shortmid}^{\flat}g^6)^{5/2}}]dx^{k_2}\} - \\ & \frac{[\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)]^2}{|\int dp_7\ {}_4^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)\ |\ (\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)}\{dp_7 + \frac{{}^{\shortmid}\partial_{i_3}[\int dp_7\ {}_4^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)]}{{}_4^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)}d\ {}^{\shortmid}x^{i_3}\}^2 + \\ & (\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)\{dE + [\ {}_1n_{k_3}(\tau) + {}_2n_{k_3}(\tau)\int dp_7 \frac{[\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)]^2}{|\int dp_7\ {}_4^{\shortmid}\Im(\tau)[\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)]|\ [(\ {}^{\shortmid}\eta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)]^{5/2}}]d\ {}^{\shortmid}x^{k_3}\}. \end{aligned}$$

The integration functions in (80) are of type (A.11) but extended to τ-dependencies and written for coordinates used in (79),

$$g_4^{[0]}(\hbar,\kappa,\tau,r,\theta),\ {}_1n_{k_1}(\hbar,\kappa,\tau,r,\theta),\ {}_2n_{k_1}(\hbar,\kappa,\tau,r,\theta);$$
$$ {}^{\shortmid}g_{[0]}^5(\hbar,\kappa,\tau,r,\theta,\varphi,p),\ {}_1n_{k_2}(\hbar,\kappa,\tau,r,\theta,\varphi,p),\ {}_2n_{k_2}(\hbar,\kappa,\tau,r,\theta,\varphi,p);$$
$$ {}^{\shortmid}g_{[0]}^7(\hbar,\kappa,\tau,r,\theta,\varphi,p,p_\varphi),\ {}_1^{\shortmid}n_{k_3}(\hbar,\kappa,\tau,p,p_\varphi),\ {}_2^{\shortmid}n_{k_3}(\hbar,\kappa,\tau,p,p_\varphi).$$

We can consider additional conditions when such generic off-diagonal gravitational interactions and nonassociative geometric evolution flows transform a prime s-metric for double conventional 4-d electro-vacuum or RNdS BHs into vacuum quasi-stationary configurations with τ-evolution, when the electromagnetic interactions are "dissipated" into a nonholonomic vacuum gravitational structure encoding star product R-flux data.

5.2.3 Computing the Bekenstein-Hawking entropy for double phase space BE configurations

We study two classes of nonassociative couples BE/BH s-metrics which are characterized by phase space Bekenstein-Hawking type thermodynamic models.

Example 1: Nonassociative τ-deformed double RNdS BHs with dissipation into BEs and Schwarzschild BHs

Any solution (80) can be decomposed in terms of χ–generating functions as for the quadratic line element (A.13) if the prime s-metric coefficients ${}_{\shortmid s}^{\flat}\mathbf{g}$ are used for a prime metric ${}_{s}^{\shortmid}\mathring{\mathbf{g}}$. In such cases, the generating and integration functions are written in κ–linearized form (A.14),

$$\begin{aligned}
\psi(\tau) &\simeq \psi(\hbar,\kappa;\tau,r,\theta)\simeq\psi_0(\hbar,\tau,r,\theta)(1+\kappa\ {}_{\psi}\chi(\hbar,\tau,r,\theta)),\ \text{for}\\
\eta_2(\tau) &\simeq \eta_2(\hbar,\kappa;\tau,r,\theta)\simeq\zeta_2(\hbar,\tau,r,\theta)(1+\kappa\chi_2(\hbar,\tau,r,\theta)),\ \text{we can consider}\ \ \eta_2(\tau)=\ \eta_1(\tau);\\
\eta_4(\tau) &\simeq \ \eta_4(\hbar,\kappa;\tau,r,\theta,\varphi)\simeq\zeta_4(\hbar,\tau,r,\theta,\varphi)(1+\kappa\ \chi_4(\hbar,\tau,r,\theta,\varphi)),\\
{}^{\shortmid}\eta^6(\tau) &\simeq {}^{\shortmid}\eta^6(\hbar,\kappa;\tau,r,\theta,\varphi,p)\simeq\ {}^{\shortmid}\zeta^6(\hbar,\kappa;\tau,r,\theta,\varphi,p)(1+\kappa\ {}^{\shortmid}\chi^6(\hbar,\kappa;\tau,r,\theta,\varphi,p)),\\
{}^{\shortmid}\eta^8(\tau) &\simeq {}^{\shortmid}\eta^8(\hbar,\kappa;\tau,r,\theta,\varphi,p,p_\varphi)\simeq\ {}^{\shortmid}\zeta^8(\hbar,\kappa;\tau,r,\theta,\varphi,p,p_\varphi)(1+\kappa\ {}^{\shortmid}\chi^8(\hbar,\kappa;\tau,r,\theta,\varphi,p,p_\varphi)).
\end{aligned}$$

We may construct τ-families of quasi-stationary solutions (A.13 with conventional horizons when the χ-polarizations satisfy the conditions

$$\begin{aligned}
\zeta_4(1+\kappa\ \chi_4)\ {}^{\flat}g_4(r) &= \zeta_4(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}-\frac{r^2}{r_{\breve{\Lambda}}^2}+\kappa\ \chi_4)=\tilde{\zeta}_4(1-\frac{r_s}{r}+\kappa\ \tilde{\chi}_4)=0, \qquad (81)\\
{}^{\shortmid}\zeta^8(1+\ {}^{p}\kappa\ {}^{\shortmid}\chi^8)\ {}_{\shortmid}^{\flat}g^8(p) &= {}^{\shortmid}\zeta^8(1-\frac{p_s}{p}+\frac{p_Q^2}{p^2}-\frac{p^2}{p^2_{{}^{\shortmid}\breve{\Lambda}}}+\kappa\ {}^{\shortmid}\chi^8)=\ {}^{\shortmid}\tilde{\zeta}^8(1-\frac{p_s}{p}+\kappa\ {}^{\shortmid}\tilde{\chi}^8)=0,
\end{aligned}$$

for non-zero $\chi_4(\tau)$ and ${}^{\shortmid}\chi^8(\tau)$; ${}^{\shortmid}\zeta^6(\tau)=1$ and $\zeta_4(\tau)\simeq 1,\ {}^{\shortmid}\zeta^8(\tau)\simeq 1$ and re-definition of the generating data,

$$\begin{aligned}
\big(\chi_4(\tau),\ \zeta_4(\tau);\ {}^{\shortmid}\chi^8(\tau),\ {}^{\shortmid}\zeta^8(\tau)\big) \ \rightarrow\ & (\tilde{\chi}_4(\tau)=\ \zeta_4(\tau)\chi_4(\tau),\tilde{\zeta}_4(\tau)=(1+\frac{r_Q^2 r_{\breve{\Lambda}}^2-r^4}{r(r-r_s)r_{\breve{\Lambda}}^2})\ \zeta_4(\tau);\\
& {}^{\shortmid}\tilde{\chi}^8(\tau)=\ {}^{\shortmid}\tilde{\zeta}^8(\tau)\ {}^{\shortmid}\chi^8(\tau),\ {}^{\shortmid}\tilde{\zeta}^8(\tau)=(1+\frac{r_Q^2 r_{\breve{\Lambda}}^2-r^4}{r(r-r_s)r_{\breve{\Lambda}}^2})\ {}^{\shortmid}\zeta^8(\tau)).
\end{aligned}$$

Viable physical models with off-diagonal solutions are generated if the integration functions for N-coefficients are chosen to tend, for instance, to zero for $r\rightarrow\infty$ and $p\rightarrow\infty$.

Geometric evolution models of two black ellipsoid, BE, phase space configurations are defined if we prescribe such generating functions:

$$
\begin{aligned}
\tilde{\chi}_4(\tau) &= {}^{e}\chi_4(\tau,r,\theta,\varphi) = 2\underline{\chi}(\tau,r,\theta)\sin(\omega_0\varphi+\varphi_0); \qquad (82)\\
{}^{\shortmid}\tilde{\chi}^6(\tau) &= {}^{\shortmid}\chi^6(\tau)=0, \text{ for } {}^{\shortmid}\tilde{\zeta}^6(\tau) = {}^{\shortmid}\zeta^6(\tau)=1;\\
{}^{\shortmid}\tilde{\chi}^8(\tau) &= {}^{e}\chi(\tau,p,p_\theta,p_\varphi) = 2\overline{\chi}(\tau,p,p_\theta)\sin({}_{\shortmid}^{p}\omega_0\ p_\varphi + \ p_\varphi^0),
\end{aligned}
$$

where $\underline{\chi}(\tau,r,\theta)$ and $\overline{\chi}(\tau,p,p_\theta)$ are smooth functions (or τ-running constants); the smooth ζ-functions can be approximated to unity, and (ω_0,φ_0) and $({}_{\shortmid}^{p}\omega_0,\ p_\varphi^0)$ are couples of constants. To define all possible horizons we have to solve the system of two independent forth order algebraic equations for r and p and stated by gravitational polarizations $\chi_4(\tau)$ and ${}^{\shortmid}\chi^8(\tau)$ (we omit such technical details). For this subclass of quasi-stationary phase space solutions, we can consider small parametric deformations and regions when

$$
r(\tau) \approx\ r_s/(1-\kappa\tilde{\chi}_4(\tau,\varphi)) \text{ and } {}^{p}r(\tau)\approx p_s/(1-\kappa\ {}^{\shortmid}\tilde{\chi}^8(\tau,p_\varphi)). \qquad (83)
$$

This describes a scenarios when phase space two RNdS BHs evolve under nonassociative geometric flows into BE deformations of certain base spacetime and co-fiber Schwarzschild solutions.

For prescribed gravitational χ-polarizations (82), the parametric formulas (83) define for rotoid configurations running on τ. Corresponding nonholonomic structures can be chosen to define certain nonassociative geometric evolution of more general black ellipsoid, BE, configurations. The corresponding parametric formulas for respective BE horizons are defined by small gravitational polarizations determined by nonassociative star product R-flux deformations. In the limits of zero eccentricity κ, such double BE configurations transform into prime double BH ones.

Putting together above formulas, we construct two BE target phase space quadratic linear element with τ–evolution,

$$
\begin{aligned}
d\,{}_{\shortmid}^{\flat}s^2_{[8d]}(\tau) &= e^{\psi_0}(1+\kappa\ {}^{\psi(\tau)}\ {}^{\shortmid}\chi(\tau))[\ {}^{\flat}g_1 dr^2 +\ {}^{\flat}g_2 d\theta^2] \qquad (84)\\
&-\{\frac{4[\partial_3(|\zeta_4(\tau)\ {}^{\flat}g_4|^{1/2})]^2}{{}^{\flat}g_4|\int dy^3\{\ {}_2\Im(\tau)\partial_3(\zeta_4(\tau)\ {}^{\flat}g_4)\}|} - \kappa[\frac{\partial_3(\chi_4(\tau)|\zeta_4(\tau)\ {}^{\flat}g_4|^{1/2})}{4\partial_3(|\zeta_4(\tau)\ {}^{\flat}g_4|^{1/2})}\\
&-\frac{\int dy^3\{\ {}_2\Im(\tau)\partial_3[(\zeta_4(\tau)\ {}^{\flat}g_4)\chi_4(\tau)]\}}{\int dy^3\{\ {}_2\Im(\tau)\partial_3(\zeta_4(\tau)\ {}^{\flat}g_4)\}}]\}\ {}^{\flat}g_3(\mathbf{e}^3(\tau))^2
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{\zeta}_4(\tau)(1-\frac{r_s}{r}+\kappa\ \tilde{\chi}_4(\tau))(\mathbf{e}^4(\tau))^2 +\ {}_{\shortmid}^{\flat}g^5 dp^2 +\ {}_{\shortmid}^{\flat}g^6 dp_\theta{}^2\\
&-\{\frac{4[\ {}^{\shortmid}\partial^7(|\ {}^{\shortmid}\zeta^8(\tau)\ \ {}_{\shortmid}^{\flat}g^8|^{1/2})]^2}{{}_{\shortmid}^{\flat}g^7|\int dp_7\{\ {}_4^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8(\tau)\ \ {}_{\shortmid}^{\flat}g^8)\}|} - \kappa[\frac{{}^{\shortmid}\partial^7(\ {}^{\shortmid}\chi^8(\tau)|\ {}^{\shortmid}\zeta^8(\tau)\ {}_{\shortmid}^{\flat}g^8|^{1/2})}{4\ {}^{\shortmid}\partial^7(|\ {}^{\shortmid}\zeta^8(\tau)\ {}_{\shortmid}^{\flat}g^8|^{1/2})} -\\
&\frac{\int dp_7\{\ {}_4^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)\ {}^{\shortmid}\chi^8(\tau)]\}}{\int dp_7\{\ {}_4^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8(\tau)\ {}_{\shortmid}^{\flat}g^8)]\}}]\}\ {}_{\shortmid}^{\flat}g^7(\ {}^{\shortmid}\mathbf{e}_7(\tau))^2 +\ {}^{\shortmid}\tilde{\zeta}^8(\tau)(1-\frac{p_s}{p}+\kappa\ {}^{\shortmid}\tilde{\chi}^8(\tau))(\ {}^{\shortmid}\mathbf{e}_8(\tau))^2,
\end{aligned}
$$

where

$$
\mathbf{e}^3(\tau) = d\varphi + [\frac{\partial_{i_1}\int dy^3\ {}_2\Im(\tau)\ \partial_3\zeta_4(\tau)}{{}^{\flat}N^3_{i_1}\ {}_2\Im(\tau)\partial_3\zeta_4(\tau)} + \kappa(\frac{\partial_{i_1}[\int dy^3\ {}_2\Im(\tau)\partial_3(\zeta_4(\tau)\chi_4(\tau))]}{\partial_{i_1}\ [\int dy^3\ {}_2\Im(\tau)\partial_3\zeta_4(\tau)]} - \frac{\partial_3(\zeta_4(\tau)\chi_4(\tau))}{\partial_3\zeta_4(\tau)})]\ {}^{\flat}N^3_{i_1}dx^{i_1},
$$

$$
\begin{aligned}
\mathbf{e}^4(\tau) &= dt + [\ {}_1n_{k_1} + 16\ {}_2n_{k_1}\int dy^3 \frac{\left(\partial_3[(\zeta_4(\tau)\ {}^{\flat}g_4)^{-1/4}]\right)^2}{|\int dy^3\partial_3[\ {}_2\Im(\tau)(\zeta_4(\tau)\ {}^{\flat}g_4)]|} +\\
&\kappa\frac{16\ {}_2n_{k_1}\int dy^3\frac{\left(\partial_3[(\zeta_4(\tau)\ {}^{\flat}g_4)^{-1/4}]\right)^2}{|\int dy^3\partial_3[\ {}_2\Im(\tau)(\zeta_4(\tau)\ {}^{\flat}g_4)]|}(\frac{\partial_3[(\zeta_4(\tau)\ {}^{\flat}g_4)^{-1/4}\chi_4)]}{2\partial_3[(\zeta_4(\tau)\ {}^{\flat}g_4)^{-1/4}]} + \frac{\int dy^3\partial_3[\ {}_2\Im(\tau)(\zeta_4(\tau)\chi_4(\tau)\ {}^{\flat}g_4)]}{\int dy^3\partial_3[\ {}_2\Im(\tau)(\zeta_4(\tau)\ {}^{\flat}g_4)]})}{{}_1n_{k_1} + 16\ {}_2n_{k_1}[\int dy^3\frac{\left(\partial_3[(\zeta_4(\tau)\ {}^{\flat}g_4)^{-1/4}]\right)^2}{|\int dy^3\partial_3[\ {}_2\Im(\tau)(\zeta_4(\tau)\ {}^{\flat}g_4)]|}]}]dx^{\acute{k}_1},
\end{aligned}
$$

$$\begin{aligned}
{}^{\shortmid}\mathbf{e}_7(\tau) &= dp_\varphi + [\frac{{}^{\shortmid}\partial_{i_3} \int dp_7 \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau))}{{}^{\shortmid}\mathring{N}_{i_3 7} \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau))} + \\
& \kappa(\frac{{}^{\shortmid}\partial_{i_3}[\int dp_7 \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)]}{{}^{\shortmid}\partial_{i_3} \ [\int dp_7 \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau))]} - \frac{{}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)}{{}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau))})] \ {}_{\shortmid}^{\flat}N_{i_3 7} d \ {}^{\shortmid}x^{i_3}, \\
{}^{\shortmid}\mathbf{e}_8(\tau) &= dE + [\ {}_1^{\shortmid}n_{i_3} + 16 \ {}_2^{\shortmid}n_{i_3} \int dp_7 \frac{\left({}^{\shortmid}\partial^7[({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)^{-1/4}]\right)^2}{|\int dp_7 \ {}_4^{\shortmid}\Im(\tau) \ \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)|} + \\
& \kappa \times 16 \ {}_2^{\shortmid}n_{i_3} \int dp_7 \frac{\left({}^{\shortmid}\partial^7[({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)^{-1/4}]\right)^2}{|\int dp_7 \ \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7[({}^{\shortmid}\zeta^8(\tau) \ \ {}_{\shortmid}^{\flat}g^8)]|} (\frac{{}^{\shortmid}\partial^7[({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)^{-1/4} \ {}^{\shortmid}\chi^8(\tau))]}{2 \ {}^{\shortmid}\partial^7[({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)^{-1/4}]} + \\
& \frac{\int dp_7 \ \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7[({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8) \ {}^{\shortmid}\chi^8(\tau)]}{\int dp_7 \ \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)}) \\
& \left({}_1^{\shortmid}n_{i_3} + 16 \ {}_2^{\shortmid}n_{i_3}[\int dp_7 \frac{\left({}^{\shortmid}\partial^7 \ [({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)^{-1/4}]\right)^2}{|\int dp_7 \ \ {}_4^{\shortmid}\Im(\tau) \ {}^{\shortmid}\partial^7({}^{\shortmid}\zeta^8(\tau) \ {}_{\shortmid}^{\flat}g^8)|}]\right)^{-1}]dx^{i_3}.
\end{aligned}$$

For any double rotoid configuration (84) with $\tau = \tau_0$, we can define and compute phase space generalizations of the Hawking temperature, T, and the Bekenstein-Hawking entropy, S, as in section 4.1.2 of [41]. Corresponding basic formulas for the 'standard' BH thermodynamics can be generalized for τ–running nonassociative Ricci solitons for effective locally anisotropic thermodynamic variables depending on angular spacetime and co-fiber coordinates:

$$\begin{aligned}
T(\tau, r, \theta, \varphi; p, p_\theta, p_\varphi) &= \frac{1}{4\pi}\left(\frac{\Omega_2}{4}\right)^{1/2} \{S_0^{-1/2}(\tau, r, \theta, \varphi) + \ {}_{\shortmid}S_0^{-1/2}(\tau, p, p_\theta, p_\varphi)\}, \text{ with} \\
S_0(\tau, r, \theta, \varphi) &= \frac{\Omega_2 \times \ (r_s)^4}{4}[1 + \frac{4\kappa}{3}\underline{\chi}(\tau, r, \theta)\sin(\omega_0\varphi + \varphi_0)] \text{ and} \\
{}_{\shortmid}S_0(\tau, p, p_\theta, p_\varphi) &= \frac{\Omega_2 \times (p_s)^4}{4}[1 + \frac{4\kappa}{3}\overline{\chi}(\tau, p, p_\theta)\sin({}_{\shortmid}^{p}\omega_0 \ p_\varphi + \ p_\varphi^0)]. \quad (85)
\end{aligned}$$

In such formulas, we used the volume of unit -sphere, $\Omega_2 = \frac{\pi^2}{(2)!}$, when the respective horizons in the 4-d spacetime and 4-d co-fiber are r_s and p_s as in formulas (79), and (82) and (83). If $\underline{\chi}\kappa \to 0$ and/or $\overline{\chi}\kappa \to 0$, the phase space double BEs transform into respective BH configurations with conventional different isotropic temperatures in the spacetime base and the co-fiber space.

Example 2: Nonassociative τ-running couples of Schwarzschild-AdS BHs and BEs deformations

Another example of phase space double BE configurations for which the concept of Bekenstein-Hawking thermodynamics is applicable consists from nonassociative star product R-flux deformations of corresponding couples of Schwarzschild-AdS BHs. Considering prime 8-d metrics (79) with

$$\begin{aligned}
{}^{\flat}g_4(r) &\to \quad {}^{\epsilon}g_4(r) = -(\epsilon - \frac{2m_+}{r} + \check{\Lambda}r^2) = -[\ {}^{\epsilon}g_1(r)]^{-1} \text{ and} \\
{}_{\shortmid}^{\flat}g^8(p) &\to \quad {}_{\shortmid}^{\epsilon}g^8(p) = -(\epsilon - \frac{2 \ {}_{\shortmid}m_+}{p} + \ {}^{\shortmid}\check{\Lambda}p^2) = -[\ {}_{\shortmid}^{\epsilon}g^5(p)]^{-1}, \quad (86)
\end{aligned}$$

for mass parameters m_+ and ${}_{\shortmid}m_+$, where $\epsilon = (+1, 0, -1)$ which corresponding, respectively, to spherical/ planar/ hyperbolic horizon geometries, we define prime s-metrics as solutions of the phase modified nonholonomic Einstein equations (3), see also (8), with nonlinear symmetries (A.8) relating certain prescribed effective sources ${}_s^{\shortmid}\mathcal{K}$ to effective cosmological constants ${}_1^{\shortmid}\Lambda_0 = {}_2^{\shortmid}\Lambda_0 = \check{\Lambda} \geq 0$ and ${}_3^{\shortmid}\Lambda_0 = {}_4^{\shortmid}\Lambda_0 = {}^{\shortmid}\check{\Lambda} \geq 0$. We can extend such s-metrics for running cosmological constants ${}_s^{\shortmid}\Lambda(\tau) = [\check{\Lambda}(\tau), {}^{\shortmid}\check{\Lambda}(\tau)] \geq 0$ as solutions of (62), parameterized in the form

$$
\begin{aligned}
d\ {}^{\epsilon}_{\shortmid}s^2_{[8d]}(\tau) &= d\ {}^{\epsilon}s^2(\tau) + d\ {}^{\epsilon}_{\shortmid}s^2(\tau), \text{ with} \qquad (87)\\
& d\ {}^{\epsilon}s^2(\tau) = {}^{\epsilon}g_1(\tau,r)dr^2 + {}^{\epsilon}g_2(r)d\theta^2 + {}^{\epsilon}g_3(r,\theta)d\varphi^2 + {}^{\epsilon}g_4(\tau,r)dt^2,\\
& d\ {}^{\epsilon}_{\shortmid}s^2 = {}^{\epsilon}_{\shortmid}g^5(\tau,p)dp^2 + {}^{\epsilon}_{\shortmid}g^6(p)dp_\theta^2 + {}^{\epsilon}_{\shortmid}g^7(p,p_\theta)dp_\varphi^2 + {}^{\epsilon}_{\shortmid}g^8(\tau,p)dE^2,
\end{aligned}
$$

when the coefficients are determined as respective metric functions (86) modified for running constants,

$$
\begin{aligned}
{}^{\epsilon}g_4(\tau,r) &= -(\epsilon - \frac{2m_+(\tau)}{r} + \check{\Lambda}(\tau)r^2) = -[\ {}^{\epsilon}g_1(\tau,r)]^{-1} \text{ and}\\
{}^{\epsilon}_{\shortmid}g^8(\tau,p) &= -(\epsilon - \frac{2\ {}_{\shortmid}m_+(\tau)}{p} + {}^{\shortmid}\check{\Lambda}(\tau)p^2) = -[\ {}^{\epsilon}_{\shortmid}g^5(\tau,p)]^{-1};
\end{aligned}
$$

and (for instance, for spherical horizon geometries) ${}^{\epsilon}g_2(r) = r^2,\ {}^{\epsilon}g_3(r,\theta) = r^2\sin^2\theta$ and ${}^{\epsilon}_{\shortmid}\mathring{g}^6(p) = p^2,\ {}^{\epsilon}_{\shortmid}\mathring{g}^7(p,p_\theta) = p^2\sin^2 p_\theta$.

We considered above a prime s-metric (79) which is not a nonassociative vacuum solution but τ-evolves into nonassociative geometric flows of quasi-stationary solutions. In this subsection, the prime s-metric (87) defines already τ-families of double BH solutions of nonassociative vacuum Einstein equations. For a fixed τ_0, such a nonassociative Ricci soliton is defined by a stationary configuration is defined by metrics with a conventional horizon radius r_+, in the base manifold, and p_+, in the co-fiber space. The thermodynamic quantities (entropy and temperature) for such couples of phase space Schwarzschild-AdS BHs are defined and computed in standard forms:

$$
\begin{aligned}
{}^{\epsilon}S_0(\tau) &= \frac{\pi\ {}^{\epsilon}\Theta}{4} r_+^2(\tau),\ {}^{\epsilon}T_0(\tau) = \frac{\epsilon + 3\check{\Lambda}(\tau)r_+^2(\tau)}{4\pi r_+(\tau)} \text{ and}\\
{}^{\epsilon}_{\shortmid}S_0(\tau) &= \frac{\pi\ {}^{\epsilon}_{\shortmid}\Theta}{4} p_+^2(\tau),\ {}^{\epsilon}_{\shortmid}T_0(\tau) = \frac{{}^{\shortmid}\epsilon + 3\ {}^{\shortmid}\check{\Lambda}(\tau)p_+^2(\tau)}{4\pi p_+(\tau)}, \qquad (88)
\end{aligned}
$$

where running of conventional mass parameters are computed following formulas

$$
m_+(\tau) = \frac{r_+(\tau)\ {}^{\epsilon}\Theta}{8}[\epsilon + \check{\Lambda}(\tau)r_+^2(\tau)] \text{ and } {}_{\shortmid}m_+(\tau) = \frac{p_+(\tau)\ {}^{\epsilon}_{\shortmid}\Theta}{8}[\ {}^{\shortmid}\epsilon + {}^{\shortmid}\check{\Lambda}(\tau)p_+^2(\tau)].
$$

In above equations, we use respective areas of constant-curvatures spaces, $\pi\ {}^{\epsilon}\Theta$ and $\pi\ {}^{\epsilon}_{\shortmid}\Theta$. For instance: ${}^{1}\Theta = 4$, for a sphere; ${}^{0}\Theta = XY$, using X and Y as sides of the torus; there is not a simple example to compute ${}^{-1}\Theta$. Here we note that following the standard BH thermodynamics for solutions with hyper-surfaces we can define and compute a conventional pressure, P, and volume, Vol (for instance, $P = \frac{3}{8\pi}\check{\Lambda}$ and $Vol = \frac{\pi\ {}^{\epsilon}\Theta}{3} r_+^3$).

Rotoid deformations with χ-polarizations (82) under nonassociative geometric information flows of families of s-metrics (87), ${}^{\epsilon}_{\shortmid}g_\alpha \to {}^{\epsilon\chi}_{\shortmid}g_\alpha$, can be generated if we change the data for the prime s-metrics, ${}^{\flat}_{\shortmid}g_\alpha = (\ {}^{\flat}g_i,\ {}^{\flat}_{\shortmid}g^a)$ (79) $\to {}^{\epsilon}_{\shortmid}g_\alpha = (\ {}^{\epsilon}g_i,\ {}^{\epsilon}_{\shortmid}g^a)$, into (84). We do not write in explicit form such quadratic linear elements, but provide the formulas for the effective locally anisotropic thermodynamic variables depending on angular spacetime and co-fiber variables:

$$
\begin{aligned}
{}^{\epsilon\chi}S(\tau,r,\theta,\varphi) &= {}^{\epsilon}S_0(\tau)\left[1 + 4\kappa\underline{\chi}(\tau,r,\theta)\sin(\omega_0\varphi + \varphi_0)\right], \qquad (89)\\
{}^{\epsilon\chi}T(\tau,r,\theta,\varphi) &= {}^{\epsilon}T_0(\tau)\left[1 + 2\kappa\underline{\chi}(\tau,r,\theta)\sin(\omega_0\varphi + \varphi_0)\right] \text{ and}\\
{}^{\epsilon\chi}_{\shortmid}S(\tau,p,p_\theta,p_\varphi) &= {}^{\epsilon}_{\shortmid}S_0(\tau)\left[1 + 4\kappa\overline{\chi}(\tau,p,p_\theta)\sin(\ {}^{p}_{\shortmid}\omega_0\ p_\varphi + \ p_\varphi^0)\right],\\
{}^{\epsilon\chi}_{\shortmid}T(\tau,p,p_\theta,p_\varphi) &= {}^{\epsilon}_{\shortmid}T_0(\tau)\left[1 + 2\kappa\overline{\chi}(\tau,p,p_\theta)\sin(\ {}^{p}_{\shortmid}\omega_0\ p_\varphi + \ p_\varphi^0)\right].
\end{aligned}
$$

For $\kappa \to 0$, these formulas transform into locally isotropic ones (88). As in the thermodynamics of moving media, we have two temperature like values, τ and ${}^{\epsilon\chi}T$. In the case of ellipsoidal deformations of couples of

Schwarzschild-AdS metrics, we have to consider two different temperature like variables ${}^{\epsilon\chi}T$ and ${}^{\epsilon\chi}_{\shortmid}T$. This is different from the case described by the formulas (85) defined and computed for prime s-metrics which are not solutions of certain nonassociative vacuum Einstein equations, but they became such ones for the target s-metrics after τ-parametric dissipation of the electromagnetic components into the N-connection coefficients.

Above phase space BHs and BEs are characterized by nonassociative locally anisotropic degrees of freedom determined by R-flux deformations with spherical/rotoid symmetries. We omit the technical details and cumbersome formulas for computing ${}^{\shortmid}_{\star}\mathfrak{g}_{\alpha_s\beta_s}(\tau) = {}^{\shortmid}_{\star}\check{\mathfrak{g}}_{\alpha_s\beta_s}(\tau) + {}^{\shortmid}_{\star}\mathfrak{a}_{\alpha_s\beta_s}(\tau)$ (21) using corresponding prime data (79), or (87), and respective off-diagonal parametric solutions for double BEs and/or BHs.

5.2.4 Geometric thermodynamic variables for nonassociative flows of RN-dS BHs

The G. Perelman thermodynamic variables (69) can be computed in explicit form for any class of τ-running quasi-stationary solutions (80) and/ or (84) defining nonassociative geometric evolution and star product R-flux deformations of double RN-dS BH configurations with ${}_1\Lambda(\tau) = {}_2\Lambda(\tau) = \check{\Lambda}$ and ${}^{\shortmid}_3\Lambda(\tau) = {}^{\shortmid}_4\Lambda(\tau) = {}^{\shortmid}\check{\Lambda}$. For simplicity, we consider a nonholonomic geometric model with fixed values of cosmological h- and c-cosmologival constants. Changing the prime data, ${}^{\shortmid}_s\mathring{\mathbf{g}} \to {}^{\flat}_{\shortmid s}\mathbf{g}$ (79), in the volume forms (72), we express

$$\begin{aligned}
{}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau) &= {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\ {}^{\shortmid}_s\Im(\tau),\ \check{\Lambda},\ {}^{\shortmid}\check{\Lambda},\ {}^{\shortmid}\eta_{\alpha_s}(\tau)\ \ {}^{\flat}_{\shortmid}g_{\alpha_s}) = \frac{1}{|\check{\Lambda}\ {}^{\shortmid}\check{\Lambda}|}\ \ {}^{\shortmid}\delta\ {}^{\shortmid}_{\eta}\mathcal{V}(\ {}^{\shortmid}_s\Im(\tau),\ {}^{\flat}_{\shortmid}g_{\alpha_s}),\ \text{where} \\
{}^{\shortmid}\delta\ {}^{\shortmid}_{\eta}\mathcal{V}(\ {}^{\shortmid}_s\Im(\tau),\ {}^{\flat}_{\shortmid}g_{\alpha_s}) &= {}^{\shortmid}\delta\ {}^{1}_{\eta}\mathcal{V}[\ {}_1\Im(\tau),\eta_2(\tau)\ \ {}^{\flat}g_2] \times {}^{\shortmid}\delta\ {}^{2}_{\eta}\mathcal{V}[\ {}_2\Im(\tau),\eta_4(\tau)\ \ {}^{\flat}g_4] \times \\
&\quad {}^{\shortmid}\delta\ {}^{3}_{\eta}\mathcal{V}[\ {}_3\Im(\tau),\ {}^{\shortmid}\eta^6(\tau)\ \ {}^{\flat}_{\shortmid}g^6] \times {}^{\shortmid}\delta\ {}^{4}_{\eta}\mathcal{V}[\ {}_4\Im(\tau),\ {}^{\shortmid}\eta^8(\tau)\ \ {}^{\flat}_{\shortmid}g^8].
\end{aligned}$$

This allows to compute, using formulas (73) and (74) , such thermodynamic variables and, respective, volume functionals:

$$\begin{aligned}
{}^{\shortmid}_s\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau) &= \int_{\tau'}^{\tau}\frac{d\tau}{64(\pi\tau)^4}\frac{\tau(\check{\Lambda}+\ {}^{\shortmid}\check{\Lambda})^2-2}{|\check{\Lambda}\ {}^{\shortmid}\check{\Lambda}|}\ {}^{\shortmid\flat}_{\eta}\mathcal{V}(\tau),\ {}^{\shortmid}_s\widehat{\mathcal{Z}}^{\star}_{\kappa}(\tau) = \exp\left[\int_{\tau'}^{\tau}\frac{d\tau}{(2\pi\tau)^4}\frac{1}{|\check{\Lambda}\ {}^{\shortmid}\check{\Lambda}|}\ {}^{\shortmid\flat}_{\eta}\mathcal{V}(\tau)\right], \qquad (90) \\
{}^{\shortmid}_s\widehat{\mathcal{E}}^{\star}_{\kappa}(\tau) &= -\int_{\tau'}^{\tau}\frac{d\tau}{128\pi^4\tau^3}\frac{\tau(\check{\Lambda}+\ {}^{\shortmid}\check{\Lambda})-2}{|\check{\Lambda}\ {}^{\shortmid}\check{\Lambda}|}\ {}^{\shortmid\flat}_{\eta}\mathcal{V}(\tau),\ {}^{\shortmid}_s\widehat{\mathcal{S}}^{\star}_{\kappa}(\tau) = -\int_{\tau'}^{\tau}\frac{d\tau}{128(\pi\tau)^4}\frac{\tau(\check{\Lambda}+\ {}^{\shortmid}\check{\Lambda})-4}{|\check{\Lambda}\ {}^{\shortmid}\check{\Lambda}|}\ {}^{\shortmid\flat}_{\eta}\mathcal{V}(\tau), \\
&\text{for}\ \ {}^{\shortmid\flat}_{\eta}\mathcal{V}(\tau) = \int_{{}^{\shortmid}_s\widehat{\Xi}} {}^{\shortmid}\delta\ {}^{\shortmid}_{\eta}\mathcal{V}(\ {}^{\shortmid}_s\Im(\tau),\ {}^{\flat}_{\shortmid}g_{\alpha_s}).
\end{aligned}$$

Such values are well-defined for τ-running nonholonomic configurations with respective η- or χ-generating functions when ${}^{\shortmid}_s\widehat{\mathcal{E}}^{\star}_{\kappa}(\tau)$ and ${}^{\shortmid}_s\widehat{\mathcal{S}}^{\star}_{\kappa}(\tau)$ can be treated as respective effective energy and entropy flow transports in a phase space media evolving in the interval $\tau' < \tau$.

5.3 Nonassociative flows of phase space Reisner-Nordström-AdS BHs

In this subsection, we construct a different class of nonassociative geometric flow deformations of RN BHs, when the effective cosmological constants are determined by negative cosmological constants and respective prime metric configurations, which correspond to different models than those stated by (77) and (79).

5.3.1 Prime metrics for higher dimension phase space RN-AdS BHs

As shown in [55], a $d = 5$ dimensional Einstein-Maxwell action

$$S = \frac{1}{16\pi G_{[5]}}\int_{V_{[5]}} d^5x\sqrt{|g_{[5]}|}[R_{[5]} - l^2_{[5]}F^2_{[4]} + \frac{12}{l^2_{[5]}}], \qquad (91)$$

with a negative constant $\Lambda_{[5]} = -6/l^2_{[5]}$ determined by the AdS radius $l_{[5]}$, can be naturally viewed (with an additional Chern-Simons term) as an effective truncation of the IIB supergravity on a 5-d sphere, $\mathbb{S}^5$. In [56], the thermodynamic geometry of 5-d Reisner-Nordström-AdS BHs existing in the theory (91) was studied for extensions to models of phase spaces determined by the scalar curvatures of the thermodynamic Weinhold/ Ruppeinder / Quevedo metrics.

In our approach to nonassociative gravity and geometric flow theory, we can consider a $d=5$ dimensional analog of the Reisner-Nordström AdS metric trivially embedded into a 8-d phase space ${}_s\mathcal{M}$,

$$d\ \breve{s}^2_{[5+3]} = {}^{\shortmid}\breve{g}_{\alpha_s}({}^{\shortmid}u^{\gamma_s})(\breve{\mathbf{e}}^{\alpha_s})^2 = \frac{d\breve{r}^2}{\breve{f}(\breve{r})} - \breve{f}(\breve{r})dt^2 + \breve{r}^2[(d^2\hat{x}^2)^2 + (d\hat{x}^3)^2 + (dp_5)^5] + (dp_6)^2 + (dp_7)^2 - dE^2, \quad (92)$$

where in natural units $\hat{x}^1 = \breve{r} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (p_5)^2}$; for simplicity, we choose $\hat{x}^2 = \hat{x}^2(x^2, x^3, p_5)$, $\hat{x}^3 = \hat{x}^3(x^2, x^3, p_5)$ and $\hat{x}^5 = \hat{x}^5(x^2, x^3, p_5)$ as coordinates for a diagonal metric on an effective 3-d Einstein phase space $V_{[3]}$ of constant scalar curvature (let say, $6\hat{k}$, for $\hat{k}=1$). The metric function in (92) is given by

$$\breve{f}(\breve{r}) = 1 - \frac{\hat{m}}{\breve{r}^2} + \frac{\breve{r}^2}{l^2_{[5]}} + \frac{\hat{q}^2}{\breve{r}^4},$$

where the integration constant $\hat{m}$ is related to the mass of BH, $\hat{M} = 3\omega_{[3]}\hat{m}/16\pi G_{[5]}$, for $\omega_{[3]}$ denoting the volume of $V_{[3]}$; and the parameter $\hat{q}$ is related to the physical charge $\hat{Q}$ of the RN-AdS BH via formula $\hat{q} = 4\pi G_{[5]}\hat{Q}/\sqrt{3}\omega_{[3]}$.

The prime metric coefficients $\breve{g}_1 = \breve{f}(\breve{r})^{-1}$, ${}^{\shortmid}\breve{g}_2 = {}^{\shortmid}\breve{g}_3 = {}^{\shortmid}\breve{g}^5 = \breve{r}^2$, $\breve{g}_4 = -\breve{f}(\breve{r})$, ${}^{\shortmid}\breve{g}^6 = {}^{\shortmid}\breve{g}^7 = -{}^{\shortmid}\breve{g}^8 = 1$ and ${}^{\shortmid}\breve{g}^{a_s}_{i_{s-1}}(\breve{r}, t, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_6, p_7, E) = 0$ from (92) can be subjected to s-adapted frame/coordinate transforms into certain data $({}^{\shortmid}\breve{\mathbf{g}}_{\alpha_s}({}^{\shortmid}u^{\gamma_s});\ {}^{\shortmid}\breve{\mathbf{N}}^{a_s}_{i_{s-1}}({}^{\shortmid}u^{\gamma_s}))$ which allow to apply the AFCDM for constructing exact and parametric solutions. The 5-d part of the 8-d metric (92) can be uplifted to ten dimensions and viewed as the near horizon geometry of $\check{N}$ rotating black D3-branes in type IIB supergravity [55, 56], when $l^4_{[10]} = 2\check{N}\ell^4_p/\pi^2 \equiv \alpha^2\check{N}$, where ℓ_p is the 10-d Planck length. The nonassociative geometric constructions with star product and R-flux deformations involve different types of constants on 8-d phase spaces. It should be noted here that we can elaborate similar nonholonomic geometric constructions with ${}_s\mathcal{M} \to {}^{\star}_s\mathcal{M}$, when for a corresponding cotangent Lorentz bundle $\mathcal{M} = T^*V, \dim V = 5$ and $\dim \mathcal{M} = 10$.

5.3.2 Nonassociative geometric κ-linear evolution of phase space RN-AdS BHs

We consider nonassociative generic off-diagonal generalizations of s-metric ${}^{\shortmid}\breve{\mathbf{g}}_{\alpha_s}$ (92) under κ-linear geometric flow evolution with fixed 8-d phases space cosmological constant ${}_s\Lambda(\tau) = \Lambda_{[5]} < 0$, for $s = 1,2,3,4$. Considering ${}^{\shortmid}\breve{g}_{\alpha_s}$ as the coefficients of the prime s-metric in (68) (instead of ${}^{\shortmid}_s\mathring{\mathbf{g}}(\tau)$) for respective effective sources ${}^{\shortmid}_s\Im(\tau)$ related via nonlinear symmetries (60) to $\Lambda_{[5]}$, we generate such a τ-family of quasi-stationary solutions of nonassociative geometric flow equations (62), ${}^{\shortmid}\breve{g}_{\alpha_s} \to {}^{\shortmid}_s\mathring{\mathbf{g}}(\tau)$, parameterized, for instance, using Φ-generating functions as in quadratic linear elements (63) with nonholonomic frames (64),

$$\begin{aligned} d\widehat{s}^2(\tau) &= e^{\psi(\hbar,\kappa;\tau,\breve{r},\hat{x}^2,\Lambda_{[5]})}[(d\breve{r})^2 + (d\hat{x}^2)^2] - \quad (93)\\ &\frac{1}{g_4^{[0]}(\tau) - \frac{({}_2\Phi(\tau))^2}{4\ \Lambda_{[5]}}} \frac{({}_2\Phi(\tau))^2[\partial_3({}_2\Phi(\tau))]^2}{|\ \Lambda_{[5]}\int dy^3({}^{\shortmid}_2\Im(\tau))[\partial_3({}_2\Phi(\tau))^2]|}(\mathbf{e}^3(\tau))^2 + \left(g_4^{[0]}(\tau) - \frac{({}_2\Phi(\tau))^2}{4\ \Lambda_{[5]}}\right)(\mathbf{e}^4(\tau))^2\\ &-\frac{1}{{}^{\shortmid}g^6_{[0]}(\tau) - \frac{({}^{\shortmid}_3\Phi(\tau))^2}{4\ \Lambda_{[5]}}} \frac{({}^{\shortmid}_3\Phi(\tau))^2[{}^{\shortmid}\partial^5({}^{\shortmid}_3\Phi(\tau))]^2}{|\ \Lambda_{[5]}\int dp_5({}^{\shortmid}_3\Im(\tau))\ {}^{\shortmid}\partial^5[({}^{\shortmid}_3\Phi(\tau))^2]|}({}^{\shortmid}\mathbf{e}_5(\tau))^2 + \left({}^{\shortmid}g^6_{[0]}(\tau) - \frac{({}^{\shortmid}_3\Phi(\tau))^2}{4\ \Lambda_{[5]}}\right)({}^{\shortmid}\mathbf{e}_6(\tau))^2\\ &-\frac{1}{{}^{\shortmid}g^8_{[0]}(\tau) - \frac{({}^{\shortmid}_4\Phi(\tau))^2}{4\ \Lambda_{[5]}}} \frac{({}^{\shortmid}_4\Phi(\tau))^2[{}^{\shortmid}\partial^7({}^{\shortmid}_4\Phi(\tau))]^2}{|\ \Lambda_{[5]}\int dp_7({}^{\shortmid}_4\Im)\ {}^{\shortmid}\partial^7[({}^{\shortmid}_4\Phi(\tau))^2]|}({}^{\shortmid}\mathbf{e}_7(\tau))^2 + \left({}^{\shortmid}g^8_{[0]}(\tau) - \frac{({}^{\shortmid}_4\Phi(\tau))^2}{4\ \Lambda_{[5]}}\right)({}^{\shortmid}\mathbf{e}_8(\tau))^2. \end{aligned}$$

In these formulas, there are used local coordinates ${}^{\shortmid}u^{\gamma_s} = (\breve{r}, t, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_6, p_7, E)$ and s-adapted frames:

$$
\begin{aligned}
\mathbf{e}^3(\tau) &= d\hat{x}^3 + \frac{\partial_{k_1} \int d\hat{x}^3 ({}_{2}^{\shortmid}\Im(\tau)) \hat{\partial}_3[({}_{2}\Phi(\tau))^2]}{({}_{2}^{\shortmid}\Im(\tau)) \ \hat{\partial}_3[({}_{2}\Phi(\tau))^2]} dx^{k_1}, \\
\mathbf{e}^4(\tau) &= dt + ({}_{1}n_{k_1}(\tau) + {}_{2}n_{k_1}(\tau) \frac{\int d\hat{x}^3 \frac{({}_{2}\Phi(\tau))^2 [\ \hat{\partial}_3({}_{2}\Phi(\tau))]^2}{|\ {}_{2}\Lambda(\tau) \int d\hat{x}^3 ({}_{2}^{\shortmid}\Im(\tau)) [\ \hat{\partial}_3({}_{2}\Phi(\tau))^2]|}}{\left| g_4^{[0]}(\tau) - \frac{({}_{2}\Phi(\tau))^2}{4\ {}_{2}\Lambda(\tau)} \right|^{5/2}}) dx^{k_1},
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}\mathbf{e}_5(\tau) &= d\hat{x}^5 + \frac{\partial_{k_2} \int d\hat{x}^5 ({}_{3}^{\shortmid}\Im(\tau)) \ \hat{\partial}[({}_{3}^{\shortmid}\Phi(\tau))^2]}{({}_{3}^{\shortmid}\Im(\tau)) \quad \hat{\partial}_5[({}_{3}^{\shortmid}\Phi(\tau))^2]} dx^{k_2}, \\
{}^{\shortmid}\mathbf{e}_6(\tau) &= dp_6 + ({}_{1}^{\shortmid}n_{k_2}(\tau) + {}_{2}^{\shortmid}n_{k_2}(\tau) \frac{\int dp_5 \frac{({}_{3}^{\shortmid}\Phi(\tau))^2 [\ {}^{\shortmid}\partial^5({}_{3}^{\shortmid}\Phi(\tau))]^2}{|\ \Lambda_{[5]} \int dp_5 ({}_{3}^{\shortmid}\Im(\tau)) [\ {}^{\shortmid}\partial^5({}_{3}^{\shortmid}\Phi(\tau))^2]|}}{\left| {}^{\shortmid}g^6_{[0]}(\tau) - \frac{({}_{3}^{\shortmid}\Phi(\tau))^2}{4\ \Lambda_{[5]}} \right|^{5/2}}) dx^{k_2},
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}\mathbf{e}_7(\tau) &= dp_7 + \frac{\partial_{k_3} \int dp_7 ({}_{4}^{\shortmid}\Im(\tau)) \quad {}^{\shortmid}\partial^7[({}_{4}^{\shortmid}\Phi(\tau))^2]}{({}_{4}^{\shortmid}\Im(\tau)) \quad {}^{\shortmid}\partial^7[({}_{4}^{\shortmid}\Phi(\tau))^2]} d\ {}^{\shortmid}x^{k_3}, \\
{}^{\shortmid}\mathbf{e}_8(\tau) &= dE + ({}_{1}^{\shortmid}n_{k_3}(\tau) + {}_{2}^{\shortmid}n_{k_3}(\tau) \frac{\int dp_7 \frac{({}_{4}^{\shortmid}\Phi(\tau))^2 [\ {}^{\shortmid}\partial^7({}_{4}^{\shortmid}\Phi(\tau))]^2}{|\ \Lambda_{[5]} \int dp_7 ({}_{4}^{\shortmid}\Im(\tau)) [\ {}^{\shortmid}\partial^7({}_{4}^{\shortmid}\Phi(\tau))^2]|}}{\left| {}^{\shortmid}g^8_{[0]}(\tau) - \frac{({}_{4}^{\shortmid}\Phi(\tau))^2}{4\ \Lambda_{[5]}} \right|^{5/2}}) d\ {}^{\shortmid}x^{k_3}.
\end{aligned}
$$

The integration functions considered above are defined in the form:

$$
\begin{aligned}
& g_4^{[0]}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2), \ {}_{1}n_{k_1}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2), \ {}_{2}n_{k_1}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2); \\
& {}^{\shortmid}g^5_{[0]}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5), \ {}_{1}n_{k_2}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5), \ {}_{2}n_{k_2}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5); \\
& {}^{\shortmid}g^7_{[0]}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_7), \ {}_{1}^{\shortmid}n_{k_3}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_7), \ {}_{2}^{\shortmid}n_{k_3}(\hbar, \kappa, \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_7).
\end{aligned}
$$

The solutions (93) can be written in terms of g-generating functions and/or η- /χ-polarization functions using nonlinear transforms (61) for

$$
\begin{aligned}
{}_{2}\Phi(\tau) &= 2\sqrt{|\ \Lambda_{[5]}\ g_4(\tau)|} = 2\sqrt{|\ \Lambda_{[5]}\ \eta_4(\tau)\breve{g}_4)|} \simeq 2\sqrt{|\ \Lambda_{[5]}\ \zeta_4(\tau)\breve{g}_4|}[1 - \frac{\kappa}{2}\chi_4(\tau)], \qquad (94) \\
{}_{3}^{\shortmid}\Phi(\tau) &= 2\sqrt{|\ \Lambda_{[5]}\ {}^{\shortmid}g^6(\tau)|} = 2\sqrt{|\ \Lambda_{[5]}\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\breve{g}^6|} \simeq 2\sqrt{|\ \Lambda_{[5]}\ {}^{\shortmid}\zeta^6(\tau)\ {}^{\shortmid}\breve{g}^6|}[1 - \frac{\kappa}{2}\ {}^{\shortmid}\chi^6(\tau)], \\
{}_{4}^{\shortmid}\Phi(\tau) &= 2\sqrt{|\ \Lambda_{[5]}\ {}^{\shortmid}g^8(\tau)|} = 2\sqrt{|\ \Lambda_{[5]}\ {}^{\shortmid}\eta^8(\tau)\ {}^{\shortmid}\breve{g}^8|} \simeq 2\sqrt{|\ \Lambda_{[5]}\ {}^{\shortmid}\zeta^8(\tau)\ {}^{\shortmid}\breve{g}^8|}[1 - \frac{\kappa}{2}\ {}^{\shortmid}\chi^8(\tau)],
\end{aligned}
$$

for a prime s-metric ${}^{\shortmid}\breve{\mathbf{g}}_{\alpha_s}$ (92). For such transforms, the generating and integration functions are written in κ–linearized form (A.14),

$$
\begin{aligned}
\psi(\tau) &\simeq \psi(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2) \simeq \psi_0(\hbar, \tau, \breve{r}, \hat{x}^2)(1 + \kappa\ {}_{\psi}\chi(\hbar, \tau, \breve{r}, \hat{x}^2)), \text{ for} \\
\eta_2(\tau) &\simeq \eta_2(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2) \simeq \zeta_2(\hbar, \tau, \breve{r}, \hat{x}^2)(1 + \kappa\chi_2(\hbar, \tau, \breve{r}, \hat{x}^2)), \text{ we can consider } \eta_2(\tau) = \eta_1(\tau); \\
\eta_4(\tau) &\simeq \eta_4(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3) \simeq \zeta_4(\hbar, \tau, \breve{r}, \hat{x}^2, \hat{x}^3)(1 + \kappa\ \chi_4(\hbar, \tau, \breve{r}, \hat{x}^2, \hat{x}^3)), \\
{}^{\shortmid}\eta^6(\tau) &\simeq {}^{\shortmid}\eta^6(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5) \simeq {}^{\shortmid}\zeta^6(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5)(1 + \kappa\ {}^{\shortmid}\chi^6(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5)), \\
{}^{\shortmid}\eta^8(\tau) &\simeq {}^{\shortmid}\eta^8(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_7) \simeq {}^{\shortmid}\zeta^8(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_7)(1 + \kappa\ {}^{\shortmid}\chi^8(\hbar, \kappa; \tau, \breve{r}, \hat{x}^2, \hat{x}^3, \hat{x}^5, p_7)).
\end{aligned}
$$

Using formulas (94), we can extract solutions with rotoid spacetime configurations determined by nonassociative star product R-flux deformation (considering χ-polarizations), or to compute volume forms (70) for η-polarizations.

5.3.3 Bekenstein-Hawking entropy of τ-running phase space RN-AdS BEs configurations

For a subclass of nonholonomic configurations, the s-metrics (93) define higher dimension BH and/or BE configurations with conventional horizons which can be characterized variables in the framework of generalized Bekenstein-Hawking thermodynamics [47, 48, 49, 50]. For simplicity, we shall consider solutions for 6-d τ-running quasi-stationary configurations evolving in a 8-d phase space, for simplicity, with trivial integration functions of type ${}_1^{\shortmid}n_{k_s} = 0$ and ${}_2^{\shortmid}n_{k_s} = 0$. The corresponding nonlinear quadratic elements are parameterized in the form:

$$\begin{aligned} d\,{}_{\shortmid}^{\chi}s^2_{[6\subset 8d]}(\tau) &= e^{\psi_0}(1+\kappa\ {}^{\psi(\tau)}\ {}^{\shortmid}\chi(\tau))[\ \breve{g}_1(\breve{r})d\breve{r}^2 + \breve{g}_2(\breve{r})(d\hat{x}^2)] \\ & -\{\frac{4[\hat{\partial}_3(|\zeta_4(\tau)\breve{g}_4(\breve{r})|^{1/2})]^2}{\breve{g}_4(\breve{r})|\int d\hat{x}^3\{\ {}_2\Im(\tau)\hat{\partial}_3(\zeta_4(\tau)\ \breve{g}_4(\breve{r}))\}|} - \kappa[\frac{\hat{\partial}_3(\chi_4(\tau)|\zeta_4(\tau)\ \breve{g}_4(\breve{r})|^{1/2})}{4\hat{\partial}_3(|\zeta_4(\tau)\ \breve{g}_4(\breve{r})|^{1/2})} \\ & -\frac{\int d\hat{x}^3\{\ {}_2\Im(\tau)\hat{\partial}_3[(\zeta_4(\tau)\ \breve{g}_4(\breve{r}))\chi_4(\tau)]\}}{\int d\hat{x}^3\{\ {}_2\Im(\tau)\hat{\partial}_3(\zeta_4(\tau)\ \breve{g}_4(\breve{r}))\}}]\}\ \breve{g}_3(\mathbf{e}^3(\tau))^2 +\ \zeta_4(\tau)(1+\kappa\ \chi_4(\tau))\breve{g}_4(\breve{r})dt^2 \end{aligned} \quad (95)$$

$$\begin{aligned} & -\{\frac{4[\hat{\partial}_5(|\ {}^{\shortmid}\zeta^6(\tau)\ \ \breve{g}^6|^{1/2})]^2}{\breve{g}_5(\breve{r})|\int d\hat{x}^5\{\ {}_3^{\shortmid}\Im(\tau)\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^6(\tau)\ \breve{g}^6)\}|} - \kappa[\frac{\hat{\partial}_5(\ {}^{\shortmid}\chi^6(\tau)|\ {}^{\shortmid}\zeta^6(\tau)\ \breve{g}^6|^{1/2})}{4\hat{\partial}_5(|\ {}^{\shortmid}\zeta^6(\tau)\ \breve{g}^6|^{1/2})} \\ - \quad & \frac{\int d\hat{x}^5\{\ {}_3^{\shortmid}\Im(\tau)\ \hat{\partial}_5[(\ {}^{\shortmid}\zeta^6(\tau)\breve{g}^6)\ {}^{\shortmid}\chi^8(\tau)]\}}{\int d\hat{x}^5\{\ {}_3^{\shortmid}\Im(\tau)\ \hat{\partial}_5[(\ {}^{\shortmid}\zeta^6(\tau)\breve{g}^6)]\}}]\}\ \breve{g}_5(\breve{r})(\mathbf{e}^5(\tau))^2 +\ {}^{\shortmid}\zeta^6(\tau)\ (1+\kappa\ {}^{\shortmid}\chi^6(\tau))(dp_6)^2 + (dp_7)^2 - dE^2, \end{aligned}$$

where

$$\mathbf{e}^3(\tau) = d\hat{x}^3 + [\frac{\hat{\partial}_{i_1}\int d\hat{x}^3\ {}_2\Im(\tau)\ \hat{\partial}_3\zeta_4(\tau)}{\breve{N}^3_{i_1}\ {}_2\Im(\tau)\hat{\partial}_3\zeta_4(\tau)} + \kappa(\frac{\hat{\partial}_{i_1}[\int d\hat{x}^3\ {}_2\Im(\tau)\hat{\partial}_3(\zeta_4(\tau)\chi_4(\tau))]}{\hat{\partial}_{i_1}\ [\int d\hat{x}^3\ {}_2\Im(\tau)\hat{\partial}_3\zeta_4(\tau)]} - \frac{\hat{\partial}_3(\zeta_4(\tau)\chi_4(\tau))}{\hat{\partial}_3\zeta_4(\tau)})]\ \breve{N}^3_{i_1}dx^{i_1},$$

$$\begin{aligned} \mathbf{e}^5(\tau) &= d\hat{x}^5 + [\frac{\hat{\partial}_{i_2}\ \int d\hat{x}^5\ {}_3^{\shortmid}\Im(\tau)\ \ \hat{\partial}_5(\ {}^{\shortmid}\zeta^6(\tau))}{{}^{\shortmid}\breve{N}^5_{i_2}\ {}_3^{\shortmid}\Im(\tau)\ \ \hat{\partial}_5(\ {}^{\shortmid}\zeta^6(\tau))} + \\ & \kappa(\frac{\hat{\partial}_{i_2}[\int d\hat{x}^5\ {}_3^{\shortmid}\Im(\tau)\ \ \hat{\partial}_5(\ {}^{\shortmid}\zeta^6(\tau)\ \ \breve{g}^6)]}{\hat{\partial}_{i_2}\ [\int d\hat{x}^5\ {}_3^{\shortmid}\Im(\tau)\ \ \hat{\partial}_5(\ {}^{\shortmid}\zeta^6(\tau))]} - \frac{\hat{\partial}_5(\ {}^{\shortmid}\zeta^6(\tau)\ \ \breve{g}^6)}{\hat{\partial}_5(\ {}^{\shortmid}\zeta^6(\tau))})]\ \ {}^{\shortmid}\breve{N}^5_{i_2}d\ {}^{\shortmid}x^{i_2}. \end{aligned}$$

A subclass of solutions (95) generates τ-families of rotoid configurations in coordinates $(\breve{r}, \hat{x}^2, \hat{x}^3)$ (as nonholonomic deformations of the phase BH solution (92)) if we chose such generating functions:

$$\chi_4(\tau) = \hat{\chi}_4(\tau, \breve{r}, \hat{x}^2, \hat{x}^3) = 2\underline{\chi}(\tau, \breve{r}, \hat{x}^2)\sin(\omega_0\hat{x}^3 + \hat{x}^3_0), \quad (96)$$

where $\underline{\chi}(\tau, \breve{r}, \hat{x}^2)$ are smooth functions (or constants), and $(\omega_0, \hat{x}^3_0)$ is a couple of constants. In a conventional 5-d phase space on shells $s = 1, 2, 3$, trivially imbedded into a 8-d phase space posses a distinct ellipsoidal type horizon with respective eccentricity κ a stated by the equations

$$\zeta_4(\tau)(1+\kappa\ \chi_4(\tau))\breve{g}_4(\breve{r}) = 0 \text{ i.e. } (1+\kappa\ \chi_4)\breve{f}(\breve{r}) = (1 - \frac{\hat{m}}{\breve{r}^2} - \frac{\Lambda_{[5]}}{6}\breve{r}^2 + \frac{\hat{q}^2}{\breve{r}^4} + \kappa\ \chi_4) = 0,$$

for $\zeta_4 \neq 0$. For small parametric deformations and configurations with $-\frac{\Lambda_{[5]}}{6}\breve{r}^2 + \frac{\hat{q}^2}{\breve{r}^4} \approx 0$, we can approximate for a fixed τ_0, $\breve{r} \simeq \hat{m}^{1/2}/(1 - \frac{\kappa}{2}\hat{\chi}_4)$. These are parametric formulas for a rotoid horizon defined by small gravitational R-flux polarizations. In the limits of zero eccentricity, such e BE configurations transform into a 5-d BH embedded into nonassociative 8-d phase space.

Extending the concept of Bekenstein-Hawking entropy for phase spaces determined by quadratic linear elements (92), we can define such thermodynamic values (computations are similar to those for formulas (8)-(15) in [56] but with different constants and following our notations):

$$
\begin{aligned}
{}^{0}\breve{S} &= \frac{{}^{0}\breve{A}}{4G_{[5]}} = \frac{\omega_{[3]}\breve{r}_h}{4G_{[5]}} \text{ and } {}^{0}\breve{T} = \frac{1}{2\pi\breve{r}_h}(\epsilon + 2\frac{\breve{r}_h^2}{l_{[5]}^2}) - \frac{2G_{[10]}^2\hat{Q}^2}{3\pi^9 l_{[5]}^8\breve{r}_h^5}, \text{ for} \\
\hat{M} &= \frac{3\omega_{[3]}\hat{m}}{16\pi G_{[5]}}(\epsilon\breve{r}_h^2 + \frac{\breve{r}_h^4}{l_{[5]}^2} + \frac{4G_{[5]}\hat{Q}^2 l_{[5]}^2}{3\pi^2\breve{r}_h^2}),
\end{aligned} \tag{97}
$$

where $\breve{r}_h$ and ${}^{0}\breve{A}$ are, respectively the horizon and area of horizon of 5-d BH, $G_{[5]} = G_{[10]}/(\pi^3 l_{[5]}^5)$ and $G_{[10]} = \ell_p^8$. Using these formulas for rotoid deformations $\breve{r}_h \to \hat{m}^{1/2}/(1 - \frac{\kappa}{2}\hat{\chi}_4)$ and ${}^{0}\breve{A} \to {}^{rot}\breve{A}$, with $\hat{\chi}_4(\tau)$ (96), we compute for respective BE configurations:

$$
\breve{S}(\tau) = {}^{0}\breve{S}(1 + \frac{\kappa}{2}\hat{\chi}_4(\tau)) \text{ and } \breve{T}(\tau) = {}^{0}\breve{T} + \kappa\left(-\frac{\epsilon}{4\pi\breve{r}_h} + \frac{\breve{r}_h}{2\pi l_{[5]}^2} - \frac{5G_{[10]}^2\hat{Q}^2}{3\pi^9 l_{[5]}^8\breve{r}_h^5}\right)\hat{\chi}_4(\tau). \tag{98}
$$

The modified Hawking temperatures $\breve{T}(\tau)$ and ${}^{0}\breve{T}$ are stated by requiring the absence of the potential conical singularity of the Euclidean BH at the horizon in the phase space.

5.3.4 G. Perelman thermodynamics of nonassociative flows of phase RN-AdS BHs

We can not apply the Bekenstein-Hawking thermodynamic paradigm in order to characterize physical properties of general classes of quasi-stationary solutions of type (93) and/or (95) excepting very special cases of nonassociative deformations, for instance, to BE configurations of type (96). The G. Perelman approach is more general and allows to define and compute statistical thermodynamic variables of type (69). Let us sketch how to compute such values for any data ${}^{\shortmid}\mathbf{\breve{g}}_{\alpha_s}$ (92), ${}^{\shortmid}_{s}\Im(\tau)$ related via nonlinear symmetries (60) to $\Lambda_{[5]}$, and nontrivial (on shells $s = 1,2,3$, see also formulas (94)). For simplicity, we can consider the same value of cosmological constant on such shells when

$$
|\ \Lambda_{[5]}\ \eta_4(\tau)\breve{g}_4)| = |\ \Lambda_{[5]}\ \zeta_4(\tau)\breve{g}_4|(1 - \kappa\chi_4(\tau)), |\ \Lambda_{[5]}\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\breve{g}^6| = |\ \Lambda_{[5]}\ {}^{\shortmid}\zeta^6(\tau)\ {}^{\shortmid}\breve{g}^6|(1 - \kappa\ {}^{\shortmid}\chi^6(\tau)) \tag{99}
$$

for a a subclass of s-metrics of type (95). We obtain such thermodynamic functionals:

$$
\begin{aligned}
{}^{\shortmid}_{s}\widehat{\mathcal{W}}^{\star}_{\kappa}(\tau) &= \int_{\tau'}^{\tau}\frac{d\tau}{32(\pi\tau)^4}\frac{2\tau\Lambda_{[5]}^2 - 1}{\Lambda_{[5]}^2}\ {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau),\ \ {}^{\shortmid}_{s}\widehat{\mathcal{Z}}^{\star}_{\kappa}(\tau) = \exp\left[\int_{\tau'}^{\tau}\frac{d\tau}{(2\pi\tau)^4}\frac{1}{\Lambda_{[5]}^2}\ {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau)\right], \\
{}^{\shortmid}_{s}\widehat{\mathcal{E}}^{\star}_{\kappa}(\tau) &= -\int_{\tau'}^{\tau}\frac{d\tau}{64\pi^4\tau^3}\frac{\tau\Lambda_{[5]} - 1}{\Lambda_{[5]}^2}\ {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau),\ {}^{\shortmid}_{s}\widehat{\mathcal{S}}^{\star}_{\kappa}(\tau) = -\int_{\tau'}^{\tau}\frac{d\tau}{64(\pi\tau)^4}\frac{\tau\Lambda_{[5]} - 2}{\Lambda_{[5]}^2}\ {}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau).
\end{aligned} \tag{100}
$$

In these formulas, we use the running phase space volume functional

$$
{}^{\shortmid}_{\eta}\breve{\mathcal{V}}(\tau) = \int_{{}^{\shortmid}_{s}\widehat{\Xi}} {}^{\shortmid}\delta\ {}^{\shortmid}_{\eta}\mathcal{V}(\ {}^{\shortmid}_{s}\Im(\tau),\ {}^{\shortmid}\breve{g}_{\alpha_s}),\ \text{for } s = 1,2,3. \tag{101}
$$

Above presented values are determined by prescribed classes of generating η-functions (99), effective generating sources ${}^{\shortmid}_{s}\Im(\tau)$, coefficients of a prime s-metric ${}^{\shortmid}\breve{g}_{\alpha_s}$ and nonholonomic distributions defining the hyper-surface ${}^{\shortmid}_{s}\widehat{\Xi}$, when the volume forms are computed

$$
\begin{aligned}
&{}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau) = {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\ {}^{\shortmid}_{s}\Im(\tau),\ \Lambda_{[5]},\ {}^{\shortmid}\eta_{\alpha_s}(\tau)\ {}^{\shortmid}\breve{g}_{\alpha_s}) = \frac{1}{\Lambda_{[5]}^2}\ {}^{\shortmid}\delta\ {}^{\shortmid}_{\eta}\mathcal{V}(\ {}^{\shortmid}_{s}\Im(\tau),\ {}^{\shortmid}\breve{g}_{\alpha_s}),\ \text{where} \\
&{}^{\shortmid}\delta\ {}^{\shortmid}_{\eta}\mathcal{V}(\ {}^{\shortmid}_{s}\Im(\tau),\ {}^{\shortmid}\breve{g}_{\alpha_s}) = {}^{\shortmid}\delta\ {}^{1}_{\eta}\mathcal{V}[{}_{1}\Im(\tau), \eta_2(\tau)\ \breve{g}_2] \times {}^{\shortmid}\delta\ {}^{2}_{\eta}\mathcal{V}[{}_{2}\Im(\tau), \eta_4(\tau)\ \breve{g}_4] \times {}^{\shortmid}\delta\ {}^{3}_{\eta}\mathcal{V}[{}_{3}\Im(\tau),\ {}^{\shortmid}\eta^6(\tau)\ {}^{\shortmid}\breve{g}^6].
\end{aligned}
$$

G. Perelman thermodynamic variables (100) can be computed in explicit form if we prescribe certain nonholonomic distributions in ${}_{\eta}^{\shortmid}\breve{\mathcal{V}}(\tau)$ (101) which results in physically viable geometric thermodynamic models. For instance, we can study solutions with more general κ-parametric deformations than those determined by rotoid generating functions (96). Prescribing generating sources ${}_{s}^{\shortmid}\Im(\tau)$ and η-generating functions for ${}^{\shortmid}\delta\ {}_{\eta}^{\shortmid}\mathcal{V}$, we define explicit thermodynamic models of deformed phase space BH objects propagating on a local temperature like parameter τ like, for instance, in the hydrodynamic of moving media. This analogy is very rough because instead of hydrodynamic flow equations we consider nonassociative geometric flow equations and certain special classes of solutions. The physical interpretation of corresponding entropy and temperature from (100) is different from that for the hyper-surface variables (98) in the Bekenstein-Hawking approach. Finally we note that the effective running of thermodynamic variables on shell cosmological constants in (90) and (100) is very different. We have to analyze such different implications, for instance, in extending the swampland conjecture for nonassociative BHs and τ-running quasi-stationary solutions.

6 Extending swampland conjectures for nonassociative geometric flows and nonholonomic BHs deformations

Connections between the swampland conjectures in string theory, QG, exact solutions in MGTs , effective QFTs and (non) commutative geometric flow equations, with a number of examples with BH solutions, were studied in a series of works [22, 23, 24, 25, 26, 27]. We also cite such papers and references therein for recent developments and applications in modern gravity, cosmology and astrophysics; and studies of various types of Hawking–Page phase transitions under Ricci, Yamanabe, Ricci-Bourguignon flows etc. It should be remembered that the general aim of the swampland program [19, 20, 21, 18] is to elaborate on mathematical and physical criteria for distinguishing (low-energy) effective classical and quantum theories, QG and MGTs, which can be completed in the UV, from other classes of theories which cannot.

The approach to nonassociative geometry and gravity [38, 39] and nonholonomic generalizations with the AFCDM for constructing exact/parametric solutions in nonassociative/noncommutative Ricci flow theories and MGTs [12, 13, 40, 41, 6, 7, 8] was developed for star product R-flux deformation models in string/M-theory. Respective fundamental nonassociative geometric flow and (modified) vacuum gravitational (Ricci soliton) equations consist very sophisticate systems of coupled nonlinear PDEs encoding nonassociative data as effective sources and generic off-diagonal terms of nonholonomic shell adapted metrics. Such equations can be integrated in very general exact/parametric forms, for instance, for quasi-stationary configurations running on τ-flow parameter as we proved in section 4, see also related results examples in section 4 of [41]. Positively, for certain classes of solutions with respective nonlinear symmetries and τ-running, or prescribed/fixed, shell effective cosmological constants, such solutions and respective nonassociative geometric thermodynamic models can be constructed to be compatible with certain refined versions of the swampland conjectures (both on higher dimension spacetimes and/or phase space). Nevertheless, generalized classes of solutions do not obligatory involve certain effective (running) cosmological constants, for instance, for nonassociative BH deformations, see subsections 5.2.4 and 5.3.4. Even for such nonassociative geometric flow configurations, we can always compute the modified G. Perelman F- and W-functionals and related thermodynamic variables, the issues of compatibility or not compatibility with the swampland ideas and conjectures have to be studied respectively for any explicit class of solutions.

In this section, we analyze and discuss two kinds of nonassociative generalized swampland and Ricci flow conjectures. First, there are outlined refined versions on the phase space of typical results obtained for high dimensional (A)dS spaces, for instance, modifications of the Black Hole Entropy Distance Conjecture (BHEDC) to the case of nonassociative black ellipsoid, BE, configurations etc. Such constructions are possible for nonassociative τ-running quasi-stationary solutions with conventional hyper-surface horizons and associated modified Bekenstein-Hawking thermodynamic models studied above in subsections 5.2.3 and 5.3.3. Second, we elaborate on some main points of this paper when the swampland program is extended to nonassociative geometric flow and MGTs using the concept of W-entropy and G. Perelman thermodynamics. There are considered certain well-defined conditions when positively swampland and nonassociative Ricci flow conjectures can formulated for such generalizations. We also analyze explicit examples of τ-running quasi-stationary solutions when it is not clear and we can't conclude without an additional analysis if such low-energy effective (encoding nonassociative data) geometric flow and gravity models can be completed into QG in the UV and distinguished from those that cannot.

6.1 The generalized distance conjecture, nonassociative Ricci flows and gravity

The goal of this subsection is to show how the swampland, gradient flows and infinite distance conjectures [22, 23] can be extended on nonassociative phase spaces for solutions of nonassociative Ricci flow/soliton equations derived in sections 2.3 and 2.4 and κ-parameterized in the form (41).

6.1.1 Phase space generalized and Weyl distances, and the AdS distance conjecture

For a phase space with star product R-flux nonassociative deformation, ${}_{s}\mathcal{M} \rightarrow {}_{s}^{\star}\mathcal{M}$, restricted to a finite region ${}_{s}^{\star}\mathcal{U} \subset {}_{s}^{\star}\mathcal{M}$, we consider a τ-family of symmetric s-metrics ${}_{s}^{\shortmid}\mathbf{g}(\tau)$ following Convention 2 (26). There are used κ-linear parametric decomposition when ${}_{\star}^{\shortmid}\check{\mathfrak{g}}_{\alpha_s\beta_s}^{[0]}(\tau) = {}_{\star}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau) = {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)$, for star product flows $\star_s(\tau)$ determined by s-adapted frames ${}^{\shortmid}\mathbf{e}_{i_s}(\tau)$ in (19) and canonical data $({}_{s}^{\shortmid}\mathbf{g}(\tau), {}_{s}^{\shortmid}\widehat{\mathbf{D}}(\tau))$. We can consider arbitrary variations of such a s-metric, or to choose a class of solutions of nonassociative geometric flow equations (41) for the data $[{}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau), {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau)]$, when the nonsymmetric ${}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}$ are computed using κ-linear parameterizations (20)–(22). On ${}_{s}^{\star}\mathcal{U}$, a path $\gamma(\tau), \tau_i \leq \tau \leq \tau_f$, is considered (for which a proper metric distance $\aleph$ is defined by ${}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)$) (5).[17] We introduce

$$
{}_{g}^{s}\aleph = \aleph[{}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau), {}_{s}^{\shortmid}\Lambda(\tau)] = \mathcal{C}\int_{\tau_i}^{\tau_f}\left(\frac{1}{\mathcal{V}_{\mathcal{U}}}\int_{\mathcal{U}} {}^{\shortmid}\mathbf{g}^{\alpha_s\mu_s}\,{}^{\shortmid}\mathbf{g}^{\beta_s\nu_s}\frac{\partial\,{}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}}{\partial\tau}\,\frac{\partial\,{}^{\shortmid}\mathbf{g}_{\mu_s\nu_s}}{\partial\tau}d\,{}^{\shortmid}\mathcal{V}ol(\tau)\right)^{1/2}d\tau \qquad (102)
$$

for the volume form $d\,{}^{\shortmid}\mathcal{V}ol(\tau)$ (36), where the constant $\mathcal{C} \sim \mathcal{O}(1)$ depending on dimension of $\mathcal{U}$ and $\mathcal{V}_{\mathcal{U}}$ is the volume of $\mathcal{U} \rightarrow {}_{s}^{\star}\mathcal{U}$. In this work, we consider that ${}_{g}^{s}\aleph$ is determined by a τ-family of κ-parametric solutions of geometric flow equations (62) for some prescribed running shell cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$.

For phase spaces, the generalized distance conjecture states that there must be an infinite tower of states with an effective mass scale ${}^{\shortmid}m(\tau)$, when

$$
{}^{\shortmid}m(\tau_f) \sim {}^{\shortmid}m(\tau_i)e^{-\hat{\alpha}|\,{}_{g}^{s}\aleph|}, \text{ where } \hat{\alpha} \sim \mathcal{O}(1). \qquad (103)
$$

This formula was introduced without labels " ${}^{\shortmid}$" for corresponding notations for higher dimension spacetimes in [22, 23], when

$$
m \sim M_p e^{-\alpha|\,{}_{g}\aleph|}. \qquad (104)
$$

In field theories, it is considered that generically the masses $m(\tau_i)$ at some initial values τ_i are of order of M_p (the Planck mass constant, determined by $\hbar$). We have to introduce such constants with additional labels also for phase spaces ${}_{s}^{\star}\mathcal{M}$, when we elaborate on (nonassociative and noncommutative) quantum geometric flow models and/or QFTs. On 4-d spacetime base with shells $s = 1, 2$, the formula ${}^{\shortmid}m$ (103) still contains nonassociative star product and R-flux data encoded into ${}_{g}^{s}\aleph$. Only for very special nonholonomic configurations, non involving

[17] In this work, we write $\aleph$ instead of $\triangle$ considered, for instance, in formula (1) from [23] for formulating the general distance conjecture. On nonassociative phase spaces, we elaborate a different system of notations when the symbol ${}^{\star}\widehat{\triangle}$ is the Laplace operator constructed for ${}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}$. Here we also note that in field theories, the parameter τ can be an arbitrary one, parameterizing some curves/geodesic etc. In this section, we are interested in models when τ can be always related to a geometric flow / temperature like parameter.

the effects of possible off-diagonal and generalized effective sources, and omitting κ-parametric terms, we may follow the assumptions for m (104). We note that the mass scale m in associative/commutative classical and quantum theories states a natural cut-off above which the effective field description is not valid. In such cases, for large distance variations in the space of metrics, when $|\ _{g}\aleph| \rightarrow \infty$, we get for QG models a respective massless tower of states. This breaks down/ invalidates the effective field theory description. In modern literature [19, 20, 21, 18, 22, 23], it is argued that for respective conditions such theories belong to the swampland. The conjectures and related results have to be revised for nonassociative geometric flow and gravity/matter field theories when exact/parametric generic off-diagonal solutions of effective nonholonomic Ricci flow and/or Ricci soliton equations are considered following the AFCDM. We formulate:

The generalized distance conjecture and claim for nonassociative phase spaces, CCL1:

a) Conjecture: *For nonassociative geometric flow, gravity, and classical field theories, and respective QG and QGTs encoding star product and R-flux data from string/M-theory, there are classes of exact/parametric solutions with nonlinear phase space symmetries of type (A.8) connecting effective sources to effective shell cosmological constants (for τ-running or fixed τ_0 quasi-stationary and/or locally anisotropic cosmological configurations with a time like coordinate), when the corresponding effective geometric/physical models belong to the swampland.*

b) Claim: *In a general context, nonassociative/ noncommutative/ nonholonomic / generic off-diagonal/ generalized (non) linear connection modifications of gravity theories contain also large classes of exact/parametric solutions (involving, or not, nonlinear symmetries of type (A.8)) defining effective τ-running, or fixed τ_0), configurations which are physically well defined and characterized by a respective modified G. Perelman thermodynamics with variables of type (50). We have to analyze additionally for which conditions such models belong or not to the swampland.*

It should be noted here that above a) Conjecture is a nonassociative phase space version of the generalized distance conjecture for (associative and commutative) d-dimensional manifolds studied in [22, 23]. In this work, we show that such a conjecture (and related ones, see next subsections) can be formulated for respective conditions on nonassociative geometric flows when $\ _{g}^{s}\aleph$(102)$\rightarrow \infty$ in the space of τ-running, or fixed τ_0, pase space s-metrics. Nevertheless, using the AFCDM, we can construct large classes of exact/parametric solutions when $\ _{g}^{s}\aleph$ is "freezed" someway, for respective nonholonomic configurations, and do not result in an infinite tower of effective phase masses $\ ^{\shortmid}m$ (103). This reflects a more rich propriety of respective classes of solutions of corresponding systems of nonlinear PDEs (describing the geometric flow evolution and/or corresponding field equations) when generic off-diagonal interactions and evolution scenarios are modeled in a more general way than in the case when the analysis is performed in the framework of diagonalizable solutions and ODEs. Such results (we provided very general classes of quasi-stationary solutions and explicit examples in the previous section) motivate the b) Claim. So, a (nonassociative) modified geometric flow, gravity, and/or classical/quantum field theory may involve large classes of solutions, and respective effective models which belong, or not to the swampland. It depends on the type of nonholonomic constraints, nonlinear symmetries and nonlinear interactions, circumstances of the considered coupling, prescribed generating functions and effective sources, and integration functions/constants. In all cases, for various models with solutions satisfying the conditions of above stated CCL1, we are able to elaborate on modified G. Perelman thermodynamic models (50), for general assumptions on nonassociative geometric flows; or (51), for (92), thermodynamic variables; and (90), or (100), for explicit examples of general nonassociative BH/BE deformations.

At the next step, we analyze how keeping the conditions of CCL1, with notions formulated for the (phase) spaces of s-metrics, it is possible to formulate a respective conjecture and claim involving the so-called Weyl distance [22] but generalized for nonassociative phase spaces. Specifically, we can consider an external conformal re-scaling of s-metrics on $\ _{s}\mathcal{M}$,

$$\ ^{\shortmid}\tilde{\mathbf{g}}_{\alpha_s\beta_s}(\bar{\tau}) = e^{2\bar{\tau}}\ ^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}, \tag{105}$$

where the (re-scaling) parameter $\bar{\tau}$, in general, is different from a geometric flow/temperature like parameter τ. Introducing (105) in (102), we compute ${}_g^s\bar{\aleph} \simeq \bar{\tau}_f - \bar{\tau}_i \simeq \bar{\tau}$. Following a) Conjecture, there is a corresponding tower of phase space states $\tilde{m}$ (103) with a scale of effective masses as $\tilde{m} \sim e^{-\hat{\alpha}\bar{\tau}}$. We obtain light masses in respective towers if ${}_g^s\bar{\aleph} \simeq \bar{\tau} \to \infty$ (we can consider also an opposite large distance limit, ${}_g^s\bar{\aleph} \simeq \bar{\tau} \to -\infty$, when light states are with masses $\tilde{m}' \sim e^{\hat{\alpha}\bar{\tau}}$). For a Weyl re-scaling, we can consider variations of the cosmological constant, ${}^{\shortmid}\bar{\Lambda}_i \leq {}^{\shortmid}\bar{\Lambda}(\bar{\tau}) \leq {}^{\shortmid}\bar{\Lambda}_f$ for corresponding families of (A)dS phase space vacua. We can express the $\bar{\tau}$-transforms of the cosmological constant for an arbitrary dimension $\check{d}$ of ${}_s\mathcal{M}$ (in this work, typical constructions are performed for $\check{d}$ =8),

$$ {}^{\shortmid}\bar{\Lambda} = -\frac{1}{2}(\check{d}-1)(\check{d}-2)M_p^2 e^{-2\bar{\tau}}. \qquad (106) $$

The phase space metric distance between any initial, ${}^{\shortmid}\Lambda_i$, and finial, ${}^{\shortmid}\Lambda_f$, values of such running cosmological constant can be approximated

$$ {}_g^s\bar{\aleph} \simeq \bar{\tau}_f - \bar{\tau}_i \simeq \log({}^{\shortmid}\Lambda_i / {}^{\shortmid}\Lambda_f), \qquad (107) $$

which states that the limit ${}^{\shortmid}\Lambda_f \to 0$ is at infinite distance with respect to the Weyl re-scaling (105) of an (A)dS phase s-metric. This corresponds to the limit $\bar{\tau} \to \infty$.

Combining the behaviour determined by the Weyl re-scaling formulas (105) - (107) and **CCL1**, we extend the AdS Distance conjecture from [22, 23] in such a form:

The AdS distance conjecture and claim for nonassociative phase spaces, CCL2:

a) Conjecture: *For models of (non) associative phase space geometric flow and gravity theories, and QG on a $\check{d}$-dimensional phase space ${}_s\mathcal{M}$ with cosmological constant ${}^{\shortmid}\bar{\Lambda}$ (106), there exists an infinite tower of phase space states with mass scale ${}^{\shortmid}m$ which, as ${}^{\shortmid}\Lambda \to 0$, behave as ${}^{\shortmid}m \sim |{}^{\shortmid}\Lambda|^{\bar{\alpha}}$, where $\bar{\alpha}$ is a positive order-one constant. Such a behavior can be described by phase space s-metrics (92) and their nonassociative quasi-stationary deformations encoding star product and R-flux data from string/M-theory with nonlinear phase space symmetries of type (A.8) connecting effective sources to effective shell cosmological constants ${}_s^{\shortmid}\Lambda(\tau) = {}^{\shortmid}\bar{\Lambda}$.*

b) Claim: *In general, in QG models with nonassociative/ noncommutative/ nonholonomic / generic off-diagonal/ generalized (non) linear connection modifications, there are large classes of exact/parametric solutions (involving, or not, nonlinear symmetries of type (A.8) when the approximation (107) is not valid. Nevertheless, corresponding classes of solutions defining effective τ-running, or fixed τ_0, configurations, are physically well defined and characterized by a respective modified G. Perelman thermodynamics with variables (50).*

We formulate the b) Claims in **CCL2** because for (non) associative geometric flow theories and gravity, and their generic off-diagonal solutions, the conformal symmetry of s-metric is not a general property. In some models, one speculates on possible duality between AdS and dS configurations (see footnote 3 in [23] on necessary exchange of the mass-towers and powers of cosmological constants). Such duality properties and re-scaling behaviour are not important for the procedure of generating exact/parametric solutions using the AFCDM.

Finally, it should be noted that we can also consider a "dual" infinite distance limit when the associated curvatures of phase space/ spacetime and (effective) cosmological constants became very large for $\bar{\tau} \to -\infty$. Such conditions for the string/M-theory and various coupling constants are analyzed in section III of [23]. In general, it is not clear if a tower of light states may appear for such models large curvature/ cosmological constants. Similar conclusions can be drawn for nonassociative phase models determined, for instance, by diagonal s-metrics of type (92) and certain κ-parametric deformations. For more general classes of solutions, limits of type $\bar{\tau} \to -\infty$ may be not obligatory correlated to certain large curvature/cosmological constants values because of generic off-diagonal interactions and imposed nonholonomic constraints on the geometric flow evolution and/or gravitational and matter field dynamics. Such issues have to be investigated for any class of well-defined physical solutions determined by respective generating functions and effective sources, integration functions and prescribed (non) linear symmetries.

6.1.2 Nonassociative Ricci flows and infinite phase space distance conjectures and claims

For associative and commutative Riemannian metrics, the geometric flow equations imply the properties that for positive Ricci curvature the manifold is contracting and for regions of negative curvature the manifold is extending. Such properties exist for space like configurations in pseudo-Riemannian geometry and nonassociative generalizations when, in general, the evolution scenarios depend on the type of solutions for respective systems of nonlinear PDEs. Let us begin with some properties of the associative and commutative R. Hamilton equations for the LC-connection ∇,

$$\frac{\partial \mathbf{g}(\tau)}{\partial \tau} = -2\ \mathcal{R}ic[\nabla](\tau). \tag{108}$$

The Ricci flow ends in certain fixed points $\tau_{\odot}$, when

$$\frac{\partial \mathbf{g}(\tau)}{\partial \tau}\ |_{\tau=\tau_{\odot}} = 0 \text{ and } \mathcal{R}ic(\tau_{\odot}) = 0,$$

i.e. a fixed point defines a flat spacetime with vanishing curvature and zero cosmological constant, $\Lambda = 0$. Because a metric with vanishing cosmological constant is at infinite Weyl distance in the space of all AdS metrics, it was conjectured [22, 23] that in QG on a family of background metrics $\mathbf{g}(\tau)$ under Ricci flows "there exists an infinite tower of states which become massless when following the flow towards a fixed point $\mathbf{g}_{\odot} = \mathbf{g}(\tau_{\odot})$ at infinite distance".

In nonassociative geometric flow theory, the equations (108) can be generalized in the form (2) for a family of nonassociative metrics $^{|}\mathfrak{g}^{\star}(\tau)$ and respective family of nonassociative LC-connections $^{|}\nabla^{\star}(\tau)$ on phase space $^{\star}\mathcal{M}$. Such nonassociative Ricci flow equations involve κ-parametric decompositions of geometric objects. Corresponding decompositions of nonsymmetric metric, canonical s-connection, and Ricci s-tensor structures are given, for instance, in formulas (20), (29) and (30); for $^{|}\nabla$, such formulas were proven in abstract and coordinate forms for LC-configurations in [39] and generalized in nonholonomic s-adapted forms in [12, 13]. The issue on definition of fixed points for nonassociative Ricci flows is more sophisticate and requests a rigorous analysis of ceratin families of exact/parametric solutions of nonlinear systems of PDEs of type (41) and their κ-linear parametric decompositions (46). For τ-families of configurations with nonlinear symmetries (A.8) and respectively prescribed running/ fixed shell cosmological constants $^{|}_{s}\Lambda(\tau)$, we can analyze fixed point properties of κ-parametric of geometric flow equations (62), when

$$\frac{\partial\ ^{|}_{s}\mathbf{g}(\tau)}{\partial \tau}\ |_{\tau=\tau_{\odot}} = 0 \text{ and } \ ^{|}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}(\tau_{\odot},\ {}_{s}\Phi(\tau_{\odot}),\ ^{|}_{s}\Im(\tau_{\odot})) = \delta^{\beta_s}{}_{\gamma_s}\ ^{|}_{s}\Lambda(\tau_{\odot}) = 0. \tag{109}$$

Here we note that fixed points determined by such equations for the canonical s-connection do not define, in general, fixed points for nonassociative geometric flows with $^{|}\nabla^{\star}(\tau)$. Nevertheless, we can extract and study properties of LC-configurations considering additional nonholonomic constraints (25),

$$^{|}_{\star s}\widehat{\mathbf{Z}}(\tau_{\odot}) = 0, \text{ which is equivalent to } \ ^{|}_{s}\widehat{\mathbf{D}}^{\star}_{|\ ^{|}_{s}\widehat{\mathbf{T}}(\tau_{\odot})=0}(\tau_{\odot}) = \ ^{|}_{\star}\nabla(\tau_{\odot}). \tag{110}$$

Above formulas and observations lead to:

The fixed points of nonassociative geometric flows conjecture and claim, CCL3:

a) Conjecture: *Consider a QG model on a family of background s-metric* $^{|}_{s}\mathbf{g}(\tau)$ *satisfying the* κ*-parametric nonassociative geometric flow equations (62). There exists an infinite tower of phase space states with zero effective masses when following the nonassociative geometric flow evolution toward a fixed point* $^{|}_{s}\mathbf{g}_{\odot} = \ ^{|}_{s}\mathbf{g}(\tau_{\odot})$ *at infinite distance.*

b) Claim: *For QG models with nonassociative star product and R-flux modifications, there are large classes of exact/parametric solutions (involving, or not, nonlinear symmetries of type (A.8)) when conditions of type (109) do not hold, for instance, for LC-configurations (110). The conditions of existence at fixed points, of infinite towers of phase spaces and behaviour of corresponding effective masses must be analyzed correspondingly for any class of solutions of κ-parametric of geometric flow equations (62) or certain their equivalents.*

We can compute the metric distance (102) along the nonassociative Ricci flows between a τ and a fixed point $\tau_{\odot}$ as function ${}_{g}^{s}\aleph(\tau,\tau_{\odot})$ determined by a background solution ${}_{s}^{\shortmid}\mathbf{g}(\tau)$ (we can compute respective nonsymmetric and symmetric components of the nonassociative s-metric) in formula (102). For such configurations, possible associated towers of phase space states as in a) Conjecture in **CCL3** scale as

$$ {}^{\shortmid}m(\tau_{\odot}) \sim {}^{\shortmid}m(\tau) e^{-\hat{\alpha}|\ {}_{g}^{s}\aleph(\tau,\tau_{\odot})|}. $$

For many examples, a fixed point will be of the form $\tau_{\odot} = \pm\infty$. The tower of nonholonomic states can only appear when the nonassociative Ricci flow evolves to a fixed point. In general, a model/ off-diagonal solution is not necessary characterized by an infinite tower of effective massless states in phase space. If ${}_{g}^{s}\aleph(\tau,\tau_{\odot}) \to \pm\infty$, a fixed point/ solution can never be reached. This means the transition between (non) associative geometric flow theories/ Ricci solitons/ gravity and/or respective classes of solutions along the geometric flows to the fixed point is discontinuous.

The a) Conjecture in **CCL3** was checked in section 2.1 of [23] (see Conjecture A in that work) for simple cases of associative and commutative Ricci flows, simple cases of Einstein spaces (for instance, for (A) dS with non-zero cosmological constant) and string theory and QG realizations. Similar behavior can be stated by phase space s-metrics (92) and their nonassociative quasi-stationary deformations encoding star product and R-flux data when nonlinear phase space symmetries (A.8) are considered for connecting effective sources to effective shell cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$, in particular, we can approximate ${}_{s}^{\shortmid}\Lambda(\tau) = {}^{\shortmid}\bar{\Lambda}$.

6.1.3 Gibbons-Hawking entropy, scalar curvatures, and generalized distances for nonassociative geometric flows

Let us consider an example of nonassociative Ricci flow equations (62) defined with nonlinear symmetries (A.8) resulting in a conventional phase space cosmological constant ${}_{s}^{\shortmid}\Lambda(\tau) = {}^{\shortmid}\Lambda_0$. Considering classes of solutions involving conventional phase space horizons, we can apply the concept of Gibbons-Hawking, GB, entropy [57] and write ${}^{\shortmid}\mathcal{S}_{GH} = ({}^{\shortmid}\Lambda_0)^{-1}$. For quantum models, ${}^{\shortmid}\mathcal{S}_{GH}$ can be related to the dimension of the Hilbert space of ${}^{\shortmid}\mathcal{H}$ for observer's causal domain [58, 59], when

$$ \dim {}^{\shortmid}\mathcal{H} = e^{1/\ {}^{\shortmid}\Lambda_0} \to \infty \text{ for } {}^{\shortmid}\Lambda_0 \to 0. $$

This property was used in [60] to relate the modified de Sitter conjecture to the GB entropy of tower of massless states (for infinite distance at ${}^{\shortmid}\Lambda_0 \to 0$). This property can be extended to nonassociative phase spaces ${}^{\star}\mathcal{M}$ for respective conditions with ${}_{s}^{\shortmid}\Lambda(\tau) = {}^{\shortmid}\Lambda_0$ and, for instance, for s-metrics of type (92). As explained above, see formula (107), we can write ${}_{g}^{s}\bar{\aleph} \simeq \log|\ {}^{\shortmid}\Lambda_0|$, which for de Sitter configurations with positive cosmological constant results in ${}_{g}^{s}\bar{\aleph} \simeq \log {}^{\shortmid}\mathcal{S}_{GH}$. We conclude that in the large distance limit the GB entropy becomes large and this leads to a large tower of light states.

The concepts of Gibbons-Hawking and/or Bekenstein-Hawking entropies can be applied only for very special classes of solutions (with conventional horizons) in geometric flow and gravity theories. In next subsections we shall consider modified G. Perelman functionals in order characterize more general classes of generic off-diagonal solutions.

Here, we analyze another type of geometric properties of the nonassociative phase space geometric distance. To work directly with formula ${}_{g}^{s}\aleph$ (102) is quite sophisticate and not completely clear how to formulate certain consistent geometric criteria to decide which families of background s-metrics can be used as backgrounds in a consistent QG theory not involving an additional tower of light states, with or not generic off-diagonal terms, in the infinite distance limit. That why we define in this work alternative, more general and more abstract distance measures, which can encode nonassociative star product R-flux contributions.

Let us consider nonassociative Ricci flows for a family of geometric data $[\ {}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau),\ {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau)]$ subjected to the condition that they are determined by a solution of nonassociative R. Hamilton equations (41). Such equations involve complex like variables and canonical nonholonomic structures modeling generic off-diagonal evolution processes and, for fixed Ricci soliton configurations, gravitational interactions subjected to non-integrable constraints. Important geometric and possible physical properties are encoded in the corresponding families of canonical nonassociative Ricci scalars ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}(\tau)$ (31). In partner works [12, 13], it is elaborated a procedure of parametric decompositions of related fundamental geometric s-objects ${}^{\shortmid}\widehat{\mathbf{R}}^{\star\mu_s}{}_{\alpha_s\beta_s\gamma_s}$ (28), ${}^{\shortmid}\widehat{\mathbf{R}}ic^{\star}_{\alpha_s\beta_s}$ (30) and ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}$ (31) with $[01, 10, 11] := \lceil \hbar, \kappa \rceil$ components, using the parametric form of the canonical s-connections (29). This procedure is summarized in by formulas (28) - (31) in Appendix.

The Conjecture B1 from [23] states that the distance $\aleph_R$ in the field space of the background Riemannian metrics along the Ricci flows on a $\check{d}$-dimensional Riemannian manifold is determined by the scalar curvature of the LC-connection, $R(\tau) := \mathcal{R}sc[\nabla](\tau)$, when at $R = 0$, there is an infinite tower of additional massless states in QG. That Conjecture do not have a straightforward extension for nonassociative geometric flows and related gravity theories because of such reasons:

1. Nonassociative phase spaces ${}_{s}^{\star}\mathcal{M}$ can be endowed with different classes of (non) linear connections and respective curvature and Ricci s-tensors, and respective scalars. There are necessary additional assumptions and physical motivations on how, for instance, the canonical geometric objects are related to similar ones for LC-configurations.

2. The generalized distance functional ${}_{g}^{s}\aleph$ (102) and, for instance, the canonical scalar curvature ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc^{\star}(\tau)$ (31), and their restrictions for LC-configurations considering additional nonholonomic constraints (25), involve complex coordinates and geometric variables. It is not clear, in general, how to find a physical interpretation for such non-quantum models even the complex variables are important in QG.

We can solve in general forms the problems stated above in paragraphs 1 and 2 if we work with canonical geometric data when the terms with complex variables are transformed into almost complex and almost symplectic ones. If necessary, such nonholonomic geometric objects can be considered as certain canonical distortions of certain LC-models. Respective κ-linear decompositions to models of parametric (real and effective associative and commutative, but nevertheless nonholonomic) geometric flows, with nonholonomic R. Hamilton equations (46), also allow to elaborate on physical viable theories with corresponding Conjectures and Claims:

The canonical distance - scalar curvature conjecture and claim for nonassociative geometric flows, CCL4:

a) Conjecture: *Consider a model of nonassociative geometric flows determined by κ-parametric R. Hamilton equations (46). The generalized distance ${}_{g}^{s}\aleph$ (102) can be defined in a form ${}_{R}^{s}\widehat{\aleph}$ determined by the canonical scalar curvature ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc[\ {}_{s}^{\shortmid}\mathbf{g}(\tau),\ {}_{s}^{\shortmid}\widehat{\mathbf{D}}(\tau),\ {}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau)]$ for certain canonical data for the equivalent system of PDEs (53). For QG models at ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc = 0$, there exists a canonical infinite tower of phase space states with zero effective masses.*

b) Claim: *For configurations with nonlinear symmetries (60) and transforms of generating functions and generating sources, $[\ {}_{s}\Psi(\tau),\ {}_{s}^{\shortmid}\Im(\tau)] \to [\ {}_{s}\Phi(\tau),\ {}_{s}^{\shortmid}\Lambda(\tau)]$, we can construct exact/parametric solutions of τ-running κ-linear modified Einstein equations (62) satisfying the conditions stated by above conjecture. LC-configurations can be extracted if additional nonholonomic constraints (25) are imposed.*

Generalizing for nonassociative canonical geometric flows on ${}^{\star}\mathcal{M}$ the formula (51) from [23], we consider an alternative definition of (102) when

$$ {}_{R}^{s}\widehat{\aleph} \simeq \log({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau_i)/ {}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau_f)). $$

For some classes of solutions, we can consider flow evolution models with $\tau_f < \tau_i$. If in the vicinity of a fixed point ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau_f) = 0$ but ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau_i) \neq 0$, we can approximate

$$ {}_{R}^{s}\widehat{\aleph}(\tau) \simeq \log({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc(\tau)) \simeq \log(\sum\nolimits_{s} {}_{s}^{\shortmid}\Lambda(\tau)), \quad (111) $$

for configurations with τ-running shell cosmological constants. The nonassociative off-diagonal geometric flux evolution and gravitational interactions define in the infinite canonical distance limit (when ${}_{R}^{s}\widehat{\aleph} \to \infty$ with ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc \to 0$) a canonical tower of effective massless states

$$ {}^{\shortmid}m \sim M_p e^{-\hat{\alpha}| \ {}_{R}^{s}\widehat{\aleph}|} \simeq ({}_{s}^{\shortmid}\widehat{\mathbf{R}}sc)^{\hat{\alpha}} \simeq (\sum\nolimits_{s} {}_{s}^{\shortmid}\Lambda)^{\hat{\alpha}}. \quad (112) $$

The statements of CCL4 can be reformulated for LC-configurations. Here we note that, in general, the condition ${}_{s}^{\shortmid}Rsc[{}^{\shortmid}\nabla] = 0$ is characterized by ${}_{s}^{\shortmid}\widehat{\mathbf{R}}sc \neq 0$ which reflects the nonholonomic, locally anisotropic and off-diagonal characters of effective masses ${}^{\shortmid}m$ (103) and/or (112). We prefer to work with canonical s-adapted geometric variables because this allows us to apply the AFCDM and construct general classes of solutions. The problem to consider ${}^{\shortmid}\nabla$ as a more fundamental than the (auxiliary) canonical s-connection, ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$, or other not/or s-adapted linear connections, has to be analysed using certain additional theoretical arguments and experimental/observational data. For pseudo-Riemannian spacetimes, the associated entropy $\mathcal{S} \simeq \exp({}_{R}^{s}\widehat{\aleph}(\tau))$ become infinite in the flat spacetime limit. This was also pointed out in [61] for the BH entropy considerations using the Bekenstein-Hawking paradigm. Restricting above formulas for base spacetime LC-configurations, we conclude that the flat space limit should be accompanied by an infinite number of massless stated and that, for instance, the Minikowki spacetime is infinitely far away from the curved manifolds along the geometric flow. This means that the flat space never be reached, and that the transition to flat space is discontinuous. Nevertheless, such results have to be revised, for instance, if we consider contributions from nonassociative star products and R-fluxes and work with more general classes of generic off-diagonal solutions.

6.1.4 Generalized distances and nonassociative geometric flow G. Perelman functionals

We have to introduce and study more complicate distance functionals and test respective claims and conjectures for nonassociative geometric flows and related classical gravity and QG models. Let us begin with original considerations for geometric flows on $\check{d}$-dimensional Riemannian manifolds, see Conjecture B2 in [23] for the F-functional $F(\tau) = F(g(\tau), f(\tau))$ (9) and (in our notations) respective distance functional $\aleph_F$. It was conjectured that in the background field space along the combined dilaton-metric flow $\aleph_F$ is determined by F-functional when in QG models related to such Ricci flows there is an infinite tower of additional massless states at $F = 0$. Here we note that $f(\tau)$ is not obligatory a dilaton type field and originally it was considered as a normalizing function [1], see also the footnote 10.

In the case of nonassociative geometric flow, we can elaborate on physical viable theories with:

The canonical distance - F-functional conjecture and claim for nonassociative geometric flows, CCL5:

a) Conjecture: *Consider a model of nonassociative geometric flows determined by a κ-parametric functional ${}_{s}^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau)$ (44). The generalized distance ${}_{g}^{s}\aleph$ (102) can be defined and computed in a form ${}_{F}^{s}\widehat{\aleph}$ determined by the canonical scalar curvature for certain canonical data $[{}_{s}^{\shortmid}\mathbf{g}(\tau), {}_{s}^{\shortmid}\widehat{\mathbf{D}}(\tau), {}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau)]$ as solutions of nonassociative geometric flow equations (53). For QG models at ${}_{s}^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star} = 0$, there exists a canonical infinite tower of phase space states with zero effective masses.*

b) Claim: *For nonassociative geometric flow models and gravitational configurations with nonlinear symmetries (60), we can construct exact/ parametric solutions of τ-parametric running κ-linear modified Einstein equations (62) satisfying, or not, the conditions stated by above conjecture. We can extract LC-configurations for additional nonholonomic constraints (25).*

We can define and compute a generalized distance functional if

$$ {}_F^s\widehat{\aleph} \simeq [\ {}_s^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau_i)/\ {}_s^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau_f)]. $$

In the case when ${}_s^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau_f)$ is a fixed point of the nonassociative flow equations (53), the canonical tower of effective massless stated scale for ${}_F^s\widehat{\aleph} \to \infty$ as

$$ {}^{\shortmid}m \sim M_p e^{-\hat{\alpha}|\ {}_F^s\widehat{\aleph}|} \simeq \left(\ {}_s^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau)\right)^{\hat{\alpha}} \simeq \left(\ {}_s^{\shortmid}\widehat{\mathbf{R}}sc(\tau) + \ {}_s^{\shortmid}\widehat{\mathbf{K}}sc(\tau)\right)^{\hat{\alpha}}. $$

A normalizing function $\widehat{f}(\tau)$ can be prescribed in a form when the distortion ${}_s^{\shortmid}\widehat{\mathbf{K}}sc(\tau)$ is absorbed in such a form that the nonassociative geometric flow is determined by ${}_F^s\widehat{\aleph} \simeq {}_R^s\widehat{\aleph} \simeq \log(\ {}_s^{\shortmid}\widehat{\mathbf{R}}sc)$ as in (111). For other type models, for instance, with f treated as a dilaton field, and considering only a base spacetime flow of dilaton, we can approximate ${}_F^s\widehat{\aleph} \simeq \log g_s$, where g_s is the string coupling (see formula (65) and footnote 8 in [23]). Nonassociative geometric flow configurations with nonlinear symmetries (60) can be characterized by ${}_F^s\widehat{\aleph} \simeq -\log|\sum_s\ {}_s^{\shortmid}\Lambda(\tau)|$, which is compatible with the conditions of previous **CCL4.**

G. Perelman [1] defined also another important functional, the W-functional, which is a "minus entropy" and can be used for treating the case of collapsing cycles of the Ricci flow theory of Riemannian metrics. In connection to the generalized distance functional, the Conjecture B3 from [23] states that for geometric flows of Riemannian metrics we can introduce a generalized distance ${}_W^s\widehat{\aleph}$ in the background field space along the Ricci flow evolution which is determined by the W-entropy. For QG models in the points where this variable vanishes, there is an infinite tower of massless states. Such constructions involve more rich nonholonomic geometric structures for nonassociative geometric flows, and results in physical models with local anisotropy.

For κ-parametric decompositions, we can introduce a re-scaling factor $\chi(\tau)$ in the definition of ${}_s^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$ (45), when

$$ \begin{aligned} & {}_s^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}[\ {}_s^{\shortmid}\mathbf{g}(\tau),\ {}_s^{\shortmid}\widehat{\mathbf{D}}(\tau), \chi(\tau);\ {}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau)] = \\ & \int_{{}_s^{\shortmid}\widehat{\Xi}} \delta\ {}^{\shortmid}u^{\gamma_s}(\chi)\sqrt{|\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\chi)|}\left[\chi\left(\ {}_s^{\shortmid}\widehat{\mathbf{R}}sc(\chi) + (\ {}_s^{\shortmid}\widehat{\mathbf{D}}(\chi)\ {}_s^{\shortmid}\widehat{f}(\chi))^2\right) + \ {}_s^{\shortmid}\widehat{f}(\chi) - 8\right]\frac{e^{-\ {}_s^{\shortmid}\widehat{f}(\chi)}}{(4\pi\chi)^4}, \end{aligned} \quad (113) $$

under constraint $\int_{{}_s^{\shortmid}\widehat{\Xi}} \delta\ {}^{\shortmid}u^{\gamma_s}(\chi)\sqrt{|\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\chi)|}e^{-\ {}_s^{\shortmid}\widehat{f}(\chi)}/(4\pi\chi)^4 = 1$. The normalizing function ${}_s^{\shortmid}\widehat{f}(\chi)$ and re-scaling factor $\chi(\tau)$ can be chosen in such a form that the resulting variational geometric flow equations involve only the canonical Ricci s-tensor and respective scalar curvature. As a result, the nonassociative geometric flow equations (46) can be written equivalently as

$$ \begin{aligned} \partial_\tau\ {}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau) &= -2\ {}^{\shortmid}\widehat{\mathbf{R}}_{\ \alpha_s\beta_s}(\tau), \\ \partial_\tau\ {}_s^{\shortmid}\widehat{f}(\tau) &= {}_s^{\shortmid}\widehat{\mathbf{R}}sc(\tau) - \widehat{\triangle}(\tau)\ {}_s^{\shortmid}\widehat{f}(\tau) + (\ {}_s^{\shortmid}\widehat{\mathbf{D}}(\tau)\ {}_s^{\shortmid}\widehat{f}(\tau))^2(\tau) + 4/\chi(\tau), \\ \partial_\tau\ \chi(\tau) &= -1. \end{aligned} \quad (114) $$

Here we note that the nonassociative star product and R-flux contributions of order $\lceil \hbar, \kappa \rceil$ are encoded into generating functions, generating sources and nonlinear symmetries (60) of s-metrics ${}^{\shortmid}\mathbf{g}_{\alpha_s\beta_s}(\tau)$. Such τ- and χ-families of s-metrics can be found as exact/parametric solutions (62) with prescribed effective running cosmological constants ${}_s^{\shortmid}\Lambda(\tau)$. Finding a solution of the nonlinear system of PDEs (114), we can introduce it in the formula (113) and compute the W-entropy of associated thermodynamic system. This functional is important for a further analysis if such a nonassociative model belongs, or not, to swampland.

Summarizing above considerations, for nonassociative geometric flows, we formulate:

The canonical distance - W-functional conjecture and claim for nonassociative geometric flows, CCL6:

a) Conjecture: *For models of nonassociative geometric flows determined by* κ *-parametric functional* ${}_{s}^{|}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$ *(113), the generalized distance* ${}_{g}^{s}\aleph$ *(102) can be defined in a form* ${}_{W}^{s}\widehat{\aleph}$ *determined by such a canonical W-entropy functional defined and computed for corresponding canonical data and solutions of nonassociative geometric flow equation (46). For QG models at* ${}_{s}^{|}\widehat{\mathcal{W}}_{\kappa}^{\star} = 0$*, there exists a canonical infinite tower of phase space states with zero effective masses.*

b) Claim: *If certain nonassociative models of geometric flows / Ricci solitons are constructed to posses nonlinear symmetries (60) to running cosmological constants* ${}_{s}^{|}\Lambda(\tau)$*, we can generate exact/ parametric solutions of* τ*-parametric running* κ*-linear modified Einstein equations (62) satisfying, or not, the conditions stated by above conjecture. LC-configurations can be extracted for additional nonholonomic constraints (25). The formulas for* ${}_{W}^{s}\widehat{\aleph}$ *can be expressed as functionals on* ${}_{s}^{|}\Lambda(\tau)$ *and respective volume forms and the conclusion that such a nonassociative geometric and thermodynamic model belongs, or not, to swampland depend on the type of prescribed generating functions and running effective cosmological constants.*

In analogy to CCL4 and CCL5, we define and compute for CCL6 such a generalized distance functional:

$$ {}_{W}^{s}\widehat{\aleph} \simeq \log({}_{s}^{|}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau_i)/ {}_{s}^{|}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau_f)). \qquad (115)$$

If ${}_{s}^{|}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau_f)$ is a fixed point of the nonassociative flow equations (114), the canonical tower of effective massless stated scale for ${}_{W}^{s}\widehat{\aleph} \to \infty$ as

$$ {}^{|}m \sim M_p e^{-\hat{\alpha}| \ {}_{W}^{s}\widehat{\aleph}|} \simeq \left({}_{s}^{|}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)\right)^{\hat{\alpha}}. \qquad (116)$$

In [23] (see formulas (88)-(95) in that work), there are provided explicit computations for functionals (115) and (116) in some cases of associative and commutative geometric flows, for instance, for high dimensional Einstein spaces and 2-d spacetime with $SO(2)$ symmetries. Another example was studied for modified gravity with F(R) Ricci flows (section 4, in that work). Here we note that more general examples and various applications for R^2-gravity with locally anisotropic cosmological and BH solutions were studied in [62]. We shall provide explicit proofs of W-entropy and related formulas for generalized distance and effective massless stated scales for explicit examples of nonassociative BH solutions in next subsections.

6.1.5 Generalized nonassociative phase space swampland conjectures for effective (A) dS phase space configurations and the Bekenstein–Hawking entropy

In [63, 26], there were studied possible physical implications of the Black Hole Entropy Distance Conjecture, BHEDC. It was postulated in the framework of a research elucidating a close connection between the infinite distance conjecture in the context of swampland and area law for the Bekenstein-Hawking entropy, S_{BH}. That conjecture sates that for BH spacetime configurations in QG, the formulas ${}_{g}^{s}\aleph$ (102) and m (104) can be written, respectively, in the form

$$ {}_{BH}^{s}\aleph \sim \log S_{BH} \text{ and } m \sim S_{BH}^{-\hat{\alpha}}, \text{ where } \hat{\alpha} \sim \mathcal{O}(1). \qquad (117)$$

For such considerations, the limit $S_{BH} \to \infty$ is identified with the Minkowski spacetime. The BHEDC was tested and generalized for concrete string setups with AdS configurations; for metrics belonging the family of RNdS solutions; multi-horizon spacetimes; for BH evaporation etc.

Nonassociative star product and R-flux deformations result in different types of nonlinear systems of PDEs characterized by more general classes of generic off-diagonal solutions, nonholonomic structures and (non) linear connections. Such solutions are generated for corresponding nonlinear symmetries and for coefficients of the fundamental geometric objects depending on all phase space coordinates and flow evolution parameter. Nevertheless, we can construct also some very special and physically important classes of solutions with conventional horizons when the nonassociative contributions are encoded into respective classes of generating functions (for instance, rotoid deformations). Our strategy for generalizing the BHEDC on nonassociative phase spaces ${}^{\star}\mathcal{M}$ is stated as follow:

The phase space black hole entropy distance conjecture and claim for nonassociative geometric flows, CCL7:

a) Conjecture: *Exact/ parametric solutions with conventional phase space hyper-surface horizons, hh, defining models of nonassociative geometric flows and/or gravity theories with κ-parametric modified R. Hamilton (46) / Ricci soliton (43) equations, and describing τ-families of star product R-flux deformed BHs. The QG models for such quasi-stationary solutions are characterized by a Bekenstein-Hawking entropy variable, ${}_s^{\shortmid}S_{hh}$ defined on corresponding phase spaces when the generalized distance ${}_{hh}^{s}\aleph \sim \log({}_s^{\shortmid}S_{hh})$ and effective mass scale for a respective tower of states is ${}_s^{\shortmid}m \sim {}_s^{\shortmid}S_{hh}^{-\hat{\alpha}}$, for $\hat{\alpha} \sim O(1)$.*

b) Claim: *For nonassociative geometric flow models and gravitational configurations with nonlinear symmetries (60) for running cosmological constants ${}_s^{\shortmid}\Lambda(\tau)$, we can construct exact/parametric solutions of τ-parametric running κ-linear modified Einstein equations (62) satisfying, or not, the conditions stated by the phase space BHEDC. We can extract BE and BH solutions for LC-configurations for additional nonholonomic constraints (25).*

Additionally, in this subsection, we provide a list of four parametric solutions for τ-families of nonassociative BE and BH solutions which are characterized by the conditions CCL7 and can be investigated as models of nonassociative geometric flows for which the paradigm of Bekenstein–Hawking thermodynamics holds true.

	Type of solution	Nonlinear quadratic element	Generalized Bekenstein-Hawking entropy, ${}_s^{\shortmid}S_{hh}$
1.	Nonassociative τ -deformed double RNdS BHs, dissipation to BEs & Schwarzschild BHs,	$d\,{}_{\shortmid}^{\flat}s^2_{[8d]}(\tau)$, see (84);	${}_s^{\shortmid}S_{hh} = S_0(\tau,r,\theta,\varphi) + {}_{\shortmid}S_0(\tau,p,p_\theta,p_\varphi)$, $S_0 = \frac{\Omega_2 \times (r_s)^4}{4}[1 + \frac{4\kappa}{3}\underline{\chi}(\tau,r,\theta)\sin(\omega_0\varphi + \varphi_0)]$, ${}_{\shortmid}S_0 = \frac{\Omega_2 \times (p_s)^4}{4}[1 + \frac{4\kappa}{3}\overline{\chi}(\tau,p,p_\theta)\sin({}_{\shortmid}^{p}\omega_0 p_\varphi + p^0_\varphi)$ see (85).
2.	Nonassociative τ-running couples of Schwarzschild-AdS BHs and BEs deformations,	$d\,{}_{\shortmid}^{\epsilon}s^2_{[8d]}(\tau)$, see (87);	${}_s^{\shortmid}S_{hh} = {}^{\epsilon\chi}S(\tau,r,\theta,\varphi) + {}_{\shortmid}^{\epsilon\chi}S(\tau,p,p_\theta,p_\varphi)$, ${}^{\epsilon\chi}S = {}^{\epsilon}S_0(\tau)[1 + \frac{4\kappa}{3}\underline{\chi}(\tau,r,\theta)\sin(\omega_0\varphi + \varphi_0)]$, ${}_{\shortmid}^{\epsilon\chi}S = {}_{\shortmid}^{\epsilon}S_0(\tau)[1 + 4\kappa\overline{\chi}(\tau,p,p_\theta)\sin({}_{\shortmid}^{p}\omega_0\ p_\varphi + p^0_\varphi)]$ see (89) and (88).
3.	$d = 5$ RN AdS metric embedded into a 8-d phase space ${}_sM$,	$d\,\breve{s}^2_{[5+3]}$, see (92);	${}^0\breve{S} = \frac{{}^0\breve{A}}{4G_{[5]}} = \frac{\omega_{[3]}\breve{r}_h}{4G_{[5]}}$, see (97).
4.	τ-running phase space RN-AdS BEs	$d\,{}_{\shortmid}^{\chi}s^2_{[6\subset 8d]}(\tau)$, see (95);	$\breve{S}(\tau) = {}^0\breve{S}(1+\frac{\kappa}{2}\hat{\chi}_4(\tau))$, see (98).

There are possible different scenarios of nonassociative geometric flow evolution defined by above outlined solutions. They depend on the type of generating functions and prescribed τ-running/fixed cosmological constants, and respective nonlinear symmetries (60). In the case 1, the off-diagonal interactions and flow evolution may transform a phase space double RNdS BH configuration into a phase system of BEs and Schwarzschild

BHs, when the thermodynamic variables are computed as rotoid type locally anisotropic deformations of respective variables in Bekenstein-Hawking thermodynamics. For certain classes of nonholonomic constraints, the corresponding ellipsoidal deformations can be stable for a fixed value τ_0 (i.e. there are defined stable nonassociative phase space Ricci solitons) and describe τ-evolution preserving the prime configuration. Such models are described by solutions of type 2. Similar scenarios can be described by solutions of type 3 and 4 but with different thermodynamic models and physical interpretations which is typical for the phase space higher dimension BEs and BHs.

6.2 Nonassociative geometric flows, swampland conjectures, and G. Perelman thermodynamics

A class of solutions for nonassociative geometric flows and/or nonassociative gravitational equations can be always characterized by an associated G. Perelman thermodynamic model, when, for instance, the conditions of CCL6 are satisfied but those stated in CCL7 are not applicable. We have to investigate explicitly (for a class of τ-quasi-stationary solutions) if and for which the conditions and statements of CCL1-CCL6 are true, not true, or undetermined. In this subsection, we study a certain important examples how the swampland conjectures have to be extended to nonassociative Ricci flows. In explicit form, we show how such constructions are related to CCL6 when the W-entropy is used for defining and computing the generalized canonical distance functional and respective tower of effective mass stated scaling, see respective formulas (115) and (116). Similar analysis on applicability and testing of CCL1-CCL5 are not presented in this work because, the formulas with canonical scalar curvature, F-functionals etc. are more cumbersome and not related directly to computation of variables of G. Perelman thermodynamics.

6.2.1 The canonical distance - W-functional conjecture and claim, CCL6, for τ-running nonassociative generating sources and quasi-stationary solutions

For any class of solutions of geometric flow equations (53) with τ-running effective sources ${}_{s}^{\shortmid}\Im(\tau)$ (52), represented in a form (55), or (65), or (68), we can compute the star product deformed W-functional ${}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}(\tau)$ (39). Let us consider a s-metric ${}_{s}^{\shortmid}\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau),\ {}_{s}\Psi(\tau),\ {}_{s}^{\shortmid}\Im(\tau)]$ for quasi-stationary solutions of type (A.10). We can compute the W-entropy and S-entropy functions as in formulas (69) but for corresponding generating functions and effective sources, and volume form ${}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau)$ (49),

$$
\begin{aligned}
{}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau) &= \int_{\tau'}^{\tau}\frac{d\tau}{(4\pi\tau)^4}\int_{{}_{s}^{\shortmid}\widehat{\Xi}}\left(\tau[\sum\nolimits_{s}\ {}_{s}^{\shortmid}\Im(\tau)]^2-8\right)\ {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau), && (118)\\
{}_{s}^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau) &= -\int_{\tau'}^{\tau}\frac{d\tau}{(4\pi\tau)^4}\int_{{}_{s}^{\shortmid}\widehat{\Xi}}\left(\tau[\sum\nolimits_{s}\ {}_{s}^{\shortmid}\Im(\tau)]-8\right)\ {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau). && (119)
\end{aligned}
$$

So, nonassociative geometric flows can be characterized by two entropy type variables: the "standard" statistical thermodynamics entropy (similar to constructions in hydrodynamic models of moving media but in terms of curvature scalars, metrics, connections) , ${}_{s}^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau)$, with $[\sum_{s}\ {}_{s}^{\shortmid}\Im(\tau)]$; and the "minus" entropy, i.e. W-entropy, ${}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$, with $[\sum_{s}\ {}_{s}^{\shortmid}\Im(\tau)]^2$.

Using (119), we can modify the formulas (117) for BHEDC and the conditions of CCL7,

$$
{}_{\mathcal{S}_{\kappa}^{\star}}^{s}\widehat{\aleph}\ \sim\log(\ {}_{s}^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau))\text{ and }\ {}_{s}^{\shortmid}m\sim(\ {}_{s}^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau))^{-\hat{\alpha}}\ \text{ for }\ \hat{\alpha}\sim O(1).
$$

Alternatively, we can follow the CCL6 and compute (115) and (116) for (118). Respectively, we obtain a generalized distance functional:

$$
{}_{W}^{s}\widehat{\aleph}\simeq\int_{\tau_f}^{\tau_i}\frac{d\tau}{(4\pi\tau)^4}\int_{{}_{s}^{\shortmid}\widehat{\Xi}}\left(\tau[\sum\nolimits_{s}\ {}_{s}^{\shortmid}\Im(\tau)]^2-8\right)\ {}^{\shortmid}\delta\ {}^{\shortmid}\mathcal{V}(\tau), \qquad (120)
$$

when the canonical tower of effective massless stated scale for ${}_{R}^{s}\widehat{\aleph} \rightarrow \infty$ as

$$ {}^{\shortmid}m \sim M_p e^{-\hat{\alpha}| \ {}_{W}^{s}\widehat{\aleph}|} \simeq \left(\int_{\tau_i}^{\tau_f} \frac{d\tau}{(4\pi\tau)^4} \int_{{}_{s}^{\shortmid}\widehat{\Xi}} \left(\tau[\sum\nolimits_s \ {}_{s}^{\shortmid}\Im(\tau)]^2 - 8\right) \ {}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}(\tau) \right)^{\hat{\alpha}}. \quad (121) $$

In general, it is not clear if such nonassociative geometric flow models belong, or not, to swampland. It depends of the data for generating functions encoded in ${}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}(\tau)$ and effective sources ${}_{s}^{\shortmid}\Im(\tau)$. We have to analyze such issues for explicit classes of solutions of (53).

6.2.2 CCL6 for τ-running effective shell cosmological constants and quasi-stationary solutions

Using nonlinear symmetries (60), with ${}_{s}^{\shortmid}\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau), \ {}_{s}\Psi(\tau), \ {}_{s}^{\shortmid}\Im(\tau)] \rightarrow \ {}_{s}^{\shortmid}\mathbf{g}[\hbar,\kappa,\tau,\psi(\tau), \ {}_{s}\Phi(\tau), \ {}_{s}^{\shortmid}\Lambda(\tau)]$, we can study the conditions of CCL5 involving quasi-stationary solutions of type (A.10). This allows us to simplify the formulas for explicit computation of W-entropy ${}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$ in terms of running/fixed effective cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$, see (69) with ${}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}(\tau)$ (70). Introducing such values, respectively, in (115) and (116), we obtain:

$$ {}_{W}^{s}\widehat{\aleph} \simeq \int_{\tau_f}^{\tau_i} \frac{d\tau}{(4\pi\tau)^4} \int_{{}_{s}^{\shortmid}\widehat{\Xi}} \left(\tau[\sum\nolimits_s \ {}_{s}^{\shortmid}\Lambda(\tau)]^2 - 8\right) \ {}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}(\tau), \quad (122) $$

when the canonical tower of effective massless stated scale for ${}_{R}^{s}\widehat{\aleph} \rightarrow \infty$ is computed as

$$ {}^{\shortmid}m \sim M_p e^{-\hat{\alpha}| \ {}_{W}^{s}\widehat{\aleph}|} \simeq \left(\int_{\tau_i}^{\tau_f} \frac{d\tau}{(4\pi\tau)^4} \int_{{}_{s}^{\shortmid}\widehat{\Xi}} \left(\tau[\sum\nolimits_s \ {}_{s}^{\shortmid}\Lambda(\tau)]^2 - 8\right) \ {}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}(\tau) \right)^{\hat{\alpha}}. \quad (123) $$

Prescribing data for ${}_{s}^{\shortmid}\Lambda(\tau)$ and ${}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}(\tau)$, we are able to generate nonassociative geometric flow scenarios which belong, or not, two swampland. Such an analysis is more simple than that for (120) and (121) because respective formulas (122) and (123) contain effective cosmological constants which can be prescribed in certain forms which allow generating viable and important physical models.

Above formulas can be computed in explicit κ-parametric form when ${}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau) = \ {}_{s}^{\shortmid}\widehat{\mathcal{W}}_{0} + \kappa \ {}_{s}^{\shortmid}\widehat{\mathcal{W}}_{1}^{\star}(\tau)$, for ${}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V} = \ {}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}_0 + \kappa \ {}^{\shortmid}\delta \ {}^{\shortmid}\mathcal{V}_1$ (75). We omit in this work such cumbersome computations and incremental formulas with κ–linear decomposition as in (76), which can be considered for solutions of type (84), with χ-polarization functions.

6.2.3 CCL6 for nonassociative flows of phase space deformed (double) RN-dS BHs

The W-entropy from (90) was computed for target s-metrics (80) with η-polarization functions defining general quasi-stationary star-product R-flux deformations of double RN-dS BHs in nonassociative phase spaces. Such configurations can be characterized by effective cosmological constants ${}_{1}^{\shortmid}\Lambda_0 = \ {}_{2}^{\shortmid}\Lambda_0 = \check{\Lambda} \geq 0$ and ${}_{3}^{\shortmid}\Lambda_0 = \ {}_{4}^{\shortmid}\Lambda_0 = \ {}^{\shortmid}\check{\Lambda} \geq 0$. This class of solutions can be extended for running cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau) = [\check{\Lambda}(\tau), \ {}^{\shortmid}\check{\Lambda}(\tau)] \geq 0$; when respective effective sources ${}_{s}^{\shortmid}\Im(\tau)$ are related via nonlinear symmetries (60) to such ${}_{s}^{\shortmid}\Lambda(\tau)$. In general, such nonassociative geometric flow deformations do not describe BH configurations and their general physical interpretation is not clear. For κ-parametric decompositions with χ-generating functions (84), we can model stable BE and BH configurations embedded self-consistently into nonassociative phase space backgrounds.

Using (90) and introducing ${}_{s}^{\shortmid}\Lambda(\tau) = [\check{\Lambda}(\tau), \ {}^{\shortmid}\check{\Lambda}(\tau)]$ in formulas (122) and (123), we compute

$$ {}_{W}^{s}\widehat{\aleph} \simeq \int_{\tau_f}^{\tau_i} \frac{d\tau}{64(\pi\tau)^4} \frac{\tau[\check{\Lambda}(\tau) + \ {}^{\shortmid}\check{\Lambda}(\tau)]^2 - 2}{|\check{\Lambda}(\tau) \ {}^{\shortmid}\check{\Lambda}(\tau)|} \ {}_{\eta}^{\shortmid\flat}\mathcal{V}(\tau), $$

when the canonical tower of effective massless stated scale for ${}_{R}^{s}\widehat{\aleph} \rightarrow \infty$ as

$$ {}^{\shortmid}m \sim M_p e^{-\hat{\alpha}| \ {}_{W}^{s}\widehat{\aleph}|} \simeq \left(\int_{\tau_i}^{\tau_f} \frac{d\tau}{64(\pi\tau)^4} \frac{\tau[\check{\Lambda}(\tau) + \ {}^{\shortmid}\check{\Lambda}(\tau)]^2 - 2}{|\check{\Lambda}(\tau) \ {}^{\shortmid}\check{\Lambda}(\tau)|} \ {}_{\eta}^{\shortmid\flat}\mathcal{V}(\tau) \right)^{\hat{\alpha}}. $$

Prescribing data for $\check{\Lambda}(\tau)$ and $^{|}\check{\Lambda}(\tau)$ and $_{\eta}^{|\flat}\mathcal{V}(\tau)$, we are able to generate nonassociative geometric flow scenarios for deforming double BH solutions in phase space, which belong, or not, to swampland. This depends on behaviour of $_{W}^{s}\widehat{\aleph}$ (if there are limits to inf, or zero) for some types of chosen functions under the integral on τ.

6.2.4 CCL6 for nonassociative flows of phase space nonholonomic deformed RN-AdS BHs

The conditions of CC6 can be tested for another class of exact/parametric solutions (93) and (95) with W-entropy (100) with volume form (101). In such a target metric, there is an effective cosmological constant $_{s}\Lambda(\tau) = \Lambda_{[5]} < 0$, for $s = 1, 2, 3, 4$; when respective effective sources $_{s}^{|}\Im(\tau)$ are related via nonlinear symmetries (60) to $\Lambda_{[5]}$. Using (100), (122) and (123), we write:

$$_{W}^{s}\widehat{\aleph} \simeq \int_{\tau_f}^{\tau_i} \frac{d\tau}{32(\pi\tau)^4} \frac{2\tau\Lambda_{[5]}^2 - 1}{\Lambda_{[5]}^2}\, _{\eta}^{|}\check{\mathcal{V}}(\tau),$$

when the canonical tower of effective massless is characterized by such a scale behaviour for $_{R}^{s}\widehat{\aleph} \to \infty$:

$$^{|}m \sim M_p e^{-\hat{\alpha}|\ _{W}^{s}\widehat{\aleph}|} \simeq \left(\int_{\tau_i}^{\tau_f} \frac{d\tau}{32(\pi\tau)^4} \frac{2\tau\Lambda_{[5]}^2 - 1}{\Lambda_{[5]}^2}\, _{\eta}^{|}\check{\mathcal{V}}(\tau) \right)^{\hat{\alpha}}.$$

Such formulas can be generalized for a τ-running cosmological constant $\Lambda_{[5]}(\tau) < 0$ with the same value for all shells. Effective volume flows $_{\eta}^{|}\check{\mathcal{V}}(\tau)$ can be adapted to nonholonomic s-distributions which allow to conclude if such nonassociative geometric flow deformed phase space RN-AdS BHs belong, or not, to swampland.

7 Discussion and conclusions

This is the fifth partner work in a series of articles devoted to the theory of nonassociative geometric and information flows, modified gravity, and applications in astrophysics and cosmology [12, 13, 40, 41]. Such a research program is motivated by important results on nonassociative geometry and physics involving theories with star product R-flux deformations in string and M-theory [34, 35, 36, 37, 38, 39]. In this paper, we postulate the main functionals and derive the geometric evolution equations for such a theory of nonassociative Ricci flows and gravity; state the necessary conditions for a general decoupling and integrability of necessary systems of nonlinear PDEs; and construct new classes of exact/parametric solutions describing nonassociative flux evolution on quasi-stationary and BH configurations. There are computed and analyzed physical properties of nonassociative phase space thermodynamic variables developing the paradigms defined (for subclasses of solutions with conventional hyper-surface horizons) by the Bekenstein-Hawking entropy, and/or following the G. Perelman W-entropy and geometric thermodynamic approach which is applicable to general geometric evolution models and generic off-diagonal solutions. Finally, we study key issues (formulated as conjectures and claims) related to nonassociative extensions and revision of the swampland program [19, 20, 21, 18, 60, 22, 23, 24, 25, 26, 27].

7.1 Summary and discussion of main results

Let us discuss how the results of this paper fit in with the Objectives (Aims 1-5) stated in section 1.2. We analyze the most important ideas and innovative methods, consider possible interpretations, and present the key points of novelty and originality:

1. Using nonassociative star product, $\star$, deformations, we define in section 2 (**first objective**) generalized G. Perelman F- and W-functionals, $_{s}^{|}\widehat{\mathcal{F}}^{\star}(\tau)$ and $_{s}^{|}\widehat{\mathcal{W}}^{\star}(\tau)$, see formulas (38) and (39); and effective

functionals ${}_{s}^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau)$ (44) and ${}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$ (45), for computing κ-parametric decompositions (on string constant κ, when τ is a temperature like parameter). Such functionals are used for deriving nonassociative versions of R. Hamilton equations describing nonassociative and noncommutative geometric flow evolution models. For self-similar configurations, they define nonassociative Ricci soliton geometries and, in particular, nonassociative vacuum gravity theories with running cosmological constants. We note that nonassociative geometric flow evolution theories and applications have not been studied in modern mathematics and physics. There were elaborated only approaches related to noncommutative geometry (a lá A. Connes) [15], geometric and quantum information flow theories and modified gravity [62, 6, 7, 8]; and, in associative/commutative form, it is performed a recent research on swampland conjectures and Ricci flows [23, 24, 26, 27]. Our nonassociative geometric models and solutions are self-consistent and well-defined in a sense that they are nonholonomic deformations with 2+2+2+... and/or (3+1)+(3+1)+.... decompositions of dimensions, encoding nonassociative geometric data, and generalizing the fundamental geometric objects and evolution equations in standard Ricci flow theory [4, 5, 1, 2, 3, 9, 10, 11]. Such geometric constructions can be performed in relativistic forms for Lorentz manifolds and their (co) tangent bundles, when the nonassociativeand noncommutative structures are determined by star product and R-flux deformations in string and M-theory.

In [38, 39], two similar models of nonassociative gravity theories were elaborated up to defining and computing the nonassociative Ricci tensor for the nonassociative star deformed Levi-Civita, LC, connection, involving (non) symmetric metric structures. The originality of nonassociative geometric methods elaborated for the nonholonomic shell oriented structures in [12, 13] consists in developing a formalism which allows us to apply the anholonomic frame and connection method, AFCDM [16, 17, 62] for constructing exact and parametric solutions in modified gravity theories, MGTs; see a brief review in Appendix A. Using the AFCDM we can prove a general decoupling and integration property of of nonassociative R. Hamilton equations (41), see section 4 and [40, 41], for related details for nonasscociative vacuum Einstein equations.

Modern mathematics states explicit limitations, in the sense of topology and geometric analysis methods, for elaborating a general nonassociative and/or noncommutative Ricci flow theory. This is because an infinite number of models of nonassociative/ noncommutative geometries can be elaborated (some of them present interest in modern mathematical physics and information theory) and it was not formulated at least an example of nonassociative/ noncommutative / relativistic generalization of the Poincaré-Thurston conjecture. Nevertheless, nonassociative geometric flow models with star product R-flux deformations are well defined (at least in κ-parametric form, in the framework of string/ M-theory, and corresponding phase space generalizations of the Einstein gravity). Applying the AFCDM, we can construct physically important solutions of respective nonlinear systems of PDEs.

2. Theories of nonassociative geometric flows and modified gravity are characterized by respective statistical thermodynamic models defined by W-entropy ${}_{s}^{\shortmid}\widehat{\mathcal{W}}^{\star}(\tau)$ and related statistical generating functional ${}_{s}^{\shortmid}\widehat{\mathcal{Z}}^{\star}(\tau)$. The novelty of the **second objective** (for section 3) consists in formulating a nonassociative geometric thermodynamic theory determined by variables (50) encoding nonassociative geometric data in the κ-linear thermodynamic variables $[\ {}_{s}^{\shortmid}\widehat{\mathcal{E}}_{\kappa}^{\star}(\tau),\ {}_{s}^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau),\ {}_{s}^{\shortmid}\widehat{\sigma}_{\kappa}^{\star}]$ (51). Such statistical thermodynamic variables are defined using fundamental (non) associative/ commutative geometric objects (the geometric measure, (non) linear derivatives, respective nonassociative Ricci tensors and scalar curvatures). They can be computed, at least in κ-parametric forms, for all classes of solutions of corresponding geometric flow/ gravitational field equations. Former results were for associative and commutative theories [23, 24, 26, 27, 63] and involving special classes of solutions with conventional horizons when the concept of Bekenstein-Hawking entropy [47, 48, 49, 50] is well-defined. Following a nonassociative generalized G. Perelman functional formalism [1, 9, 10, 11, 7], we can elaborate on a new statistical/ geometric thermodynamic paradigm for classical and quantum gravity and information theories, and study non-

linear models of quasi-stationary and locally anisotropic cosmological evolution in MGTs, see details in [62, 6, 7, 8, 17, 62].

3. In our approach, the nonassociative geometric and gravity theories with star product R-flux deformations are defined on nonholonomic phase spaces modelled as cotangent Lorentz bundles, ${}_{s}^{\star}\mathcal{M} = T_{s}^{*}V$, with nonholonomic shell, s, dyadic splitting of dimensions, 2+2+2+2. Following the **third objective** of this work, in section 4, we show how the AFCDM developed for nonassociative gravity theories in [13, 40, 41] can be applied for generating τ-evolving classes of quasi-stationary solutions of nonassociative geometric flow equations written in κ-parametric form (46). An original and important result consists from the proof that, for well-defined nonholonomic geometric and relativistic physical assumptions, such equations can be written in two equivalent forms:

   - 1] as τ-running canonically s-deformed Einstein equations (53), with effective sources ${}^{\shortmid}\Im_{\alpha_s\beta_s}(\tau)$ encoding nonassociative geometric data; and
   - 2] as equivalent families of modified vacuum gravitational equation (62), with effective τ-running shell cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$.

   Such functional representations of κ-parametric nonassociative R. Hamilton equations are important because they allow us to use directly the AFCDM (as for the nonholonomic Einstein equations but with additional τ-dependence) and generate various classes of generic off-diagonal solutions. The method simplifies substantially if there are considered, for instance, some general classes of quasi-stationary solutions involving nonlinear symmetries of type (60) (for generating locally anisotropic cosmological solutions, we have to consider certain dual nonlinear symmetries).

   We constructed in general off-diagonal forms such exact/parametric nonassociative geometric flow solutions for τ-evolving quasi-stationary configurations: with effective sources, (55); with running cosmological constants (63); and for various types of s-metric generating and polarizations functions, see (65), or (68). The coefficients of such (non) symmetric metrics and (non) linear connections depend, in general, on all conventional space like, co-fiber coordinates, for certain Killing type phase space symmetries, being determined by corresponding τ-families of generating and integration functions, generating effective sources and running cosmological constants as in (57).

   The AFCDM is an innovative geometric and analytic method for constructing exact/parametric solutions of physically important systems of nonlinear PDEs in (non) associative/ commutative MGTs and geometric flow generalizations. It allows to generate vacuum and non-vacuum metrics for very general off-diagonal ansatz not using special assumptions for diagonalizable metrics resulting into some associated systems of nonlinear ODEs (such details, for GR, are presented in [43, 44, 45, 46]). Another important difference from other analytic and numeric methods is that using the AFCDM we work with certain canonical s-adapted connections which allow to decouple necessary types of nonlinear systems of equations. We have to impose additional nonholonomic constraints of type (25) in order to extract LC-configurations and model their nonassociative geometric flow evolution. It should be noted that the AFCDM can be applied for constructing exact/parametric solutions of nonassociative vacuum gravitational equations with coefficients proportional to the complex unity. Such κ-parametric decompositions are for nonassociative Ricci tensors are considered, for instance, in [38, 39], when only the real coefficients are taken for analyzing possible consequences for classical models. It is not clear what physical importance may have such complex solutions in classical theories but, positively, they can be used to study quasi-classical approximations in QG and string/ M-theory. In our approach, we can redefine all (non) associative/ geometric constructions using almost complex/symplectic structures and related real nonholonomic geometric models [12, 13, 40].

4. The **fourth objective** of this work, in section 5, is twofold on its novelty of results and methods:

   1] There are constructed and analyzed most important physical properties of two new classes of τ-running nonassociative phase space black hole, BH, black ellipsoid, BE, and other types of more general quasi-stationary deformations. Such generic off-diagonal solutions are classified with respect to two different types of primary phase space s-metrics defined by double Schwarzschild - AdS BHs and, in the second case, phase space generalizations of Reisner-Norström BHs, when the generated target s-metrics have different interpretations. For certain types of nonholonomic constraints and generating functions, there are generated τ-evolving BE type solutions with conventional phase space hyper-surfaces, when the concept of Bekenstein–Hawking is applicable, and we compute respective thermodynamic variables in sections 5.2.3 and 5.3.3.

   2] Nevertheless, general models with nonassociative and generic off-diagonal Ricci flow evolution of physically important quasi-stationary solutions do not contain hyper-surfaces and we have to apply more general concepts of nonassociative generalized G. Perelman thermodynamics. The formulas for respective thermodynamic variables (73) are defined by effective running shell cosmological constants, ${}_{s}^{|}\Lambda(\tau)$ and certain volume forms depending on the prescribed nonholonomic structure. Such values are computed in explicit forms for corresponding classes of nonassociative Ricci flow deformed BHs solutions, see subsections 5.2.4 and 5.3.4. They functionally depend on generating and integration functions, which can be chosen in such forms when the configurations are well-defined as effective relativistic thermodynamic variables, preserving certain physical important properties under τ-evolution, to belong, or not to swampland models etc.

5. We study nonassociative modifications of the swampland program [19, 20, 21, 18, 60], which consists the **fifth objective** of this paper, see section 6. As motivated in modern literature on high energy physics and gravity [15, 6, 17, 7, 8, 22, 23, 24, 25, 26, 27], the Ricci flow is closely related to the RG flow with respect to the energy scale in the underlying two-dimensional non-linear sigma-models and various modified gravity and string theories. This provides a connection between RG, Ricci flows, and the swampland idea.

   In this work, we formulate a theory of nonassociative geometric flows using generalizations of G. Perelman functionals and related thermodynamic models and develop the AFCDM of constructing new classes of τ-evolving nonassociative quasi-stationary star-deformed BH and BE solutions. In general, the thermodynamic physical properties of such nonassociative geometric flow and gravity generic off-diagonal solutions can not be investigated following only the Bekenstein-Hawking entropy paradigm. We analyze how the a series of important swampland and Ricci flow conjectures can be extended to nonassociative geometric flows and respective extensions to the modified G. Perelman thermodynamic models. Then, we argue that we can prescribe certain subclasses of nonholonomic constraints, generating and integration functions when corresponding models satisfy the conditions of swampland conjectures studied in [23, 24, 25, 26, 27]. Corresponding phase space generalizations and modifications involving respective generalized distance functionals and effective mass scale of towers of states are stated respectively in subsections 6.1 and 6.2 and numbered as a) Conjectures 1-7. We provide and analyze four examples of τ-running quasi-stationary phase space solutions for nonassociative BH configurations with conventional hyper-surface horizons when the concept of generalized Bekenstein–Hawking entropy is applicable, see the end of subsection 6.1. This proves that for certain defined geometric conditions the swampland conjectures can be extended to conventional phase space and generalized for such nonassociative geometric flow and gravitational configurations.

   For more general nonassociative κ-parametric off-diagonal solutions, characterized by respective W-entropy functionals, we need a more tedious analysis in order to conclude if a corresponding nonassociative geometric flow model belongs, or not, to swampland following b) Claims 1-7. We formulate such criteria in the form of Conjectures and Claims, CCL, 1-7. In subsection 6.2, we provide explicit formulas for computing nonassociative and off–diagonal modifications of general distances and towers of effective mass states for various classes of τ-running nonassociative quasi-stationary equations following the CCL5 and effective cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$.

7.2 Conclusions, validity of claims and results, and perspectives

Discussing Objectives 1-5 for nonassociative Ricci flow theory and applications, we have advocated for modifications of the Swampland Program summarized in a series of Conjectures and Claims, CCL:

- **CCL1: The generalized distance conjecture and claim for nonassociative phase space**. We conclude that for very special nonholonomic/ diagonal configurations and additional assumptions on the type of nonlinear symmetries for the κ-parametric distance there are limits of the general distance functional ${}_{g}^{s}\aleph$ (102)$\rightarrow\infty$ resulting in infinite towers of effective phase masses ${}^{\shortmid}m$ (103). This extends for nonassociative phase space the generalized distance conjecture studied in [22, 23] for (associative and commutative) d-dimensional manifolds. In this work, we show that using the AFCDM, we can construct new classes of exact/parametric solutions when, for nonassociative configurations, when ${}_{g}^{s}\aleph$ and ${}^{\shortmid}m$ may have a different behaviour because generic off-diagonal interactions and a more rich nonholonomic structure of phase spaces and nonassociative geometric flux evolution. This claims additional investigations for any class of off-diagonal solutions in order to understand if they result in effective models which belong, or not, to swampland.

- **CCL2: The AdS distance conjecture and claim for nonassociative phase spaces**. Another important conclusion of this work is that combining the conditions of CCL1 with the notion of (conformal) Weyl re-scaling of nonassociative phase space metrics, we can extend the AdS Distance conjecture in a form including a part a) as a conjecture for a subclass of phase space quasi-stationary s-metrics with κ- and τ parametric nonlinear symmetries (60) connecting effective sources to effective running cosmological constants. A part b) Claim in CCL2 is motivated because the conformal transforms and symmetries are relevant only to very special classes of solutions encoding nonassociative data and generic off-diagonal geometric flow evolution scenarios and/or nonassociative gravitational interactions. In general, we need an additional analysis in order to determine if a class of parametric solutions for nonassociative Ricci flows belong, or not, to swampland.

- **CCL3: The fixed points of nonassociative geometric flows conjecture and claim**. This concludes that in the nonassociative geometric flow theory elaborated in this work there exist parametric solutions defining infinite towers with effective zero masses points for flow evolution toward a fixed point at infinite distance. The a) Conjecture in CCL3 can be checked on nonassociative phase spaces, for instance, for s-metrics (92) in a form similar to that for associative and commutative Ricci flows in [23]. For more general classes of nonassociative quasi-stationary off-diagonal solutions, we need a more rigorous analysis for chosen classes of generating functions and effective cosmological constants. Different nonholonomic configurations may result, or not, in well defined effective classical and QG models.

- **CCL4: The canonical distance - scalar curvature conjecture and claim for nonassociative geometric flows**. The AFCDM allows to decouple (modified) Ricci flow and gravitational equations and construct exact/parametric solutions if the canonical s-connection is used instead of the LC-connection. An original result of the subsection 6.1.3 consists in deriving formulas for the generalized canonical distance and related tower of effective mass states in terms of the canonical scalar curvatures and running shell cosmological constants (see constructions related to (111) and (112)). So, in principle, we can check if some classes of parametric solutions belong or not to swampland if we formulate all results in terms of the canonical s-connections. We can extract LC-configurations imposing additional nonholonomic constraints but the corresponding geometric constructions and analysis of possible physical implications depend on the type of generated families of quasi-stationary solutions. This imposes certain limitations for testing in general LC-form the CCL4. Nevertheless, such formulas involving the canonical s-curvature allow to express the necessary results in terms of ${}_{s}^{\shortmid}\Lambda(\tau)$ which is important for testing, for instance, the CCL6 and CCL7 (see below).

- **CCL5: The canonical distance - F-functional conjecture and claim for nonassociative geometric flows**. The idea to define the general distance functional in terms of F- and W-functionals was elaborated for associative and commutative geometric flows, see Conjectures B2 and B3 in [23]. The subsection 6.1.4 consists an original extension of those constructions for nonassociative geometric flows determined by respective κ-parametric generalizations of G. Perelman's functionals. Using ${}_{s}^{\shortmid}\widehat{\mathcal{F}}_{\kappa}^{\star}(\tau)$ (44), we can compute the generalized distance and state the conditions of existence of canonical infinite tower of phase space states with zero effective masse as in a) Conjecture of CCL5. The part b) Claim states that in general, solving respective τ- and κ-parametric equations we can express necessary formulas in terms of respective F-functionals and decide if a class of solutions belong or not two swampland and speculate on how LC-configurations can be extracted. Nevertheless, we conclude that the constructions with generalized F-functionals are quite sophisticate when we study the statistic and geometric thermodynamic properties of modified (nonassociative and/or other types) geometric flows and Ricci soliton configurations and, in particular of quasi-stationary and nonholonomic BH solutions.

- **CCL6: The canonical distance - W-functional conjecture and claim for nonassociative geometric flows**. The a) Conjecture and b) Claim parts are similar to those stated for CCL5 but using the star deformed W-functional. We derived the formulas ${}_{W}^{s}\widehat{\aleph} \simeq \log({}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau_i)/ {}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau_f))$ (115) and ${}^{\shortmid}m \sim M_p e^{-\hat{\alpha}|\ {}_{W}^{s}\widehat{\aleph}|} \simeq \left({}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)\right)^{\hat{\alpha}}$ (116), for ${}_{W}^{s}\widehat{\aleph} \to \infty$, which are very important and simplify the procedure for analyzing if certain classes of exact/ parametric solutions belong, or not, to swampland. Using nonlinear symmetries, we can express ${}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau_i)$ as a functional of running cosmological constants and volume forms and, then, we can speculate on classes of generating functions which drive a class of solutions to swampland, or inversely, keep it with some well-defined properties. For associative and commutative Ricci flows, such formulas were conjectured in [23].

- **CCL7: The phase space black hole entropy distance conjecture and claim for nonassociative geometric flows**. An original approach to swampland and BH physics was elaborated in [63, 26] exploiting possible connections between the distance conjecture and area low, involving Ricci flow conjectures, for the Bekenstein-Hawking entropy. The statements of the Black Hole Entropy Distance Conjecture, BHEDC, can be extended to certain classes of nonassociative phase solutions for geometric flows/ Ricci solitons if the s-metrics contain certain hyper-surface configurations, as we concluded in a) Conjecture of CCL7. Nevertheless, even for such type of quasi-stationary κ- and τ-running solutions related via nonlinear symmetries to running cosmological constants, we may need an additional analysis (the b) Claim of CCL7) in order to decide if such a solution belong to swampland or not. We list a table with 4 examples of such nonlinear quadratic elements and respective formulas for phase generalized Bekenstein-Hawking entropy computed, for instance, for rotoid configurations.

Summarizing CCL1-CCL7, we conclude that, in general, the solutions of nonassociative geometric flow and Ricci soliton equations can be characterized by a corresponding modified G. Perelman thermodynamic model, when the conditions of CCL6 can be analyzed with respect to the purposes of the Swampland Program, even the statements of CCL7 are not applicable. Here we note that we can define and compute two types of entropy variables: the "minus" entropy, i.e. W-entropy, ${}_{s}^{\shortmid}\widehat{\mathcal{W}}_{\kappa}^{\star}(\tau)$ and ${}_{s}^{\shortmid}\widehat{\mathcal{S}}_{\kappa}^{\star}(\tau)$, with effective shell sources $\sum_s {}_{s}^{\shortmid}\Im(\tau)$ (119). Using nonlinear symmetries / transforms (60), such formulas allow us to modify respectively the statements of CCL6 in certain forms which are applicable to compute the generalized distance functional and the effective mass stated scale in terms of effective running cosmological constants. We present such important formulas for quasi-stationary solutions, see (122) and (123). Together with CCL6, the modified CCL7 is tested by explicit examples of nonassociative quasi-stationary geometric flow deformations of phase space double nonholonomic deformed RN-dS BHs and nonholonomic deformed RN-AdS BHs, see respective formulas in subsections 6.2.3 and 6.2.4.

Let us discuss the proofs and validity of results and claims of this article stated as Issues 1-5, **Is1-5**:

- **Is1:** *Postulating nonassociative geometric flow and vacuum gravitational equations.* It is not possible to formulate a general theory of nonassociative or noncommutative geometric flows or MGTs involving different types of non-Riemannian metric and (non) linear connection structures. For a subclass of theories determined by R-flux deformations [12, 13, 40, 41], it is possible to define star variants of Riemannian and Ricci tensors and perform corresponding κ–linear decompositions of geometric objects (generalizing in nonholonomic form the constructions from [38, 39]). The geometric constructions are very different for other models involving octonionic or supersymmetric variables, spectral triples etc. [15, 33, 34, 35, 36, 37, 70]. Using ${}_{s}^{\shortmid}\widehat{\mathcal{R}}ic^{\star}$ and ${}_{s}^{\shortmid}\widehat{\mathcal{R}}sc^{\star}$ from (27), we can postulate the nonassociative geometric flow equations (2), or (41), describing the evolution of τ-families $\left({}_{\star}^{\shortmid}\mathfrak{g}_{\alpha_s\beta_s}(\tau),\ {}_{s}^{\shortmid}\widehat{\mathbf{D}}^{\star}(\tau)\right)$ (as we considered in subsection 2.3.1). For some subclasses of nonassociative Ricci solitons, we obtain nonassociative generalizations of the Einstein equations (43), or (32), which can be formulated independently in abstract symbolic geometric form as in [43] but for nonassociative geometric s-objects.

- **Is2:** *Postulating nonassociative Perelman's functionals and thermodynamic variables.* Considering ${}_{s}^{\shortmid}\widehat{\mathcal{R}}ic^{\star}$ and ${}_{s}^{\shortmid}\widehat{\mathcal{R}}sc^{\star}$, and respective normalizing functions and volume forms, we can define in abstract geometric form the F- and W-functionals (38) and (39). Such Lyapunov type functionals consist nonholonomic relativistic generalizations and star product deformations of standard Perelman functionals (9) for Ricci flows of Riemannian metrics. The nonassociative W-functional (39) is similar to a "minus" nonassociative entropy, which allowed us to define geometrically (generalizing the constructions from [1]) nonassociative versions of statistical/ geometric thermodynamic variables (50). Such nonassociative geometric thermodynamic models can be defined in abstract symbolic geometric form and corresponding physical motivations are provided at the end of subsection 2.3.2, see also below the paragraphs Is3 - Is5.

- **Is3:** *Problems and a cure for formulating variational procedures for deriving nonassociative geometric flow and nonassociative gravitational field equations.* One of the fundamental G. Perelman's result [1] consisted in a proof that for Riemannian geometries the R. Hamilton equations (108) can be derived using variational procedures for F- or W-functionals (9). Those geometric constructions were used for elaborating rigorous proofs of the Poincaré-Thurston conjecture [1], see details in monographs [9, 10, 11]. Unfortunately, it is not possible to formulate a self-consistent variational principle for general nonassociative deformations determined by a general twist product and in other types of nonassociative and noncommutative theories. Here, we note that one can be defined an infinite number of nonassociative and noncommutative differential and integral calculi which is different from the commutative (pseudo) Riemannian geometry. So, it is not possible to formulate some general forms of nonassociative the Poincaré-Thurston conjecture and use "twisted" variational constructions for corresponding proofs. Nevertheless, this does not prohibit us to elaborate on physically important and well-defined models of nonassociative geometric flows. We can postulate in symbolic geometric form nonassociative geometric flow equations (2), or (41), and then to study their thermodynamic properties using associated nonassociative F-and W-functionals. For such nonassociative geometric constructions, we can perform always parametric decompositions and compute all terms for κ-deformed R. Hamilton equations (46) and Perelman's functionals (44) and (45). Working with κ–linear deformed theories, we can always elaborate self-consistent variational procedures, when corresponding nonassociative geometric flow/ gravitational equations are characterized by κ-linear thermodynamic variables (51).

- **Is4:** *Physically important nonvariational theories.* There are various physical arguments (similarly to those for models of hydrodynamics with parametric turbulence, or for thermodynamic models with parametric phase transitions, multi-diffusion and branch decomposition and evolution) to study nonassociative geometric flow theories. In such cases, the definition of self-consistent variational procedures is not uniquely defined but depends on some prescribed (non) linear / associative / commutative symmetries, on the order of parametric decompositions and corresponding nonholonomic s-distributions. We emphasize that the variational procedure described in subsection 2.3.2 can be performed in a quite general form if we begin with a corresponding commutative canonical s-adapted variational nonholonomic configuration and subject the constructions to κ-parametric deformations following the Convention 2 (26). Such a variational procedure can be performed recurrently for higher orders on κ. Unfortunately, to prove some general convergence conditions summarizing all terms on powers of κ is an unsolved mathematical problem, which is typical for nonlinear functional analysis. Theoretically, we can consider an inverse situation and begin with general nonassociative formulas for flow equations (2) and formal functionals (38) and (39). Then, we can perform adapted κ-parametric decompositions involving imaginary and real terms. Unfortunately, in such cases, we are not able to elaborate a general method for constructing exact/parametric solutions but following the inverse procedure we can always apply the AFCDM as we proved in previous sections (for quasi-stationary solutions) and in partner works [13, 40, 41]. This is a generic property for any nonassociative/ noncommutative differential and integral calculus.

- **Is5:** *Validity through exact and parametric solutions of physically important nonlinear systems of PDE encoding nonassociative data.* So, haven defined corresponding fundamental nonassociative geometric s-objects like a star deformed metric, a canonical s-connection and respective Ricci s-tensor and canonical scalar, we can always postulate geometrically certain nonassociative geometric flow equations and F- and W-functionals. Such values are well-defined in all orders on κ but for twisted theories there is not a general nonassociative variational proof for nonassociative geometric flow/ gravitational equations from some generalized Perelman/ action functionals. For applications in modern physics and quantum information theory, we can consider only κ-linear constructions, apply the AFCDM and construct physically important solutions (star-deformed black holes, wormholes, locally anisotropic cosmological s-metrics), and speculate on verifiable models encoding nonassociative data. Recurrently, we can construct nonassociative and noncommutative solutions with higher orders on κ and $\hbar$ but the technically the resulting formulas are much cumbersome. For any such higher order parametric configurations, we can define and compute effective F- and W-functionals, and respective nonassociative thermodynamic variables. Re-defining the measures (for respective normalizing s-functions and nonholonomic s-distributions), we suppose that a corresponding variational procedure can be formulated for any stated polynomial order on κ and $\hbar$ (like we considered in subsection 2.3.2). But this do not provide us a general well-defined nonassociative variational theory with twist product. We can construct and study physical properties of realistic nonassociative theories considering physically important solutions encoding κ-linear data.

Finally, with respect to above Is1-5, we note that (in certain similar forms) such problems exist in all classical gravity and QG theories. For instance, the classical Einstein equations are linear on $\varkappa = 8\pi G/c^4$, were G is the Newtonian constant of gravitation and c is the light velocity constant. We do not have yet a well-defined theory of QG, but it is aways possible to elaborate on physically important effects proportional to $\varkappa$ and $\hbar$ and consider higher orders on such constants involving (square) curvature terms etc. Such effective models are commutative or noncommutative. R-flux deformations from string theory result in nonassociative geometric configurations determined by twist products. This results in ambiguities for constructing general self-consistent principles, involves complex terms for curvatures and Ricci tensors for some parametric decompositions on κ and $\hbar$ like in [38, 39, 12] etc. This reflects our not complete knowledge about string theory and QG. The priority of nonholonomic s-adapted definition of star product (19) is that using the Convention 2 (26) we can define and compute recurrently parametric R-flux deformations of some (associative/commutative) variatonal equations and apply the AFCDM in order to construct physically important solutions of nonassociative geometric flows and vacuum gravitational equations. In general form, such nonassociative solutions are characterized by respective modified Perelman (thermodynamic) functionals and variables. This states a new paradigm for formulating and investigating possible physical implications of nonassociative theories and provides a general computational method when nonassociative effects can be computed and the validity of certain claims can be verified at least (for simplicity) in the linear approximation on κ.

Above conclusions and theoretical/ computational tests of CCL1-CCL7 support the **The Main Hypothesis, MH,** of this work (formulated at the end of subsection 1.2) *that the Swampland Program has to be revised/ modified in order to elaborate explicit criteria how to include nonassociative and noncommutative geometric flows, QM, QFTs, and MGTs in elaborating QG theories. The corresponding Conjectures and Claims allows us to select self-consistent nonassociative geometric and physical models (defined by generic off-diagonal solutions) encoding at least in parametric form nonassociative star product and R-flux data for string / M-theory. In low-energy limits, such configurations can be completed into QG in the UV forms and distinguished from another classes of theories/models/ solutions which do not have such properties.*

The MH modifies a series of future research purposes stated in our partner works [12, 13, 40, 41]. We list and speculate on five perspective directions of research involving nonassociative geometric flow methods, classical and quantum gravity, and quantum information models[18]:

- **Q1a**: *Nonassociative Einstein-Yang-Mills-Higgs systems.* A full investigation of nonassociative geometric flow and gravity theories should involve flux evolution and field equations with nontrivial matter sources and other types of nonassociative/ noncommutative structures (for instance, octonions etc.) which are not necessarily determined by R-flux deformations but based on other types of star product and/or nonassociative algebraic structures. One of the next steps is to formulate and study models with nonassociative nonholonomic deformations and geometric flow evolution of Einstein-Yang-Mills-Higgs systems resulting in nonassociative gravity and matter field theories with nonsymmetric metrics, generalized connections and nonzero sources. Such nonassociative models should generalize in nonassociative form the Einstein-Eisenhart-Moffat theories [64, 65, 66, 66, 68], see respective nonholonomic and phase space Ricci flow constructions in [69]. References [12, 13, 40, 41] provide a series of new ideas on further nonassociative/ nonsymmetric metric / nonholonomic developments when the AFCDM can be applied for constructing various classes of exact/parametric solutions (for instance, defining locally anisotropic wormholes, BHs and BEs, solitonc hierarchies, nonassociative quasi-periodic cosmological structures etc.).

- **Q2a:** *Nonassociative Finsler-Lagrange-Hamilton geometric flows/gravity, their almost symplectic models, and deformation quantization.* Nonassociative geometric flow and gravity theories determined by a star product of type (19) on tensor products of cotangent bundles consist of a class of nonassociative generalizations of the Finsler-Lagrange-Hamilton geometry which have generalized metrics/connections depending on velocity/ momentum-like coordinates. In relativistic/ noncommutative and commutative/ supersymmetric / fractional etc. variants, there are status reports and reviews of results [70, 8]. The importance of Finsler-like geometric objects and nonholonomic variables (they can be defined even on Lorentz manifold enabled with a nonolonomic fibered structure) is that they can be reformulated as some equivalent almost Kaehler/ symplectic and almost complex variables. This allows us to elaborate on realistic classical models for (non) associative/ commutative geometric flow and gravity theories (to apply complex variables for quantum models and almost complex variables for classical models) and apply/develop methods of deformation and/or geometric quantization. A more sophisticate program is to formulate and study models of QG involving nonassociative/ noncommutative Lagrange-Hamilton structures and/or almost symplectic variables.

[18]for instance, Q1a is used for modifications involving nonassociative Ricci flow of the query Q1 from [41]; we also state a new query Q5.

- **Q3a:** *Elaborating Bekenstein-Hawking and G. Perelman thermodynamic and locally anisotropic kinetic models for physically important solutions in nonassociative classical and quantum theories.* The sections 2 and 3 of this work provide a solution of query Q3 stated at the end of [41]. Here we remember that the nonassociative gravity with star product and R-flux deformations was formulated up to defining and computing the nonassociative Ricci tensor in [38, 39]. This was enough to develop models of nonassociative/ noncommutative Ricci flows but those abstract geometric and/or coordinate frame constructions based on LC-connection do not allow to motivate respective generalizations of R. Hamilton equations, prove decoupling and integration properties of such geometric flow equations and find exact/ parametric solutions. To study possible physical implications we had to elaborate a respective nonholonomic formalism with dyadic shell structures, see details in [12, 13] and Appendices to this work. Such constructions were motivated by finding physically important exact/parametric solutions and computing respective Bekenstein-Hawking and/or G. Perelman thermodynamic variables in section 4 of [41] and section 5 in this paper.

- **Q4a:** *The program on nonassociative geometric and quantum information flow theories.* We point again to perspectives and importance of a research program to extend in nonassociative forms the geometric flow information theory [6, 7] to theories with nonassociative qubits and entanglement, conditional entropies etc. Such new directions in modern QFT, strings and gravity, theory of quantum computers have deep roots in nonassociative quantum mechanics and gauge models [30, 31, 33, 37] and motivations from noncommutative geometry, string and M-theory [15, 34, 35, 36, 38, 39].

- **Q5:** *Cosmological models and dark energy and dark matter physics encodding quasi periodic structure and data for nonassociative theories.* In this work, we constructed and studied the main properties of a series of new classes of nonassociative exact/ parametric solutions for τ-running quasi-stationary physically important solutions defining star product and R-flux deformed BHs, BEs etc. The AFCDM can be also developed and applied in certain dual forms when there are generated locally anisotropic solutions [72, 17, 71]. Extending such geometric and analytic methods, we plan to elaborate on a series of works on phase space and spacetime nonassociative quasi-crystal configurations and related web filaments, nonassociative quasi-periodic geometric flow evolution and pattern forming structures in accelerating and inflationary cosmology and dark energy and dark matter physics. Such models are generic off-diagonal (for certain classes of holonomic configurations, the s-metrics can be diagonalized) encoding nonassociative star product deformations. They are characterized by respective generalized G. Perelman thermodynamic variables and put their imprints and requests for modifications of the Swampland Program.

We shall develop on above directions and report on progress for queries Q1a-Q4a,Q5 in future works.

Acknowledgments: This is the second work together with [41] supported by a Fulbright senior fellowship for SV and hosted by the physics department at California State University at Fresno, USA. Some results are related also to a research program proposed for the PNRR-Initiative 8 of the Romanian Ministry of Research, Innovation and Digitization and for a visiting program to be decided by CASLMU München. It is also a partner work to [12, 13, 40] extending to nonassociative geometric flows former research programs and fellowships at CERN (Geneva, Switzerland) and Max Planck Institut für Physik / Werner Heisenberg Institut, München (Germany). SV is also grateful to professors D. Lüst, N. Mavromatos, J. Moffat, Yu. A. Seti, P. Stavrinos, and M. V. Tkach for hosting visits and collaborations. Authors thank the referee for very important critics and requests to provide substantial modifications and proofs on s-adapted variational calculus for nonassociative geometric flow equations performed in section 2.3.

A The anholonomic frame and connection deformation method, AFCDM

We outline the AFCDM for constructing exact and parametric solutions for nonlinear systems of PDEs (16). Details with proofs and methods in commutative gravity theories are provided in [7, 8] and, for recent nonassociative generalizations, [12, 13, 40, 41]. In this section, we consider effective sources parameterized in the form (35) encoding nonassociative star product and R-flux deformations. Such PDEs can't be integrated in some general off-diagonal forms if we follow standard methods from GR [46, 43, 45, 44] when solutions are found for some special diagonal ansatz of metrics transforming the (modified) Einstein equations into certain systems of nonlinear ordinary differential equations, ODE. For diagonalizable metrics, there are imposed some higher order symmetries (spherical, cylindrical type etc.) which allow to integrate the resulting ODEs in very general forms. The solutions are classified by respective integrations constants. This is used, for instance, for constructing black hole, BH, solutions. The AFCDM is a more genera geometric method for generating exact and parametric solutions, when a necessary type of auxiliary connection (for instance, the canonical s-connection ${}_{s}^{\shortmid}\widehat{\mathbf{D}}$) is used. This allows us to decouple and integrate systems of (modified) geometric flow and gravitational field equations defined by generic off-diagonal metrics, (non-) Riemannian connections etc. with coefficients depending, in principle, on all spacetime and phase space coordinates. Different classes of such exact/ parametric solutions are determined by respective classes of generating functions, generating sources, integrating functions and constants, and/or decompositions certain small physical parameters (for instance, on string and Plank constants) etc.

To apply the AFCDM the quadratic linear element for generating quasi-stationary solutions is parameterized by such an off-diagonal ansatz for s-metrics:

$$d\widehat{s}^2 = g_{i_1}(x^{k_1})(dx^{i_1})^2 + g_{a_2}(x^{i_1},y^3)(\mathbf{e}^{a_2})^2 + {}^{\shortmid}g^{a_3}(x^{i_2},p_5)({}^{\shortmid}\mathbf{e}_{a_3})^2 + {}^{\shortmid}g^{a_4}({}^{\shortmid}x^{i_3},E)({}^{\shortmid}\mathbf{e}_{a_4})^2, \quad (A.1)$$

for $i_1, j_1, k_1 ... = 1,2; a_2 = 3,4; i_2 = (i_1, b_2), b_2 = 3,4; i_3 = (i_2, b_3), b_3 = 5,6; b_4 = 7,8$ etc.; and coordinates $x^{i_1} = (x^1, x^2); u^{i_2} = x^{i_2} = (x^{i_1}, y^{b_2}), y^4 = t; {}^{\shortmid}u^{i_3} = {}^{\shortmid}x^{i_3} = (x^{i_2}, p_{b_3}); {}^{\shortmid}u^{i_4} = ({}^{\shortmid}x^{i_3}, p_{b_4}), p_8 = E$. The dual bases in (A.1),

$$\mathbf{e}^{a_2} = dy^{a_2} + N^{a_2}_{k_1}(x^{i_1}, y^2)dx^{k_1},\ {}^{\shortmid}\mathbf{e}_{a_3} = dp_{a_3} + {}^{\shortmid}N_{a_3 k_2}(x^{i_2}, p_6)dx^{k_2},\ {}^{\shortmid}\mathbf{e}_{a_4} = dp_{a_4} + {}^{\shortmid}N_{a_4 k_3}({}^{\shortmid}x^{i_3}, E)d\,{}^{\shortmid}x^{k_3},$$

are determined by respective N-connection coefficients,

$$\begin{aligned} N^3_{k_1} &= w_{k_1}(x^{i_1}, y^3), N^4_{k_1} = n_{k_1}(x^{i_1}, y^3); \\ {}^{\shortmid}N_{5k_2} &= n_{k_2}(x^{i_2}, p_5),\ {}^{\shortmid}N_{6k_2} = w_{k_2}(x^{i_2}, p_5);\ {}^{\shortmid}N_{7k_3} = n_{k_3}(x^{i_3}, E),\ {}^{\shortmid}N_{8k_3} = w_{k_3}(x^{i_3}, E). \end{aligned} \quad (A.2)$$

We emphasize that above coefficients defined with respect to s-adapted frames do not depend on $u^4 = y^4 = x^4 = t$, i.e. there is a Killing vector ∂_t on shell $s = 2$; do not depend on ${}^{\shortmid}u^6 = p_6$, i.e. there is a Killing symmetry on ${}^{\shortmid}\partial^6$ on shell $s = 3$; and do not depend on ${}^{\shortmid}u^7 = p_7$, i.e. there is a Killing symmetry on ${}^{\shortmid}\partial^7$ on shell $s = 4$. In similar forms, changing parameterizations of coordinates and coefficients, we can consider ansatz with Killing symmetry on ∂_3 instead of ∂_4; on ${}^{\shortmid}\partial^5$ instead of ${}^{\shortmid}\partial^6$; on ${}^{\shortmid}\partial^8$ instead of ${}^{\shortmid}\partial^7$. Such parameterizations allow to decouple and integrate in explicit form various classes of vacuum and non-vacuum modified gravitational equations.

A.1 Off-diagonal quasi-stationary solutions with effective sources

Tedious computations provided in [13] (re-defined in real momentum coordinates ${}^{\shortmid}u^{\alpha}$) prove that quasi-stationary solutions of the nonassociative parametric vacuum gravitational equations (16) with effective sources (35) are defined by respective s-metric (A.1) and N-connection (A.2) coefficients:

$$\begin{aligned} g_1(x^{i_1}) &= g_2(x^{i_1}) = e^{\psi(\hbar,\kappa;x^{k_1})}, \\ g_3(x^{i_1}, y^3) &= \frac{[\partial_3({}_2\Psi)]^2}{4({}^{\shortmid}_2\mathcal{K})^2\{g_4^{[0]} - \int dy^3 \frac{\partial_3[({}_2\Psi)^2]}{4({}^{\shortmid}_2\mathcal{K})}\}},\ g_4(x^{i_1}, y^3) = g_4^{[0]} - \int dy^3 \frac{\partial_3[({}_2\Psi)^2]}{4({}^{\shortmid}_2\mathcal{K})}, \end{aligned} \quad (A.3)$$

$$
\begin{aligned}
{}^{\shortmid}g^5(x^{i_1},y^3,p_5) &= \frac{[\,{}^{\shortmid}\partial^5(\,{}^{\shortmid}_3\Psi)]^2}{4(\,{}^{\shortmid}_3\mathcal{K})^2\{\,{}^{\shortmid}g^6_{[0]} - \int dp_5 \frac{{}^{\shortmid}\partial^5[(\,{}^{\shortmid}_3\Psi)^2]}{4(\,{}^{\shortmid}_3\mathcal{K})}\}},\ {}^{\shortmid}g^6(x^{i_1},y^3,p_5) = {}^{\shortmid}g^6_{[0]} - \int dp_5 \frac{{}^{\shortmid}\partial^5[(\,{}^{\shortmid}_3\Psi)^2]}{4(\,{}^{\shortmid}_3\mathcal{K})}, \\
{}^{\shortmid}g^8(x^{i_1},y^3,p_5,p_7) &= \frac{[\,{}^{\shortmid}\partial^7(\,{}^{\shortmid}_4\Psi)]^2}{4(\,{}^{\shortmid}_4\mathcal{K})^2\{\,{}^{\shortmid}g^8_{[0]} - \int dp_7 \frac{{}^{\shortmid}\partial^7[(\,{}^{\shortmid}_4\Psi)^2]}{4(\,{}^{\shortmid}_4\mathcal{K})}\}},\ {}^{\shortmid}g^8(x^{i_1},y^3,p_5,p_7) = {}^{\shortmid}g^8_{[0]} - \int dp_7 \frac{{}^{\shortmid}\partial^7[(\,{}^{\shortmid}_4\Psi)^2]}{4(\,{}^{\shortmid}_4\mathcal{K})};
\end{aligned}
$$

$$
\begin{aligned}
\text{and } N^3_{k_1} &= w_{k_1}(x^{i_1},y^3) = \frac{\partial_{k_1}(\,_2\Psi)}{\partial_3(\,_2\Psi)}, \\
N^4_{k_1} &= n_{k_1}(x^{i_1},y^3) = \,_1n_{k_1} + \,_2n_{k_1}\int dy^3 \frac{\partial_3[(\,_2\Psi)^2]}{4(\,{}^{\shortmid}_2\mathcal{K})^2|g^{[0]}_4 - \int dy^3 \frac{\partial_3[(\,_2\Psi)^2]}{4(\,{}^{\shortmid}_2\mathcal{K})}|^{5/2}};
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}N_{5k_2} &= w_{k_2}(x^{i_2},p_5) = \frac{\partial_{k_2}(\,{}^{\shortmid}_3\Psi)}{{}^{\shortmid}\partial^5(\,{}^{\shortmid}_3\Psi)}, \\
{}^{\shortmid}N_{6k_2} &= n_{k_2}(x^{i_2},p_5) = \,{}^{\shortmid}_1n_{k_2} + \,{}^{\shortmid}_2n_{k_2}\int dp_5 \frac{{}^{\shortmid}\partial^5[(\,{}^{\shortmid}_3\Psi)^2]}{4(\,{}^{\shortmid}_3\mathcal{K})^2|\,{}^{\shortmid}g^6_{[0]} - \int dp_5 \frac{{}^{\shortmid}\partial^5[(\,{}^{\shortmid}_3\Psi)^2]}{4(\,{}^{\shortmid}_3\mathcal{K})}|^{5/2}};
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}N_{7k_3} &= w_{k_3}(x^{i_2},p_5,p_7) = \frac{{}^{\shortmid}\partial_{k_3}(\,{}^{\shortmid}_4\Psi)}{{}^{\shortmid}\partial^7(\,{}^{\shortmid}_4\Psi)}, \\
{}^{\shortmid}N_{8k_3} &= n_{k_3}(x^{i_2},p_5,p_7) = \,{}^{\shortmid}_1n_{k_3} + \,{}^{\shortmid}_2n_{k_3}\int dp_7 \frac{{}^{\shortmid}\partial^7[(\,{}^{\shortmid}_4\Psi)^2]}{4(\,{}^{\shortmid}_4\mathcal{K})^2|\,{}^{\shortmid}g^8_{[0]} - \int dp_7 \frac{{}^{\shortmid}\partial^7[(\,{}^{\shortmid}_4\Psi)^2]}{4(\,{}^{\shortmid}_4\mathcal{K})}|^{5/2}},
\end{aligned}
$$

Above coefficients are functionals on such functions and parameters:

$$
\begin{aligned}
&\text{generating functions: } \psi(\hbar,\kappa;x^{k_1});\ _2\Psi(\hbar,\kappa;x^{k_1},y^3);\ {}^{\shortmid}_3\Psi(\hbar,\kappa;x^{k_2},p_5);\ {}^{\shortmid}_4\Psi(\hbar,\kappa;\,{}^{\shortmid}x^{k_3},p_7); \qquad (A.4)\\
&\text{generating sources: } {}^{\shortmid}_1\mathcal{K}(\hbar,\kappa;x^{k_1});\ {}^{\shortmid}_2\mathcal{K}(\hbar,\kappa;x^{k_1},y^3);\ {}^{\shortmid}_3\mathcal{K}(\hbar,\kappa;x^{k_2},p_5);\ {}^{\shortmid}_4\mathcal{K}(\hbar,\kappa;\,{}^{\shortmid}x^{k_3},p_7);\\
&\text{integrating functions: } g^{[0]}_4(\hbar,\kappa;x^{k_1}),\ _1n_{k_1}(\hbar,\kappa;x^{j_1}),\ _2n_{k_1}(\hbar,\kappa;x^{j_1});\\
&\quad {}^{\shortmid}g^6_{[0]}(\hbar,\kappa;x^{k_2}),\ _1n_{k_2}(\hbar,\kappa;x^{j_2}),\ _2n_{k_2}(\hbar,\kappa;x^{j_2});\ {}^{\shortmid}g^8_{[0]}(\hbar,\kappa;\,{}^{\shortmid}x^{j_3}),\ {}^{\shortmid}_1n_{k_3}(\hbar,\kappa;\,{}^{\shortmid}x^{j_3}),\ {}^{\shortmid}_2n_{k_3}(\hbar,\kappa;\,{}^{\shortmid}x^{j_3}).
\end{aligned}
$$

We emphasize that $\psi(\hbar,\kappa;x^{k_1})$ is determined on shell $s=1$ as a solution of 2-d Poisson equation, $\partial^2_{11}\psi+\partial^2_{22}\psi = 2\ _1\mathcal{K}$, and may encode certain nonassociative data if $\ _1\mathcal{K}$ contains nonholonomic dependencies on such ones.

Any generic off-diagonal ansatz (A.3) defines a class of exact solutions determined by generating data (A.4) and depend in parametric form on $\hbar,\kappa$ for any nonassociative star product and R-flux data encoded in ${}^{\shortmid}_s\mathcal{K}$. Corresponding quasi-stationary configurations are also characterized by nontrivial coefficients of respective nonsymmetric metrics ${}^{\shortmid}_\star\mathfrak{a}_{\alpha_s\beta_s}$ computed by introducing in (22) the s-metric and N-connection coefficients for (A.3). Such solutions are with nontrivial nonholonomic torsion but can be constrained to subclasses of generating data which solve the conditions (25), see also (18), and allow to extract LC-configurations, see details in section 5.3.3 of [13].

A.2 Nonlinear symmetries and solutions with effective cosmological constants

Quasi-stationary solutions posses an important nonlinear symmetry which allow to formulate them in different functional forms emphasizing certain classes of effective sources and cosmological constants, different types of generating functions and parametric decompositions, which is important for finding other classes of solutions and investigating their physical properties.

We can study nonassociative nonholonomic deformations of a **prime** s-metric ${}_{s}^{\shortmid}\mathring{\mathbf{g}}$ (which may be, or not, a solution of some (modified) gravitational equations) into a **target** s-metric ${}_{s}^{\shortmid}\mathbf{g}$ defining a quasi-stationary solution (A.3) on ${}_{s}^{\star}\mathcal{M}$,

$$ {}_{s}^{\shortmid}\mathring{\mathbf{g}} \rightarrow {}_{s}^{\shortmid}\mathbf{g} = [\ {}^{\shortmid}g_{\alpha_s} = {}^{\shortmid}\eta_{\alpha_s}\ {}^{\shortmid}\mathring{g}_{\alpha_s},\ {}^{\shortmid}N_{i_{s-1}}^{a_s} = {}^{\shortmid}\eta_{i_{s-1}}^{a_s}\ {}^{\shortmid}\mathring{N}_{i_{s-1}}^{a_s}]. \qquad (A.5)$$

Such nonholonomic s-deformations can be described in terms of gravitational polarization functions (η-polarizations), when the target s-metrics are parameterized

$$ {}^{\shortmid}\eta_{\alpha_s}(\hbar, \kappa, x^{i_{s-1}}, p_{a_s}) \ \text{ and }\ {}^{\shortmid}\eta_{i_{s-1}}^{a_s}(\hbar, \kappa, x^{i_{s-1}}, p_{a_s}). \qquad (A.6)$$

For κ-linear s-deformations, we can introduce χ-polarizations,

$$\begin{aligned} &[\ {}^{\shortmid}g_{\alpha_s} = {}^{\shortmid}\zeta_{\alpha_s}(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s})\ {}^{\shortmid}\mathring{g}_{\alpha_s},\ {}^{\shortmid}N_{i_s}^{a_s} = {}^{\shortmid}\zeta_{i_{s-1}}^{a_s}(1+\kappa\ {}^{\shortmid}\chi_{i_{s-1}}^{a_s})\ {}^{\shortmid}\mathring{N}_{i_{s-1}}^{a_s}], \text{ when} \qquad (A.7)\\ &{}^{\shortmid}\eta_{\alpha_s} = {}^{\shortmid}\zeta_{\alpha_s}(\hbar, x^{i_{s-1}}, p_{a_s})[1+\kappa\ \ {}^{\shortmid}\chi_{\alpha_s}(\hbar, x^{i_{s-1}}, p_{a_s})] \ \text{ and}\\ &{}^{\shortmid}\eta_{i_{s-1}}^{a_s} = {}^{\shortmid}\zeta_{i_{s-1}}^{a_s}(\hbar, x^{i_{s-1}}, p_{a_s})[1+\kappa\ {}^{\shortmid}\chi_{i_{s-1}}^{a_s}(\hbar, x^{i_{s-1}}, p_{a_s})]. \end{aligned}$$

In detailed forms, general quasi-stationary deformations to solutions of type (A.3) determined by gravitational polarizations of type (A.6) and (A.7) are studied in section 2.3 of [13] and appendix A.2 of [41].

Any target s-metric ${}_{s}^{\shortmid}\mathbf{g}$ (A.3) satisfies on shells $s = 2, 3, 4$ certain nonlinear symmetries which allow to re-define the generating functions and relate the effective sources to certain effective shell cosmological constants,

$$\begin{aligned} (\ {}_{s}\Psi,\ {}_{s}^{\shortmid}\mathcal{K}) &\leftrightarrow (\ {}_{s}^{\shortmid}\mathbf{g},\ {}_{s}^{\shortmid}\mathcal{K}) \leftrightarrow (\ {}_{s}^{\shortmid}\eta\ {}^{\shortmid}\mathring{g}_{\alpha_s} \sim {}^{\shortmid}\zeta_{\alpha_s}(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s})\ {}^{\shortmid}\mathring{g}_{\alpha_s},\ {}_{s}^{\shortmid}\mathcal{K}) \leftrightarrow \qquad (A.8)\\ (\ {}_{s}\Phi,\ {}_{s}^{\shortmid}\Lambda_0) &\leftrightarrow (\ {}_{s}^{\shortmid}\mathbf{g},\ {}_{s}^{\shortmid}\Lambda_0) \leftrightarrow (\ {}_{s}^{\shortmid}\eta\ {}^{\shortmid}\mathring{g}_{\alpha_s} \sim {}^{\shortmid}\zeta_{\alpha_s}(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s})\ {}^{\shortmid}\mathring{g}_{\alpha_s},\ {}_{s}^{\shortmid}\Lambda_0). \end{aligned}$$

In explicit form, such nonlinear transforms are defined by equations

$$\frac{\partial_3[(\ {}_2\Psi)^2]}{{}_{2}^{\shortmid}\mathcal{K}} = \frac{\partial_3[(\ {}_2\Phi)^2]}{{}_2\Lambda_0}, \frac{{}^{\shortmid}\partial^5[(\ {}_{3}^{\shortmid}\Psi)^2]}{{}_{3}^{\shortmid}\mathcal{K}} = \frac{{}^{\shortmid}\partial^5[(\ {}_{3}^{\shortmid}\Phi)^2]}{{}_{3}^{\shortmid}\Lambda_0}, \frac{{}^{\shortmid}\partial^8[(\ {}_{4}^{\shortmid}\Psi)^2]}{{}_{4}^{\shortmid}\mathcal{K}} = \frac{{}^{\shortmid}\partial^8[(\ {}_{4}^{\shortmid}\Phi)^2]}{{}_{4}^{\shortmid}\Lambda_0}.$$

In integral forms, we obtain such transforms:

$$\begin{aligned} (\ {}_2\Psi)^2 &= (\ {}_2\Lambda_0)^{-1}\int dy^3(\ {}_{2}^{\shortmid}\mathcal{K})\partial_3[(\ {}_2\Phi)^2] \ \text{ and/or } (\ {}_2\Phi)^2 = \ {}_2\Lambda_0\int dy^3(\ {}_{2}^{\shortmid}\mathcal{K})^{-1}\partial_3[(\ {}_2\Psi)^2],\\ (\ {}_{3}^{\shortmid}\Psi)^2 &= (\ {}_{3}^{\shortmid}\Lambda_0)^{-1}\int d\ p_5(\ {}_{3}^{\shortmid}\mathcal{K})[(\ {}_{3}^{\shortmid}\Phi)^2] \text{ and/or } (\ {}_{3}^{\shortmid}\Phi)^2 = \ {}_{3}^{\shortmid}\Lambda_0\int dp_5(\ {}_{3}^{\shortmid}\mathcal{K})^{-1}[(\ {}_{3}^{\shortmid}\Psi)^2],\\ (\ {}_{4}^{\shortmid}\Psi)^2 &= (\ {}_{4}^{\shortmid}\Lambda_0)^{-1}\int d\ p_7(\ {}_{4}^{\shortmid}\mathcal{K})[(\ {}_{4}^{\shortmid}\Phi)^2] \text{ and/or } (\ {}_{4}^{\shortmid}\Phi)^2 = \ {}_{4}^{\shortmid}\Lambda_0\int dp_7\ (\ {}_{4}^{\shortmid}\mathcal{K})^{-1}[(\ {}_{4}^{\shortmid}\Psi)^2]. \end{aligned}$$

The generating functions/ sources/ cosmological constants and gravitational polarization functions (A.6) and (A.7) can be re-defined for various geometric and analytic purposes when the nonlinear symmetries are re-written in other equivalent forms:

$$\begin{aligned} \partial_3[(\ {}_2\Psi)^2] &= -\int dy^3(\ {}_{2}^{\shortmid}\mathcal{K})\partial_3 g_4 \simeq -\int dy^3(\ {}_{2}^{\shortmid}\mathcal{K})\partial_3(\ {}^{\shortmid}\eta_4\ \ \mathring{g}_4) \simeq -\int dy^3(\ {}_{2}^{\shortmid}\mathcal{K})\partial_3[\ {}^{\shortmid}\zeta_4(1+\kappa\ {}^{\shortmid}\chi_4)\ \mathring{g}_4],\\ (\ {}_2\Phi)^2 &= -4\ {}_2\Lambda_0 g_4 \simeq -4\ {}_2\Lambda_0\ {}^{\shortmid}\eta_4\ \ \mathring{g}_4 \simeq -4\ {}_2\Lambda_0\ {}^{\shortmid}\zeta_4(1+\kappa\ {}^{\shortmid}\chi_4)\ \mathring{g}_4; \end{aligned}$$

$$\begin{aligned} {}^{\shortmid}\partial^5[(\ {}_{3}^{\shortmid}\Psi)^2] &= -\int dp_5(\ {}_{3}^{\shortmid}\mathcal{K})\ {}^{\shortmid}\partial^5\ {}^{\shortmid}g^6 \simeq -\int dp_5(\ {}_{3}^{\shortmid}\mathcal{K})\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\eta^6\ {}^{\shortmid}\mathring{g}^6) \simeq -\int dp_5(\ {}_{3}^{\shortmid}\mathcal{K})\ {}^{\shortmid}\partial^5[\ {}^{\shortmid}\zeta^6(1+\kappa\ {}^{\shortmid}\chi^6)\ \mathring{g}^6],\\ (\ {}_{3}^{\shortmid}\Phi)^2 &= -4\ {}_{3}^{\shortmid}\Lambda_0\ {}^{\shortmid}g^6 \simeq\ -4\ {}_{3}^{\shortmid}\Lambda_0\ {}^{\shortmid}\eta^6\ {}^{\shortmid}\mathring{g}^6 \simeq -4\ {}_{3}^{\shortmid}\Lambda_0\ {}^{\shortmid}\zeta^6(1+\kappa\ {}^{\shortmid}\chi^6)\ {}^{\shortmid}\mathring{g}^6; \end{aligned}$$

$$\begin{aligned}
{}^{\shortmid}\partial^7[({}_4^{\shortmid}\Psi)^2] &= -\int dp_7({}_4^{\shortmid}\mathcal{K})\,{}^{\shortmid}\partial^7\,{}^{\shortmid}g^8 \simeq -\int dp_7({}_4^{\shortmid}\mathcal{K})\,{}^{\shortmid}\partial^7({}^{\shortmid}\eta^8\,{}^{\shortmid}\mathring{g}^8) \simeq -\int dp_7({}_4^{\shortmid}\mathcal{K})\,{}^{\shortmid}\partial^8[{}^{\shortmid}\zeta^8(1+\kappa\,{}^{\shortmid}\chi^8)\,\mathring{g}^8],\\
({}_4^{\shortmid}\Phi)^2 &= -4\,{}_4^{\shortmid}\Lambda_0\,{}^{\shortmid}g^8 \simeq\ -4\,{}_4^{\shortmid}\Lambda_0\,{}^{\shortmid}\eta^8\,{}^{\shortmid}\mathring{g}^8 \simeq -4\,{}_4^{\shortmid}\Lambda_0\,{}^{\shortmid}\zeta^8(1+\kappa\,{}^{\shortmid}\chi^8)\,{}^{\shortmid}\mathring{g}^8.
\end{aligned}$$

The formulas for nonlinear symmetries (A.8) allow to express solutions (A.3) in different equivalent forms and/or for different approximations via corresponding functionals on generating sources, effective cosmological constants, and generating functions:

$$\begin{aligned}
{}_s^{\shortmid}\mathbf{g} &= {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,{}_s\Psi,{}_s^{\shortmid}\mathcal{K}] = {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,{}_s\Phi,\ {}_s^{\shortmid}\Lambda_0] \qquad (A.9)\\
&\simeq {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_s^{\shortmid}\mathcal{K},g_4,\ {}^{\shortmid}g^6,\ {}^{\shortmid}g^8]\\
&\simeq {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_s^{\shortmid}\Lambda_0,g_4,\ {}^{\shortmid}g^6,\ {}^{\shortmid}g^8]\\
&\simeq {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_s^{\shortmid}\mathcal{K},\ {}^{\shortmid}\eta_4\ \ \mathring{g}_4,\ {}^{\shortmid}\eta^6\,{}^{\shortmid}\mathring{g}^6,\ {}^{\shortmid}\eta^8\,{}^{\shortmid}\mathring{g}^8,\mathring{g}_3,\ {}^{\shortmid}\mathring{g}^5,\ {}^{\shortmid}\mathring{g}^7]\\
&\simeq {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_s^{\shortmid}\Lambda_0,\ {}_s^{\shortmid}\mathcal{K},\ {}^{\shortmid}\eta_4\ \ \mathring{g}_4,\ {}^{\shortmid}\eta^6\,{}^{\shortmid}\mathring{g}^6,\ {}^{\shortmid}\eta^8\,{}^{\shortmid}\mathring{g}^8,\mathring{g}_3,\ {}^{\shortmid}\mathring{g}^5,\ {}^{\shortmid}\mathring{g}^7]\\
&\simeq {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_s^{\shortmid}\mathcal{K},\ {}^{\shortmid}\zeta_4,\ {}^{\shortmid}\chi_4,\ \mathring{g}_4;\ \ {}^{\shortmid}\zeta^6,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\mathring{g}^6;\ \ {}^{\shortmid}\zeta^8,\ {}^{\shortmid}\chi^8,\ {}^{\shortmid}\mathring{g}^8,\mathring{g}_3,\ {}^{\shortmid}\mathring{g}^5,\ {}^{\shortmid}\mathring{g}^7]\\
&\simeq {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_s^{\shortmid}\Lambda_0,\ {}_s^{\shortmid}\mathcal{K},\ {}^{\shortmid}\zeta_4,\ {}^{\shortmid}\chi_4,\ \mathring{g}_4;\ \ {}^{\shortmid}\zeta^6,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\mathring{g}^6;\ \ {}^{\shortmid}\zeta^8,\ {}^{\shortmid}\chi^8,\ {}^{\shortmid}\mathring{g}^8,\mathring{g}_3,\ {}^{\shortmid}\mathring{g}^5,\ {}^{\shortmid}\mathring{g}^7].
\end{aligned}$$

We conclude that using functional data we can construct a class of solutions for certain prescribed effective sources ${}_s^{\shortmid}\mathcal{K}$ and generating functions ${}_s\Psi$. To elaborate on solutions in classical and quantum gravity and information theory, it can be useful to consider equivalent (or almost equivalent, with "$\simeq$", for some decompositions on a small parameter) representations of such solutions when they are connected to certain effective cosmological constants ${}_s^{\shortmid}\Lambda_0$ and generating functions ${}_s\Phi$. We can consider configurations when some coefficients of s-metrics are taken as explicit generating functions, for instance, $g_4,\ {}^{\shortmid}g^6,\ {}^{\shortmid}g^8$. In another case, some η-polarizations are prescribed as generating functions, for instance, we can use ${}^{\shortmid}\eta_4,\ {}^{\shortmid}\eta^6,\ {}^{\shortmid}\eta^8$. In [40, 41], nonassociative star product and R-flux deformations of black hole solutions into black ellipsoid configurations were studied using κ-linear s-deformations determined by generating functions ${}^{\shortmid}\chi_4,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\chi^8$.

A.3 Solutions with effective cosmological constants

We can transform the nonassociative vacuum s-metrics (A.3), ${}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,{}_s\Psi,{}_s^{\shortmid}\mathcal{K}] \rightarrow {}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,{}_s\Phi,\ {}_s^{\shortmid}\Lambda_0]$ following the first line in the functional representations (A.9):

$$\begin{aligned}
g_1(x^{k_1}) &= g_2(x^{k_1}) = g_1[\psi] = g_2[\psi] = e^{\psi(\hbar,\kappa;x^{k_1})}, \qquad (A.10)\\
g_3(x^{k_1},y^3) &= g_3[\,{}_2\Phi] = -\frac{1}{g_4[\,{}_2\Phi]}\frac{(\,{}_2\Phi)^2[\partial_3(\,{}_2\Phi)]^2}{|\,{}_2\Lambda_0\int dy^3({}_2^{\shortmid}\mathcal{K})[\partial_3(\,{}_2\Phi)^2]|},\\
g_4(x^{k_1},y^3) &= g_4[\,{}_2\Phi] = g_4^{[0]} - \frac{(\,{}_2\Phi)^2}{4\,{}_2\Lambda_0};\\
{}^{\shortmid}g^5(x^{i_2},p_5) &= {}^{\shortmid}g^5[\,{}_3^{\shortmid}\Phi] = -\frac{1}{{}^{\shortmid}g^6[\,{}_3^{\shortmid}\Phi]}\frac{({}_3^{\shortmid}\Phi)^2[\,{}^{\shortmid}\partial^5({}_3^{\shortmid}\Phi)]^2}{|\,{}_3^{\shortmid}\Lambda_0\int dp_5({}_3^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^5[({}_3^{\shortmid}\Phi)^2]|},\\
{}^{\shortmid}g^6(x^{i_2},p_5) &= {}^{\shortmid}g^6[\,{}_3^{\shortmid}\Phi] = g^6_{[0]} - \frac{({}_3^{\shortmid}\Phi)^2}{4\,{}_3^{\shortmid}\Lambda_0};\\
{}^{\shortmid}g^7(x^{i_2},p_5,p_7) &= {}^{\shortmid}g^7[\,{}_4^{\shortmid}\Phi] = -\frac{1}{{}^{\shortmid}g^8[\,{}_4^{\shortmid}\Phi]}\frac{({}_4^{\shortmid}\Phi)^2[\,{}^{\shortmid}\partial^7({}_4^{\shortmid}\Phi)]^2}{|\,{}_4^{\shortmid}\Lambda_0\int dp_7({}_4^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^7[({}_4^{\shortmid}\Phi)^2]|},\\
{}^{\shortmid}g^8(x^{i_2},p_5,p_8) &= {}^{\shortmid}g^8[\,{}_4^{\shortmid}\Phi] = g^8_{[0]} - \frac{({}_4^{\shortmid}\Phi)^2}{4\,{}_4^{\shortmid}\Lambda_0};
\end{aligned}$$

$$
\begin{aligned}
N_{3i_1}(x^{k_1},y^3) &= w_{i_1}[\,_2\Phi] = \frac{\partial_{i_1}\ \int dy^3(\,_2^{\shortmid}\mathcal{K})\ \partial_3[(\,_2\Phi)^2]}{(\,_2^{\shortmid}\mathcal{K})\ \partial_3[(\,_2\Phi)^2]}, \\
N_{4k_1}(x^{i_1},y^3) &= n_{k_1}[\,_2\Phi] = \,_1n_{k_1} + \,_2n_{k_1}\int dy^3\ \frac{g_3[\,_2\Phi]}{|\ g_4[\,_2\Phi]|^{3/2}} \\
&= \ _1n_{k_1} + \,_2n_{k_1}\int dy^3 \frac{(\,_2\Phi)^2[\,\partial_3(\,_2\Phi)]^2}{|\,_2\Lambda_0\int dy^3(\,_2^{\shortmid}\mathcal{K})[\,\partial_3(\,_2\Phi)^2]|}\left|g_4^{[0]} - \frac{(\,_2\Phi)^2}{4\,_2\Lambda_0}\right|^{-5/2};
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}N_{5k_2}(x^{i_2},p_5) &= w_{k_2}[\,_3^{\shortmid}\Phi] = \frac{\partial_{k_2}\ \int dp_5(\,_3^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^5[(\,_3^{\shortmid}\Phi)^2]}{(\,_3^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^5[(\,_3^{\shortmid}\Phi)^2]}, \\
{}^{\shortmid}N_{6k_2}(x^{i_2},p_5) &= n_{k_2}[\,_3^{\shortmid}\Phi] = \,_1^{\shortmid}n_{k_2} + \,_2^{\shortmid}n_{k_2}\int dp_5\ \frac{{}^{\shortmid}g^5[\,_3^{\shortmid}\Phi]}{|\ {}^{\shortmid}g^6[\,_3^{\shortmid}\Phi]|^{3/2}} \\
&= \ _1^{\shortmid}n_{k_2} + \,_2^{\shortmid}n_{k_2}\int dp_5 \frac{(\,_3^{\shortmid}\Phi)^2[\,{}^{\shortmid}\partial^5(\,_3^{\shortmid}\Phi)]^2}{|\,_3^{\shortmid}\Lambda_0\int dp_5(\,_3^{\shortmid}\mathcal{K})[\,{}^{\shortmid}\partial^5(\,_3^{\shortmid}\Phi)^2]|}\left|\,{}^{\shortmid}g^6_{[0]} - \frac{(\,_3^{\shortmid}\Phi)^2}{4\,_3^{\shortmid}\Lambda_0}\right|^{-5/2};
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}N_{7k_3}(x^{i_2},p_5,p_7) &= w_{k_3}[\,_4^{\shortmid}\Phi] = \frac{\partial_{k_3}\ \int dp_7(\,_4^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^7[(\,_4^{\shortmid}\Phi)^2]}{(\,_4^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^7[(\,_4^{\shortmid}\Phi)^2]}. \\
{}^{\shortmid}N_{8k_3}(x^{i_2},p_5,p_7) &= n_{k_3}[\,_4^{\shortmid}\Phi] = \,_1^{\shortmid}n_{k_3} + \,_2^{\shortmid}n_{k_3}\int dp_7\ \frac{{}^{\shortmid}g^7[\,_4^{\shortmid}\Phi]}{|\ {}^{\shortmid}g^8[\,_4^{\shortmid}\Phi]|^{3/2}} \\
&= \ _1^{\shortmid}n_{k_3} + \,_2^{\shortmid}n_{k_3}\int dp_7 \frac{(\,_4^{\shortmid}\Phi)^2[\,{}^{\shortmid}\partial^7(\,_4^{\shortmid}\Phi)]^2}{|\,_4^{\shortmid}\Lambda_0\int dp_7(\,_4^{\shortmid}\mathcal{K})[\,{}^{\shortmid}\partial^7(\,_4^{\shortmid}\Phi)^2]|}\left|\,{}^{\shortmid}g^8_{[0]} - \frac{(\,_4^{\shortmid}\Phi)^2}{4\,_4^{\shortmid}\Lambda_0}\right|^{-5/2}.
\end{aligned}
$$

In formulas for above coefficients, there are used such conventions:
for indices: $i_1, j_1, k_1, ... = 1,2; i_2, j_2, k_2, ... = 1,2,3,4; i_3, j_3, k_3, ... = 1,2,...6; x^3 = \varphi, y^4 = t,\ p_8 = \ E$; and

$$
\begin{aligned}
&\text{generating functions: } \psi(\hbar,\kappa,x^{k_1});\ \,_2\Phi(\hbar,\kappa,x^{k_1}y^3);\ \,_3^{\shortmid}\Phi(\hbar,\kappa,x^{k_2},p_6);\ \,_4^{\shortmid}\Phi(\hbar,\kappa,\ {}^{\shortmid}x^{k_3},p_7); \\
&\text{generating sources: } \,_1^{\shortmid}\mathcal{K}(\hbar,\kappa,x^{k_1});\ \,_2^{\shortmid}\mathcal{K}(\hbar,\kappa,x^{k_1},y^3);\ \,_3^{\shortmid}\mathcal{K}(\hbar,\kappa,x^{k_2},p_5);\ \,_4^{\shortmid}\mathcal{K}(\hbar,\kappa,\ {}^{\shortmid}x^{k_3},p_7); \\
&\text{integration functions: } g_4^{[0]}(\hbar,\kappa,x^{k_1}),\ \,_1n_{k_1}(\hbar,\kappa,x^{j_1}),\ \,_2n_{k_1}(\hbar,\kappa,x^{j_1}); \qquad\qquad (A.11) \\
&{}^{\shortmid}g^6_{[0]}(\hbar,\kappa,x^{k_2}),\ \,_1n_{k_2}(\hbar,\kappa,x^{j_2}),\ \,_2n_{k_2}(\hbar,\kappa,x^{j_2});\ {}^{\shortmid}g^8_{[0]}(\hbar,\kappa,\ {}^{\shortmid}x^{j_3}),\ \,_1^{\shortmid}n_{k_3}(\hbar,\kappa,\ {}^{\shortmid}x^{j_3}),\ \,_2^{\shortmid}n_{k_3}(\hbar,\kappa,\ {}^{\shortmid}x^{j_3}).
\end{aligned}
$$

Such functional representations of off-diagonal solutions allow to encode possible contributions from effective cosmological constants when certain dynamics of effective sources is re-distributed into off-diagonal terms of s-metrics. Nevertheless, the contributions from $\,_s^{\shortmid}\mathcal{K}$ are not completely excluded being present in integrals for certain s-connection coefficients like $g_3,\ {}^{\shortmid}g^5,\ {}^{\shortmid}g^7$ and all N-connection coefficients.

Changing the generating functions and generating sources/ cosmological constants data, $[\,_s\Psi, \,_s^{\shortmid}\mathcal{K}] \rightarrow [\,_s\Phi,\ \,_s^{\shortmid}\Lambda_0]$, we re-express the data for quasi-stationary solutions of the nonassociative parametric vacuum gravitational equations (16) with effective sources (35), defined by (A.3) as solutions for ${}^{\shortmid}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}[\,_s\Psi, \,_s^{\shortmid}\mathcal{K},...] = \delta^{\beta_s}{}_{\gamma_s}\ \,_s^{\shortmid}\mathcal{K}$ (3), into solutions of type (A.10) of ${}^{\shortmid}\widehat{\mathbf{R}}^{\beta_s}{}_{\gamma_s}[\,_s\Phi,\ \,_s^{\shortmid}\Lambda_0, \,_s^{\shortmid}\mathcal{K},...] = \delta^{\beta_s}{}_{\gamma_s}\ \ \,_s^{\shortmid}\Lambda_0$ (8). The functional structure of geometric objects is subjected to certain (A.8) transforms when the data for effective sources $\,_s^{\shortmid}\mathcal{K}$ are kept into off-diagonal N-connection terms but the left side of modified Einstein equations is stated with effective cosmological constants $\,_s^{\shortmid}\Lambda_0$.

A.4 Using some coefficients of s-metrics as generating functions

We can consider

$$
\begin{aligned}
g_4(x^{k_1},y^3) &= g_4[\,_2\Psi,\ \,_2^{\shortmid}\mathcal{K}] = g_4[\,_2\Phi,\ \,_2\Lambda_0];\ {}^{\shortmid}g^6(x^{i_2},p_5) = {}^{\shortmid}g^6[\,_3\Psi,\ \,_3^{\shortmid}\mathcal{K}] = {}^{\shortmid}g^6[\,_3^{\shortmid}\Phi,\ \,_3\Lambda_0]; \\
{}^{\shortmid}g^8(x^{i_2},p_5,p_7) &= \ {}^{\shortmid}g^8[\,_4\Psi,\ \,_4^{\shortmid}\mathcal{K}] = {}^{\shortmid}g^8[\,_4^{\shortmid}\Phi,\ \,_4\Lambda_0],
\end{aligned}
$$

from (A.3) and (A.10) as generating functions for a s-metric (A.1) and N-coefficients (A.2). In the first case, expressing ${}_s\Psi = {}_s\Psi[{}_s^{\shortmid}\mathcal{K}, g_4, {}^{\shortmid}g^6, {}^{\shortmid}g^8]$, we obtain such parameterizations of quasi-stationary solutions:

$$
\begin{aligned}
& g_1(x^{k_1}) = g_2(x^{k_1}) = g_1[\psi] = g_2[\psi] = e^{\psi(\hbar,\kappa;x^{k_1})}, \qquad (A.12)\\
& g_3(x^{k_1},y^3) = -\frac{(\partial_3 g_4)^2}{|\int dy^3 \partial_3[({}_2^{\shortmid}\mathcal{K})g_4]|\ g_4}, g_4(x^{k_1},y^3) \text{ is a generating function on shell } s=2;
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}g^5(x^{i_2},p_5) &= -\frac{[{}^{\shortmid}\partial^5({}^{\shortmid}g^6)]^2}{|\int dp_5\ {}^{\shortmid}\partial^5[({}_3^{\shortmid}\mathcal{K})\ {}^{\shortmid}g^6]\ |\ {}^{\shortmid}g^6}, \ {}^{\shortmid}g^6(x^{i_2},p_5) \text{ is a generating function on shell } s=3,\\
{}^{\shortmid}g^7(x^{i_2},p_5,p_7) &= -\frac{[{}^{\shortmid}\partial^7({}^{\shortmid}g^8)]^2}{|\int dp_7\ \ {}^{\shortmid}\partial^7[({}_4^{\shortmid}\mathcal{K})\ {}^{\shortmid}g^8]\ |\ \ {}^{\shortmid}g^8}, \ {}^{\shortmid}g^8(x^{i_2},p_5,p_7) \text{ is a generating function on shell } s=4,;
\end{aligned}
$$

$$
\begin{aligned}
N_{3i_1}(x^{k_1},y^3) &= w_{i_1}[g_4] = \frac{\partial_{i_1}[\int dy^3({}_2^{\shortmid}\mathcal{K})\ \partial_3 g_4]}{({}_2^{\shortmid}\mathcal{K})\ \partial_3 g_4},\\
N_{4k_1}(x^{i_1},y^3) &= n_{k_1}[\ g_4] = {}_1n_{k_1} + {}_2n_{k_1}\int dy^3 \frac{(\partial_3 g_4)^2}{|\int dy^3\partial_3[({}_2^{\shortmid}\mathcal{K})g_4]|\ [g_4]^{5/2}}\ ;
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}N_{5k_2}(x^{i_2},p_5) &= w_{k_2}[{}^{\shortmid}g^6] = \frac{\partial_{k_2}[\int dp_5({}_3^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^5({}^{\shortmid}g^6)]}{({}_3^{\shortmid}\mathcal{K})\ {}^{\shortmid}\partial^5({}^{\shortmid}g^6)},\\
{}^{\shortmid}N_{6k_2}(x^{i_2},p_5) &= n_{k_2}[{}^{\shortmid}g^6] = {}_1^{\shortmid}n_{k_2} + {}_2^{\shortmid}n_{k_2}\int dp_5 \frac{[{}^{\shortmid}\partial^5({}^{\shortmid}g^6)]^2}{|\int dp_5\ \ {}^{\shortmid}\partial^5[({}_3^{\shortmid}\mathcal{K})\ {}^{\shortmid}g^6]|\ [{}^{\shortmid}g^6]^{5/2}};
\end{aligned}
$$

$$
\begin{aligned}
{}^{\shortmid}N_{7k_3}(x^{i_2},p_5,p_7) &= w_{k_3}[{}^{\shortmid}g^8] = \frac{{}^{\shortmid}\partial_{k_3}[\int dp_7(\ {}_4^{\shortmid}\mathcal{K})\ \ {}^{\shortmid}\partial^7({}^{\shortmid}g^8)]}{({}_4^{\shortmid}\mathcal{K})\ {}^{\shortmid}\partial^7({}^{\shortmid}g^8)},\\
{}^{\shortmid}N_{7k_3}(x^{i_2},p_5,p_7) &= n_{k_3}[{}^{\shortmid}g^8] = {}_1^{\shortmid}n_{k_3} + {}_2^{\shortmid}n_{k_3}\int dp_7 \frac{[{}^{\shortmid}\partial^7({}^{\shortmid}g^8)]^2}{|\int dp_7\ \ {}^{\shortmid}\partial^7[({}_4^{\shortmid}\mathcal{K})\ {}^{\shortmid}g^8]|\ [{}^{\shortmid}g^8]^{5/2}}\ ,
\end{aligned}
$$

The s-coefficients (A.12) define quasi-stationary solutions of type ${}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi, {}_s^{\shortmid}\mathcal{K}, g_4, {}^{\shortmid}g^6, {}^{\shortmid}g^8]$ from (A.9).

Above coefficients can be re-defined to include functional dependencies on effective cosmological constants ${}_s^{\shortmid}\Lambda$ if we begin with (A.10) and express ${}_s\Phi = {}_s\Phi[{}_s^{\shortmid}\Lambda_0, {}_s^{\shortmid}\mathcal{K}, g_4, {}^{\shortmid}g^6, {}^{\shortmid}g^8]$. This way we generate a solution of type ${}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi, {}_s^{\shortmid}\Lambda_0, {}_s^{\shortmid}\mathcal{K}, g_4, {}^{\shortmid}g^6, {}^{\shortmid}g^8]$ from (A.9).

A.5 Parametric quasi-stationary gravitational polarizations and κ-linear flows

To model off-diagonal deformations of a prescribed prime metric into target ones, ${}_s^{\shortmid}\mathbf{\mathring{g}} = [{}^{\shortmid}\mathring{g}_{\alpha_s}, {}^{\shortmid}\mathring{N}^{a_s}_{i_{s-1}}] \rightarrow {}_s^{\shortmid}\mathbf{g}$ (A.5) by η-polarizations (A.6), we can consider as generating functions such values:

$$\psi(\hbar,\kappa;x^{k_1}),\ {}^{\shortmid}\eta_4(x^{k_1},y^3),\ {}^{\shortmid}\eta^6(x^{i_2},p_5),\ {}^{\shortmid}\eta^8(x^{i_2},p_5,p_7).$$

This allows us to compute all polarization functions ${}^{\shortmid}\eta_{\alpha_s}(\hbar,\kappa,x^{i_{s-1}},p_{a_s})$ and ${}^{\shortmid}\eta^{a_s}_{i_{s-1}}(\hbar,\kappa,x^{i_{s-1}},p_{a_s})$ following the standard AFCDM. The explicit form of η-polarizations depend, for instance, on the type of target quasi-stationary solutions we search. If the target is of type (A.3), we generate s-metrics with functional dependence ${}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi, {}_s^{\shortmid}\mathcal{K}, {}^{\shortmid}\eta_4\ \mathring{g}_4, {}^{\shortmid}\eta^6\ {}^{\shortmid}\mathring{g}^6, {}^{\shortmid}\eta^8\ {}^{\shortmid}\mathring{g}^8, \mathring{g}_3, {}^{\shortmid}\mathring{g}^5, {}^{\shortmid}\mathring{g}^7]$ stated by classifications (A.9). In explicit form, with complex variables, such a s-metric was constructed in appendix B.2 to [13], see formulas (B.4) in that work. We can introduce effective cosmological constants ${}_s^{\shortmid}\Lambda_0$ and generate s-metrics with quasi-stationary dependence ${}_s^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi, {}_s^{\shortmid}\Lambda_0, {}_s^{\shortmid}\mathcal{K}, {}^{\shortmid}\eta_4\ \mathring{g}_4, {}^{\shortmid}\eta^6\ {}^{\shortmid}\mathring{g}^6, {}^{\shortmid}\eta^8\ {}^{\shortmid}\mathring{g}^8, \mathring{g}_3, {}^{\shortmid}\mathring{g}^5, {}^{\shortmid}\mathring{g}^7]$ if such a target metric is of type (A.10). More general classes of solutons describing nonassociative geometric flows of quasi-stationary metrics with η-polarizations, effective sources ${}_{s}^{\shortmid}\Im(\tau)$ (52) and running cosmological constants ${}_{s}^{\shortmid}\Lambda(\tau)$ are studied in section 4.5.

In [13, 40, 41], we constructed quasi-stationary 4-d and 8-d solutions of nonassociative vacuum gravitational equations when the prime metrics are certain black hole, BH, metrics and the target s-metrics for black ellipsoid, BE, s-metrics are generated as κ-linear s-deformations determined by generating functions ${}^{\shortmid}\chi_4,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\chi^8$, see (A.7). The coefficients of such s-metrics, with functional dependence of type ${}_{s}^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_{s}^{\shortmid}\mathcal{K},\ {}^{\shortmid}\zeta_4,\ {}^{\shortmid}\chi_4,\ \mathring{g}_4;\ \ {}^{\shortmid}\zeta^6,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\mathring{g}^6;\ \ {}^{\shortmid}\zeta^8,\ {}^{\shortmid}\chi^8,\ {}^{\shortmid}\mathring{g}^8,\mathring{g}_3,\ {}^{\shortmid}\mathring{g}^5,\ {}^{\shortmid}\mathring{g}^7]$ from (A.9, were computed in a general form involving complex coordinates in appendix B.3, formulas (B.7) of [13]. This way, we generate a class of κ-parametric target quasi-stationary s-metrics of type (A.3). In a similar form, we can construct κ-parametric target quasi-stationary s-metrics of type (A.10) with dependencies on ${}_{s}^{\shortmid}\Lambda_0$. Such generic off-diagonal configurations are described by functionals of type ${}_{s}^{\shortmid}\mathbf{g}[\hbar,\kappa,\psi,\ {}_{s}^{\shortmid}\Lambda_0,\ {}_{s}^{\shortmid}\mathcal{K},\ {}^{\shortmid}\zeta_4,\ {}^{\shortmid}\chi_4,\ \mathring{g}_4;\ \ {}^{\shortmid}\zeta^6,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\mathring{g}^6;\ \ {}^{\shortmid}\zeta^8,\ {}^{\shortmid}\chi^8,\ {}^{\shortmid}\mathring{g}^8,\mathring{g}_3,\ {}^{\shortmid}\mathring{g}^5,\ {}^{\shortmid}\mathring{g}^7]$ from (A.9). We note also that nonassociative BE target solutions are generated for special rotoid polarizations of ${}^{\shortmid}\chi_4,\ {}^{\shortmid}\chi^6,\ {}^{\shortmid}\chi^8$ as we studied in [40, 41]. The goal of this appendix is to generalize such solutions in real variables for nonassociative geometric flow with small parametric deformations depending on τ-parameter.

In a τ-running family of quasi-stationary s-metrics of type (68), the η-polarizations are expressed as κ-linear functions when s-metric and N-connection coefficients of families of prime s-metrics are transformed into respective families of target ones,

$$ {}_{s}^{\shortmid}\mathring{\mathbf{g}}(\tau)\rightarrow\ {}_{s}^{\shortmid\kappa}\mathbf{g}(\tau)=[\ {}^{\shortmid}g_{\alpha_s}(\tau)=\ {}^{\shortmid}\zeta_{\alpha_s}(\tau)(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s}(\tau))\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau),\ {}^{\shortmid}N_{i_s}^{a_s}(\tau)=\ {}^{\shortmid}\zeta_{i_{s-1}}^{a_s}(\tau)(1+\kappa\ {}^{\shortmid}\chi_{i_{s-1}}^{a_s}(\tau))\ {}^{\shortmid}\mathring{N}_{i_{s-1}}^{a_s}(\tau)]. $$

The ζ- and χ-coefficients for deformations the η-polarization generating functions (67) are respectively κ-linearized as data

$$ \begin{array}{rcl} \psi(\tau) & \simeq & \psi(\hbar,\kappa;\tau,x^{k_1})\simeq\psi_0(\hbar,\tau,x^{k_1})(1+\kappa\ {}_{\psi}\chi(\hbar,\tau,x^{k_1})),\ \text{for} \\ \eta_2(\tau) & \simeq & \eta_2(\hbar,\kappa;\tau,x^{k_1})\simeq\zeta_2(\hbar,\tau,x^{k_1})(1+\kappa\chi_2(\hbar,\tau,x^{k_1})),\ \text{we can consider}\ \ \eta_2(\tau)=\ \eta_1(\tau); \\ {}^{\shortmid}\eta_4(\tau) & \simeq & {}^{\shortmid}\eta_4(\hbar,\kappa;\tau,x^{k_1},y^3)\simeq\ {}^{\shortmid}\zeta_4(\hbar,\tau,x^{k_1},y^3)(1+\kappa\ {}^{\shortmid}\chi_4(\hbar,\tau,x^{k_1},y^3)), \\ {}^{\shortmid}\eta^6(\tau) & \simeq & {}^{\shortmid}\eta^6(\hbar,\kappa;\tau,x^{i_2},p_5)\simeq\ {}^{\shortmid}\zeta^6(\hbar,\kappa;\tau,x^{i_2},p_5)(1+\kappa\ {}^{\shortmid}\chi^6(\hbar,\kappa;\tau,x^{i_2},p_5)), \\ {}^{\shortmid}\eta^8(\tau) & \simeq & {}^{\shortmid}\eta^8(\hbar,\kappa;\tau,x^{i_2},p_5,p_7)\simeq\ {}^{\shortmid}\zeta^8(\hbar,\kappa;\tau,x^{i_2},p_5,p_7)(1+\kappa\ {}^{\shortmid}\chi^8(\hbar,\kappa;\tau,x^{i_2},p_5,p_7)). \end{array} \quad \text{(A.13)} $$

In above formulas, $\psi(\tau)$ and $\eta_2(\tau)=\ \eta_1(\tau)$ are related to define a τ-family of solutions 2-d Poisson equation $\partial_{11}^2\psi(\tau)+\partial_{22}^2\psi(\tau)=2\ {}_1\Im(\tau)$.

For parameterizations (A.13), we can re-write the geometric evolution of quasi-stationary s-metrics in a small κ-parametric form with χ-generating functions (for simplicity, we do not write in this formula the phase space coordinate and parametric τ-dependence of coefficients):

$$ \begin{array}{rcl} d\ {}^{\shortmid}\widehat{s}^2(\tau) & = & {}^{\shortmid}\widehat{g}_{\alpha_s\beta_s}(\hbar,\kappa;\tau,x^k,y^3,p_{a_3},p_{a_4};g_4(\tau),\ {}^{\shortmid}g^6(\tau),\ {}^{\shortmid}g^8(\tau),\ {}_{s}^{\shortmid}\Im(\tau))d\ {}^{\shortmid}u^{\alpha_s}d\ {}^{\shortmid}u^{\beta_s} \\ & = & e^{\psi_0}(1+\kappa\ {}^{\psi}\ {}^{\shortmid}\chi)[(dx^1)^2+(dx^2)^2] \end{array} $$

$$ \begin{array}{c} -\{\frac{4[\partial_3(|\zeta_4\mathring{g}_4|^{1/2})]^2}{\mathring{g}_3|\int dy^3\{\ {}_2\Im\partial_3(\zeta_4\mathring{g}_4)\}|}-\kappa[\frac{\partial_3(\chi_4|\zeta_4\mathring{g}_4|^{1/2})}{4\partial_3(|\zeta_4\mathring{g}_4|^{1/2})}-\frac{\int dy^3\{\ {}_2\Im\partial_3[(\zeta_4\mathring{g}_4)\chi_4]\}}{\int dy^3\{\ {}_2\Im\partial_3(\zeta_4\mathring{g}_4)\}}]\}\mathring{g}_3 \\ \{dy^3+[\frac{\partial_{i_1}\ \int dy^3\ {}_2\Im\ \partial_3\zeta_4}{(\mathring{N}_{i_1}^3)\ {}_2\Im\partial_3\zeta_4}+\kappa(\frac{\partial_{i_1}[\int dy^3\ {}_2\Im\partial_3(\zeta_4\chi_4)]}{\partial_{i_1}\ [\int dy^3\ {}_2\Im\partial_3\zeta_4]}-\frac{\partial_3(\zeta_4\chi_4)}{\partial_3\zeta_4})]\mathring{N}_{i_1}^3dx^{i_1}\}^2 \end{array} $$

$$+\zeta_4(1+\kappa\ \chi_4)\ \mathring{g}_4\{dt+[(\mathring{N}^4_{k_1})^{-1}[\ {}_1n_{k_1}+16\ {}_2n_{k_1}[\int dy^3\frac{\left(\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}]\right)^2}{|\int dy^3\partial_3[\ {}_2\Im(\zeta_4\mathring{g}_4)]|}] \tag{A.14}$$

$$+\kappa\frac{16\ {}_2n_{k_1}\int dy^3\frac{\left(\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}]\right)^2}{|\int dy^3\partial_3[\ {}_2\Im(\zeta_4\mathring{g}_4)]|}(\frac{\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}\chi_4)]}{2\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}]}+\frac{\int dy^3\partial_3[\ {}_2\Im(\zeta_4\chi_4\mathring{g}_4)]}{\int dy^3\partial_3[\ {}_2\Im(\zeta_4\mathring{g}_4)]})}{{}_1n_{k_1}+16\ {}_2n_{k_1}[\int dy^3\frac{\left(\partial_3[(\zeta_4\mathring{g}_4)^{-1/4}]\right)^2}{|\int dy^3\partial_3[\ {}_2\Im(\zeta_4\mathring{g}_4)]|}]}]\mathring{N}^4_{k_1}dx^{\acute{k}_1}\}^2+$$

$$-\{\frac{4[\ {}^{\shortmid}\partial^5(|\ {}^{\shortmid}\zeta^6\ \ {}^{\shortmid}\mathring{g}^6|^{1/2})]^2}{{}^{\shortmid}\mathring{g}^5|\int dp_5\{\ {}^{\shortmid}_3\Im\ \ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)]\}|}-\kappa[\frac{{}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)}{{}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^5)}-\frac{\partial_{i_2}[\int dp_5\ {}^{\shortmid}_3\Im\ \ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6\ \ {}^{\shortmid}\mathring{g}^6)]}{\partial_{i_2}\ [\int dp_5\ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6)]}]\}\ {}^{\shortmid}\mathring{g}^5$$

$$\{dp_5+[\frac{\partial_{i_2}\ \int dp_5\ {}^{\shortmid}_3\Im\ \ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6)}{(\ {}^{\shortmid}\mathring{N}_{i_2 5})\ {}^{\shortmid}_3\Im\ \ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6)}+\kappa(\frac{\partial_{i_2}[\int dp_5\ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)]}{\partial_{i_2}\ [\int dp_5\ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6)]}-\frac{{}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)}{{}^{\shortmid}\partial^5(\ {}^{\shortmid}\zeta^5)})](\ {}^{\shortmid}\mathring{N}_{i_2 5})dx^{i_2}\}$$

$$+\ {}^{\shortmid}\zeta^6(1+\kappa\ {}^{\shortmid}\chi^6)\ \ {}^{\shortmid}\mathring{g}^6\{dp_5+[\ (\ {}^{\shortmid}\mathring{N}_{i_2 6})^{-1}[\ {}^{\shortmid}_1n_{i_2}+16\ {}^{\shortmid}_2n_{i_2}[\int dp_5\{\frac{\left(\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)^{-1/4}]\right)^2}{|\int dp_5\ \ {}^{\shortmid}\partial^5[\ {}^{\shortmid}_3\Im(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)]|}]+$$

$$+\kappa\frac{16\ {}^{\shortmid}_2n_{i_2}\int dp_5\frac{\left(\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)^{-1/4}]\right)^2}{|\int dp_5\ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)]|}(\frac{{}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)^{-1/4}\ {}^{\shortmid}\chi^6)]}{2\ \ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)^{-1/4}]}+\frac{\int dp_5\ \ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)\ {}^{\shortmid}\chi^6]}{\int dp_5\ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)]})}{{}^{\shortmid}_1n_{i_2}+16\ {}^{\shortmid}_2n_{i_2}[\int dp_5\frac{\left(\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)^{-1/4}]\right)^2}{|\int dp_5\ \ {}^{\shortmid}_3\Im\ {}^{\shortmid}\partial^5[(\ {}^{\shortmid}\zeta^6\ {}^{\shortmid}\mathring{g}^6)]|}]}]](\ {}^{\shortmid}\mathring{N}_{i_2 6})dx^{i_2}\}^2$$

$$-\{\frac{4[\ {}^{\shortmid}\partial^7(|\ {}^{\shortmid}\zeta^8\ \ {}^{\shortmid}\mathring{g}^8|^{1/2})]^2}{{}^{\shortmid}\mathring{g}^7|\int dp_7\{\ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8\ \ {}^{\shortmid}\mathring{g}^8)\}|}-\kappa[\frac{{}^{\shortmid}\partial^7(\ {}^{\shortmid}\chi^8|\ {}^{\shortmid}\zeta^8\ \ {}^{\shortmid}\mathring{g}^8|^{1/2})}{4\ {}^{\shortmid}\partial^7(|\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8|^{1/2})}-\frac{\int dp_7\{\ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)\ {}^{\shortmid}\chi^8]\}}{\int dp_7\{\ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)]\}}]\}\ {}^{\shortmid}\mathring{g}^7$$

$$\{dp_7+[\frac{{}^{\shortmid}\partial_{i_3}\ \int dp_7\ {}^{\shortmid}_4\Im\ \ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8)}{(\ {}^{\shortmid}\mathring{N}_{i_3 7})\ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8)}+\kappa[\frac{{}^{\shortmid}\partial_{i_3}[\int dp_7\ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)]}{{}^{\shortmid}\partial_{i_3}\ [\int dp_7\ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8)]}-\frac{{}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)}{{}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8)}]](\ {}^{\shortmid}\mathring{N}_{i_3 7})d\ {}^{\shortmid}x^{i_3}\}^2$$

$$+\ {}^{\shortmid}\zeta^8(1+\kappa\ {}^{\shortmid}\chi^8)\ \ {}^{\shortmid}\mathring{g}^8\{dp_7+[(\ {}^{\shortmid}\mathring{N}_{i_3 8})^{-1}[\ {}^{\shortmid}_1n_{i_3}+16\ {}^{\shortmid}_2n_{i_3}[\int dp_7\{\frac{\left(\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)^{-1/4}]\right)^2}{|\int dp_7\ {}^{\shortmid}_4\Im\ \ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)|}]$$

$$+\kappa\frac{16\ {}^{\shortmid}_2n_{i_3}\int dp_7\frac{\left(\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)^{-1/4}]\right)^2}{|\int dp_7\ \ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)]|}(\frac{{}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)^{-1/4}\ \ {}^{\shortmid}\chi^8)]}{2\ \ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)^{-1/4}]}+\frac{\int dp_7\ \ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7[(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)\ {}^{\shortmid}\chi^8]}{\int dp_7\ \ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)})}{{}^{\shortmid}_1n_{i_3}+16\ {}^{\shortmid}_2n_{i_3}[\int dp_7\frac{\left(\ {}^{\shortmid}\partial^7\ [(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)^{-1/4}]\right)^2}{|\int dp_7\ \ {}^{\shortmid}_4\Im\ {}^{\shortmid}\partial^7(\ {}^{\shortmid}\zeta^8\ {}^{\shortmid}\mathring{g}^8)|}]}]]\ (\ {}^{\shortmid}\mathring{N}_{i_3 8})dx^{i_3}\}^2.$$

Quasi-stationary solutions of type (A.14) can be constructed for κ-parametric decompositions of τ-running gravitational polarization functions beginning with quadratic linear elements (63) with nonholonomic frames (64) and respective nonlinear transforms (60) and (61), when

$$\begin{aligned}[\ {}^{\shortmid}_s\mathbf{g}(\tau),\ {}_s\Psi(\tau),\ \ {}^{\shortmid}_s\Im(\tau)] &\leftrightarrow [\ {}^{\shortmid}_s\mathbf{g}(\tau),\ \ {}_s\Phi(\tau),\ \ {}^{\shortmid}_s\Im(\tau),\ {}^{\shortmid}_s\Lambda(\tau)]\leftrightarrow[\ {}^{\shortmid}_s\mathbf{g}(\tau),\ \ {}^{\shortmid}_s\eta(\tau)\ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau),\ \ {}^{\shortmid}_s\Im(\tau),\ {}^{\shortmid}_s\Lambda(\tau)]\\ &\leftrightarrow [\ {}^{\shortmid}_s\mathbf{g}(\tau),\ {}^{\shortmid}\zeta_{\alpha_s}(\tau)(1+\kappa\ {}^{\shortmid}\chi_{\alpha_s}(\tau))\ \ {}^{\shortmid}\mathring{g}_{\alpha_s}(\tau),\ \ {}^{\shortmid}_s\Im(\tau),\ {}^{\shortmid}_s\Lambda(\tau)].\end{aligned}$$

We omit details on such technical constructions but present examples in next appendix section and some applications, for instance, for double BE and BHs phase space flow solutions in section (5.2).

If we fix $\tau=\tau_0$ for self-similar configurations of s-metrics (A.14), we generate κ-parametric solutions for nonassociative Ricci solitons (47). Such solutions were constructed for nonassociative vacuum Einstein equations with effective sources ${}^{\shortparallel}_s\mathcal{K}$ and complex variables in [13] (see appendix B.3 with formulas (B.7) in that work).

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